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GENERALIZED SUPPLEMENTED MODULES

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Abstract. Let R be a ring and M a right R-module. It is shown that: (1) M is Artinian if and only if M is a GAS-module and satisfies DCC on generalized supplement submodules and on small submodules; (2) if M satisfies ACC on small submodules, then M is a lifting module if and only if M is a GAS-module and every generalized supplement submodule is a direct summand of M if and only if M satisfies (P^*); (3) R is semilocal if and only if every cyclic module is a GWS-module.

1. Introduction and Preliminaries

In this note all rings are associative with identity and all modules are unital right modules unless otherwise specified.

The concepts of generalized (amply) supplemented modules were introduced in [13] to characterize semiperfect modules, (semi)perfect rings. It is well known that a module M is Artinian if and only if M is an amply supplemented module and satisfies DCC on supplement submodules and on small submodules. In Section 2, we show that a module M is Artinian if and only if M is a GAS-module and satisfies DCC on generalized supplement submodules and on small submodules. It is also proven that a module M with ACC on small submodules is a lifting module if and only if M is a GAS-module and every generalized supplement is a direct summand of M if and only if M satisfies (P^*). In Section 3, we define the concept of a WGS-module and prove that a ring R is semilocal if and only if every cyclic right R-module is a WGS-module.

Let R be a ring and M a module. $N \leq M$ will mean N is a submodule of M. $Rad(M)$ will denote the Jacobson radical of M. A submodule E of M is called

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essential in M (notation $E \leq_e M$) if $E \cap A \neq 0$ for any non-zero submodule A of M. Dually, a submodule S of M is called *small* in M (notation $S \ll M$) if $M \neq S + T$ for any proper submodule T of M. Let N and L be submodules of M, N is called a *supplement* of L in M if $N + L = M$ and N is minimal with respect to this property, or equivalently, $M = N + L$ and $N \cap L \ll N$. N is said to be a *supplement submodule* of M if N is a supplement of some submodule of M. M is called an *amply supplemented* module if for any two submodules A and B of M with $A + B = M$, B contains a supplement of A. M is called a *supplemented module* (see [8]) if for each submodule A of M there exists a submodule B of M such that $M = A + B$ and $A \cap B \ll B$. M is called a *weakly supplemented module* if for each submodule A of M there exists a submodule B of M such that $M = A + B$ and $A \cap B \ll M$. M is called a *hollow* module if every proper submodule of M is small in M. M has *property (P^*)* (see [2]) if for any submodule N of M, there exists a direct summand K of M such that $K \leq N$ and $N/K \leq Rad(M/K)$. The notions which are not explained here will be found in [12].

Lemma 1.1. (see [12, 41.1]) *Let M be a module and K a supplement submodule of M. Then $K \cap Rad(M) = Rad(K)$.*

Lemma 1.2. (see [3, Proposition 5.20]) *Suppose that $K_1 \leq M_1 \leq M$, $K_2 \leq M_2 \leq M$, and $M = M_1 \oplus M_2$. Then $K_1 \oplus K_2 \leq_e M_1 \oplus M_2$ if and only if $K_1 \leq_e M_1$ and $K_2 \leq_e M_2$.*

Theorem 1.3. (see [2, Theorem 5]) *Let R be any ring and M a module. Then $Rad(M)$ is Artinian if and only if M satisfies DCC on small submodules.*

2. GS-Modules and GAS-Modules

Let M be a module. If U, $U' \leq M$ and $M = U + U'$ then U' is called a *generalized supplement* of U in case $U \cap U' \leq Rad(U')$. Clearly each supplement is a generalized supplement. M is called a *generalized supplemented module* or briefly a GS-module (see [13]) in case each submodule U has a generalized supplement U'. For example, (amply) supplemented modules, hollow modules and modules with (P^*) are GS-modules. M is called a *generalized amply supplemented module* or briefly a GAS-module in case $M = U + V$ implies that U has a generalized supplement $U' \leq V$. U is called a generalized supplement submodule if U is a generalized supplement of some submodule of M.

We start with the following.

Proposition 2.1. *Let M be a GS-module and L a submodule of M with $L \cap Rad(M) = 0$. Then L is semisimple. In particular, a GS-module M with $Rad(M) = 0$ is semisimple.*

Proof. Let L' be any submodule of L. Since M is a GS-module, there exists $L'' \leq M$ such that $L' + L'' = M$ and $L' \cap L'' \leq Rad(L'')$. Thus $L = L \cap M = L \cap (L' + L'') = L' + L \cap L''$. Since $L' \cap L'' \leq Rad(L'')$ and $L' \cap L \cap L'' = L' \cap L'' \leq L \cap Rad(L'') \leq L \cap Rad(M) = 0$, $L = L' \oplus (L \cap L'')$. So L is semisimple.

Proposition 2.2. *Let M be a GAS-module and K a direct summand of M. Then K is a GAS-module.*

Proof. Since K is a direct summand of M, there exists $K' \leq M$ such that $M = K \oplus K'$. Suppose that $K = C + D$, then $M = D + (C \oplus K')$. Since M is a GAS-module, there exists $P \leq D$ such that $M = P + (C \oplus K')$ and $P \cap (C \oplus K') \leq Rad(P)$. Therefore $K = K \cap M = K \cap (P + (C \oplus K')) = P + C$ and $P \cap C = P \cap (C \oplus K') \leq Rad(P)$, as required.

Proposition 2.3. *Let M be a GS-module. Then $M = N \oplus L$ for some semisimple N and some module L with essential radical.*

Proof. For $Rad(M)$, there exists $N \leq M$ such that $N \cap Rad(M) = 0$ and $N \oplus Rad(M) \leq_e M$. Since M is a GS-module, there exists $L \leq M$ such that $N + L = M$ and $N \cap L \leq Rad(L)$. Since $N \cap L = N \cap (N \cap L) \leq N \cap Rad(L) \leq N \cap Rad(M) = 0$, $M = N \oplus L$. By Proposition 2.1, N is semisimple. Thus $Rad(M) = Rad(N) \oplus Rad(L) = Rad(L)$. Since $N \oplus Rad(M) \leq_e M = N \oplus L$, i.e., $N \oplus Rad(L) \leq_e M = N \oplus L$, $Rad(L) \leq_e L$ by Lemma 1.2. This completes the proof.

Proposition 2.4. *Let $M_1, U \leq M$ and M_1 be a GS-module. If $M_1 + U$ has a generalized supplement in M, then so does U.*

Proof. Since $M_1 + U$ has a generalized supplement in M, there exists $X \leq M$ such that $X + (M_1 + U) = M$ and $X \cap (M_1 + U) \leq Rad(X)$. For $(X + U) \cap M_1$, since M_1 is a GS-module, there exists $Y \leq M_1$ such that $(X + U) \cap M_1 + Y = M_1$ and $(X + U) \cap Y \leq Rad(Y)$. Thus we have $X + U + Y = M$ and $(X + U) \cap Y \leq Rad(Y)$, that is, Y is a generalized supplement of $X + U$ in M. Next, we will show that $X + Y$ is a generalized supplement of U in M. It is clear that $(X + Y) + U = M$, so it suffices to show that $(X + Y) \cap U \leq Rad(X + Y)$. Since $Y + U \leq M_1 + U$, $X \cap (Y + U) \leq X \cap (M_1 + U) \leq Rad(X)$. Thus $(X+Y) \cap U \leq X \cap (Y+U) + Y \cap (X+U) \leq Rad(X) + Rad(Y) \leq Rad(X+Y)$, as required.

Proposition 2.5. *Let M_1 and M_2 be GS-modules. If $M = M_1 + M_2$, then M is a GS-module.*

Proof. Let U be a submodule of M. Since $M_1 + M_2 + U = M$ trivially has a generalized supplement in M, $M_2 + U$ has a generalized supplement in M by Proposition 2.4. Thus U has a generalized supplement in M by Proposition 2.4 again. So M is a GS-module.

Proposition 2.6. *If M is a GS-module, then*

(1) *Every finitely M-generated module is a GS-module.*

(2) *$M/Rad(M)$ is semisimple.*

Proof. (1) From Proposition 2.5, we know that every finite sum of GS-modules is a GS-module. Next we will show that every factor module of a GS-module is again a GS-module.

Let M be a GS-module and M/N any factor module of M. For any submodule L of M containing N, since M is a GS-module, there exists $K \leq M$ such that $L + K = M$ and $L \cap K \leq Rad(K)$. Thus $M/N = L/N + (N + K)/N$ and $(L/N) \cap ((N + K)/N) = (N + (L \cap K))/N \leq Rad((N + K)/N)$, that is, $(N + K)/N$ is a generalized supplement of L/N in M/N, as required.

(2) Let N be any submodule of M containing $Rad(M)$. Then there exists a generalized supplement K of N in M, i.e., $M = N + K$ and $N \cap K \leq Rad(K) \leq Rad(M)$. Thus $M/Rad(M) = N/Rad(M) \oplus (K + Rad(M))/Rad(M)$, and so every submodule of $M/Rad(M)$ is a direct summand. Therefore $M/Rad(M)$ is semisimple.

Let M be a module and $N \leq M$. N is said to have generalized ample supplements in M if for every submodule L such that $M = N + L$, N has a generalized supplement in L.

Proposition 2.7. *Let M be a module and $M = U_1 + U_2$. If U_1, U_2 have generalized ample supplements in M, then $U_1 \cap U_2$ also has generalized ample supplements in M.*

Proof. Let $V \leq M$ and $U_1 \cap U_2 + V = M$. Then $U_1 = U_1 \cap U_2 + (V \cap U_1)$ and $U_2 = U_1 \cap U_2 + (V \cap U_2)$, so $M = U_1 + V \cap U_2$ and $M = U_2 + V \cap U_1$. Since U_1, U_2 have generalized ample supplements in M, there exist $V_2' \leq V \cap U_2$ and $V_1' \leq V \cap U_1$ such that $U_1 + V_2' = M$ and $U_1 \cap V_2' \leq Rad(V_2')$, and $U_2 + V_1' = M$ and $U_2 \cap V_1' \leq Rad(V_1')$. Thus $V_1' + V_2' \leq V$ and $U_1 = U_1 \cap U_2 + V_1'$ and $U_2 = U_1 \cap U_2 + V_2'$. Therefore $(U_1 \cap U_2) + (V_1' + V_2') = M$ and $(U_1 \cap U_2) \cap (V_1' + V_2') = (U_2 \cap V_1') + (U_1 \cap V_2') \leq Rad(V_1' + V_2')$. This completes the proof.

Theorem 2.8. *Let M be a module and $U \leq M$. The following statements are equivalent.*

(1) *There is a decomposition $M = X \oplus X'$ with $X \leq U$ and $X' \cap U \leq Rad(X')$.*

(2) *There is an idempotent* $e \in End(M)$ *with* $e(M) \leq U$ *and* $(1-e)U \leq Rad((1-e)(M))$.
(3) *There is a direct summand* X *of* M *with* $X \leq U$ *and* $U/X \leq Rad(M/X)$.
(4) U *has a generalized supplement* V *in* M *such that* $V \cap U$ *is a direct summand of* U.

Proof. $(1) \Longrightarrow (2)$ For a decomposition $M = X \oplus X'$, there is an idempotent $e \in End(M)$ with $e(M) = X$ and $(1-e)(M) = X'$. Since $X \leq U$, we have $(1-e)(U) = U \cap (1-e)(M) \leq Rad((1-e)(M))$.

$(2) \Longrightarrow (3)$ Take $X = e(M)$.

$(3) \Longrightarrow (1)$ If $M = X \oplus X'$ and $U/X \leq Rad(M/X)$, then $U = X + (X' \cap U)$, $X' \cap U \simeq U/X \leq Rad(M/X) \leq M/X \simeq X'$, hence $X' \cap U \leq Rad(X')$.

$(1) \Longrightarrow (4)$ By assumption, X' is a generalized supplement of U in M and $U = X \oplus (X' \cap U)$, as required.

$(4) \Longrightarrow (1)$ Let V be a generalized supplement of U in M with $U = X \oplus (V \cap U)$. Then $M = U + V = X + (V \cap U) + V = X + V$ and $X \cap V = (X \cap U) \cap V = X \cap (V \cap U) = 0$ (for $X \leq U$), i.e., X is a direct summand of M. This completes the proof.

Corollary 2.9. *A module* M *has* (P^*) *if and only if for any submodule* N *of* M *there exist submodules* K, K' *of* M *such that* $M = K \oplus K'$, $K \leq N$ *and* $N \cap K' \leq Rad(K')$.

Lemma 2.10. *Let* U, V *be submodules of* M *and* V *a generalized supplement of* U *in* M. *If* U *is a maximal submodule of* M, *then* $U \cap V = Rad(V)$ *is a unique maximal submodule of* V.

Proof. Since $V/(U \cap V) \simeq M/U$, $U \cap V$ *is a maximal submodule of* V, *and hence* $Rad(V) \leq U \cap V$. *Since* $U \cap V \leq Rad(V)$, $U \cap V = Rad(V)$, *as desired.*

Let M be a module and $N \leq M$. N is said to be a *cofinite submodule* (see [1]) of M if M/N is finitely generated.

Theorem 2.11. *Let* M *be a module. Then the following statements are equivalent.*

(1) M *is a sum of hollow submodules and* $Rad(M) \ll M$.
(2) *Every proper submodule of* M *is contained in a maximal one and every cofinite submodule of* M *has a generalized supplement in* M.
(3) M *is an irredundant sum of local modules and* $Rad(M) \ll M$.

Proof. $(1) \Longleftrightarrow (3)$ Let $M = \Sigma_I L_i$ and L_i $(i \in I)$ be hollow submodules of M. Then $M/Rad(M) = \Sigma_I(L_i+Rad(M))/Rad(M)$. Since $Rad(L_i) \leq L_i \cap Rad(M)$ and $(L_i + Rad(M))/Rad(M) \simeq L_i/(L_i \cap Rad(M))$, these factors are simple or zero. Thus we have $M/Rad(M) = \oplus_{I'}(L_i + Rad(M))/Rad(M)$. Therefore $M = \Sigma_{I'} L_i$ is an irredundant sum of local submodules L_i $(i \in I' \subseteq I)$ (for $Rad(M) \ll M$), as required.

$(3) \Longrightarrow (2)$ It is clear that $M/Rad(M)$ is semisimple. Since $Rad(M) \ll M$, every proper submodule of M is contained in a maximal submodule by [12, 21.6]. Let $K \leq M$ with M/K finitely generated. Then there are finitely many local submodules $L_1, \cdots, L_n$ such that $M = K + L_1 + \cdots + L_n$. By Proposition 2.5, $L_1 + \cdots + L_n$ is a GS-module. Thus K has a generalized supplement in M by Proposition 2.4.

$(2) \Longrightarrow (1)$ Since every proper submodule of M is contained in a maximal submodule, $Rad(M) \ll M$ holds by [12, 21.6]. Let H be the sum of all hollow submodules of M and assume $H \neq M$. By assumption, there exists a maximal submodule N of M such that $H \leq N$ and N has a generalized supplement L in M. By Lemma 2.10, L is local (hollow). Thus $L \leq H \leq N$, a contradiction. Hence $H = M$.

Corollary 2.12. *Let M be a finitely generated module. Then the following statements are equivalent.*

(1) *M is a GS-module.*

(2) *Every maximal submodule of M has a generalized supplement in M.*

(3) *M is a sum of hollow submodules.*

(4) *M is an irredundant sum of local submodules.*

Proposition 2.13. *Let M be a module. If every submodule of M is a GS-module, then M is a GAS-module.*

Proof. Let L, $N \leq M$ and $M = N + L$. By assumption, there is $H \leq L$ such that $(L \cap N) + H = L$ and $(L \cap N) \cap H = N \cap H \leq Rad(H)$. Thus $H + N \geq H + (L \cap N) = L$ and hence $H + N \geq N + L = M$. Therefore $M = H + N$, as required.

Corollary 2.14. *Let R be any ring. Then the following statements are equivalent.*

(1) *Every module is a GAS-module.*

(2) *Every module is a GS-module.*

A module M is said to be π-*projective* if for every two submodules U, V of M with $U + V = M$ there exists $f \in End(M)$ with Im$f \leq U$ and Im$(1 - f) \leq V$.

Theorem 2.15. *Let M be a module. If M is a π-projective GS-module, then M is a GAS-module.*

Proof. Let A, B be submodules of M such that $M = A + B$. Since M is π-projective, there exists an endomorphism e of M such that $e(M) \leq A$ and $(1-e)(M) \leq B$. Note that $(1-e)(A) \leq A$. Let C be a generalized supplement of A in M. Then $M = e(M)+(1-e)(M) = e(M)+(1-e)(A+C) \leq A+(1-e)(C) \leq M$, so that $M = A + (1 - e)(C)$. Note that $(1 - e)(C)$ is a submodule of B. Let $y \in A \cap (1 - e)(C)$. Then $y \in A$ and $y = (1 - e)(x) = x - e(x)$ for some $x \in C$. Next $x = y + e(x) \in A$, so that $y \in (1 - e)(A \cap C)$. But $A \cap C \leq Rad(C)$ gives that $A \cap (1 - e)(C) = (1 - e)(A \cap C) \leq Rad((1 - e)(C))$. Thus $(1 - e)(C)$ is a generalized supplement of A in M. It follows that M is a GAS-module.

Theorem 2.16. *Let M be a module. Then M is Artinian if and only if M is a GAS-module and satisfies DCC on generalized supplement submodules and on small submodules.*

Proof. The necessity is clear. Conversely, suppose that M is a GAS-module which satisfies DCC on generalized supplement submodules and on small submodules. Then $Rad(M)$ is Artinian by Theorem 1.3. Next, it suffices to show that $M/Rad(M)$ is Artinian. Let N be any submodule of M containing $Rad(M)$. Then there exists a generalized supplement K of N in M, i.e., $M = N + K$ and $N \cap K \leq Rad(K) \leq Rad(M)$. Thus $M/Rad(M) = (N/Rad(M)) \oplus ((K + Rad(M))/Rad(M))$ and so every submodule of $M/Rad(M)$ is a direct summand. Therefore $M/Rad(M)$ is semisimple.

Now suppose that $Rad(M) \leq N_1 \leq N_2 \leq N_3 \leq \cdots$ is an ascending chain of submodules of M. Because M is a GAS-module, there exists a descending chain of submodules $K_1 \geq K_2 \geq \cdots$ such that K_i is a generalized supplement of N_i in M for each $i \geq 1$. By hypothesis, there exists a positive integer t such that $K_t = K_{t+1} = K_{t+2} = \cdots$. Because $M/Rad(M) = N_i/Rad(M) \oplus (K_i + Rad(M))/Rad(M)$ for all $i \geq t$, it follows that that $N_t = N_{t+1} = \cdots$. Thus $M/Rad(M)$ is Noetherian, and hence finitely generated. So $M/Rad(M)$ is Artinian, as desired.

Corollary 2.17. *Let M be a finitely generated GAS-module. Then M is Artinian if and only if M satisfies DCC on small submodules.*

Proof. "$\Longleftarrow$" Since $M/Rad(M)$ is semisimple and M is finitely generated, $M/Rad(M)$ is Artinian. Now that M satisfies DCC on small submodules, $Rad(M)$ is Artinian by Theorem 1.3. Thus M is Artinian.

" $\Longrightarrow$ " is clear.

Remark 2.18. Let R be a ring. If R_R is a GAS-module, then R is a right Artinian ring if and only if R satisfies DCC on small right ideals. Thus a right perfect ring which satisfies DCC on small right ideals is a right Artinian ring.

Let M be a module. M is called a *lifting* module if for any submodule N of M, there exists a direct summand K of M such that $K \leq M$ and $N/K \ll M/K$, equivalently, for every submodule N of M there exist submodules K, K' of M such that $M = K \oplus K'$, $K \leq N$ and $N \cap K' \ll K'$. M is called a *quasi-discrete* module if M is lifting and has (D_3) (i.e., for any pair of direct summands K, L of M with $M = K + L$, $K \cap L$ is a direct summand of M).

Theorem 2.19. *Let M be a module with ACC on small submodules. Then*

(1) *M is a GAS-module and every generalized supplement is a direct summand of M if and only if M is a lifting module.*

(2) *M satisfies (P^*) if and only if M is a lifting module.*

(3) *If M is a π-projective GS-module, then M is a quasi-discrete module.*

Proof. (1) " $\Longrightarrow$ " Let $M = A + B$. Since M is a GAS-module, there exists $C \leq B$ such that $M = A + C$ and $A \cap C \leq Rad(C)$. Since M satisfies ACC on small submodules, $Rad(C)$ is Noetherian by [2, Proposition 2], and hence $Rad(C)$ is finitely generated. Thus $Rad(C) \ll C$ by [7, Corollary 9.1.3], and [12, 19.3] and C is a supplement of A. Therefore M is an amply supplemented module. Since every supplement submodule is a generalized supplement submodule, every supplement is a direct summand of M by assumption. Thus M is lifting.

" $\Longleftarrow$ " Since M is lifting, M is an amply supplemented module, and hence M is a GAS-module. Let A be a generalized supplement submodule, i.e., there exists $B \leq M$ such that $M = A + B$ and $A \cap B \leq Rad(A)$. By an argument analogous to that of " $\Longrightarrow$ ", we know that A is a supplement of B. So A is a direct summand of M by assumption, as desired.

(2) " $\Longleftarrow$ " is clear.

" $\Longrightarrow$ " It suffices to prove that every factor module of M satisfies ACC on small submodules. Let A be any submodule of M and $B_1/A \leq B_2/A \leq \cdots$ where each $B_i/A \ll M/A$. From the proof of (1), we know that M is a supplemented module. Let C be a supplement of A in M. Then $M/A = (A+C)/A \simeq C/(A\cap C)$. Since B_i/A is small in M/A, $B_i/A \simeq D_i/(A\cap C) \ll C/(A\cap C)$ for some D_i. Next we prove that $D_i \ll M$. Let $D_i + E = M$. Then $(D_i + (E + A \cap C))/(A \cap C) =$

$M/(A\cap C)$. Hence $E + A\cap C = M$ and $E = M$. Thus we have $D_1 \leq D_2 \leq \cdots$. Since M satisfies ACC on small submodules, there exists n such that $D_k = D_{k+1}$ for all $k \geq n$. Thus $B_k/A = B_{k+1}/A$ for all $k \geq n$. Therefore M/A satisfies ACC on small submodules, as required.

(3) From the proof of (1), we know that M is a supplemented module. The rest is obvious.

Remark 2.20. Let M be a GS (GAS)-module and $Rad(M)$ be Noetherian (or M satisfies ACC on small submodules). Then M is a supplemented (an amply supplemented) module.

Example 3.21. Let R be an incomplete rank one discrete valuation ring with quotient field K. Then the module $M = K \oplus K$ is a GS-module but not a GAS-module. In fact, if M is a GAS-module, it is an amply supplemented module by Remark 2.20 (for M is Noetherian). This is a contradiction (see [9, Lemma A. 5]).

3. WGS-Modules

In this section, we define the concept of a weakly generalized supplemented module (or briefly a WGS-module) and prove that a ring R is semilocal if and only if every cyclic module is a WGS-module.

Definition 3.1. A module M is said to be a generalized weakly supplemented or briefly a WGS-module if for any submodule $N \leq M$, there exists $L \leq M$ such that $M = N + L$ and $N \cap L \leq Rad(M)$.

Let P and M be modules. An epimorphism $f : P \longrightarrow M$ is called a *cover* (see [10, 13]) of M in case $Kerf \ll P$. An epimorphism $f : P \longrightarrow M$ with $Kerf \ll P$ is called a projective cover of M in case P is projective.

Proposition 3.2. *Let M be a WGS-module. Then*

(1) *Every supplement submodule of M is a WGS-module.*

(2) *If $f : N \longrightarrow M$ is a cover of M, N is also a WGS-module.*

(3) *Every factor module of M is a WGS-module.*

Proof. (1) Let K be a supplement in M. For any submodule $N \leq K$, since M is a WGS-module, there exists $L \leq M$ such that $M = N + L$ and $N \cap L \leq Rad(M)$. Thus $K = K \cap M = K \cap (N + L) = N + (K \cap L)$ and

$N \cap (K \cap L) = N \cap L = K \cap (N \cap L) \leq K \cap Rad(M) = Rad(K)$ by Lemma 1.1. Therefore K is a GWS-module.

(2) Let $f : N \longrightarrow M$ be a cover of M. For any $L \leq N$, we have $f(L) \leq M$. Since M is a WGS-module, there exists $P \leq M$ such that $f(L) + P = M$ and $f(L) \cap P \leq Rad(M)$. Thus $N = f^{-1}(M) = f^{-1}(f(L) + P) = Kerf + L + f^{-1}(P) = L + f^{-1}(P)$ (for $Kerf \ll N$) and $f^{-1}(Rad(M)) \geq f^{-1}(f(L) \cap P) \geq L \cap f^{-1}(P)$. By [3, Proposition 9.15], $f^{-1}(Rad(M)) = Kerf + Rad(N) = Rad(N)$. Therefore $L \cap f^{-1}(P) \leq Rad(N)$, as required.

(3) Let N be any submodule of M and L/N any submodule of M/N. For $L \leq M$, there exists $K \leq M$ such that $L + K = M$ and $K \cap L \leq Rad(M)$ since M is a WGS-module. Thus $M/N = L/N + (K + N)/N$. Let $f : M \to M/N$ be a canonical epimorphism. Since $K \cap L \leq Rad(M)$, $(L/N) \cap ((K + N)/N) = (L \cap (K + N))/N = (N + (K \cap L))/N = f(L \cap K) \leq f(Rad(M)) \leq Rad(M/N)$, this completes the proof.

Corollary 3.3. *Let M be a module and $N \ll M$. Then M is a WGS-module if and only if M/N is a WGS-module.*

Proof. It follows from Proposition 3.2.

Proposition 3.4. *Let M be finitely generated. Then M is a WGS-module if and only if M is a weakly supplemented module.*

Proof. " $\Longleftarrow$ " is clear.

" $\Longrightarrow$ " For any submodule N of M, there exists $L \leq M$ such that $N + L = M$ and $N \cap L \leq Rad(M)$ since M is a WGS-module. Since M is finitely generated, $Rad(M) \ll M$. Thus $N \cap L \ll M$, as desired.

Proposition 3.5. *Suppose M is finitely generated and $f : P \longrightarrow M$ a projective cover of M. If M is a weakly supplemented module, then so is P.*

Proof. Since $P/Kerf \simeq M$ is finitely generated, there is a finitely generated submodule P' of P such that $P' + Kerf = P$. Since $Kerf \ll P$, $P' = P$. Thus P is finitely generated. By Propositions 3.2 and 3.4, P is a weakly supplemented module.

Lemma 3.6. *Let K, $M_1 \leq M$ and M_1 be a WGS-module. If $M_1 + K$ has a generalized weak supplement in M, then so does K.*

Proof. By assumption, there exists $N \leq M$ such that $(M_1 + K) + N = M$ and $N \cap (M_1 + K) \leq Rad(M)$. Since M_1 is a WGS-module, there exists a submodule $L \leq M_1$ such that $M_1 \cap (N + K) + L = M_1$ and $L \cap (N + K) \leq Rad(M_1)$. Thus

$M = K + N + L$ and $K \cap (N + L) \leq (K + M_1) \cap N + L \cap (N + K) \leq Rad(M)$, that is, $N + L$ is a generalized weak supplement of K in M.

Proposition 3.7. *Let $M = M_1 + M_2$. If M_1 and M_2 are WGS-modules, then M is a WGS-module.*

Proof. Let N be a submodule of M. Since $M_1 + M_2 + N = M$ trivially has a generalized supplement in M, $M_2 + N$ has a generalized supplement in M by Lemma 3.6. Thus N has a generalized supplement in M by Lemma 3.6 again. So M is a WGS-module.

Theorem 3.8. *Let M be a module and $Rad(M) \ll M$. The following statements are equivalent.*

(1) *M is a WGS-module.*

(2) *$M/Rad(M)$ is semisimple.*

(3) *There is a decomposition $M = M_1 \oplus M_2$ such that M_1 is semisimple, $Rad(M) \leq_e M_2$ and $M_2/Rad(M)$ is semisimple.*

Proof. $(1) \Longrightarrow (2)$ Let L be any submodule of M containing $Rad(M)$. Since M is a WGS-module, there exists $N \leq M$ such that $N + L = M$ and $N \cap L \leq Rad(M)$. Thus $M/Rad(M) = L/Rad(M) + (N + Rad(M))/Rad(M)$ and $L/Rad(M) \cap (N + Rad(M))/Rad(M) = (L \cap N + Rad(M))/Rad(M) = 0$. So $M/Rad(M) = L/Rad(M) \oplus (N + Rad(M))/Rad(M)$, as required.

$(2) \Longrightarrow (1)$ For any submodule $N \leq M$, since $M/Rad(M)$ is semisimple, there exists a submodule $L \leq M$ containing $Rad(M)$ such that $M/Rad(M) = (N + Rad(M))/Rad(M) \oplus L/Rad(M)$. Thus $M = N + Rad(M) + L$. Since $Rad(M) \ll M$, $M = N + L$. $N \cap L \leq Rad(M)$ is obvious.

$(2) \Longrightarrow (3)$ Let M_1 be a complement of $Rad(M)$ in M. Then $M_1 \simeq (M_1 \oplus Rad(M))/Rad(M)$ is a direct summand of $M/Rad(M)$, and hence it is semisimple. Therefore, there exists a semisimple submodule $M_2/Rad(M)$ such that $(M_1 \oplus Rad(M))/Ra$
$d(M) \oplus M_2/Rad(M) = M/Rad(M)$. Thus $M_1 + M_2 = M$ and $M_1 \cap M_2 \leq Rad(M) \cap M_1 = 0$ implies $M = M_1 \oplus M_2$. Since $M_1 \oplus Rad(M) \leq_e M = M_1 \oplus M_2$, $Rad(M) \leq_e M_2$ by Lemma 1.2.

$(3) \Longrightarrow (2)$ is clear.

Theorem 3.9. *Let R be a ring. The following statements are equivalent.*

(1) *R is semilocal.*

(2) *Every module with small radical is a WGS-module.*

(3) *Every finitely generated module is a WGS-module.*

(4) *Every cyclic module is a WGS-module.*

Proof. (1)$\Longrightarrow$(2) Since for any module M there exist a set Λ and an epimorphism $f : R^{(\Lambda)} \longrightarrow M$ with $f(Rad(R^{(\Lambda)})) \leq Rad(M)$ and $R^{(\Lambda)}/Rad(R^{(\Lambda)}) \simeq (R/J(R))^{(\Lambda)}$, we obtain an epimorphism $\delta : R^{(\Lambda)}/Rad(R^{(\Lambda)}) \longrightarrow M/Rad(M)$. Thus $M/Rad(M)$ is semisimple, and so M is a WGS-module by Theorem 3.8.

(2)$\Longrightarrow$ $(3) \Longrightarrow (4)$ are clear.

$(4) \Longrightarrow (1)$ It is known that a ring R is semilocal if and only if R_R is weakly supplemented. The rest is obvious by Proposition 3.4.

The following example shows that a WGS-modules need not be a GS-module.

Example 3.11. Consider the ring $R = \mathbb{Z}_{p,q}$=$\{\frac{a}{b} \mid a, b \in \mathbb{Z},\ b \neq 0,\ p \nmid b$ and $q \nmid b\}$, where R is a commutative uniform semilocal Noetherian domain. Thus R_R is a WGS-module by Theorem 3.10. Since R_R is Noetherian, it satisfies ACC on small submodules. If R_R is a GS-module, then R_R is a supplemented module by Remark 2.20, this is a contradiction (see [1, Example 2.17]).

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REFERENCES

1. R. Alizade and E. Bugukasik, Cofinitely Weakly Supplemented Modules, *Comm. Algebra*, **31** (2003), 5377-5390.

2. I. Al-Khazzi and P. F. Smith, Modules with Chain Conditions on Superfluous Submodules, *Comm. Algebra*, **19** (1991), 2331-2351.

3. F. W. Anderson and K. R. Fuller, *Rings and Categories of Modules*, Springer-Verlag, New York, 1974.

4. N. V. Dung, D. V. Huynh, P. F. Simth and R. Wisbauer, *Extending Modules*, Pitman Reacher Notes in Math. Ser. 313, Longman Sci. and Tech., 1994

5. A. Harmanci, D. Keskin and P. F. Smith, On $\oplus$-Supplemented Modules, *Acta Math. Hungar*, **83** (1999), 161-169.

6. J. Hausen, Supplemented Modules over Dedekind Domains, *Pacific J. Math.*, **100** (1982), 387-402.

7. F. Kasch, *Modules and Rings*, London New York, 1982.

8. D. Keskin, On Lifting Modules, *Comm. Algebra*, **28** (2000), 3427-3440.

9. S. H. Mohamed and B. J. Müller, *Continuous and Discrete Modules*, London Math. Soc. Lecture Note Series, 147, Cambridge University Press, 1990.

10. S. Nakahara, A Generalization of Semiperfect Modules, *Osaka J. Math.*, **20** (1983), 43-50.

11. K. Varadarajan, Modules with Supplements, *Pacific J. Math.*, **82** (1979), 559-564.

12. R. Wisbauer, *Foundations of Module and Ring Theory*, Gordon and Breach, Philadelphia, 1991.

13. W. Xue, Characterizations of Semiperfect and Perfect Rings, *Publications Matematiques*, **40** (1996), 115-125.

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