This full text paper was peer reviewed at the direction of IEEE Communications Society subject matter experts for publication in the IEEE Globecom 2010 proceedings.

Generalized Sum-Product Algorithm for Joint Channel Decoding and Physical-Layer Network Coding in Two-Way Relay Systems

Dirk Wübben and Yidong Lang
Department of Communications Engineering, University of Bremen, Otto-Hahn-Allee, 28359 Bremen, Germany
Email: {wuebben, lang}@ant.uni-bremen.de, Phone: +49 421 218 62385

***Abstract*—In this paper a physical-layer network coded two-way relay system applying Low-Density Parity-Check (LDPC) codes for error correction is considered, where two sources A and B desire to exchange information with each other by the help of a relay R. The critical process in such a system is the calculation of the network-coded transmit word at the relay on basis of the superimposed channel-coded words of the two sources. For this joint channel-decoding and network-encoding task a generalized Sum-Product Algorithm (SPA) is developed. This novel iterative decoding approach outperforms other recently proposed schemes as demonstrated by simulation results.**

I. Introduction

The concept of network coding affords to increase the throughput of networks by allowing the intermediate nodes to perform operations on the incoming data [1]. The basic idea can be applied to wireless two-way relay systems, where two sources A and B desire to exchange information with each other by the help of a relay R. In a direct approach A and B send their information one after the other to the relay and R transmits the XOR of both received messages to the sources in a third time slot exploring the broadcast nature of the wireless channel [2]. As both sources know what they have transmitted to R previously, they can estimate the message of the other source by simple XOR operation. Thus, the number of time slots for information exchange is reduced to two by allowing A and B to transmit their information simultaneously to R. This scheme is called physical-layer network coding (PLNC) as the information of both sources are combined during the transmission [3]. In PLNC a relay is not required to decode the information of the two sources explicitly, but it can map the received signal directly to a network encoded signal to be relayed. Such modulation-demodulation approaches neglecting the impact of channel coding have been considered in [3], [4]. Extensions for Joint Channel decoding and physical-layer Network Coding (JCNC) have been presented in [5], [6]. Applying the same linear channel code at both source nodes, the XOR of both source codewords is also a valid codeword. Thus, the received signal can be decoded to the XOR of the source information at the relay without changing the decoding algorithm. As this approach looses information, decoding based on the arithmetic-sum (AS-JCNC) of the source codewords was proposed for Repeat Accumulate (RA) codes in [7] and for LDPC codes in [8], [9] requiring an adopted channel decoder at the relay. However, as AWGN channels have been considered so far, the arithmetic-sum contains only three different values. In this paper we consider fading channels leading to four different undisturbed receive values. For this system we develop a Generalized Sum-Product Algorithm (G-SPA) over the Galois field $\mathbb{F}_4$ [10], [11] resulting in an improved performance.

The remainder of this paper is organized as follows. In Section II the two-way relaying system is introduced. In Section III common schemes for joint channel decoding and physical-layer network coding from the literature are reviewed. Our new approach is developed in Section IV and the performance of the different schemes is compared in Section V. The paper is finished by a summary in Section VI.

This work was supported in part by the German Research Foundation (DFG) under grant Wu 499/7 and the Central Research Funding, University of Bremen, Germany, under grant 01/129/07.

II. System Model

Fig. 1 illustrates a two-way relay system with two sources A and B, and one relay R. Both sources wish to exchange information with each other by the help of R as no direct link is present. It is assumed, that all nodes operate in half-duplex mode, i.e., they can not receive and transmit simultaneously

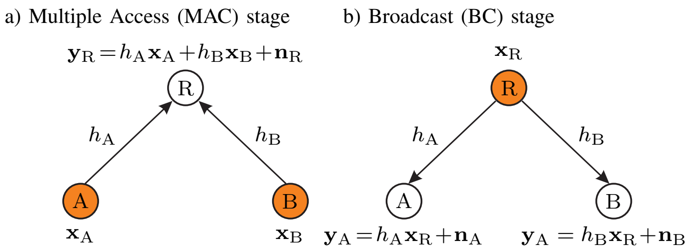

Fig. 1. Two source A and B exchange information with each other by the relay R. The communication consists of a MAC and a BC stage.

Let $\mathbf{b}_\mathrm{A}$ and $\mathbf{b}_\mathrm{B}$ denote the binary information words of length K of A and B, respectively. This information is encoded by the same linear channel code Γ with code rate $R_c = K/N$ into the codewords $\mathbf{c}_\mathrm{A} = \Gamma(\mathbf{b}_\mathrm{A})$ and $\mathbf{c}_\mathrm{B} = \Gamma(\mathbf{b}_\mathrm{B})$ of length N, denoted as *source codewords*. Afterwards, the codewords

This full text paper was peer reviewed at the direction of IEEE Communications Society subject matter experts for publication in the IEEE Globecom 2010 proceedings.

are BPSK-modulated to $\mathbf{x}_{\mathrm{A}} = \mathcal{M}\{\mathbf{c}_{\mathrm{A}}\}$ and $\mathbf{x}_{\mathrm{B}} = \mathcal{M}\{\mathbf{c}_{\mathrm{B}}\}$ according to the mapping rule $0 \to 1$ and $1 \to -1$.

The two-way relaying transmission consists of two stages: multiple access (MAC) and broadcast (BC). In the MAC stage, both sources A and B transmit $\mathbf{x}_{\mathrm{A}}$ and $\mathbf{x}_{\mathrm{B}}$ to the relay R simultaneously. Thus, the received signal at R is given by the linear superposition of the transmit signals weighted by the fading coefficients h_{A} and h_{B}, i.e.,

$$\mathbf{y}_{\mathrm{R}} = h_{\mathrm{A}}\mathbf{x}_{\mathrm{A}} + h_{\mathrm{B}}\mathbf{x}_{\mathrm{B}} + \mathbf{n}_{\mathrm{R}} \ . \tag{1}$$

The elements of the noise vector $\mathbf{n}_{\mathrm{R}}$ at the relay are i.i.d zero-mean complex Gaussian random variables with variance σ_n^2. For network coding the relay performs an estimation with respect to the XOR of the source codewords $\mathbf{c}_{\mathrm{A}\oplus\mathrm{B}} = \mathbf{c}_{\mathrm{A}} \oplus \mathbf{c}_{\mathrm{B}}$ based on $\mathbf{y}_{\mathrm{R}}$. This estimated *relay codeword* $\mathbf{c}_{\mathrm{R}} = \hat{\mathbf{c}}_{\mathrm{A}\oplus\mathrm{B}}$ is again BPSK-modulated and the relay broadcasts $\mathbf{x}_{\mathrm{R}} = \mathcal{M}\{\mathbf{c}_{\mathrm{R}}\}$ towards both sources A and B in the BC stage. For simplicity it is assumed, that the transmission channels are reciprocal, the relay R transmits with the same power as both sources, and the noise variance is again σ_n^2. Thus, the received signals at A and B are given by $\mathbf{y}_{\mathrm{A}} = h_{\mathrm{A}}\mathbf{x}_{\mathrm{R}} + \mathbf{n}_{\mathrm{A}}$ and $\mathbf{y}_{\mathrm{B}} = h_{\mathrm{B}}\mathbf{x}_{\mathrm{R}} + \mathbf{n}_{\mathrm{B}}$. Both sources A and B can then estimate the information $\hat{\mathbf{b}}_{\mathrm{R,A}}$ and $\hat{\mathbf{b}}_{\mathrm{R,B}}$ from $\mathbf{y}_{\mathrm{A}}$ and $\mathbf{y}_{\mathrm{B}}$, respectively. Since both sources know what they have transmitted in the MAC stage, the information from the other source can be obtained by simple XOR, i.e., $\hat{\mathbf{b}}_{\mathrm{B}} = \hat{\mathbf{b}}_{\mathrm{R,A}} \oplus \mathbf{b}_{\mathrm{A}}$ and $\hat{\mathbf{b}}_{\mathrm{A}} = \hat{\mathbf{b}}_{\mathrm{R,B}} \oplus \mathbf{b}_{\mathrm{B}}$. For the critical step of channel decoding and physical-layer network encoding at the relay, a new decoding algorithm for $\mathbf{y}_{\mathrm{R}} \to \mathbf{c}_{\mathrm{R}}$ is derived in this paper. This new algorithm makes full use of the channel codes in combination with the PLNC scheme.

i	c_{A}	c_{B}	$c_{\mathrm{A}\oplus\mathrm{B}}$	c_{AB}	x_{A}	x_{B}	s_{AB}
0	0	0	0	0	1	1	$h_{\mathrm{A}} + h_{\mathrm{B}}$
1	1	0	1	1	-1	1	$-h_{\mathrm{A}} + h_{\mathrm{B}}$
2	0	1	1	D	1	-1	$h_{\mathrm{A}} - h_{\mathrm{B}}$
3	1	1	0	$1{+}D$	-1	-1	$-h_{\mathrm{A}} - h_{\mathrm{B}}$

TABLE I

Mapping rules for code bits (c_{A}, c_{B}) and transmit signals (x_{A}, x_{B}).

Subsequently, some basic relations between the different occurring signals and their probabilities are given. The codeword $\mathbf{c}_{\mathrm{A}} = [c_{\mathrm{A}}(1) \ \ldots \ c_{\mathrm{A}}(N)]$ of source A consists of N symbols $c_{\mathrm{A}}(n)$. In order to ease the notation, the time index n will be avoided whenever possible, i.e., c_{A} denotes an arbitrary symbol of the source codeword $\mathbf{c}_{\mathrm{A}}$. The same argument holds for the elements of the remaining vectors. Tab. I summarizes the basic relationships between the occurring code symbols c_{A}, c_{B} and the corresponding BPSK signals x_{A}, x_{B}. For the later derivations we also include the XOR $c_{\mathrm{A}\oplus\mathrm{B}} = c_{\mathrm{A}} \oplus c_{\mathrm{B}}$ of the code symbols and the four different noise-free signal levels at the receiver side $s_{\mathrm{AB}} \in \mathcal{S}_{\mathrm{AB}}$ with $\mathcal{S}_{\mathrm{AB}} = \{h_{\mathrm{A}}+h_{\mathrm{B}}, -h_{\mathrm{A}}+h_{\mathrm{B}}, h_{\mathrm{A}}-h_{\mathrm{B}}, -h_{\mathrm{A}}-h_{\mathrm{B}}\}$ following (1). Furthermore, the quaternary symbol $c_{\mathrm{AB}} = c_{\mathrm{A}} + c_{\mathrm{B}}D$ is defined as a short hand notation for the four different combinations of c_{A} and c_{B}, i.e, $c_{\mathrm{AB}} \in \mathcal{C}_{\mathrm{AB}}$ with Galois Field $\mathcal{C}_{\mathrm{AB}} = \mathbb{F}_4 = \{0, 1, D, 1{+}D\}$ (in polynomial description with indeterminate D). Please note that $c_{\mathrm{AB}} = \mathcal{C}_{\mathrm{AB}}(i)$ and $s_{\mathrm{AB}} = \mathcal{S}_{\mathrm{AB}}(i)$ represent the i-th event $(0 \leq i \leq 3)$ in $\mathbb{F}_4$ and in the receive signal space, respectively.

The a-priori probabilities for $c_{\mathrm{AB}} = \mathcal{C}_{\mathrm{AB}}(i)$ and $s_{\mathrm{AB}} = \mathcal{S}_{\mathrm{AB}}(i)$ are given for equally likely code symbols by $\Pr\{c_{\mathrm{AB}} = \mathcal{C}_{\mathrm{AB}}(i)\} = \Pr\{s_{\mathrm{AB}} = \mathcal{S}_{\mathrm{AB}}(i)\} = \frac{1}{4}$. The probability density for y_{R} given the noise-free receive signal $s_{\mathrm{AB}} \in \mathcal{S}_{\mathrm{AB}}$ can be calculated by

$$p\{y_{\mathrm{R}}|s_{\mathrm{AB}}{=}\mathcal{S}_{\mathrm{AB}}(i)\} = \frac{1}{\pi\sigma_n^2}\exp\left(-\frac{|y_{\mathrm{R}} - \mathcal{S}_{\mathrm{AB}}(i)|^2}{\sigma_n^2}\right) \ . \tag{2}$$

Thus, the probability that the signal $s_{\mathrm{AB}} = \mathcal{S}_{\mathrm{AB}}(i)$ was transmitted given the current receive signal y_{R} is

$$\begin{aligned} P_i &= \Pr\{c_{\mathrm{AB}} = \mathcal{C}_{\mathrm{AB}}(i)|y_{\mathrm{R}}\} = \Pr\{s_{\mathrm{AB}} = \mathcal{S}_{\mathrm{AB}}(i)|y_{\mathrm{R}}\} \\ &= p\{y_{\mathrm{R}}|s_{\mathrm{AB}} = \mathcal{S}_{\mathrm{AB}}(i)\}\frac{\Pr\{s_{\mathrm{AB}} = \mathcal{S}_{\mathrm{AB}}(i)\}}{\Pr\{y_{\mathrm{R}}\}} \\ &= p\{y_{\mathrm{R}}|s_{\mathrm{AB}} = \mathcal{S}_{\mathrm{AB}}(i)\}\frac{1}{C} \ . \end{aligned} \tag{3}$$

As the sum over all probabilities P_i should be 1, the constant $C = 4\Pr\{y_{\mathrm{R}}\}$ in (3) can be calculated and is used to normalize the probabilities P_i.

III. Common Decoding Schemes

In this section two common approaches to perform the decoding at the relay are repeated shortly.

A. Separated Channel Decoding (SCD)

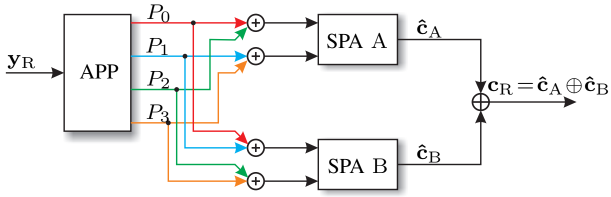

Fig. 2. Block diagram for parallel Separated Channel Decoding (P-SCD) for estimation of $\mathbf{c}_{\mathrm{A}}$ and $\mathbf{c}_{\mathrm{B}}$ and subsequent network encoding $\mathbf{c}_{\mathrm{R}} = \hat{\mathbf{c}}_{\mathrm{A}} \oplus \hat{\mathbf{c}}_{\mathrm{B}}$.

The estimation of the source information at the relay can be interpreted as the traditional multiple access problem, which aims to estimate $\mathbf{c}_{\mathrm{A}}$ and $\mathbf{c}_{\mathrm{B}}$ explicitly by Separated Channel Decoding (SCD). One simple approach performs a decoding of $\mathbf{c}_{\mathrm{A}}$ and $\mathbf{c}_{\mathrm{B}}$ on basis of the receive word $\mathbf{y}_{\mathrm{R}}$ in parallel (called P-SCD) as shown in Fig. 2. The *a-posteriori probabilities* (APPs) for c_{A} given the receive signal y_{R} are given by

$$\begin{aligned} \Pr\{c_{\mathrm{A}}{=}0|y_{\mathrm{R}}\} &= \Pr\{c_{\mathrm{AB}}{=}0|y_{\mathrm{R}}\} + \Pr\{c_{\mathrm{AB}}{=}D|y_{\mathrm{R}}\} \\ &= P_0 + P_2 \end{aligned} \tag{4a}$$

$$\begin{aligned} \Pr\{c_{\mathrm{A}}{=}1|y_{\mathrm{R}}\} &= \Pr\{c_{\mathrm{AB}}{=}1|y_{\mathrm{R}}\} + \Pr\{c_{\mathrm{AB}}{=}1{+}D|y_{\mathrm{R}}\} \\ &= P_1 + P_3 \end{aligned} \tag{4b}$$

and can be calculated using (3). For decoding, the APP vector

$$[\Pr\{c_{\mathrm{A}} = 0|y_{\mathrm{R}}\} \ \ \Pr\{c_{\mathrm{A}} = 1|y_{\mathrm{R}}\}] = [P_0 + P_2 \ \ P_1 + P_3] \tag{5}$$

or the corresponding Log-Likelihood Ratio (LLR)

$$\lambda_{\mathrm{A}} = \ln\left(\frac{\Pr\{c_{\mathrm{A}} = 0|y_{\mathrm{R}}\}}{\Pr\{c_{\mathrm{A}} = 1|y_{\mathrm{R}}\}}\right) = \ln\left(\frac{P_0 + P_2}{P_1 + P_3}\right) \tag{6}$$

978-1-4244-5637-6/10/$26.00 ©2010 IEEE

is fed to the Sum-Product Algorithm (SPA). At the output the estimate $\hat{\mathbf{c}}_{\mathrm{A}}$ for the codeword transmitted by source A is achieved. Similarly, the APPs for c_{B} are given by $\Pr\{c_{\mathrm{B}}=0|y_{\mathrm{R}}\} = P_0 + P_1$, $\Pr\{c_{\mathrm{B}}=1|y_{\mathrm{R}}\} = P_2 + P_3$ and the APP vector $[\Pr\{c_{\mathrm{B}}=0|y_{\mathrm{R}}\}\ \Pr\{c_{\mathrm{B}}=1|y_{\mathrm{R}}\}] = [P_0+P_1\ P_2+P_3]$ or the LLR $\lambda_{\mathrm{B}} = \ln\left(\frac{\Pr\{c_{\mathrm{B}}=0|y_{\mathrm{R}}\}}{\Pr\{c_{\mathrm{B}}=1|y_{\mathrm{R}}\}}\right) = \ln\left(\frac{P_0+P_1}{P_2+P_3}\right)$ is fed to the decoder B to yield the estimate $\hat{\mathbf{c}}_{\mathrm{B}}$. Finally, the decoder output vectors $\hat{\mathbf{c}}_{\mathrm{A}}$ and $\hat{\mathbf{c}}_{\mathrm{B}}$ are combined to achieve the *relay codeword* $\mathbf{c}_{\mathrm{R}} = \hat{\mathbf{c}}_{\mathrm{A}} \oplus \hat{\mathbf{c}}_{\mathrm{B}}$. Thus, at the relay common network coding is performed as presented in [2].

Alternatively, for successive decoding (called S-SCD) the decoding result of the channel with the larger fading gain is subtracted form the receive signal and a common decoding for the second codeword with respect to this interference reduced signal is performed.

B. Joint Channel Decoding and Physical-Layer Network Coding (JCNC)

For physical-layer network coding the relay is asked to generate from the receive signal $\mathbf{y}_{\mathrm{R}}$ a network coded symbol $\mathbf{x}_{\mathrm{R}}$ being a function of $\mathbf{c}_{\mathrm{A}}$ and $\mathbf{c}_{\mathrm{B}}$ (or $\mathbf{x}_{\mathrm{A}}$ and $\mathbf{x}_{\mathrm{B}}$). To generate this relay codeword $\mathbf{c}_{\mathrm{R}}$ it is not necessary that the relay knows the source codewords $\mathbf{c}_{\mathrm{A}}$ and $\mathbf{c}_{\mathrm{B}}$ explicitly as observed in [3].

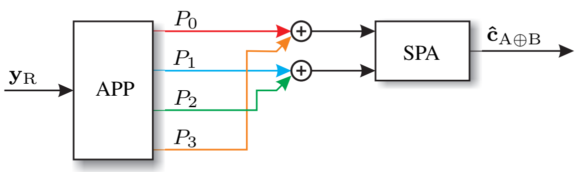

Fig. 3. Block diagram for Joint Cannel decoding and physical-layer Network Coding (JCNC)

Since $\mathbf{c}_{\mathrm{A}}$ and $\mathbf{c}_{\mathrm{B}}$ are codewords of the same linear channel code Γ, the modulo-2 sum $\mathbf{c}_{\mathrm{A}\oplus\mathrm{B}} = \mathbf{c}_{\mathrm{A}} \oplus \mathbf{c}_{\mathrm{B}}$ is also a valid codeword of Γ. Thus, Joint Channel decoding and physical-layer Network coding (JCNC) aims to estimate that codeword $\hat{\mathbf{c}}_{\mathrm{A}\oplus\mathrm{B}}$ that caused the observation $\mathbf{y}_{\mathrm{R}}$ most likely using a standard decoding algorithm, i.e., by SPA for LDPC codes. To this end, the APPs for $c_{\mathrm{A}\oplus\mathrm{B}} = 0$ and $c_{\mathrm{A}\oplus\mathrm{B}} = 1$ have to be calculated for each index $1 \le n \le N$ of the codeword with respect to the corresponding observation y_{R}. The APPs

$$\Pr\{c_{\mathrm{A}\oplus\mathrm{B}}=0|y_{\mathrm{R}}\} = \Pr\{c_{\mathrm{AB}}=0|y_{\mathrm{R}}\} + \Pr\{c_{\mathrm{AB}}=1+D|y_{\mathrm{R}}\} = P_0 + P_3 \tag{7a}$$

$$\Pr\{c_{\mathrm{A}\oplus\mathrm{B}}=1|y_{\mathrm{R}}\} = \Pr\{c_{\mathrm{AB}}=1|y_{\mathrm{R}}\} + \Pr\{c_{\mathrm{AB}}=D|y_{\mathrm{R}}\} = P_1 + P_2 \tag{7b}$$

can again be calculated using (3). For decoding the APP vector

$$[\Pr\{c_{\mathrm{A}\oplus\mathrm{B}}=0|y_{\mathrm{R}}\}\ \Pr\{c_{\mathrm{A}\oplus\mathrm{B}}=1|y_{\mathrm{R}}\}] = [P_0+P_3\ P_1+P_2] \tag{8}$$

or the corresponding LLR

$$\lambda_{c_{\mathrm{A}\oplus\mathrm{B}}} = \ln\left(\frac{\Pr\{c_{\mathrm{A}\oplus\mathrm{B}}=0|y_{\mathrm{R}}\}}{\Pr\{c_{\mathrm{A}\oplus\mathrm{B}}=1|y_{\mathrm{R}}\}}\right) = \ln\left(\frac{P_0+P_3}{P_1+P_2}\right) \tag{9}$$

is fed to the SPA as shown in Fig. 3. At the output the estimate for the relay codeword $\mathbf{c}_{\mathrm{R}} = \hat{\mathbf{c}}_{\mathrm{A}\oplus\mathrm{B}}$ is achieved which is then transmitted to both sources after BPSK-modulation.

The basic idea of this approach is the estimation of the XOR of the two source vectors using a common decoder. However, this direct decoding $\mathbf{y}_{\mathrm{R}} \to \hat{\mathbf{c}}_{\mathrm{A}\oplus\mathrm{B}}$ discards useful information provided by the two channel codes [7]. In order to improve the decoding at the relay, an improved decoding algorithm is presented in the next section.

IV. Generalized Joint Channel Decoding and Physical-Layer Network Coding (G-JCNC)

A. General Approach

Instead of decoding the source signals separately as presented in Section III-A or by decoding the XOR as in Section III-B, we propose to decode the two codes jointly within a Generalized Sum-Product Algorithm (G-SPA). Thus, first the decoding $\mathbf{y}_{\mathrm{R}} \to \hat{\mathbf{c}}_{\mathrm{AB}}$ with respect to $\mathbb{F}_4$ is performed and then the physical layer network coding $\hat{\mathbf{c}}_{\mathrm{AB}} \to \mathbf{c}_{\mathrm{R}}$ by a corresponding mapping rule is executed as shown in Fig. 4. This approach fully exploits all available information about the superimposed receive signal as well as the code structure of both channel codes.

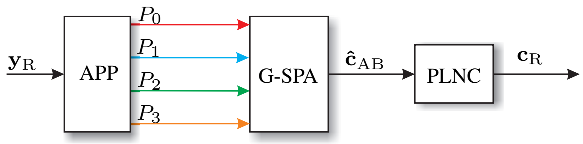

Fig. 4. Block diagram for Generalized Joint Channel decoding and physical-layer Network Coding (G-JCNC)

The basic idea goes back to the Arithmetic-Sum JCNC (AS-JCNC) approach presented in [7]–[9], where the authors restricted the analysis to AWGN channels. In contrast to our approach, this results in only three different undisturbed signal levels $\mathcal{S}'_{\mathrm{AB}} = \{-2, 0, 2\}$ where $s'_{\mathrm{AB}} = 0$ is true for the two cases where $x_{\mathrm{A}} \neq x_{\mathrm{B}}$. However, for fading channels the gains h_{A} and h_{B} are usually different and the knowledge due to the four different receive signal levels should be used in the decoding process by means of a modified SPA over $\mathbb{F}_4$.

In order to derive this G-SPA the channel encoding and physical layer encoding process is considered. Basically, each code symbol of a linear channel code consists of the modulo-2 sum of some information bits, e.g., the codebit $c_{\mathrm{A}}(n)$ of Γ is given by the sum of the k-th and the ℓ-th information bits $c_{\mathrm{A}}(n) = b_{\mathrm{A}}(k) \oplus b_{\mathrm{A}}(\ell)$. As the same code is used at both sources, the n-th quaternary signal calculates as

$$\begin{aligned} c_{\mathrm{AB}}(n) &= c_{\mathrm{A}}(n) + c_{\mathrm{B}}(n)D & \text{(10a)} \\ &= (b_{\mathrm{A}}(k) \oplus b_{\mathrm{A}}(\ell)) + (b_{\mathrm{B}}(k) \oplus b_{\mathrm{B}}(\ell))\,D & \text{(10b)} \\ &= (b_{\mathrm{A}}(k) + b_{\mathrm{B}}(k)D) \oplus (b_{\mathrm{A}}(\ell) + b_{\mathrm{B}}(\ell)D) & \text{(10c)} \\ &= b_{\mathrm{AB}}(k) \oplus b_{\mathrm{AB}}(\ell)\,. & \text{(10d)} \end{aligned}$$

Thus, $c_{\mathrm{AB}}(n)$ is simply given by the sum of the quaternary information symbols $b_{\mathrm{AB}}(k) = b_{\mathrm{A}}(k) + b_{\mathrm{B}}(k)D$ and

$b_{\mathrm{AB}}(\ell) = b_{\mathrm{A}}(\ell) + b_{\mathrm{B}}(\ell)D$ in $\mathbb{F}_4$. Similarly, if a code bit equals the modulo-2 sum of more than two information bits, the symbol $c_{\mathrm{AB}}(n)$ is given by the sum of the corresponding quaternary symbols $b_{\mathrm{AB}}(\cdot)$ in $\mathbb{F}_4$. Based on this observation, the overall encoding process of $\mathbf{b}_{\mathrm{AB}} = \mathbf{b}_{\mathrm{A}} + \mathbf{b}_{\mathrm{B}}D \rightarrow \mathbf{c}_{\mathrm{AB}}$ can be interpreted as a LDPC code over $\mathbb{F}_4$ with the restriction, that the elements of the resulting parity check matrix are either 0 or $1+D$. Consequently, a SPA for $\mathbb{F}_4$ can be used for decoding [11]. In a similar way also the sum of several quaternary code symbols $c_{\mathrm{AB}}(\cdot)$ has to be executed in $\mathbb{F}_4$.

B. Messages and Initialization

The G-SPA determines iteratively the a-posteriori probability of each message symbol $c_{\mathrm{AB}}(n)$ and it is conveniently described over a factor graph which depicts the relations between the variable nodes and the check nodes defined by the parity check matrix $\mathbf{H}$ of the LDPC code [10].

The probability mass function for a quaternary random variable can be represented by the probability vector $\mathbf{p} = [p_0 \;\; p_1 \;\; p_2 \;\; p_3]$ where p_i denotes the probability that the value of the variable is $\mathcal{C}_{\mathrm{AB}}(i)$ with $i \in \{0,1,2,3\}$ and $p_0+p_1+p_2+p_3 = 1$ holds. Within the G-SPA these probability vectors are exchanged between the variable nodes and the check nodes as messages. The initial message of variable node $c_{\mathrm{AB}}(n)$ given the received signal $y_{\mathrm{R}}(n)$ equals

$$\mathbf{p} = \begin{bmatrix} P_0 & P_1 & P_2 & P_3 \end{bmatrix} , \tag{11}$$

with probabilities $P_i = \Pr\{c_{\mathrm{AB}} = \mathcal{C}_{\mathrm{AB}}(i)|y_{\mathrm{R}}\}$ given in (3). Within the G-SPA the same message updating rules at the variable and the check nodes are used as discussed in [10]. Consistently, the update functions at the variable nodes and at the check nodes are defined as VAR and CHK, respectively. Subsequently, the discussion will be restricted to nodes of degree three, i.e., the nodes are connected by three edges. Messages from the variable nodes (or check nodes) with degree of greater than three can be calculated by

$$\mathrm{VAR}(\mathbf{p},\mathbf{q},\cdots) = \mathrm{VAR}(\mathbf{p},\mathrm{VAR}(\mathbf{q},\mathrm{VAR}(\cdot,\cdot)) \tag{12a}$$
$$\mathrm{CHK}(\mathbf{p},\mathbf{q},\cdots) = \mathrm{CHK}(\mathbf{p},\mathrm{CHK}(\mathbf{q},\mathrm{CHK}(\cdot,\cdot)) , \tag{12b}$$

where $\mathbf{p}$ and $\mathbf{q}$ denote corresponding input messages of the variable nodes (or check nodes) [10].

C. Output Message of Variable Nodes

When the two input messages $\mathbf{p} = [p_0 \; p_1 \; p_2 \; p_3]$ and $\mathbf{q} = [q_0 \; q_1 \; q_2 \; q_3]$ arrive at the variable node $c_{\mathrm{AB}}(n)$, the probability that the code symbol $c_{\mathrm{AB}}(n)$ is $\mathcal{C}_{\mathrm{AB}}(i)$, $0 \le i \le 3$, is given by

$$\begin{aligned}&\Pr\{c_{\mathrm{AB}}(n)=\mathcal{C}_{\mathrm{AB}}(i)|\mathbf{p},\mathbf{q}\}\\ &= \frac{\Pr\{\mathbf{p},\mathbf{q}|c_{\mathrm{AB}}(n)=\mathcal{C}_{\mathrm{AB}}(i)\}\Pr\{c_{\mathrm{AB}}(n)=\mathcal{C}_{\mathrm{AB}}(i)\}}{\Pr\{\mathbf{p},\mathbf{q}\}}\\ &= \frac{\Pr\{c_{\mathrm{AB}}(n)=\mathcal{C}_{\mathrm{AB}}(i)|\mathbf{p}\}\Pr\{c_{\mathrm{AB}}(n)=\mathcal{C}_{\mathrm{AB}}(i)|\mathbf{q}\}\Pr\{\mathbf{p}\}\Pr\{\mathbf{q}\}}{\Pr\{c_{\mathrm{AB}}(n)=\mathcal{C}_{\mathrm{AB}}(i)\}\Pr\{\mathbf{p},\mathbf{q}\}}\\ &= \beta p_i q_i ,\end{aligned} \tag{13}$$

where $\beta = \frac{\Pr\{\mathbf{p}\}\Pr\{\mathbf{q}\}}{\Pr\{c_{\mathrm{AB}}(n)=\mathcal{C}_{\mathrm{AB}}(i)\}\Pr\{\mathbf{p},\mathbf{q}\}}$ is a normalization factor. Since the sum of the probabilities (13) should be 1 over all i, the normalization factor equals $\beta = 1/(p_0q_0 + p_1q_1 + p_2q_2 + p_3q_3)$. Thus, the output message of the variable node is

$$\mathrm{VAR}(\mathbf{p},\mathbf{q}) = \beta \begin{bmatrix} p_0q_0 & p_1q_1 & p_2q_2 & p_3q_3 \end{bmatrix} . \tag{14}$$

D. Output Message of Check Nodes

A specific parity check equation is satisfied, if the $\mathbb{F}_4$ sum of the corresponding quaternary symbols equals zero, i.e., $c_{\mathrm{AB}}(k) \oplus c_{\mathrm{AB}}(\ell) \oplus c_{\mathrm{AB}}(n) = 0$. Assume the two input message vectors from the variable nodes $c_{\mathrm{AB}}(k)$ and $c_{\mathrm{AB}}(\ell)$ are $\mathbf{p} = [p_0 \; p_1 \; p_2 \; p_3]$ and $\mathbf{q} = [q_0 \; q_1 \; q_2 \; q_3]$, respectively. The probability that the parity check equation is satisfied under the assumption that $c_{\mathrm{AB}}(n)$ is fixed to $\mathcal{C}_{\mathrm{AB}}(0)$ equals

$$\begin{aligned}\Pr\{c_{\mathrm{AB}}(n)=0|\mathbf{p},\mathbf{q}\} &= \Pr\{c_{\mathrm{AB}}(k)=0, c_{\mathrm{AB}}(\ell)=0|\mathbf{p},\mathbf{q}\}\\ &\quad+\Pr\{c_{\mathrm{AB}}(k)=1, c_{\mathrm{AB}}(\ell)=1|\mathbf{p},\mathbf{q}\}\\ &\quad+\Pr\{c_{\mathrm{AB}}(k)=D, c_{\mathrm{AB}}(\ell)=D|\mathbf{p},\mathbf{q}\}\\ &\quad+\Pr\{c_{\mathrm{AB}}(k)=1+D, c_{\mathrm{AB}}(\ell)=1+D|\mathbf{p},\mathbf{q}\}\\ &= p_0q_0 + p_1q_1 + p_2q_2 + p_3q_3 .\end{aligned} \tag{15}$$

In a similar way, the probabilities $\Pr\{c_{\mathrm{AB}}(n) = \mathcal{C}_{\mathrm{AB}}(i)|\mathbf{p},\mathbf{q}\}$ are obtained for $i = 1, 2, 3$

$$\begin{aligned}\Pr\{c_{\mathrm{AB}}(n)=1|\mathbf{p},\mathbf{q}\} &= p_0q_1 + p_1q_0 + p_2q_3 + p_3q_2\\ \Pr\{c_{\mathrm{AB}}(n)=D|\mathbf{p},\mathbf{q}\} &= p_0q_2 + p_1q_3 + p_2q_0 + p_3q_1\\ \Pr\{c_{\mathrm{AB}}(n)=1+D|\mathbf{p},\mathbf{q}\} &= p_0q_3 + p_1q_2 + p_2q_1 + p_3q_0 .\end{aligned} \tag{16}$$

Finally, the message vector out of one check node equals

$$\mathrm{CHK}(\mathbf{p},\mathbf{q}) = \begin{bmatrix} p_0q_0 + p_1q_1 + p_2q_2 + p_3q_3\\ p_0q_1 + p_1q_0 + p_2q_3 + p_3q_2\\ p_0q_2 + p_1q_3 + p_2q_0 + p_3q_1\\ p_0q_3 + p_1q_2 + p_2q_1 + p_3q_0 \end{bmatrix}^T . \tag{17}$$

E. Finalization and PLNC Mapping

The decoding is stopped if all parity check equations are fulfilled or the maximum number of iterations is reached. Otherwise, the algorithm proceeds with steps *C* and *D* for further iterations until one of these conditions is fulfilled. At the end the decoding algorithm generates the APP vector $\mathbf{p}$ with $p_i = \Pr\{c_{\mathrm{AB}} = \mathcal{C}_{\mathrm{AB}}(i)|\mathbf{y}_{\mathrm{R}}\}$ for each code symbol $c_{\mathrm{AB}}(n)$ and the PLNC mapping is done by

$$c_{\mathrm{R}}(n) = \hat{c}_{\mathrm{AB}}(n) = \begin{cases} 1 & \text{if } \operatorname*{argmax}_i p_i = 1 \text{ or } 2\\ 0 & \text{else} \end{cases} . \tag{18}$$

V. Simulation Results

In this section, the performance of the proposed G-JCNC scheme is compared to the separated channel decoding schemes P-SCD and S-SCD of Sec. III-A, the JCNC of Sec. III-B and the arithmetic-sum JCNC scheme developed for AWGN channels [8]. Optimized LDPC codes for codeword length $N = 1000$ and code rate $R_c = 0.4$ are used [12]. All SPA decoders perform 10 iterations and perfect synchronization of all nodes is assumed. For simulations, a normalized fading channel is considered, where the channel for A is always $h_{\mathrm{A}} = 1$ and the channel for B is uniformly distributed on the unit circle, i.e., $h_{\mathrm{B}} = \exp(j\phi)$ with $\phi \sim U(-\pi, \pi)$, but remains constant for one transmission block. Thus, both channels have the same reliability. This enables an investigation of the decoding gain of the different approaches neglecting the influence of fading or diversity.

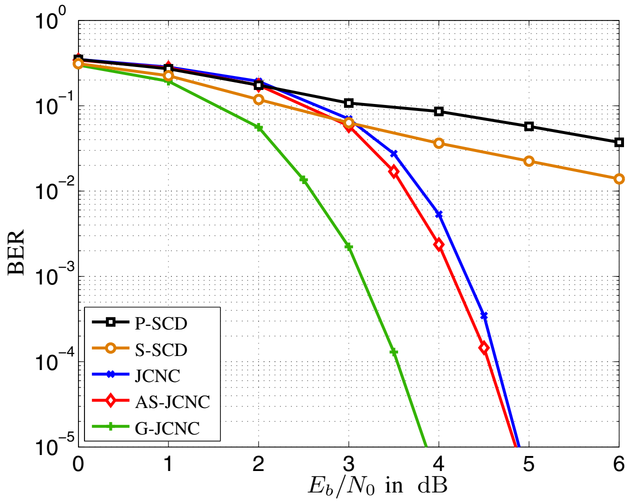

Fig. 5. BER for parallel and successive separated channel decoding (P-SCD and S-SCD), joint channel decoding and physical-layer network coding (JCNC), arithmetic-sum JCNC (AS-JCNC), and generalized JCNC (G-JCNC).

Fig. 5 shows the end-to-end bit error rate (BER) averaged over both sources A and B for varying E_b/N_0. The separated decoding approaches P-SCD and S-SCD are not able to estimate the relay codeword sufficiently leading to degraded performance. The approaches from the literature JCNC and AS-JCNC lead to much better results. However, the new approach G-JCNC significantly outperforms all other schemes under investigation. In comparison to the JCNC and the AS-JCNC scheme a gain of approximately $1\,\mathrm{dB}$ for BER of 10^{-4} is achieved.

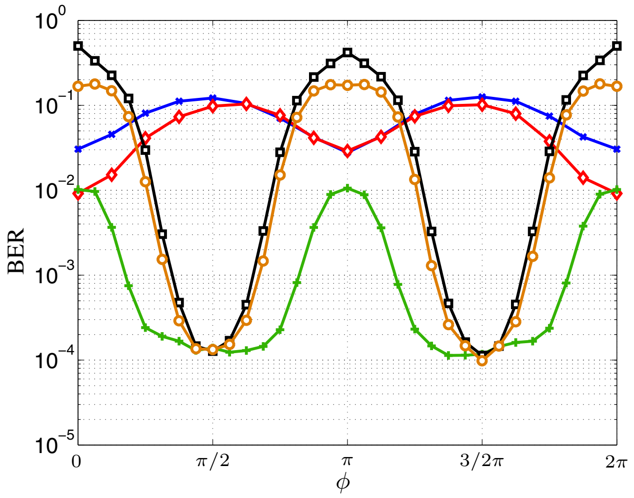

Fig. 6. BERs for a fixed $E_b/N_0 = 3\,\mathrm{dB}$ versus the angle ϕ.

For fixed $E_b/N_0 = 3\,\mathrm{dB}$, where all schemes perform reasonable well, Fig. 6 shows the BERs versus the angle ϕ. Both separated decoding schemes perform very well if the two channels do not interfere much, i.e. for $\phi \approx \pi/2$ $\phi \approx 3/2\pi$. Otherwise, the performance degrades significantly. JCNC and AS-JCNC are less effected by the angle, however do not reach low BERs. The most robust scheme with respect to the angle is our new approach based on the G-SPA.

The G-JCNC approach has recently been extended for the transmission of QPSK signals leading to an G-SPA over $\mathbb{F}_{16}$ [13]. Based on this framework, extensions for other modulation alphabets are straightforward. Furthermore, [13] contains the discussion of OFDM relaying systems with arbitrary fading coefficients.

VI. Summary

In this paper joint channel decoding and physical-layer network coding in two-way relay systems was investigated. The new decoding approach Generalized Joint Channel decoding and physical-layer Network Coding (G-JCNC) was presented to estimate the XOR of the two source codewords at the relay from the superimposed receive signal. To this end a Generalized Sum-Product Algorithm (G-SPA) was derived which performs decoding with respect to $\mathbb{F}_4$. The simulation results show a significant performance improvement in comparison to the schemes from the literature.

References

[1] R. Ahlswede, N. Cai, S.-Y.R. Li, and R.W. Yeung, "Network Information Flow," *IEEE Trans. on Information Theory*, vol. 46, no. 4, pp. 1204–1216, July 2000.
[2] C. Fragouli, J.Y. Boudec, and J. Widmer, "Network Coding: An Instant Primer," *ACM SIGCOMM Computer Communication Review*, vol. 36, no. 1, pp. 63–68, Mar. 2006.
[3] S. Zhang, S.C. Liew, and P. Lam, "Hot Topic: Physical Layer Network Coding," in *Proc. International Conference on Mobile Computing and Networking (MobiCom)*, Los Angeles, CA, USA, 2006, pp. 358–365.
[4] P. Popovski and H. Yomo, "The AntiPackets Can Increase the Achievable Throughput of a Wireless MultiHop Network," in *IEEE Proc. Int. Conference on Communications (ICC)*, Istanbul, Turkey, June 2006.
[5] S. Zhang, S.C. Liew, and P.P. Lam, "Physical Layer Network Coding," "http://arxiv.org/abs/0704.2475".
[6] A. Zhan and C. He, "Joint Design of Channel Coding and Physical Network Coding for Wireless Network," in *IEEE Proc. Int. Conference on Neural Networks and Signal Processing*, Zhejiang, China, June 2008.
[7] S. Zhang and S.C. Liew, "Channel Coding and Decoding in a Relay System Operated with Physical-Layer Network Coding," *IEEE Journal on Selected Areas in Commununications*, vol. 27, no. 5, pp. 788–796, Oct. 2009.
[8] Y. Lang, D. Wübben, and K.D. Kammeyer, "An Improved Physical Layer Network Coding Scheme for Two-Way Rrelay Systems," in *Proc. International ITG Workshop on Smart Antennas (WSA)*, Bremen, Germany, Feb. 2010.
[9] Y. Lang and D. Wübben, "Generalized Joint Channel Coding and Physical Network Coding for Two-way Relay Systems," in *IEEE Proc. Vehicular Technology Conference (VTC)*, Taipei, Taiwan, May 2010.
[10] F.R. Kschischang, B.J. Frey, and H.A. Loeliger, "Factor Graphs and the Sum-Product Algorithm," *IEEE Trans. on Information Theory*, vol. 47, no. 2, pp. 498–519, Feb. 2001.
[11] M.C. Davey and D. MacKay, "Low Density Parity Check Codes over $GF(q)$," *IEEE Communications Letters*, vol. 2, no. 6, pp. 165–167, June 1998.
[12] R. Urbanke, "LDPC: Optimization of LDPC Codes for BP Decoder," online: http://ipgdemos.epfl.ch/ldpcopt/.
[13] D. Wübben, "Joint Channel Decoding and Physical-Layer Network Coding in Two-Way QPSK Rrelay Systems by a Generalized Sum-Product Algorithm," in *Proc. 7th International Symposium on Wireless Communication Systems (ISWCS)*, York, UK, Sept. 2010.