

Generalized second law of thermodynamic in modified teleparallel theory

M. Zubair^{1,a}, Sebastian Bahamonde^{2,b}, Mubasher Jamil^{3,c}

¹ Department of Mathematics, COMSATS Institute of Information Technology Lahore, Lahore, Pakistan

² Department of Mathematics, University College London, Gower Street, London WC1E 6BT, UK

³ Department of Mathematics, School of Natural Sciences (SNS), National University of Sciences and Technology (NUST), H-12, Islamabad, Pakistan

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Abstract This study is conducted to examine the validity of the generalized second law of thermodynamics (GSLT) in flat FRW for modified teleparallel gravity involving coupling between a scalar field with the torsion scalar T and the boundary term $B = 2\nabla_\mu T^\mu$. This theory is very useful, since it can reproduce other important well-known scalar field theories in suitable limits. The validity of the first and second law of thermodynamics at the apparent horizon is discussed for any coupling. As examples, we have also explored the validity of those thermodynamics laws in some new cosmological solutions under the theory. Additionally, we have also considered the logarithmic entropy corrected relation and discuss the GSLT at the apparent horizon.

1 Introduction

The rapid growth of observational measurements on expansion history reveals the expanding paradigm of the universe. This fact is based on accumulative observational evidence mainly from Type Ia supernova and other renowned sources [1–3]. The expanding phase implicates the presence of a repulsive force which compensates the attractiveness property of gravity on cosmological scales. This phenomenon may be translated as the existence of exotic matter components and most acceptable understanding for such enigma is termed dark energy (DE); it has a large negative pressure. Various DE models and modified theories of gravity have been proposed to incorporate the role of DE in cosmic expansion history (for reviews see [4–6]).

In contrast to Einstein's relativity and its proposed modifications where the source of gravity is determined by curvature scalar terms, another formulation has been presented which comprises a torsional formulation as gravity source [7–11]. This theory is labeled TEGR (teleparallel equivalent of general relativity) and it is determined by a Lagrangian density involving a zero curvature Weitzenböck connection instead of a zero torsion Levi-Civita connection with the vierbein as a fundamental tool. The Weitzenböck connection is a specific connection which characterizes a globally flat space-time endowed with a non-zero torsion tensor. Using that connection, one can construct an alternative and equivalent theory of GR. The latter appears, since the scalar torsion only differs by a boundary term $B = \frac{2}{e} \partial_\mu (e T^\mu)$ with the scalar curvature by the relationship $R = -T + B$, making the two variations of the Einstein–Hilbert and TEGR actions the same. Thus, these two theories have the same field equations. However, these two theories have different geometrical interpretations, since in TEGR the torsion acts as a force; meanwhile in GR, the gravitational effects are understood due to the curved space-time. TEGR is then further extended to a generalized form by the inclusion of a $f(T)$ function in the Lagrangian density (as $f(R)$ is the extension of GR) and it has been tested cosmologically by numerous researchers [12–14]. It is important to mention that $f(T)$ and $f(R)$ are no longer equivalent theories, and in order to consider the equivalent teleparallel theory of $f(R)$, one needs to consider a more generalized function $f(T, B)$, incorporating the boundary term in the action [15]. In [16], the authors studied some cosmological features (reconstruction method, thermodynamics and stability) within $f(T, B)$ gravity and, in [17], some cosmological solutions were found using the Noether symmetry approach. Additionally, it has been proved that when one considers Gauss–Bonnet higher order terms, an additional boundary term B_G (related to the contorsion ten-

^a e-mails: mzubairkk@gmail.com; drmzubair@ciitlahore.edu.pk

^b e-mail: sebastian.beltran.14@ucl.ac.uk

^c e-mail: mjamil@sns.nust.edu.pk

sor) needs to be introduced to find the equivalent teleparallel modified Gauss–Bonnet $f(R, G)$ theory [18].

Later, Harko et al. [19] proposed a comprehensive form of this theory by involving a non-minimal torsion matter interaction in the Lagrangian density. In Ref. [20], Zubair and Waheed recently have investigated the validity of energy constraints for some specific $f(T)$ models and discussed the feasible bounds of involved arbitrary parameters. They also discussed the validity of the generalized second law of thermodynamic in the cosmological constant regime [21].

Another very much studied approach in modified theories of gravity is to change the matter content of the universe by adding a scalar field in the matter sector. These models have been considered several times in cosmology, using different kinds of scalar fields such as quintessence, quintom, k-essence, etc. (see [22–24]). Moreover, we can also extend that idea by adding a coupling between the scalar field and the gravitational sector (see [25–27]) where, cosmologically speaking, we can have new interesting results such as the possibility of crossing the phantom barrier. Motivated by these theories, other interesting modified theories of gravity have also been discussed in the literature [28–30] on the cosmological landscape. Recently, Bahamonde and Wright [31] presented a new model of teleparallel gravity by introducing a scalar field non-minimally coupled to both the torsion T and the boundary term $B = 2\nabla_\mu T^\mu$. It is shown that such a theory can describe the non-minimal coupling to torsion and also the non-minimal coupling to scalar curvature under certain limits.

Black hole thermodynamics suggests that there is a fundamental connection between gravitation and thermodynamics [32]. Hawking radiation [33], the proportionality relation between the temperature and surface gravity, and also the connection between horizon entropy and the area of a black hole [34] support this idea. Jacobson [35] was the first to deduce the Einstein field equations from the Clausius relation, $T_h d\hat{S}_h = \delta\hat{Q}$, together with the fact that the entropy is proportional to the horizon area. In the case of a general spherically symmetric space-time, it was shown that the field equations can be constituted as the first law of thermodynamics (FLT) [36]. The relation between the FRW equations and the FLT was shown in [37] for $T_h = 1/2\pi R_A$, $S_h = \pi R_A^2/G$. The investigation of the validity of thermodynamical laws in GR as well as modified theories has been carried out by numerous researchers in the literature [38–53].

In this work we focus on the validity of the thermodynamical laws in a modified teleparallel gravity involving a non-minimal coupling between both torsion scalar and the boundary term with a scalar field.

In Sect. 1, we give a brief introduction of this theory and then we derive the respective field equation for flat FRW geometry with a perfect fluid as the matter contents. In Sects. 2 and 3, we formulate the first and second law of thermody-

namics at the apparent horizon for any coupling. Additionally, as examples, we study the validity of the thermodynamics laws using new cosmological solutions based on power-law and exponential-law cosmology (see Appendix A). In Sect. 4, we discuss the validity of GSLT for the entropy functional with quantum corrections. In each case, suitable limits of the parameters are chosen in order to visualize the validity of GSLT in quintessence, scalar field non-minimally coupled to torsion (known as teleparallel dark energy) and non-minimally coupled to the scalar curvature theories. Finally, in Sect. 5, a discussion of the work is presented.

In the following paper, the notation used is the same as in [31], where the tetrad and the inverse of the tetrad fields are denoted by a lower letter e_μ^a and a capital letter E_a^μ , respectively, with the $(+, -, -, -)$ metric signature.

2 Teleparallel quintessence with a non-minimal coupling to a boundary term

In this study we consider the modified teleparallel model which involves a scalar field non-minimally coupled to the torsion T and a boundary term defined in terms of a divergence of the torsion vector, $B = \frac{2}{\epsilon} \partial_\mu (e T^\mu)$, where $e = \det(e_\mu^a)$. The action of this theory is given by

$$S = \int \left[\frac{1}{2\kappa^2} (f(\phi)T + g(\phi)B) + \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi) + L_m \right] e d^4x, \quad (1)$$

where L_m determines the matter contents, $\kappa^2 = 8\pi G$, $V(\phi)$ is the energy potential and $f(\phi)$ and $g(\phi)$ are coupling functions. For simplicity, we will use the notation NMC-(B+T) to refer to this theory ($f(\phi) \neq g(\phi) \neq 0$). In Ref. [31], the authors consider a special case of this action where $f(\phi) = 1 + \kappa^2 \xi \phi^2$ and $g(\phi) = \kappa^2 \chi \phi^2$. A non-minimally coupled scalar field with the boundary term B (NMC-B) is recovered if $\xi = 0$. Using dynamical system techniques, the cosmology in NMC-B was studied in [31]. If $\chi = 0$, we get the same action as in [25], which is known as “teleparallel dark-energy theory” (TDE). By choosing $\chi = -\xi$ we obtain scalar field models non-minimally coupled to the Ricci scalar that hereafter, for simplicity we will label as NMC-R [26]. In addition, the so-called “Minimally coupled quintessence” theories arise when we take $\chi = \xi = 0$ [27].

Variation of the action (1) with respect to the tetrad field yields the following field equations:

$$\frac{2}{\kappa^2} f(\phi) \left[e^{-1} \partial_\mu (e S_a^{\mu\nu}) - E_a^\lambda T^\rho{}_{\mu\lambda} S_\rho{}^{\nu\mu} - \frac{1}{4} E_a^\nu T \right] - E_a^\nu \left[\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi) \right]$$

$$\begin{aligned}
 &+ E_a^\mu \partial^\nu \phi \partial_\mu \phi + \frac{1}{\kappa^2} \left[2(\partial_\mu f(\phi) + \partial_\mu g(\phi)) E_a^\rho S_\rho^{\mu\nu} \right. \\
 &\left. + E_a^\nu \square g(\phi) - E_a^\mu \nabla^\nu \nabla_\mu g(\phi) \right] = \mathcal{T}_a^\nu, \tag{2}
 \end{aligned}$$

where $\square = \nabla_\alpha \nabla^\alpha$; ∇_α is the covariant derivative linked with the Levi-Civita connection symbol and \mathcal{T}_a^ν is the matter-energy momentum tensor.

If we vary the action (1) with respect to the scalar field, that yields the following modified Klein–Gordon equation:

$$\square \phi + V'(\phi) = \frac{1}{2\kappa^2} \left(f'(\phi)T + g'(\phi)B \right). \tag{3}$$

Throughout this paper, a prime denotes differentiation with respect to ϕ . We will assume the homogeneous and isotropic flat FRW metric in Euclidean coordinates defined by

$$ds^2 = dt^2 - a^2(t)(dx^2 + dy^2 + dz^2), \tag{4}$$

where $a(t)$ represents the scale factor and the corresponding tetrad components are $e_\mu^i = (1, a(t), a(t), a(t))$. It is important to mention that despite $f(T)$ gravity not being invariant under local Lorentz transformations, the latter tetrad is a “good tetrad” to consider, since it does not constrain its field equations [54]. Hence, this tetrad can be used safely within this scalar field theory too.

The energy-momentum tensor of matter is defined as

$$\mathcal{T}_{\mu\nu} = (\rho_m + p_m)u_\mu u_\nu - p_m g_{\mu\nu}, \tag{5}$$

where u_μ is the four velocity of the fluid and ρ_m and p_m define the matter energy density and pressure, respectively. Using the tetrad components for the flat FRW metric, the field Eq. (2) leads to

$$3H^2 f(\phi) = \kappa^2 \left(\rho_m + V(\phi) + \frac{1}{2} \dot{\phi}^2 \right) + 3H \dot{\phi} g'(\phi), \tag{6}$$

$$\begin{aligned}
 3H^2 f(\phi) + 2\dot{H} f(\phi) &= -\kappa^2 \left(p_m - V(\phi) + \frac{1}{2} \dot{\phi}^2 \right) \\
 - 2H \dot{\phi} f'(\phi) &+ (g''(\phi) \dot{\phi}^2 + g'(\phi) \ddot{\phi}). \tag{7}
 \end{aligned}$$

Here, $H = \dot{a}(t)/a(t)$ is the Hubble parameter and dots and primes denote differentiation with respect to the time coordinate and the argument of the function, respectively. We can also rewrite these equations in a fluid representation,

$$3H^2 = \kappa_{\text{eff}}^2 \rho_{\text{eff}}, \tag{8}$$

$$2\dot{H} = -\kappa_{\text{eff}}^2 (\rho_{\text{eff}} + p_{\text{eff}}), \tag{9}$$

where $\kappa_{\text{eff}}^2 = \frac{\kappa^2}{f(\phi)}$, $\rho_{\text{eff}} = \rho_m + \rho_\phi$ is the total energy density and $p_{\text{eff}} = p_m + p_\phi$ is the total pressure. The energy density and the pressure of the scalar field ρ_ϕ and p_ϕ are, respectively, defined as follows:

$$\rho_\phi = \frac{1}{2} \dot{\phi}^2 + V(\phi) + \frac{3}{\kappa^2} H \dot{\phi} g'(\phi),$$

$$\begin{aligned}
 p_\phi &= \frac{1}{2} \dot{\phi}^2 - V(\phi) \\
 &+ \frac{1}{\kappa^2} \left(2H \dot{\phi} f'(\phi) - (g''(\phi) \dot{\phi}^2 + g'(\phi) \ddot{\phi}) \right). \tag{10}
 \end{aligned}$$

In this theory the standard continuity equation reads

$$\dot{\rho}_{\text{eff}} + 3H(\rho_{\text{eff}} + p_{\text{eff}}) = 0, \tag{11}$$

$$\dot{\rho}_m + 3H(\rho_m + p_m) = 0. \tag{12}$$

Hereafter, we will assume a standard equation of state for the matter given by a barotropic equation $p_m = w\rho_m$, with w being the state parameter. If we use the above equation, we can directly find that the energy density becomes

$$\rho_m = \rho_0 a(t)^{-3(1+w)}, \tag{13}$$

where ρ_0 is an integration constant. It is proper to mention that, for a flat FRW metric, the torsion scalar and the boundary term are

$$T = -6H^2, \tag{14}$$

$$B = -18H^2 - 6\dot{H}, \tag{15}$$

and hence the Ricci scalar is recovered via $R = -T + B = -12H^2 - 6\dot{H}$.

Finally, the equation for the scalar field, the so-called Klein–Gordon equation, takes the form

$$\ddot{\phi} + 3H\dot{\phi} - \frac{1}{2\kappa^2} \left(f'(\phi)T + g'(\phi)B \right) + V'(\phi) = 0. \tag{16}$$

Note that the Klein–Gordon equation can also be obtained directly from the field equations (6) and (7), so that it is not an extra equation.

3 Thermodynamics in modified teleparallel theory

In this section we are interested to explore the general thermodynamic laws in the framework of the theory studied in [31], where the authors considered a quintessence theory non-minimally coupled between both a torsion scalar T and the boundary term B with the scalar field (NMC-B+T). The main aim of the next sections will be to formulate the first and second laws of thermodynamics in this theory. The complete and general thermodynamics law will be derived. After that, we will use the cosmological solutions found in Appendix A to study some interesting examples to visualize if they do or do not satisfy the thermodynamic laws.

3.1 First law of thermodynamics

This section is devoted to an investigation of the validity of the first law of thermodynamics in NMC-(B+T) at the apparent horizon for a flat FRW universe.

The condition $h^{\alpha\beta}\partial_\alpha R_A\partial_\beta R_A = 0$ gives the radius R_A of apparent horizon for flat FRW metric by

$$R_A = \frac{1}{H}.$$

The associated temperature is $T_h = \kappa_{sg}/2\pi$, where the surface gravity κ_{sg} is given by [55]

$$\begin{aligned} \kappa_{sg} &= \frac{1}{2\sqrt{-h}}\partial_\alpha(\sqrt{-h}h^{\alpha\beta}\partial_\beta R_A) = -\frac{1}{R_A}\left(1 - \frac{\dot{R}_A}{2HR_A}\right) \\ &= -\frac{R_A}{2}(2H^2 + \dot{H}). \end{aligned} \tag{17}$$

By using Eq. (9) and (III A) we easily get

$$\frac{f(\phi)}{G}dR_A = 4\pi R_A^3 H(\rho_{\text{eff}} + p_{\text{eff}})dt. \tag{18}$$

Now, multiplying both sides of this equation by a factor $-2\pi R_A T_h = 1 - \dot{R}_A/(2HR_A)$, we can rewrite the above equation as follows:

$$\begin{aligned} T_h d\left(\frac{Af(\phi)}{4G}\right) &= -(4\pi R_A^3 H dt - 2\pi R_A^2 dR_A) \\ &\quad \times (\rho_{\text{eff}} + p_{\text{eff}}) + \frac{\pi}{G}R_A^2 T_h df(\phi), \end{aligned} \tag{19}$$

where we have used $A = 4\pi R_A^2$. Thus, we can identify the entropy as the quantity which is multiplied by T_h , namely

$$S_h = \frac{Af(\phi)}{4G}. \tag{20}$$

Now, we define energy of the universe within the apparent horizon. The Misner–Sharp energy is defined as $E = R_A/(2G_{\text{eff}})$, which can be written as

$$E = \frac{R_A f(\phi)}{2G}. \tag{21}$$

In terms of the volume $V = 4\pi R_A^3/3$, we find that the energy density is given by

$$\bar{E} = \frac{3H^2 f(\phi)}{8\pi G}V \equiv \rho_{\text{eff}}V. \tag{22}$$

Taking the differential of the energy relation, we get

$$d\bar{E} = \frac{R_A}{2G}df(\phi) + 4\pi\rho_{\text{eff}}R_A^2dR_A - 4\pi HR_A^3(\rho_{\text{eff}} + p_{\text{eff}})dt. \tag{23}$$

Combining Eqs. (19) and (23), it results in

$$\begin{aligned} T_h dS_h &= d\bar{E} - 2\pi R_A^2(\rho_{\text{eff}} - p_{\text{eff}})dR_A \\ &\quad + \frac{R_A}{2G}(2\pi R_A T_h - 1)df(\phi). \end{aligned} \tag{24}$$

By defining the work density, we get

$$\bar{W} = -\frac{1}{2}\left(T^{(M)\alpha\beta}h_{\alpha\beta} + \bar{T}^{(de)\alpha\beta}h_{\alpha\beta}\right) = \frac{1}{2}(\rho_{\text{eff}} - p_{\text{eff}}). \tag{25}$$

Here $\bar{T}^{(de)\alpha\beta}h_{\alpha\beta}$ is the energy density of the dark components. Using the above definition of the work density in Eq. (24), we obtain

$$T_h dS_h = d\bar{E} - \bar{W}dV + \frac{R_A}{2G}(2\pi R_A T_h - 1)df(\phi), \tag{26}$$

which can be rewritten as

$$T_h dS_h + T_h dS_p = d\bar{E} - \bar{W}dV, \tag{27}$$

where $dS_p = -(R_A/(2GT_h))(2\pi R_A T_h - 1)df(\phi)$, which is the first law of thermodynamics in this teleparallel theory. The extra term dS_p defined in Eq. (27) can be treated as an entropy production term in non-equilibrium thermodynamics. In gravitational theories such as Einstein, Gauss–Bonnet and Lovelock gravities [56–58], the usual first law of thermodynamics is satisfied by the respective field equations. In fact, these theories do not involve any surplus term in universal form of the first law of thermodynamics, i.e., $TdS = dE - WdV$.

Initially Akbar and Cai used this approach to discuss thermodynamic laws in $f(R)$ gravity [59]. It is shown that an additional entropy term is produced, to be compared to other modified theories. Later Bamba et al. [41,42,60,61] developed the first law of thermodynamics in Palatini $f(R)$, $f(T)$, $f(R, \phi, X)$ (where $X = -1/2g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi$ is the kinetic term of a scalar field ϕ) and $f(R, \phi, X, G)$ (where $G = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ is the Gauss–Bonnet invariant) theories and formulated an additional entropy production term. A similar approach is applied to discuss the thermodynamic laws in $f(R, T)$, $f(R, L_m)$ and $f(R, T, R_{\mu\nu}T^{\mu\nu})$ theories, and one can see that the presence of non-equilibrium entropy production terms is necessary in such theories [46,47,62]. Bamba et al. [41,42,60,61] have shown that one can manipulate the FRW equations in order to redefine the entropy relation, which results in an equilibrium description of thermodynamics so that the first law of thermodynamics takes the form $TdS = dE - WdV$. Moreover, in all these theories it has been the case that the usual form of the first law of thermodynamics, i.e., $TdS_{\text{eff}} = dE - WdV$, can be obtained by defining the general entropy relation as a sum of an horizon entropy and an entropy production term.

Here, we may define the effective entropy term being the sum of horizon entropy and entropy production term as $S_{\text{eff}} = S_h + S_p$ so that Eq. (27) can be rewritten as

$$T_h dS_{\text{eff}} = d\bar{E} - \bar{W}dV, \tag{28}$$

where S_{eff} is the effective entropy related to the contributions involving a scalar field non-minimally coupled to the torsion T and a boundary term at the apparent horizon of FRW space-time.

3.2 Generalized second law of thermodynamics

According to the generalized second law of thermodynamic (GSLT), the entropy of matter and energy sources inside the horizon plus the entropy associated with a boundary of the horizon must be non-decreasing. In the previous section we have shown that the usual first law of thermodynamics does not hold in this theory. Therefore, to study GSLT, we would use the modified first law of thermodynamics. In fact the generalized entropy relation satisfies the condition

$$\dot{S}_{\text{tot}} = \dot{S}_h + \dot{S}_p + \dot{S}_{\text{in}} \geq 0, \tag{29}$$

where \dot{S}_h represents the entropy associated with the horizon, \dot{S}_p represents the entropy production term and \dot{S}_{in} is the sum of all entropy components inside the horizon.

Let us proceed with the modified first law of thermodynamics,

$$T_i dS_i = dE_i + p_i dV - T_i dS_p, \tag{30}$$

which can be represented as

$$T_{\text{in}} \dot{S}_{\text{in}} = (\rho_i + p_i) 4\pi R_h^2 (\dot{R}_h - H R_h) + \frac{4}{3} \pi R_h^3 \dot{Q}_i - T_i \dot{S}_p, \tag{31}$$

where R_h represents the radius of the horizon, T_{in} denotes the temperature for all the components inside the horizon and Q_i is an interaction term for the i th component. Summing up the total entropy inside the horizon, we find

$$T_{\text{in}} \dot{S}_{\text{in}} = (\rho_{\text{eff}} + p_{\text{eff}}) 4\pi R_h^2 (\dot{R}_h - H R_h) - T_{\text{in}} \dot{S}_p. \tag{32}$$

Here,

$$\sum_i Q_i = 0, \quad \sum_i (\rho_i + p_i) = \rho_{\text{eff}} + p_{\text{eff}}.$$

Now, let us study the validity of GSLT at the apparent horizon. In de Sitter space-time $R_A = 1/H$ and the future event horizon becomes the same as the Hubble horizon. The time derivative of (20) results in

$$\dot{S}_h = \frac{\pi}{H^2 G} \left\{ \dot{\phi} f'(\phi) - \frac{2\dot{H}}{H} f(\phi) \right\}. \tag{33}$$

Here, we set the thermal equilibrium with $T_{\text{in}} = T_h$, so that Eq. (32) implies

$$\dot{S}_{\text{in}} + \dot{S}_p = \frac{1}{T_h} (\rho_{\text{eff}} + p_{\text{eff}}) 4\pi R_A^2 (\dot{R}_A - H R_A). \tag{34}$$

After simplification, we get

$$\dot{S}_{\text{in}} = \frac{4\pi}{G} \frac{\dot{H}(\dot{H} + H^2)}{(\dot{H} + 2H^2)H^3}. \tag{35}$$

Hence, Eq. (29) implies the relation of GSLT of the form

$$\dot{S}_{\text{tot}} = \frac{\pi}{G H^2} \left(\frac{4\dot{H}(\dot{H} + H^2)}{H(\dot{H} + 2H^2)} + \dot{\phi} f'(\phi) - \frac{2\dot{H}}{H} f(\phi) \right) \geq 0. \tag{36}$$

In the following sections we will see if the GSLT is satisfied for some interesting cosmological solutions that can be constructed from our solutions found before (see Sects. A.1–A.3).

3.2.1 Specific model: power-law solutions for $f(\phi) = 1 + \xi \kappa^2 \phi^2$ and $g(\phi) = \chi \kappa^2 \phi^2$

In this section we will analyze if the GSLT is valid for different power-law models. In Sect. A.1, we found some specific solutions to the specific coupling $f(\phi) = 1 + \xi \kappa^2 \phi^2$ and $g(\phi) = \chi \kappa^2 \phi^2$. Here, we will focus our study for quartic ($m = -1$) and inverse potentials ($m = 2/3$). Additionally, by setting the coupling constants ξ and χ , we will also focus our study on different non-minimally coupled scalar–tensor theories.

(a) Power-law potential with $\chi \neq \frac{1}{4}$

For this case, the energy potential reads

$$V(\phi) = \frac{1}{2} \phi^{2-2/m} \phi_0^{2/m} \left(m^2 + 12mn\chi - 6n^2\xi \right),$$

$$\chi \neq \frac{1}{4}, \quad m = \frac{2n(2\xi + 3\chi) + 2\chi \pm \sqrt{4(2n\xi + (3n+1)\chi)^2 + 8n\xi(1-4\chi)}}{2(4\chi-1)}.$$

In Fig. 1, we present the evolution of energy density ρ and potential $V(\phi)$ for the power-law case. It can be seen that $V(\phi)$ is a decreasing function of time, where we set $\xi \leq 0$.

Here, we set the validity condition for GSLT in terms of the parameters m , ξ and χ . We choose m to show the particular representation of potential.

(b) Case $m = -1$

This choice of m corresponds to a quartic potential, $V(\phi) \propto \phi^4$. Here, we find the relation for n of the form $n = \frac{1-6\chi}{6\xi+\chi}$ and the constraint $n > 1$ results in the following condition:

$$\left(\chi < \frac{1}{6} \ \& \ -\chi < \xi < \frac{1}{6}(1 - 12\chi) \right)$$

$$\| \left(\chi > \frac{1}{6} \ \& \ \frac{1}{6}(1 - 12\chi) < \xi < -\chi \right). \tag{37}$$

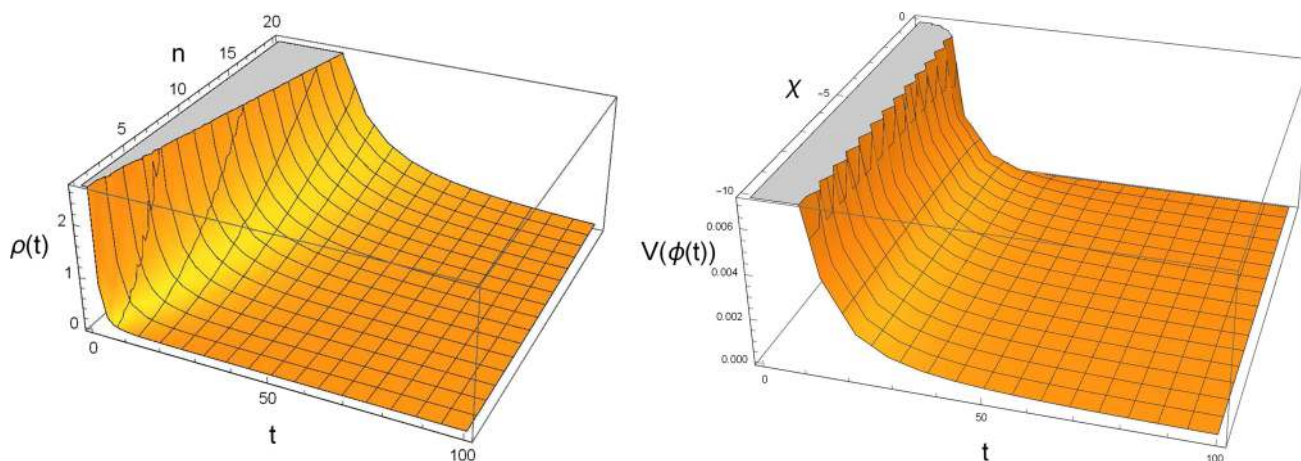


Fig. 1 For the power-law potentials, the energy density and the scalar potential are plotted against time. We chose $\phi_0 = \kappa = 1, n = 2$ ($a(t) \propto t^2$) and $\xi = 0.1$

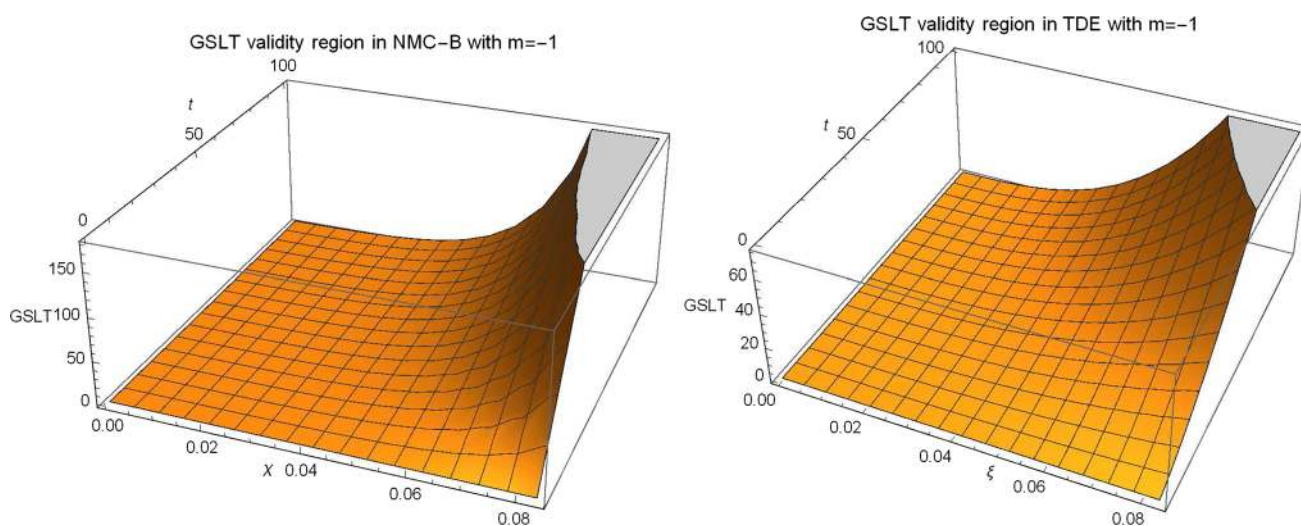


Fig. 2 Validity of GSLT for power-law potential with $\chi \neq \frac{1}{4}$. The figure on *left* (*right*) shows the validity for NMC-B(TDE) theory

Here, we discuss the specific cases NMC-B ($\xi = 0$) and TDE ($\chi = 0$). The validity of GSLT for a quartic potential is shown in Fig. 2. For NMC-B theory, the GSLT is valid in the range $0 < \chi < \frac{1}{12}$ and in the case of TDE we need $0 < \xi < \frac{1}{6}$. These constraints are set in accordance with the condition of power-law solutions, $n > 1$.

We also the evolution of GSLT in Fig. 3, where we set $n = 2$ and choose the particular values for χ and ξ . The black curve corresponds to NMC-B with $\chi = \frac{1}{18}$ and the blue curve represents the plot of GSLT for TDE with $\xi = \frac{1}{18}$.

(c) Case $m = 2/3$

In this case one can recover the inverse potential i.e., $V(\phi) \propto \phi^{-1}$. Here, we show the validity of GSLT in Figs. 4 and 5. In Fig. 4, the left plot corresponds to NMC-B, which shows the validity in the range $-\frac{1}{8} < \chi < 0$ and the right plot corresponds to TDE with $-\frac{2}{3} < \xi < 0$. It can be seen that

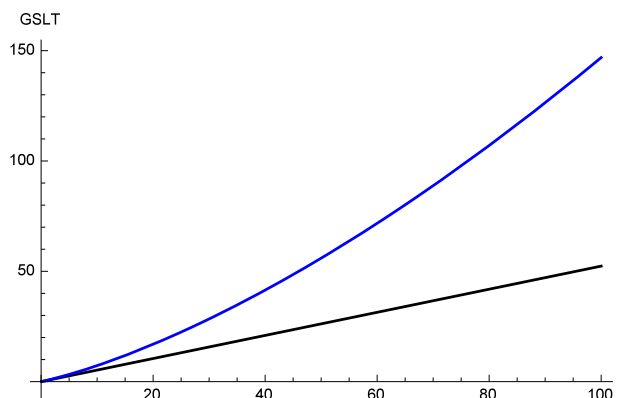


Fig. 3 Validity of GSLT for power-law potential with $n = 2$

GSLT is violated in case of TDE, whereas it is valid in both NMC-B and NMC-R.

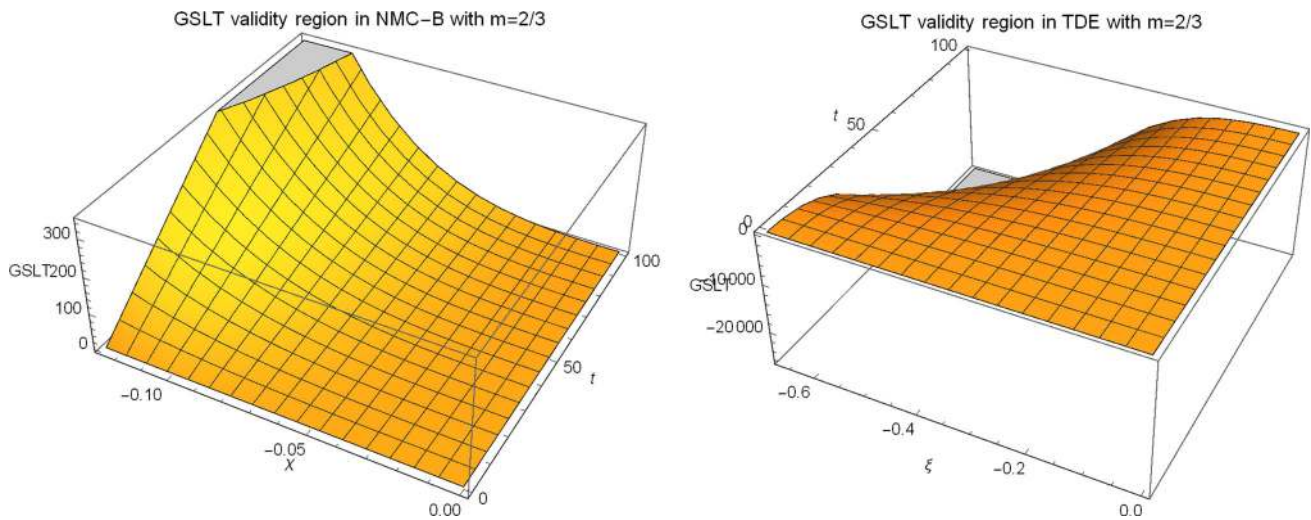


Fig. 4 Validity of GSLT for power-law potential for $m = 2/3$ in NMCB and TDE theories

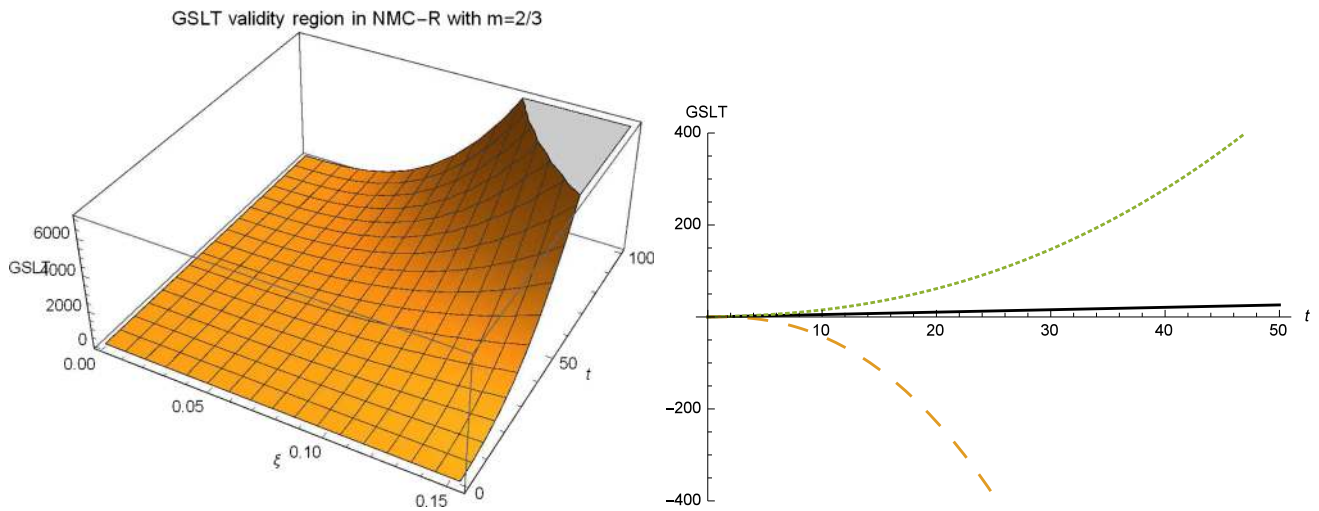


Fig. 5 Left plot shows the validity of GSLT in NMC-R for $0 < \xi < \frac{2}{13}$ and right plot sums up the validity in the case of inverse potential. The curves for NMC-B ($\chi = \frac{-1}{17}$), TDE ($\xi = \frac{-1}{3}$) and NMC-R ($\xi = \frac{1}{14}$) are represented by solid, dashed and dotted lines, respectively

(d) Power-law potential with $\chi = \frac{1}{4}$
 For this case the potential takes the following form:

$$V(\phi) = \frac{n^2 \xi (8\xi + 3) (3n^2 (8\xi + 3) - 1) \phi^{-\frac{3n+1}{2n\xi} - 2} \phi_0^{\frac{3n+1}{2n\xi} + 4}}{(n(8\xi + 3) + 1)^2},$$

$$\chi = \frac{1}{4}, \quad m = \frac{4n\xi}{8\xi n + 3n + 1}, \quad n \neq -\frac{1}{8\xi + 3}.$$

For this model, we have a fixed value of χ , so we cannot discuss the TDE and NMC-B theories. In the case of NMC-R ($\chi = -\xi$), we find that this representation does not show realistic results as we need to fix $n < 0$ for inverse and quadratic potentials.

Moreover, in the case of NMC-(B+R), we explore the validity for quartic potential, where we need to set $\xi =$

$-\frac{(3n+1)}{12n}$. Finally, we have GSLT dependence on n , and validity is shown in Fig. 6.

3.2.2 Specific model: exponential solutions for $f(\phi) = 1 + \kappa^2 \alpha e^{m\phi}$ and $g(\phi) = \kappa^2 \beta e^{k\phi}$

This section is devoted to an analysis of the question if the GSLT is satisfied for some exponential solutions found in Sect. A.3. Since mostly all the solutions are similar, we will only analyze the case where $w = \frac{2-\phi_0}{3n} - 1$ (see Sect. A.3.2). In this case, the energy potential contains three exponentials that can represent different kind of potentials. For example, by taking $m = \frac{\sqrt{3}\kappa\phi_0}{\sqrt{2(\phi_0-1)}}$, we find

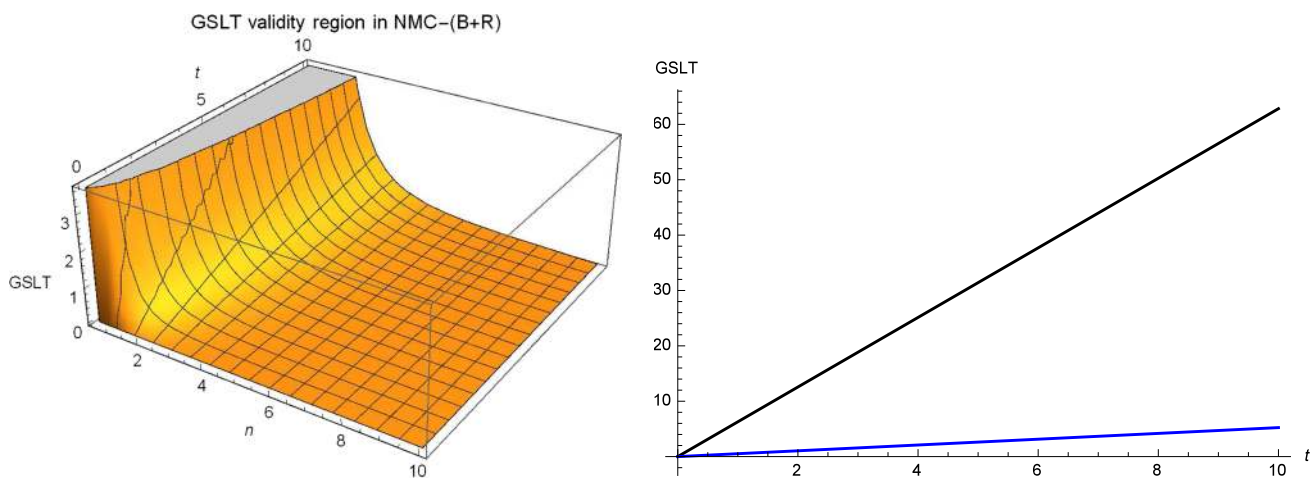


Fig. 6 Here, we show the validity for NMC-(B+R) theory with quartic potential. In this case we find that GSLT can be met for $n > 0.5$. In the left plot we show the validity region, and its behavior for particular choices is shown on the right side. Black and blue curves correspond to $n = 1, 20$

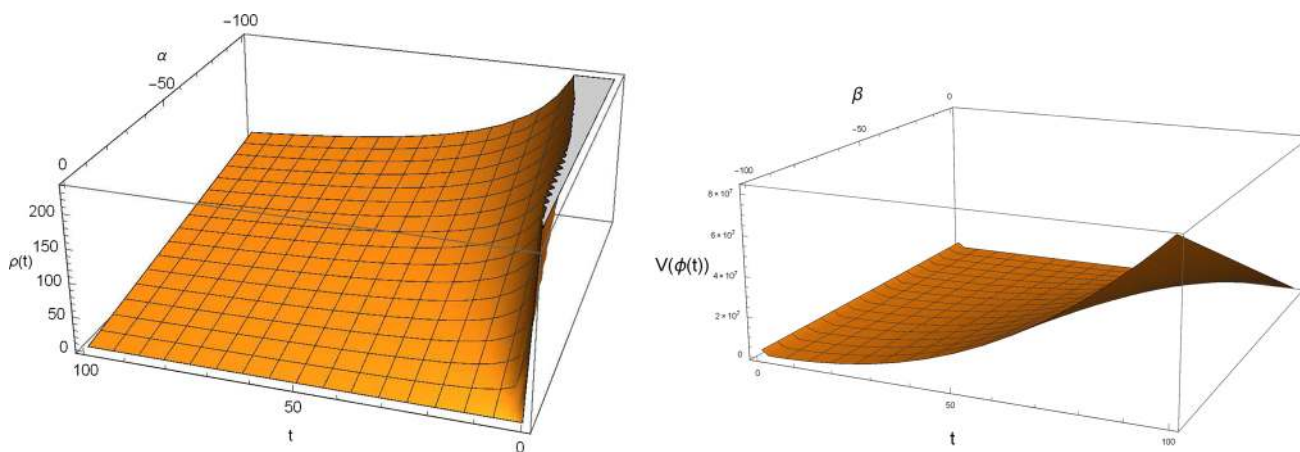


Fig. 7 For the exponential-law model, the energy density and the scalar potential are plotted against time. We chose $\phi_0 = 1.5, \kappa = 1, m = 1$ and $a_0 = 1$

$$V(\phi) = \frac{(\phi_0 - 2)(\phi_0 - 1)e^{-\frac{\sqrt{6}\kappa\phi}{\sqrt{\phi_0-1}}}}{3\kappa^2} - \frac{24\beta\kappa^2(24\kappa^2 + m^2)e^{\frac{6\kappa^2\phi}{m} - \frac{m\phi}{4}}}{m^4}, \tag{39}$$

$$- \frac{(\phi_0 - 1)\phi_0(\alpha(\phi_0 - 1) + 3\beta(\phi_0 - 2))e^{\frac{\sqrt{\frac{3}{2}}\kappa\phi(\phi_0-2)}{\sqrt{\phi_0-1}}}}{3(\phi_0 - 2)}, \tag{38}$$

which is known as a double exponential potential. This kind of potential has been widely studied in the literature. From Fig. 7, it is observed that the scalar potential increases as the field increases. Thus the cosmic acceleration will be driven by the potential energy of the scalar field.

Another interesting example that can be constructed is by taking $\phi_0 = 4$ and $\alpha = \frac{24\kappa^2 - m^2}{48\kappa^4}$, which gives us the following potential:

$$V(\phi) = \frac{16(m^2 - 24\kappa^2) \sinh\left(\frac{m\phi}{2}\right)}{m^4}$$

which is an interesting kind of potential studied in the literature (see [63, 64]). The exponential function models generally lead to an accelerating expansion behavior of the universe.

Let us now study the GSLT for this kind of potential. In this model, Eq. (36) becomes

$$\dot{S}_{\text{tot}} = \frac{\pi t \left(\frac{2}{2n-1} + \alpha(\phi_0 + 2)t^{\phi_0} \right)}{n^2} \geq 0, \quad \phi_0 \neq 2, \tag{40}$$

where for simplicity we took $\kappa = G = 1$. Clearly, the above inequality will hold depending on the values of the parameters ϕ_0, n and α . Since $n > 1$, if $\phi_0 \geq -2$ and $\alpha > 0$, GSLT will be valid at any time. Additionally, if $\phi_0 \leq -2$ and $\alpha < 0$, the GSLT will be always true. For all the other cases, the validity of GSLT will depend on time. Figures 8 and 9 show the behavior of the GSLT inequality for different values of the parameters. For $\phi_0 > 0$ and $\alpha < 0$, the inequal-

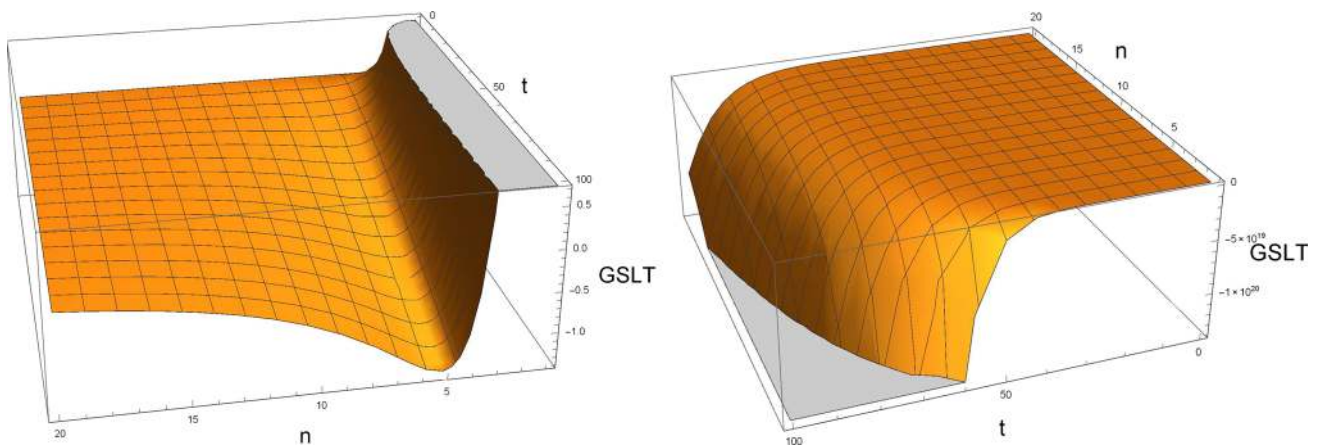


Fig. 8 Validity of GSLT for exponential coupling for $\alpha = -0.1$. The figure on the *right* (*left*) shows the case where $\phi_0 = 10$ ($\phi_0 = 0.1$)

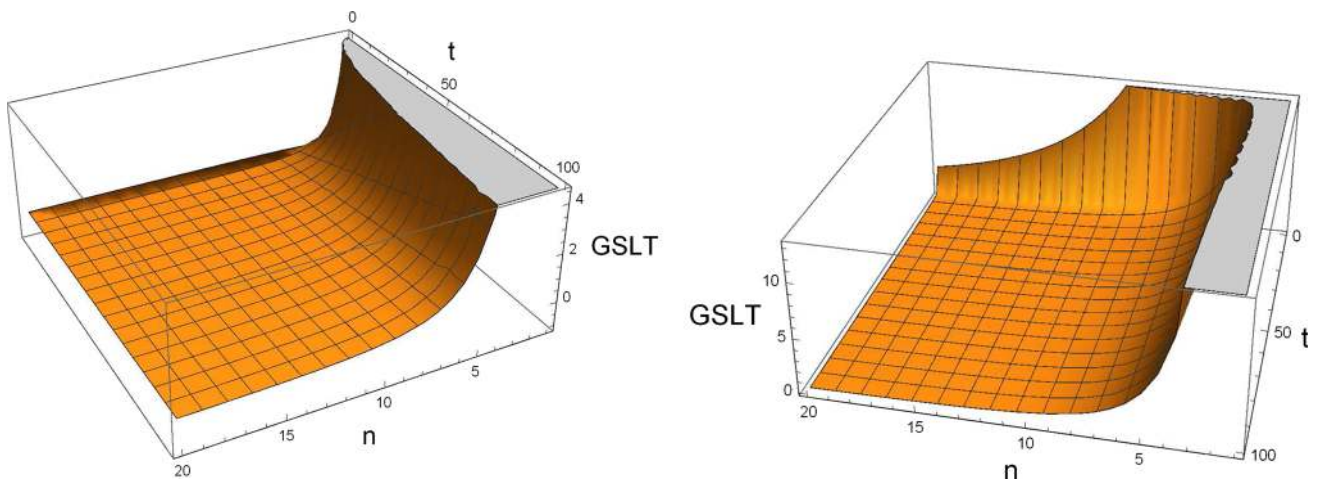


Fig. 9 Validity of GSLT for exponential coupling for $\phi_0 = -1.4$. The figure on the *right* (*left*) shows the case where $\alpha = -100$ ($\alpha = -0.1$)

ity will be more constrained for bigger $|\phi_0|$. Additionally, it can be seen that GSLT will be always true at very late times for $-2 < \phi_0 < 0$.

4 Generalized second law of thermodynamics with logarithmic entropy corrections

The entropy–area relation involving quantum corrections leads to curvature corrections in the Einstein–Hilbert action [65,66]. The logarithmic corrected entropy is defined through the relation [67–70]

$$S = \frac{Af(\phi)}{4G} + \alpha \ln \frac{Af(\phi)}{4G} + \beta \frac{4G}{Af(\phi)} + \gamma, \tag{41}$$

where α, β and γ are dimensionless constants; however, the exact values of these constants are yet to be determined. These corrections arise in black hole entropy in loop quantum gravity due to thermal equilibrium fluctuations and quantum

fluctuations [71,72]. In [70] Sadjadi and Jamil investigated the validity of GSLT for FRW space-time with logarithmic correction. They found that in a (super-) accelerated universe the GSL is valid whenever $\alpha(<) > 0$, leading to a (negative) positive contribution from logarithmic correction to the entropy. In the following section, we will present the validity of the GSLT with modified entropy relations involving logarithmic corrections.

First note that the time derivative of Eq. (41) yields

$$\dot{S} = \left(\frac{\dot{A}}{A} + \frac{f'}{f} \dot{\phi} \right) \frac{Af}{4G} \left[1 + \alpha \left(\frac{4G}{Af} \right) - \beta \left(\frac{4G}{Af} \right)^2 \right]. \tag{42}$$

In the case of the apparent horizon, Eq. (41) implies

$$\dot{S}_A = \left(\frac{-2\dot{H}}{H} + \frac{f'}{f} \dot{\phi} \right) \frac{\pi f}{GH^2} \left[1 + \alpha \left(\frac{GH^2}{f\pi} \right) - \beta \left(\frac{GH^2}{f\pi} \right)^2 \right]. \tag{43}$$

Here, we used the fact that the Hubble horizon is $R_A = 1/H$. Now, by using Eqs. (35) and (43), we get

$$\begin{aligned} \dot{S}_{\text{tot}} = & \left(\frac{-2\dot{H}}{H} + \frac{f'}{f} \dot{\phi} \right) \frac{\pi f}{GH^2} \\ & \times \left[1 + \alpha \left(\frac{GH^2}{f\pi} \right) - \beta \left(\frac{GH^2}{f\pi} \right)^2 \right] \\ & + \frac{4\pi}{G} \frac{\dot{H}(\dot{H} + H^2)}{(\dot{H} + 2H^2)H^3}, \end{aligned} \quad (44)$$

which can be represented in terms of effective components as follows:

$$\begin{aligned} \dot{S}_{\text{tot}} = & \left(\kappa(w_{\text{eff}} + 1) \sqrt{\frac{3\rho_{\text{eff}}}{f(\phi(t))}} + \frac{f'}{f} \dot{\phi} \right) \frac{3\pi f(\phi(t))^2}{G\kappa^2\rho_{\text{eff}}} \\ & \times \left[1 + \alpha \left(\frac{G\kappa^2\rho_{\text{eff}}}{3\pi f(\phi(t))^2} \right) - \beta \left(\frac{G\kappa^2\rho_{\text{eff}}}{3\pi f(\phi(t))^2} \right)^2 \right] \\ & + \frac{6\pi(w_{\text{eff}} + 1)(3w_{\text{eff}} + 1)}{G\kappa(1 - 3w_{\text{eff}})} \sqrt{\frac{3f(\phi(t))}{\rho_{\text{eff}}}} \geq 0. \end{aligned} \quad (45)$$

The GSL with quantum corrections have natural implications in the studies of the very early universe since quantum corrections are directly linked with high energy and short distance scales.

5 Conclusions

The non-minimally coupling models in cosmology are frequently used to study guaranteed late time acceleration, phantom crossing and the existence of finite time future singularity. In this paper, the thermodynamic study is executed in modified teleparallel theory which involves scalar field non-minimally coupled to both the torsion and a boundary term [31]. One interesting factor in this theory is that in a suitable limit, one can recover very well-known theories of gravity such as quintessence, teleparallel dark energy and non-minimally coupled scalar field with the Ricci scalar R .

In this paper, we investigated the GSLT for an expanding universe with apparent horizon. The laws of thermodynamics are universally valid and hence they must apply in a modified form to the whole universe. The existence of an apparent horizon in a space-time provides the opportunity to formulate the first law and the generalized second law of thermodynamics. According to the laws of thermodynamics, the first law must always be true for a physical system (provided it is non-dissipative); however, the GSLT holds exactly using the apparent horizon. We obtained these laws in the non-minimal coupled teleparallel theory with

a boundary term and a scalar field. Our results generalize many previous GSLT studies, while extending them to the logarithmic corrected entropy–area law. Moreover, GSLT with quantum corrections have natural implications in the studies of the very early universe, since quantum corrections are directly linked with high energy and short distance scales.

We explored the existence of power-law solutions for two different choices of Lagrangian coefficients, which are functions of the scalar field, $f(\phi)$ and $g(\phi)$. In the first case we set $f(\phi) = 1 + \kappa^2\xi\phi^2$ and $g(\phi) = \kappa^2\chi\phi^2$ and choose the power-law forms of scale factor and scalar field to discuss possible forms of the power-law potential $V(\phi)$ (see [28]). Here, one can recover the quadratic, quartic, inverse and Ratra–Peebles potentials. We also discuss the Brans–Dicke theory as a particular case in power-law cosmology. Moreover, we also discuss the power-law solutions for the choice of $f(\phi) = 1 + \kappa^2\alpha e^{m\phi}$ and $g(\phi) = \kappa^2\beta e^{k\phi}$ with $\phi(t) = \frac{\phi_0}{m} \log(t)$. Here, we explored different forms of $V(\phi)$ depending on the equation of state parameter.

We showed the behavior of matter density ρ and potential $V(\phi)$ for the power-law potential; it can be seen that both are decreasing functions of time as shown in Fig. 1. In our discussion of the validity of GSLT, we considered the quartic and inverse potentials in all the viable special cases like TDE, NMC-B, NMC-R and NMC-(B+R) and presented results in Figs. 2, 3, 4, 5 and 6. In Figs. 2 and 3, we found that GSLT is true for NMC-B (with $0 < \xi < \frac{1}{6}$) and TDE (with $0 < \chi < \frac{1}{12}$). For the inverse potential, we find that GSLT is valid for NMC-B (with $-\frac{1}{8} < \chi < 0$) and TDE (with $-\frac{2}{3} < \xi < 0$), whereas it is violated for TDE, as shown in Figs. 4 and 5. For the choice of $\chi = \frac{1}{4}$, we found the validity region only in the case of NMC-(B+R) with $n > \frac{1}{2}$ as shown in Fig. 6. For the exponential forms of the Lagrangian coefficients, we found that GSLT is always true for $(\phi_0 \leq -2 \ \& \ \alpha < 0)$ and $(\phi_0 \geq -2 \ \& \ \alpha > 0)$, whereas for other choices of the parameters the validity of GSLT is time dependent. Finally, we also formulated the GSLT for logarithmic entropy corrections.

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A New cosmological solutions

A.1 Power-law solutions for $f(\phi) = 1 + \kappa^2 \xi \phi^2$ and $g(\phi) = \kappa^2 \chi \phi^2$

In this section we are going to find analytical power-law solutions for the NMC-(B+T) case where the coupling functions are

$$f(\phi) = 1 + \kappa^2 \xi \phi^2, \tag{A1}$$

$$g(\phi) = \kappa^2 \chi \phi^2, \tag{A2}$$

where ξ and χ are coupling constants. As we discussed before, for this specific choice we can recover NMC-B ($\xi = 0$), TDE ($\chi = 0$), NMC-R ($\xi = -\chi$) and quintessence theories ($\chi = \xi = 0$). Let us now study power-law cosmology where the scale function and the scalar field take a power-law form

$$\rho_0 = \frac{3n^2}{\kappa^2}, \tag{A9}$$

$$m = \begin{cases} \frac{2n(2\xi + 3\chi) + 2\chi \pm \sqrt{4(2n\xi + (3n + 1)\chi)^2 + 8n\xi(1 - 4\chi)}}{2(4\chi - 1)}, & \chi \neq \frac{1}{4}, \\ \frac{4n\xi}{8\xi n + 3n + 1}, & \chi = \frac{1}{4}, n \neq -\frac{1}{8\xi + 3}. \end{cases} \tag{A10}$$

$$a(t) = a_0 t^n, \quad n > 1, \tag{A3}$$

$$\phi(t) = \phi_0 t^m, \tag{A4}$$

Since $n \neq 0$, the exponents of the scalar field of the first and second terms on the r.h.s. must match otherwise the above equation will not hold. Therefore, the state parameter needs to be

$$w = \frac{2}{3n} - 1. \tag{A7}$$

Using this condition, Eq. (A6) becomes

$$\frac{2\phi_0^{2/m} \phi^{-2/m} (\kappa^2 \rho_0 a_0^{-2/n} - 3n^2)}{3n} = \kappa^2 \phi_0^{2/m} \phi^{2-2/m} (m^2(4\chi - 1) - 2m(2n\xi + 3n\chi + \chi) + 2n\xi). \tag{A8}$$

This condition will be true for different choices of the parameters. For simplicity, we will assume the case that $a_0 = 1$, which gives

Thus, the energy potential (A5) and the energy density are simplified to

$$\rho_m(t) = \frac{3n^2}{\kappa^2} t^{-2}, \tag{A11}$$

$$V(\phi) = \begin{cases} \frac{1}{2} \phi^{2-2/m} \phi_0^{2/m} (m^2 + 12mn\chi - 6n^2\xi), & \chi \neq \frac{1}{4}, m = \frac{2n(2\xi+3\chi)+2\chi \pm \sqrt{4(2n\xi+(3n+1)\chi)^2+8n\xi(1-4\chi)}}{2(4\chi-1)}, \\ \frac{n^2\xi(8\xi+3)(3n^2(8\xi+3)-1)\phi^{-\frac{3n+1}{2n\xi}-2}\phi_0^{\frac{3n+1}{2n\xi}+4}}{(n(8\xi+3)+1)^2}, & \chi = \frac{1}{4}, m = \frac{4n\xi}{8\xi n + 3n + 1}, n \neq -\frac{1}{8\xi + 3}. \end{cases} \tag{A12}$$

where a_0, ϕ_0, m and n are constants. Under these ansatzes, from (6) we directly find that the energy potential becomes

$$V(\phi) = -\rho_0 a_0^{-3(w+1)} \phi^{\frac{2-3n(w+1)}{m} - \frac{2}{m} \frac{3n(w+1)}{m}} - \frac{1}{2} \phi^{2-\frac{2}{m}} \phi_0^{2/m} (m^2 + 12mn\chi - 6n^2\xi) + \frac{3n^2 \phi^{-2/m} \phi_0^{2/m}}{\kappa^2}. \tag{A5}$$

Now, by replacing the above expression in (7) we find that the parameters need to satisfy

$$\kappa^2 \phi_0^{2/m} \phi^{2-2/m} [m^2(4\chi - 1) - 2m(2n\xi + 3n\chi + \chi) + 2n\xi] = \kappa^2 \rho_0 (w + 1) a_0^{-3(w+1)} \phi_0^{\frac{3n(w+1)}{m}} \phi^{-\frac{3n(w+1)}{m}} + 2n\phi_0^{2/m} \phi^{-2/m}. \tag{A6}$$

For the first potential ($\chi \neq 1/4$), we can recover a self-interacting scalar field case $V(\phi) \propto \phi^4$ if $m = -1$, which is the case where the coupling parameters satisfied $\xi = \frac{-6n\chi - 6\chi + 1}{6n}$. It corresponds to the inverse potential if $m = 2/3$ with the additional constraint $\xi = -\frac{2(1-\chi+9n\chi)}{3n}$ and the Ratra–Peebles potential is recovered by choosing $0 < m < 1$.

A.2 Power-law solutions for Brans–Dicke: $f(\phi) = \kappa^2 \xi \phi^2$ and $g(\phi) = \kappa^2 \chi \phi^2$

It is easy to see that if we choose

$$f(\phi) = \kappa^2 \xi \phi^2, \tag{A13}$$

$$g(\phi) = \kappa^2 \chi \phi^2, \tag{A14}$$

our action becomes a Brans–Dicke one in a canonical form with the Brans–Dicke parameter being $w_{BD} = 1$. The latter theory has been widely studied in the literature, so that it is an important coupling to consider.

To find solutions, we will follow the same approach as the previous section. Moreover, since the equations are so similar, we will not state all the steps. It is easy to see that if $w = -1 + \frac{2(1-m)}{3n} \neq -1$ we obtain the following solutions:

$$\rho_m(t) = \begin{cases} \frac{3n\phi_0^2 t^{2m-2} (m^2(1-4\chi) + 2m(2n\xi + 3n\chi + \chi) - 2n\xi)}{2(m-1)} & \rho_0 = \frac{3n\phi_0^2 a_0^{\frac{2}{n}} - \frac{2m}{n} (-4m^2\chi + m^2 + 4mn\xi + 6mn\chi + 2m\chi - 2n\xi)}{2(m-1)}, \\ \rho_0 & n = \frac{2\chi-1}{2(\xi+3\chi)}, \quad m = 1. \end{cases}$$

$$V(\phi) = \begin{cases} -\frac{m\phi^{2-\frac{2}{m}} \phi_0^{2/m} (m^2 + m(3n-1) + 6n(n(\xi+3\chi) - \chi))}{2(m-1)} & \rho_0 = \frac{3n\phi_0^2 a_0^{\frac{2}{n}} - \frac{2m}{n} (-4m^2\chi + m^2 + 4mn\xi + 6mn\chi + 2m\chi - 2n\xi)}{2(m-1)}, \\ \frac{-2\xi^2(2\rho_0 + \phi_0^2) - 3\xi(8\rho_0\chi + (4\chi^2 + 4\chi - 1)\phi_0^2) + 18\chi^2((1-4\chi)\phi_0^2 - 2\rho_0)}{4(\xi+3\chi)^2} & n = \frac{2\chi-1}{2(\xi+3\chi)}, \quad m = 1. \end{cases}$$

needs to hold. For the specific case where $\rho_0 \neq 0$, we have three possible cases for the state parameter,

$$w = \begin{cases} \frac{2m - k\phi_0}{3mn} - 1, \\ \frac{2 - \phi_0}{3n} - 1, \\ -1. \end{cases} \tag{A21}$$

A.3 Exponential solutions for $f(\phi) = 1 + \kappa^2\alpha e^{m\phi}$ and $g(\phi) = \kappa^2\beta e^{k\phi}$

In this section, we will assume that the coupling functions are exponential:

$$f(\phi) = 1 + \kappa^2\alpha e^{m\phi}, \tag{A15}$$

$$g(\phi) = \kappa^2\beta e^{k\phi}, \tag{A16}$$

where α, β, m and k are parameters. The cosmology for the specific case where $\beta = -\alpha$ and $k = m$ was studied in [73]. Additionally, we assume that the scale factor and the scalar field behave as

$$a(t) = a_0 t^n, \quad n > 1, \tag{A17}$$

$$\phi(t) = \frac{\phi_0}{m} \log(t), \tag{A18}$$

where n, m, ϕ_0 and a_0 are constants. By using (6), we find that the energy potential becomes

$$V(\phi) = -\rho_0 a_0^{-3(w+1)} e^{-\frac{3mn(w+1)\phi}{\phi_0}} - e^{-\frac{2m\phi}{\phi_0}} \times \left(\frac{3\beta k n \phi_0 e^{k\phi}}{m} + \frac{\phi_0^2}{2m^2} - \frac{3n^2}{\kappa^2} \right) + 3\alpha n^2 e^{\frac{m\phi(\phi_0-2)}{\phi_0}}. \tag{A19}$$

Now, if we replace this equation in (7), we find that

$$2\alpha\kappa^2 m^2 n (\phi_0 - 1) a_0^{3w+3} e^{\frac{m\phi(3n(w+1)+\phi_0)}{\phi_0}} + a_0^{3w+3} \left(\kappa^2 \phi_0^2 - 2m^2 n \right) e^{\frac{3mn(w+1)\phi}{\phi_0}} = \beta\kappa^2 k \phi_0 a_0^{3w+3} (k\phi_0 - m(3n+1)) e^{\phi \left(k + \frac{3mn(w+1)}{\phi_0} \right)} - \kappa^2 m^2 \rho_0 (w+1) e^{\frac{2m\phi}{\phi_0}} \tag{A20}$$

Now, we will explore all the possible solutions under these state parameters. In general, all the solutions are very similar but they are not exactly the same.

A.3.1 Case 1: $w = \frac{2m-k\phi_0}{3mn} - 1$

For this special case, from (A20) the other parameters must satisfy ($k \neq 2m$)

$$\rho_0 = -\frac{3\beta\kappa^2 k (3\kappa^2 - 2km + 2m^2) a_0^{\frac{2m(2m-k)}{\kappa^2}}}{4m^4(2m-k)},$$

$$\phi_0 = 1, \quad n = \frac{\kappa^2}{2m^2}. \tag{A22}$$

Note that for the case $k = 2m$ we find that $n = 1/3$, which is not allowed. Hence, if $k \neq 2m$ we find that the energy density and energy potential for the first state parameter, w_1 , are given by

$$\rho_m(t) = \frac{\beta k (-3\kappa^2 + 2km - 2m^2) (3\kappa^2 + 2km - 4m^2) t^{\frac{k}{m}-2}}{4m^4(k-2m)}, \tag{A23}$$

$$V(\phi) = \frac{3\beta k (2\kappa^2 m^2 - 3\kappa^4) e^{k\phi-2m\phi}}{4m^4(k-2m)} + \frac{3\alpha\kappa^4 e^{-m\phi}}{4m^4} + \frac{(3\kappa^2 - 2m^2) e^{-2m\phi}}{4m^4}. \tag{A24}$$

The special case of non-minimal coupling with the Ricci scalar can be obtained by taking $k = m$ and $\beta = -\alpha$, which makes the potential become

$$V(\phi) = \frac{3\beta\kappa^2 (\kappa^2 - m^2) e^{-m\phi}}{2m^4} + \frac{(3\kappa^2 - 2m^2) e^{-2m\phi}}{4m^4}. \tag{A25}$$

A.3.2 Case 2: $w = \frac{2-\phi_0}{3n} - 1$

For this case, we need to have $\phi_0 \neq 2$ and also from (A20) the parameters must obey

$$\rho_0 = \frac{3\alpha\kappa^4(\phi_0 - 1)\phi_0^4 a_0^{\frac{4m^2}{\kappa^2\phi_0^2} - \frac{2m^2}{\kappa^2\phi_0}}}{2m^4(\phi_0 - 2)},$$

$$k = \frac{3mn + m}{\phi_0}, \quad n = \frac{\kappa^2\phi_0^2}{2m^2}, \tag{A26}$$

and hence the energy density and the potential take the following form:

$$\rho_m(t) = \frac{3\alpha\kappa^4(\phi_0 - 1)\phi_0^4 t^{\phi_0 - 2}}{2m^4(\phi_0 - 2)}, \tag{A27}$$

$$V(\phi) = -\frac{3\alpha\kappa^4\phi_0^5 e^{m\phi - \frac{2m\phi}{\phi_0}}}{4m^4(\phi_0 - 2)}$$

$$- \frac{3\beta(3\kappa^4\phi_0^4 + 2\kappa^2m^2\phi_0^2) e^{\frac{3\kappa^2\phi\phi_0}{2m} - \frac{m\phi}{\phi_0}}}{4m^4}$$

$$+ \frac{(3\kappa^2\phi_0^4 - 2m^2\phi_0^2) e^{-\frac{2m\phi}{\phi_0}}}{4m^4}. \tag{A28}$$

A.3.3 Case 3: $w = -1$ (dark-energy fluid)

For a dark-energy fluid with $w = -1$, from (A20) the parameters need to be

$$\phi_0 = 1, \quad k = 3mn + m, \quad n = \frac{\kappa^2}{2m^2}, \tag{A29}$$

and therefore the energy density is a constant ρ_0 and the potential becomes

$$V(\phi) = \frac{3\alpha\kappa^4 e^{-m\phi}}{4m^4} - \frac{3\beta(3\kappa^4 + 2\kappa^2m^2) e^{\phi(\frac{3\kappa^2}{2m} + m) - 2m\phi}}{4m^4}$$

$$+ \frac{(3\kappa^2 - 2m^2) e^{-2m\phi}}{4m^4} - \rho_0. \tag{A30}$$

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