Generalized semiconfined harmonic oscillator model with a position-dependent effective mass

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Abstract

By using a point canonical transformation starting from the constant-mass Schrödinger equation for the isotonic potential, it is shown that a semiconfined harmonic oscillator model with a position-dependent mass in the BenDaniel-Duke setting and the same spectrum as the standard harmonic oscillator can be easily constructed and extended to a semiconfined shifted harmonic oscillator, which could result from the presence of a uniform gravitational field. A further generalization is proposed by considering a *m*-dependent positiondependent mass for 0 < m < 2 and deriving the associated semiconfined potential. This results in a family of position-dependent mass and potential pairs, to which the original pair belongs as it corresponds to m = 1. Finally, the potential that would result from a general von Roos kinetic energy operator is presented and the examples of the Zhu-Kroemer and Mustafa-Mazharimousavi settings are briefly discussed.

Keywords: Schrödinger equation; position-dependent mass; harmonic oscillator; point canonical transformation

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1 Introduction

There is much interest in the Schrödinger equation wherein the constant mass is replaced by a position-dependent mass (PDM), because the latter has many applications in problems occurring in several fields of physics [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12]. As it has been shown [13] that the PDM Schrödinger equation is equivalent to two other unconventional Schrödinger equations, namely the Schrödinger equation resulting from the use of deformed commutation relations [14, 15, 16], as well as that in curved space [17, 18, 19], this has reinforced the interest in its study.

As a consequence, much attention has been devoted to finding exact solutions of PDM Schrödinger equations because they may provide a conceptual understanding of some physical phenomena, as well as a testing ground for some approximation schemes. The generation of PDM and potential pairs leading to such exact solutions has been achieved by various methods (see, e.g., [20] and references quoted therein). One of the most powerful techniques for such a purpose consists in applying a point canonical transformation (PCT) to an exactly solvable constant-mass Schrödinger equation [21, 22]. Recently, such an approach has proved its efficiency again by providing a straightforward generalization [23] of a harmonic oscillator model wherein both the mass and the angular frequency are dependent on the position [24].

The purpose of the present paper is to re-examine a new model of semiconfined harmonic oscillator with a mass that varies with position, which has the striking property of having the same spectrum as the standard harmonic oscillator model [25]. By using the PCT method, we plan to prove that one can find a family of PDM and semiconfined potential pairs corresponding to such a spectrum and to which the original PDM and semiconfined harmonic oscillator pair belongs.

The paper is organized as follows. In Sec. 2, the model of [25] is reviewed and shown to be derivable by applying the PCT technique to the constant-mass isotonic oscillator model [26, 27]. In Sec. 3, an extension of the model is proposed by starting from a more general PDM and determining the associated semiconfined potential. Finally, Sec. 4 contains some comments.

2 Semiconfined harmonic oscillator model and its derivation by the PCT technique

Jafarov and Van der Jeugt recently determined the exact solution of a PDM semiconfined harmonic oscillator model, characterized by the Schrödinger equation [25]

$$\left(-\frac{d}{dx}\frac{1}{M(x)}\frac{d}{dx} + V_{\text{eff}}(x)\right)\psi_n(x) = E_n\psi_n(x),\tag{1}$$

where the kinetic energy operator has the BenDaniel-Duke form [28] and the potential has the harmonic oscillator form

$$V_{\text{eff}}(x) = \frac{1}{4}M(x)\omega^2 x^2,$$
 (2)

except that the mass

$$M(x) = \begin{cases} \left(1 + \frac{x}{a}\right)^{-1} & \text{if } -a < x < +\infty, \\ +\infty & \text{if } x \le -a, \end{cases}$$
(3)

with a > 0, depends on the position in such a way that $V_{\text{eff}}(-a) = +\infty$ and $\lim_{x\to+\infty} V_{\text{eff}}(x) = +\infty.^{1}$

By directly solving the differential equation (1), they found that the spectrum of this semiconfined model is that of the standard harmonic oscillator,

$$E_n = \omega \left(n + \frac{1}{2} \right), \qquad n = 0, 1, 2, \dots,$$

$$\tag{4}$$

with corresponding wavefunctions

$$\psi_n(x) = C_n \left(1 + \frac{x}{a} \right)^{\frac{1}{2}\omega a^2} e^{-\frac{1}{2}\omega a(x+a)} L_n^{(\omega a^2)} \left(\omega a^2 \left(1 + \frac{x}{a} \right) \right), \qquad -a < x < +\infty,$$
(5)

expressed in terms of Laguerre polynomials $L_n^{(\alpha)}(z)$ and vanishing at x = -a and $x \to +\infty$, as it should be. Here C_n is a normalization coefficient given by

$$C_n = (\omega a^2)^{\frac{1}{2}(\omega a^2 + 1)} \sqrt{\frac{n!}{a\Gamma(\omega a^2 + n + 1)}}.$$
 (6)

¹Note that we have adopted here units wherein $\hbar = 2m_0$ in the original paper.

These results may be alternatively derived by applying a PCT to the constantmass Schrödinger equation for the isotonic oscillator [26, 27]

$$\left(-\frac{d^2}{du^2} + U(u)\right)\phi_n(u) = \epsilon_n\phi_n(u),\tag{7}$$

where

$$U(u) = \frac{1}{4}\bar{\omega}^2 u^2 + \frac{g}{u^2}, \qquad g > 0, \qquad 0 < u < +\infty,$$
(8)

$$\epsilon_n = \bar{\omega}(2n + \alpha + 1), \qquad \alpha = \frac{1}{2}\sqrt{1 + 4g},$$
(9)

and

$$\phi_n(u) \propto u^{\alpha + \frac{1}{2}} e^{-\frac{1}{4}\bar{\omega}u^2} L_n^{(\alpha)} \left(\frac{1}{2}\bar{\omega}u^2\right).$$
(10)

A PCT transforming an equation such as (7) into a PDM equation of type (1) [21, 22] consists in making a change of variable

$$u(x) = \bar{a}v(x) + \bar{b}, \qquad v(x) = \int^x \sqrt{M(x')} \, dx',$$
 (11)

and a change of function

$$\phi_n(u(x)) \propto [M(x)]^{-1/4} \psi_n(x).$$
 (12)

The potential $V_{\text{eff}}(x)$ and the energy eigenvalues E_n of the PDM Schrödinger equation are then given in terms of the potential and the energy eigenvalues of the constant-mass one by

$$V_{\text{eff}}(x) = \bar{a}^2 U(u(x)) + \frac{M''}{4M^2} - \frac{7M'^2}{16M^3} + \bar{c}, \qquad (13)$$

and

$$E_n = \bar{a}^2 \epsilon_n + \bar{c},\tag{14}$$

where a prime denotes derivation with respect to x and \bar{a} , \bar{b} , \bar{c} are three real constants.

In the present case, from (3) and (11), we directly obtain

$$v(x) = 2a\sqrt{1+\frac{x}{a}}.$$
(15)

and

$$\frac{M''}{4M^2} - \frac{7M'^2}{16M^3} = \frac{1}{16a^2} \left(1 + \frac{x}{a}\right)^{-1} \tag{16}$$

for $-a < x < +\infty$. With the choice $\bar{a} = \sqrt{\frac{\omega}{2\bar{\omega}}}$, $\bar{b} = 0$, we get for the change of variable (11)

$$u(x) = a\sqrt{\frac{2\omega}{\bar{\omega}}}\sqrt{1+\frac{x}{a}}$$
(17)

and the change of function (12), together with (10), leads to

$$\psi_n(x) = C_n \left(1 + \frac{x}{a} \right)^{\alpha/2} e^{-\frac{1}{2}\omega a(x+a)} L_n^{(\alpha)} \left(\omega a^2 \left(1 + \frac{x}{a} \right) \right), \tag{18}$$

where C_n turns out to be

$$C_n = (\omega a^2)^{\frac{1}{2}(\alpha+1)} \sqrt{\frac{n!}{a\Gamma(\alpha+n+1)}}.$$
(19)

Furthermore, on assuming $\bar{c} = -\frac{\omega}{2}\alpha$, the transformed potential (13) becomes

$$V_{\text{eff}}(x) = \frac{a\omega^2}{4(x+a)} \left(x+a-\frac{\alpha}{a\omega}\right)^2,\tag{20}$$

with corresponding eigenvalues E_n given by (4).

If we compare these results with those of [25], we notice that we have obtained the same energy spectrum (4), but with generalized potential and wavefunctions, since the latter depend on an extra parameter α absent in [25]. By taking $\alpha = a^2 \omega$, the original results are retrieved, but for other values of α , the potential (20) describes a semiconfined shifted harmonic oscillator. Note that such a potential might be interpreted as a semiconfined harmonic oscillator in a uniform gravitational field as was done for a shifted harmonic oscillator with another type of PDM [29, 30].

3 Family of generalized semiconfined oscillator models

A further generalization of the model of [25] can be obtained by changing the PDM (3) into a PDM depending on some parameter m taking values in the interval 0 < m < 2,

$$M(x) = \begin{cases} \left(1 + \frac{x}{a}\right)^{-m} & \text{if } -a < x < +\infty, \\ +\infty & \text{if } x \le -a, \end{cases}$$
(21)

and determining the associated potential $V_{\text{eff}}(x)$ with the assumption that the starting constant-mass Schrödinger equation remains as given in (7) and (8). The results of Sec. 2 will then correspond to the m = 1 special case.

Equations (15) and (16) are now replaced by

$$v(x) = \frac{2a}{2-m} \left(1 + \frac{x}{a}\right)^{1-\frac{m}{2}}$$
(22)

and

$$\frac{M''}{4M^2} - \frac{7M'^2}{16M^3} = -\frac{1}{16a^2}m(3m-4)\left(1+\frac{x}{a}\right)^{m-2},\tag{23}$$

respectively. On keeping the same values for \bar{a} , \bar{b} , and \bar{c} as in Sec. 2, we get a new change of variable

$$u(x) = \frac{a}{2-m} \sqrt{\frac{2\omega}{\bar{\omega}}} \left(1 + \frac{x}{a}\right)^{1-\frac{m}{2}},\tag{24}$$

but the resulting energy eigenvalues remain given by (4). From (13), however, the resulting potential turns out to be *m*-dependent and given by

$$V_{\text{eff}}(x) = \frac{a^m \omega^2}{4(2-m)^2} (x+a)^{2-m} + \frac{[(m-2)\alpha - (m-1)][(m-2)\alpha + m-1]}{4a^m (x+a)^{2-m}} - \frac{1}{2} \omega \alpha.$$
(25)

This is also the case for the wavefunctions, which become

$$\psi_n(x) = C_n \left(1 + \frac{x}{a} \right)^{-\frac{m}{2}(\alpha+1) + \alpha + \frac{1}{2}} e^{-\frac{1}{2(2-m)^2} \omega a^2 \left(1 + \frac{x}{a} \right)^{2-m}} \times L_n^{(\alpha)} \left(\frac{\omega a^2}{(2-m)^2} \left(1 + \frac{x}{a} \right)^{2-m} \right),$$
(26)

where

$$C_n = \left(\frac{\omega a^2}{(2-m)^2}\right)^{\frac{1}{2}(\alpha+1)} \sqrt{\frac{(2-m)n!}{a\Gamma(\alpha+n+1)}}.$$
 (27)

The new *m*-dependent potential (25) will be a semiconfined potential provided it goes to $+\infty$ for $x \to +\infty$ and $x \to -a$. The former condition is automatically satisfied, but the latter imposes that

$$\alpha > \frac{m-1}{2-m},\tag{28}$$



Figure 1: Plot of the semiconfined potential (25) in terms of x for m = 1 (black line), $m = \frac{1}{2}$ (red line), and $m = \frac{3}{2}$ (green line). The parameter values are $\omega = 1$, a = 2, and $\alpha = 4$.

which implies a restriction for m values such that $\frac{m-1}{2-m} > \frac{1}{2}$, i.e., for those in the interval $\frac{4}{3} < m < 2$. In such a case, the wavefunctions (26) vanish for $x \to +\infty$ and $x \to -a$, as it should be. The minimum of the potential occurs for

$$x_{\min} = -a + \left\{ \frac{(2-m)^2}{a^m \omega} \sqrt{\alpha^2 - \left(\frac{m-1}{2-m}\right)^2} \right\}^{1/(2-m)}$$
(29)

and is given by

$$(V_{\text{eff}})_{\min} = \frac{1}{2}\omega \left\{ \sqrt{\alpha^2 - \left(\frac{m-1}{2-m}\right)^2} - \alpha \right\}.$$
(30)

It is therefore slightly negative, except for m = 1 for which it vanishes.

In Fig. 1, we show the dependence of the semiconfined potential (25) on m. The black line corresponds to the original semiconfined harmonic oscillator (2).

4 Comments

In the present paper, we have first shown that the PCT method applied to the constant-mass Schrödinger equation for the isotonic oscillator allows us to easily retrieve the results of [25] and to extend them in order to describe a semiconfined shifted harmonic oscillator, which might be interpreted as a semiconfined harmonic oscillator in a uniform gravitational field.

In a second step, we have obtained a further generalization by considering a m-dependent PDM for 0 < m < 2 and by deriving the corresponding semiconfined potential with the same spectrum as the standard harmonic oscillator. We have therefore constructed a family of PDM and potential pairs, to which the original pair belongs as it corresponds to m = 1.

In [25], the BenDaniel-Duke ordering [28] was chosen for the momentum and mass operators. One finds, however, in the literature, several other orderings, which are special cases of the von Roos general two-parameter form of the kinetic energy operator [31], for which the Schrödinger operator writes

$$\begin{cases} -\frac{1}{2} \left[M(x)^{\xi} \frac{d}{dx} M(x)^{\eta} \frac{d}{dx} M(x)^{\zeta} + M(x)^{\zeta} \frac{d}{dx} M(x)^{\eta} \frac{d}{dx} M(x)^{\xi} \right] + V_{\rm vR}(x) \end{cases} \psi_n(x) \\ = E_n \psi_n(x), \tag{31}$$

where ξ , η , ζ are some real parameters restricted by the condition $\xi + \eta + \zeta = -1$. In particular, the BenDaniel-Duke ordering corresponds to $\xi = \zeta = 0$, $\eta = -1$ and the relation between the potentials in (1) and (31) is given by

$$V_{\rm vR}(x) = V_{\rm eff}(x) - \frac{1}{2}(1+\eta)\frac{M''}{M^2} + [\xi(\xi+\eta+1)+\eta+1]\frac{M'^2}{M^3}.$$
 (32)

For the mass chosen in (21), the latter becomes

$$V_{\rm vR}(x) = V_{\rm eff}(x) + \left\{ -\frac{1}{2}(1+\eta)m(m+1) + [\xi(\xi+\eta+1)+\eta+1]m^2 \right\} \frac{(x+a)^{m-2}}{a^m}.$$
 (33)

It is worth noting, in particular, the Zhu-Kroemer [32] and Mustafa-Mazharimousavi [33] orderings, which pass the de Souza Dutra and Almeida test [34] as good orderings. The former corresponds to $\xi = \zeta = -\frac{1}{2}$, $\eta = 0$, and leads to replacing (33) by

$$V_{\rm ZK}(x) = \frac{a^m \omega^2}{4(2-m)^2} (x+a)^{2-m} + \frac{[(m-2)\alpha - 1][(m-2)\alpha + 1]}{4a^m (x+a)^{2-m}} - \frac{1}{2}\omega\alpha, \quad (34)$$

while the latter is associated with $\xi = \zeta = -\frac{1}{4}$, $\eta = -\frac{1}{2}$, and gives rise to

$$V_{\rm MM}(x) = \frac{a^m \omega^2}{4(2-m)^2} (x+a)^{2-m} + \frac{(m-2)^2(\alpha^2 - \frac{1}{4})}{4a^m (x+a)^{2-m}} - \frac{1}{2}\omega\alpha.$$
 (35)

These potentials have a behaviour very similar to that of $V_{\text{eff}}(x)$, since they are semiconfined for α restricted to $\alpha > 1/(2 - m)$ or for any value of α (> 1/2 by definition (9)), respectively. The place of the minimum and its value are given by (29) and (30) provided $\sqrt{\alpha^2 - [(m-1)/(2-m)]^2}$ is replaced by $\sqrt{\alpha^2 - 1/(2-m)^2}$ or $\sqrt{\alpha^2 - 1/4}$.

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