# Generalized Surface Quasi-Geostrophic Equations with Singular Velocities

DONGHO CHAE
*Chung-Ang University*

PETER CONSTANTIN
*Princeton University*

DIEGO CÓRDOBA
*Consejo Superior de Investigaciones Cientificas*

FRANCISCO GANCEDO
*Universidad de Sevilla*

AND

JIAHONG WU
*Oklahoma State University*

**Abstract**

This paper establishes several existence and uniqueness results for two families of active scalar equations with velocity fields determined by the scalars through very singular integrals. The first family is a generalized surface quasi-geostrophic (SQG) equation with the velocity field $u$ related to the scalar $\theta$ by $u = \nabla^{\perp}\Lambda^{\beta-2}\theta$, where $1 < \beta \leq 2$ and $\Lambda = (-\Delta)^{1/2}$ is the Zygmund operator. The borderline case $\beta = 1$ corresponds to the SQG equation and the situation is more singular for $\beta > 1$. We obtain the local existence and uniqueness of classical solutions, the global existence of weak solutions, and the local existence of patch-type solutions. The second family is a dissipative active scalar equation with $u = \nabla^{\perp}(\log(I - \Delta))^{\mu}\theta$ for $\mu > 0$, which is at least logarithmically more singular than the velocity in the first family. We prove that this family with any fractional dissipation possesses a unique local smooth solution for any given smooth data. This result for the second family constitutes a first step towards resolving the global regularity issue recently proposed by K. Ohkitani.
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## 1 Introduction

This paper studies solutions of generalized surface quasi-geostrophic (SQG) equations with velocity fields given by more singular integral operators than the

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Riesz transforms. Recall the inviscid SQG equation

$$\partial_t \theta + u \cdot \nabla \theta = 0, \qquad (1.1)$$
$$u = \nabla^{\perp} \psi \equiv (-\partial_{x_2}, \partial_{x_1}) \psi, \quad \Lambda \psi = \theta,$$

where $\Lambda = (-\Delta)^{1/2}$ is the Zygmund operator, $\theta = \theta(x,t)$ is a scalar function, $u$ denotes the two-dimensional velocity field, and $\psi$ the stream function. Clearly, $u$ can be represented in terms of the Riesz transforms of $\theta$, namely,

$$u = (-\mathcal{R}_2, \mathcal{R}_1)\theta \equiv (-\partial_{x_2} \Lambda^{-1}, \partial_{x_1} \Lambda^{-1})\theta.$$

Equation (1.1), its counterpart with fractional dissipation, and several closely related generalizations have recently been investigated very extensively, and significant progress has been made on fundamental issues concerning solutions of these equations (see, e.g., [1, 3, 5, 7, 9, 14, 15, 16] and many more).

Our goal here is to understand solutions of the SQG-type equations with velocity fields determined by even more singular integral operators. Attention is focused on two generalized SQG equations. The first one assumes the form

$$\partial_t \theta + u \cdot \nabla \theta = 0, \qquad (1.2)$$
$$u = \nabla^{\perp} \psi, \quad \Delta \psi = \Lambda^{\beta} \theta,$$

where $\beta$ is a real parameter satisfying $1 < \beta \le 2$. Here the spatial domain is either the whole plane $\mathbb{R}^2$ or the two-dimensional periodic box $\mathbb{T}^2$, and the fractional Laplacian operator $(-\Delta)^{\alpha}$ is defined through the Fourier transform

$$\widehat{(-\Delta)^{\alpha} f}(\xi) = |\xi|^{2\alpha} \widehat{f}(\xi).$$

The borderline case $\beta = 1$ of (1.2) is the SQG equation (1.1), while (1.2) with $\beta = 0$ is the well-known two-dimensional Euler vorticity equation with $\theta$ representing the vorticity (see, e.g., [10]). The second generalized SQG equation under study is the dissipative active scalar equation

$$\partial_t \theta + u \cdot \nabla \theta + \kappa (-\Delta)^{\alpha} \theta = 0, \qquad (1.3)$$
$$u = \nabla^{\perp} \psi, \quad \psi = (\log(I - \Delta))^{\mu} \theta,$$

where $\kappa > 0$, $\alpha > 0$, and $\mu > 0$ are real parameters, and $(\log(I - \Delta))^{\mu}$ denotes the Fourier multiplier operator defined by

$$\widehat{(\log(I - \Delta))^{\mu} f}(\xi) = (\log(1 + |\xi|^2))^{\mu} \widehat{f}(\xi).$$

Equation (1.3) is closely related to (1.2). In fact, both (1.2) with $\beta = 2$ and (1.3) with $\kappa = 0$ and $\mu = 0$ formally reduce to the trivial linear equation

$$\partial_t \theta + \nabla^{\perp} \theta \cdot \nabla \theta = 0 \quad \text{or} \quad \partial_t \theta = 0.$$

For $\mu > 0$, the velocity field $u$ in (1.3) is at least logarithmically more singular than those in (1.2).

We establish four main results for the existence and uniqueness of solutions to the equations defined in (1.2) and in (1.3) with a given initial data

$$\theta(x,0) = \theta_0(x).$$

We now preview these results. Our first main result establishes the local existence and uniqueness of smooth solutions to (1.2) associated with any given smooth initial data. More precisely, we have the following theorem:

THEOREM 1.1. *Consider* (1.2) *with* $1 < \beta \leq 2$. *Assume that* $\theta_0 \in H^m(\mathbb{R}^2)$ *with* $m \geq 4$. *Then there exists* $T = T(\|\theta_0\|_{H^m}) > 0$ *such that* (1.2) *has a unique solution* $\theta$ *on* $[0,T]$. *In addition,* $\theta \in C([0,T]; H^m(\mathbb{R}^2))$.

*Remark* 1.2. As mentioned previously, when $\beta = 2$, $\psi = \theta$, and $u = \nabla^\perp\theta$, then (1.2) reduces to the trivial equation

$$\partial_t\theta = 0 \quad \text{or} \quad \theta(x,t) = \theta_0(x).$$

Therefore, (1.2) with $\beta = 2$ has a global steady-state solution.

For $1 < \beta < 2$, the velocity $u$ is determined by a very singular integral of $\theta$, and $\nabla u$ is not known to be bounded in $L^\infty$. As a consequence, the nonlinear term cannot be directly bounded. To deal with this difficulty, we rewrite the nonlinear term in the form of a commutator to explore the extra cancellation. In order to prove Theorem 1.1, we need to derive a suitable commutator estimate (see Proposition 2.1).

Our second main result proves the local existence and uniqueness of smooth solutions to (1.3). In fact, the following theorem holds:

THEOREM 1.3. *Consider the active scalar equation* (1.3) *with* $\kappa > 0$, $\alpha > 0$, *and* $\mu > 0$. *Assume the initial data* $\theta_0 \in H^4(\mathbb{R}^2)$. *Then there exists* $T > 0$ *such that* (1.3) *has a unique solution* $\theta \in C([0,T]; H^4(\mathbb{R}^2))$.

We remark that the velocity field $u$ in (1.3) is determined by

$$u = \nabla^\perp(\log(I - \Delta))^\mu\theta \quad \text{with } \mu > 0,$$

which is even logarithmically more singular than that in (1.2) with $\beta = 2$, namely, the trivial steady-state case. In a recent lecture [11], K. Ohkitani argued that (1.3) with $\kappa = 0$ may be globally well-posed based on numerical computations. Theorem 1.3 is a first step towards positively confirming his prediction.

Again the difficulty arises from the nonlinear term. In order to obtain a local (in time) bound for $\|\theta\|_{H^4}$, we need to rewrite the most singular part in the nonlinear term as a commutator. This commutator involves the logarithm of Laplacian, and it appears that no $L^2$-bound for such a commutator is currently available. By applying Besov space techniques, we are able to prove the following bound for such commutators:

PROPOSITION 1.4. *Let $\mu \geq 0$. Let $\partial_x$ denote a partial derivative, either $\partial_{x_1}$ or $\partial_{x_2}$. Then, for any $\delta > 0$ and $\epsilon > 0$,*

$$\|[(\ln(I-\Delta))^{\mu}\partial_x, g]f\|_{L^2} \leq C_{\mu,\epsilon,\delta}\left(1+\left(\ln\left(1+\frac{\|f\|_{\dot{H}^\delta}}{\|f\|_{L^2}}\right)\right)^{\mu}\right)\|f\|_{L^2}\,\|g\|_{H^{2+3\epsilon}},$$

*where $C_{\mu,\epsilon,\delta}$ is a constant depending on $\mu$, $\epsilon$, and $\delta$ only, $\dot{H}^\delta$ denotes the standard homogeneous Sobolev space, and the brackets denote the commutator, namely,*

$$[(\ln(I-\Delta))^{\mu}\partial_x, g]f = (\ln(I-\Delta))^{\mu}\partial_x(fg) - \big((\ln(I-\Delta))^{\mu}\partial_x f\big)g.$$

Our third main result assesses the global existence of weak solutions to (1.2). Our consideration is restricted to the setting of periodic boundary conditions. The weak solution is essentially in the distributional sense and its precise definition is as follows: $\mathbb{T}^2$ in the definition denotes the two-dimensional periodic box.

DEFINITION 1.5. Let $T > 0$. A function $\theta \in L^\infty([0,T]; L^2(\mathbb{T}^2))$ is a weak solution of (1.2) if, for any test function $\phi \in C_c^\infty([0,T) \times \mathbb{T}^2)$, the following integral equation holds:

$$\int_0^T \int_{\mathbb{T}^2} \theta(\partial_t \phi + u \cdot \nabla\phi)\,dx\,dt = \int_{\mathbb{T}^2} \theta_0(x)\phi(x,0)\,dx. \tag{1.4}$$

Although the velocity $u$ is more singular than the scalar $\theta$ and the nonlinear term above could not make sense, it is well-defined due to a commutator hidden in the equation (see Section 4). We prove that any mean-zero $L^2$ data leads to a global (in time) weak solution. That is, we have the following theorem:

THEOREM 1.6. *Assume that $\theta_0 \in L^2(\mathbb{T}^2)$ has mean* 0*, namely,*

$$\int_{\mathbb{T}^2} \theta_0(x)\,dx = 0.$$

*Then* (1.2) *has a global weak solution in the sense of Definition* 1.5.

This result is an extension of Resnick's work [12] on the inviscid SQG equation (1.1). However, for $1 < \beta < 2$, the velocity is more singular, and we need to write the nonlinear term as a commutator in terms of the stream function $\psi$. More details can be found in the proof of Theorem 1.6 in Section 4.

Our last main result establishes the local well-posedness of the patch problem associated with the active scalar equation (1.2). This result extends Gancedo's previous work for (1.2) with $0 < \beta \leq 1$ [6]. Since $\beta$ is now in the range $(1,2)$, $u$ is given by a more singular integral and demands a regular function and more sophisticated manipulation. The initial data is given by

$$\theta_0(x) = \begin{cases} \theta_1, & x \in \Omega, \\ \theta_2, & x \in \mathbb{R}^2 \setminus \Omega, \end{cases} \tag{1.5}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded domain. We parametrize the boundary of $\Omega$ by $x = x_0(\gamma)$ with $\gamma \in \mathbb{T} = [-\pi, \pi]$ so that

$$|\partial_\gamma x_0(\gamma)|^2 = A_0,$$

where $2\pi\sqrt{A_0}$ is the length of the contour. In addition, we assume that the curve $x_0(\gamma)$ does not cross itself and there is a lower bound on $|\partial_\gamma x_0(\gamma)|$, namely,

$$\frac{|x_0(\gamma) - x_0(\gamma - \eta)|}{|\eta|} > 0, \quad \forall \gamma, \eta \in \mathbb{T}. \tag{1.6}$$

Alternatively, if we define

$$F(x)(\gamma, \eta, t) = \begin{cases} \frac{|\eta|}{|x(\gamma,t) - x(\gamma-\eta,t)|} & \text{if } \eta \neq 0, \\ \frac{1}{|\partial_\gamma x(\gamma,t)|} & \text{if } \eta = 0, \end{cases} \tag{1.7}$$

then (1.6) is equivalent to

$$F(x_0)(\gamma, \eta, 0) < \infty \quad \forall \gamma, \eta \in \mathbb{T}. \tag{1.8}$$

The solution of (1.2) corresponding to the initial data in (1.5) can be determined by studying the evolution of the boundary of the patch. As derived in [6], the parametrization $x(\gamma, t)$ of the boundary $\partial\Omega(t)$ satisfies

$$\partial_t x(\gamma, t) = C_\beta(\theta_1 - \theta_2) \int_T \frac{\partial_\gamma x(\gamma, t) - \partial_\gamma x(\gamma - \eta, t)}{|x(\gamma, t) - x(\gamma - \eta, t)|^\beta} \, d\eta, \tag{1.9}$$

where $C_\beta$ is a constant depending on $\beta$ only. For $\beta \in (1, 2)$, the integral on the right of (1.9) is singular. Since the velocity in the tangential direction does not change the shape of the curve, we can modify (1.9) in the tangential direction so that we get an extra cancellation.

More precisely, we consider the modified equation

$$\begin{aligned} \partial_t x(\gamma, t) &= \\ &C_\beta(\theta_1 - \theta_2) \int_T \frac{\partial_\gamma x(\gamma, t) - \partial_\gamma x(\gamma - \eta, t)}{|x(\gamma, t) - x(\gamma - \eta, t)|^\beta} \, d\eta + \lambda(\gamma, t)\partial_\gamma x(\gamma, t) \end{aligned} \tag{1.10}$$

with $\lambda(\gamma, t)$ so chosen that

$$\partial_\gamma x(\gamma, t) \cdot \partial_\gamma^2 x(\gamma, t) = 0 \quad \text{or} \quad |\partial_\gamma x(\gamma, t)|^2 = A(t),$$

where $A(t)$ denotes a function of $t$ only. A similar calculation as in [6] leads to the following explicit formula for $\lambda(\gamma, t)$:

$$\begin{aligned} \lambda(\gamma, t) &= C\,\frac{\gamma + \pi}{2\pi} \int_{\mathbb{T}} \frac{\partial_\gamma x(\gamma, t)}{|\partial_\gamma x(\gamma, t)|^2} \cdot \partial_\gamma \left( \int_{\mathbb{T}} \frac{\partial_\gamma x(\gamma, t) - \partial_\gamma x(\gamma - \eta, t)}{|x(\gamma, t) - x(\gamma - \eta, t)|^\beta} \, d\eta \right) d\gamma \\ &\quad - C \int_{-\pi}^{\gamma} \frac{\partial_\gamma x(\eta, t)}{|\partial_\gamma x(\eta, t)|^2} \cdot \partial_\eta \left( \int_{\mathbb{T}} \frac{\partial_\gamma x(\eta, t) - \partial_\gamma x(\eta - \xi, t)}{|x(\eta, t) - x(\eta - \xi, t)|^\beta} \, d\xi \right) d\eta, \end{aligned} \tag{1.11}$$

where $C = C_\beta(\theta_1 - \theta_2)$.

We establish the local well-posedness of the contour dynamics equation (CDE) given by (1.10) and (1.11) corresponding to an initial contour

$$x(\gamma, 0) = x_0(\gamma)$$

satisfying (1.8). More precisely, we have the following theorem:

THEOREM 1.7. *Let $x_0(\gamma) \in H^k(\mathbb{T})$ for $k \geq 4$ and $F(x_0)(\gamma, \eta, 0) < \infty$ for any $\gamma, \eta \in \mathbb{T}$. Then there exists $T > 0$ such that the CDE given by* (1.10) *and* (1.11) *has a solution $x(\gamma, t) \in C([0, T]; H^k(\mathbb{T}))$ with $x(\gamma, 0) = x_0(\gamma)$.*

This theorem is proven by obtaining an inequality of the form

$$\frac{d}{dt}(\|x\|_{H^4} + \|F(x)\|_{L^\infty}) \leq C(\|x\|_{H^4} + \|F(x)\|_{L^\infty})^{9+\beta}.$$

The ingredients involved in the proof include appropriate combination and cancellation of terms. The detailed proof is provided in Section 5.

## 2 Local Smooth Solutions

This section proves Theorem 1.1, which assesses the local (in time) existence and uniqueness of solutions to (1.2) in $H^m$ with $m \geq 4$.

For $1 < \beta \leq 2$, the velocity $u$ is determined by a very singular integral of $\theta$ and the nonlinear term cannot be directly bounded. To deal with this difficulty, we rewrite the nonlinear term in the form of a commutator to explore the extra cancellation. The following proposition provides a $L^2$-bound for the commutator:

PROPOSITION 2.1. *Let $s$ be a real number. Let $\partial_x$ denote a partial derivative, either $\partial_{x_1}$ or $\partial_{x_2}$. Then*

$$\|[\Lambda^s \partial_x, g] f\|_{L^2(\mathbb{R}^2)} \leq C\big(\|\Lambda^s f\|_{L^2} \, \|\widehat{\Lambda g}(\eta)\|_{L^1} + C\|f\|_{L^2} \|\widehat{\Lambda^{1+s} g}(\eta)\|_{L^1}\big),$$

*where $C$ is a constant depending on $s$ only. In particular, by Sobolev embedding, for any $\epsilon > 0$, there exists $C_\epsilon$ such that*

$$\|[\Lambda^s \partial_x, g] f\|_{L^2(\mathbb{R}^2)} \leq C_\epsilon(\|\Lambda^s f\|_{L^2} \, \|g\|_{H^{2+\epsilon}} + \|f\|_{L^2} \, \|g\|_{H^{2+s+\epsilon}}).$$

Since this commutator estimate itself appears to be interesting, we provide a proof for this proposition.

PROOF. The Fourier transform of $[\Lambda^s \partial_x, g] f$ is given by

$$\widehat{[\Lambda^s \partial_x, g] f}(\xi) = \int_{\mathbb{R}^2} (|\xi|^s \xi_j - |\xi - \eta|^s (\xi - \eta)_j) \widehat{f}(\xi - \eta) \widehat{g}(\eta) d\eta. \tag{2.1}$$

where $j = 1$ or 2. It is easy to verify that, for any real number $s$,

$$\big||\xi|^s \xi_j - |\xi - \eta|^s (\xi - \eta)_j\big| \leq C \max\{|\xi|^s, |\xi - \eta|^s\} \, |\eta|. \tag{2.2}$$

In fact, we can write

$$
\begin{aligned}
|\xi|^s\xi_j - |\xi-\eta|^s(\xi-\eta)_j &= \int_0^1 \frac{d}{d\rho}(|A|^s A_j) \\
&= \int_0^1 (|A|^s \eta_j \; + \; s|A|^{s-2}(A\cdot\eta)A_j)\,d\rho,
\end{aligned} \tag{2.3}
$$

where $A(\rho,\xi,\eta) = \rho\xi + (1-\rho)(\xi-\eta)$. Therefore,

$$
\big||\xi|^s\xi_j - |\xi-\eta|^s(\xi-\eta)_j\big| \le (1+|s|)|\eta| \int_0^1 |A|^s\,d\rho.
$$

For $s \ge 0$, it is clear that

$$
|A|^s \le \max\{|\xi|^s, |\xi-\eta|^s\}.
$$

When $s < 0$, $F(x) = |x|^s$ is convex and

$$
\begin{aligned}
|A|^s = |\rho\xi + (1-\rho)(\xi-\eta)|^s &\le \rho|\xi|^s + (1-\rho)|\xi-\eta|^s \\
&\le \max\{|\xi|^s, |\xi-\eta|^s\}.
\end{aligned}
$$

To obtain the bound in Proposition 2.1, we first consider the case when $s \ge 0$. Inserting (2.2) into (2.1) and using the basic inequality $|\xi|^s \le 2^{s-1}(|\xi-\eta|^s + |\eta|^s)$, we have

$$
\begin{aligned}
&\big|\widehat{[\Lambda^s\partial_x, g]f}(\xi)\big| \\
&\le C|\xi|^s \int_{\mathbb{R}^2} |\widehat{f}(\xi-\eta)|\,|\eta\widehat{g}(\eta)|d\eta + C\int_{\mathbb{R}^2} \big||\xi-\eta|^s\widehat{f}(\xi-\eta)\big|\,|\eta\widehat{g}(\eta)|d\eta \\
&\le C\int_{\mathbb{R}^2} \big||\xi-\eta|^s\widehat{f}(\xi-\eta)\big|\,|\eta\widehat{g}(\eta)|d\eta + C\int_{\mathbb{R}^2} |\widehat{f}(\xi-\eta)|\,\big||\eta|^{1+s}\widehat{g}(\eta)\big|d\eta.
\end{aligned} \tag{2.4}
$$

By Plancherel's theorem and Young's inequality for convolution,

$$
\|[\Lambda^s\partial_x, g]f\|_{L^2} \le C\|\Lambda^s f\|_{L^2}\|\widehat{\Lambda g}(\eta)\|_{L^1} + C\|f\|_{L^2}\|\widehat{\Lambda^{1+s}g}(\eta)\|_{L^1}.
$$

Applying the embedding inequality

$$
\||\eta|^{1+s}\widehat{g}(\eta)\|_{L^1(\mathbb{R}^2)} \le C_\epsilon \|g\|_{H^{2+s+\epsilon}(\mathbb{R}^2)},
$$

we have, for $s \ge 0$,

$$
\begin{aligned}
&\|[\Lambda^s\partial_x, g]f\|_{L^2(\mathbb{R}^2)} \le \\
&\qquad C_\epsilon\big(\|\Lambda^s f\|_{L^2(\mathbb{R}^2)}\,\|g\|_{H^{2+\epsilon}(\mathbb{R}^2)} + \|f\|_{L^2(\mathbb{R}^2)}\,\|g\|_{H^{2+s+\epsilon}(\mathbb{R}^2)}\big).
\end{aligned}
$$

The case when $s < 0$ is handled differently. We insert (2.3) into (2.1) and change the order of integration to obtain

$$
\widehat{\big[\Lambda^s\partial_x, g\big]f}(\xi) = H_1 + H_2,
$$

where

$$H_1 = \int_0^1 \int_{\mathbb{R}^2} |A|^s \widehat{f}(\xi-\eta)\eta_j \widehat{g}(\eta) d\eta\, d\rho, \tag{2.5}$$

$$H_2 = s\int_0^1 \int_{\mathbb{R}^2} |A|^{s-2}(A\cdot\eta)A_j \widehat{f}(\xi-\eta)\widehat{g}(\eta) d\eta\, d\rho. \tag{2.6}$$

Using the fact that $F(x) = |x|^s$ with $s < 0$ is convex, we have

$$\begin{aligned} |A|^s = |(\xi-\eta)+\rho\eta|^s &= (1+\rho)^s \left| \frac{1}{1+\rho}(\xi-\eta) + \frac{\rho}{1+\rho}\eta \right|^s \\ &\le (1+\rho)^s \left( \frac{1}{1+\rho}|\xi-\eta|^s + \frac{\rho}{1+\rho}|\eta|^s \right) \\ &= (1+\rho)^{s-1}|\xi-\eta|^s + \rho(1+\rho)^{s-1}|\eta|^s. \end{aligned}$$

Inserting this inequality into (2.5), we obtain

$$\begin{aligned} |H_1| \le &\int_0^1 (1+\rho)^{s-1}\, d\rho \int_{\mathbb{R}^2} \big||\xi-\eta|^s \widehat{f}(\xi-\eta)\big|\, |\eta\widehat{g}(\eta)| d\eta \\ &+ \int_0^1 \rho(1+\rho)^{s-1}\, d\rho \int_{\mathbb{R}^2} |\widehat{f}(\xi-\eta)|\, \big||\eta|^{1+s}\, |\widehat{g}(\eta)|\big|\eta. \end{aligned}$$

Applying Young's inequality for convolution, Plancherel's theorem, and Sobolev's inequality, we have

$$\begin{aligned} \|H_1\|_{L^2} &\le C\|\Lambda^s f\|_{L^2} \|\widehat{\Lambda g}(\eta)\|_{L^1} + C\|f\|_{L^2}\, \|\widehat{\Lambda^{1+s} g}(\eta)\|_{L^1} \\ &\le C_\epsilon \|\Lambda^s f\|_{L^2}\, \|g\|_{H^{2+\epsilon}} + C_\epsilon \|f\|_{L^2}\, \|g\|_{H^{2+s+\epsilon}}. \end{aligned}$$

To bound $H_2$, it suffices to notice that

$$|H_2| \le |s| \int_0^1 \int_{\mathbb{R}^2} |A|^s\, |\widehat{f}(\xi-\eta)|\, |\eta\widehat{g}(\eta)| d\eta\, d\rho.$$

Therefore, $\|H_2\|_{L^2}$ admits the same bound as $\|H_1\|_{L^2}$. This completes the proof of Proposition 2.1. □

With this commutator estimate at our disposal, we are ready to prove Theorem 1.1.

PROOF OF THEOREM 1.1. This proof provides a local (in time) a priori bound for $\|\theta\|_{H^m}$. Once the local bound is established, the construction of a local solution can be obtained through standard procedure such as successive approximation. We shall omit the construction part to avoid redundancy.

We consider the case when $m = 4$. The general case can be dealt with in a similar manner. By $\nabla \cdot u = 0$,

$$\frac{1}{2}\frac{d}{dt}\|\theta(\cdot,t)\|_{L^2}^2 = 0 \quad \text{or} \quad \|\theta(\cdot,t)\|_{L^2} = \|\theta_0\|_{L^2}.$$

Let $\sigma$ be a multi-index with $|\sigma| = 4$. Then,

$$\frac{1}{2}\frac{d}{dt}\|D^\sigma\theta\|_{L^2}^2 = -\int D^\sigma\theta\, D^\sigma(u\cdot\nabla\theta)dx,$$

where $\int$ means the integral over $\mathbb{R}^2$; we shall omit $dx$ when there is no confusion. Clearly, the right-hand side can be decomposed into $I_1 + I_2 + I_3 + I_4 + I_5$ with

$$I_1 = -\int D^\sigma\theta\, D^\sigma u\cdot\nabla\theta\, dx,$$

$$I_2 = -\sum_{\substack{|\sigma_1|=3\\ \sigma_1+\sigma_2=\sigma}} \int D^\sigma\theta\, D^{\sigma_1}u\cdot D^{\sigma_2}\nabla\theta\, dx,$$

$$I_3 = -\sum_{\substack{|\sigma_1|=2\\ \sigma_1+\sigma_2=\sigma}} \int D^\sigma\theta\, D^{\sigma_1}u\cdot D^{\sigma_2}\nabla\theta\, dx,$$

$$I_4 = -\sum_{\substack{|\sigma_1|=1\\ \sigma_1+\sigma_2=\sigma}} \int D^\sigma\theta\, D^{\sigma_1}u\cdot D^{\sigma_2}\nabla\theta\, dx,$$

$$I_5 = \int D^\sigma\theta\, u\cdot\nabla D^\sigma\theta\, dx.$$

The divergence-free condition $\nabla\cdot u = 0$ yields $I_5 = 0$. We now estimate $I_1$. For $1 < \beta < 2$, $D^\sigma u = \nabla^\perp\Lambda^{-2+\beta}D^\sigma\theta$ with $|\sigma| = 4$ cannot be bounded directly in terms of $\|\theta\|_{H^4}$. We rewrite $I_1$ as a commutator. For this we observe that for any skew-adjoint operator $A$ in $L^2$ (i.e., $(Af,g)_{L^2} = -(f,Ag)_{L^2}$ for all $f,g \in L^2$), we have $\int fA(f)g\,dx = -\int fA(gf)dx$, and therefore

$$\int fA(f)g\,dx = -\frac{1}{2}\int\{fA(gf) - fgA(f)\}\,dx = -\frac{1}{2}\int f[A,g]f\,dx. \tag{2.7}$$

Applying this fact to $I_1$ with $A := \Lambda^{-2+\beta}\nabla^\perp$, $f := D^\sigma\theta$, and $g := \nabla\theta$, one obtains

$$I_1 = \frac{1}{2}\int D^\sigma\theta[\Lambda^{-2+\beta}\nabla^\perp\cdot,\nabla\theta]D^\sigma\theta\,dx.$$

By Hölder's inequality and Proposition 2.1 with $s = -2+\beta < 0$, we have

$$\begin{aligned}|I_1| &\le C_\epsilon\|D^\sigma\theta\|_{L^2}(\|D^\sigma\theta\|_{L^2} + \|\Lambda^{-2+\beta}D^\sigma\theta\|_{L^2})\|\theta\|_{H^{3+\epsilon}}\\ &\le C\|D^\sigma\theta\|_{L^2}\,\|\theta\|_{H^4}^2.\end{aligned}$$

The estimate for $I_2$ is easy. By Hölder's and Sobolev's inequalities,

$$|I_2| \le C\|D^\sigma\theta\|_{L^2}\,\|\theta\|_{H^{2+\beta}}\,\|\theta\|_{H^4}.$$

By Hölder's inequality and the Gagliardo-Nirenberg inequality,

$$
\begin{aligned}
|I_3| &\le C \sum_{\substack{|\sigma_1|=2\\ \sigma_1+\sigma_2=4}} \|D^\sigma\theta\|_{L^2}\,\|D^{\sigma_1}u\|_{L^4}\,\|D^{\sigma_2}\nabla\theta\|_{L^4}\\
&\le C\,\|D^\sigma\theta\|_{L^2}\,\|\theta\|_{H^{\beta+1}}^{1/2}\,\|\theta\|_{H^{\beta+2}}^{1/2}\,\|\theta\|_{H^3}^{1/2}\,\|\theta\|_{H^4}^{1/2}\\
&\le C\,\|D^\sigma\theta\|_{L^2}\,\|\theta\|_{H^3}\,\|\theta\|_{H^4}.
\end{aligned}
$$

By Hölder's and Sobolev's inequalities,

$$
\begin{aligned}
|I_4| &\le C \sum_{\substack{|\sigma_1|=1\\ \sigma_1+\sigma_2=4}} \|D^\sigma\theta\|_{L^2}\,\|D^{\sigma_1}u\|_{L^\infty}\,\|D^{\sigma_2}\nabla\theta\|_{L^2}\\
&\le C\,\|D^\sigma\theta\|_{L^2}\,\|\theta\|_{H^{\beta+2}}\,\|\theta\|_{H^4}.
\end{aligned}
$$

For $1<\beta<2$, the bounds above yields

$$\frac{d}{dt}\|\theta\|_{H^4}^2 \le C\,\|\theta\|_{H^4}^3.$$

This inequality allows us to obtain a local (in time) bound for $\|\theta\|_{H^4}$.

In order to get uniqueness, one could check the evolution of two solutions with the same initial data. With a similar approach, we find

$$\frac{d}{dt}\|\theta_2-\theta_1\|_{H^1} \le C(\|\theta_2\|_{H^4}+\|\theta_1\|_{H^4})\|\theta_2-\theta_1\|_{H^1}.$$

An easy application of the Gronwall inequality provides $\theta_2=\theta_1$. This concludes the proof of Theorem 1.1. □

## 3 The Case Logarithmically Beyond $\beta = 2$

This section focuses on the dissipative active scalar equation defined in (1.3), and the goal is to prove Theorem 1.3.

As mentioned in the introduction, the major difficulty in proving this theorem is due to the fact that the velocity $u$ is determined by a very singular integral of $\theta$. To overcome this difficulty, we rewrite the nonlinear term in the form of a commutator to explore the extra cancellation. The commutator involves the logarithm of the Laplacian, and we need a suitable bound for this type of commutator. The bound is stated in Proposition 1.4, but we restate it here.

PROPOSITION 3.1. *Let $\mu\ge 0$. Let $\partial_x$ denote a first partial, i.e., either $\partial_{x_1}$ or $\partial_{x_2}$. Then, for any $\delta>0$ and $\epsilon>0$,*

$$\|[(\ln(I-\Delta))^\mu\partial_x, g]f\|_{L^2} \le C_{\mu,\epsilon,\delta}\left(1+\left(\ln\left(1+\frac{\|f\|_{\dot H^\delta}}{\|f\|_{L^2}}\right)\right)^\mu\right)\|f\|_{L^2}\,\|g\|_{H^{2+3\epsilon}},$$

*where $C_{\mu,\epsilon,\delta}$ is a constant depending on $\mu$, $\epsilon$, and $\delta$ only, and $\dot H^\delta$ denotes the standard homogeneous Sobolev space.*

*Remark* 3.2. The constant $C_{\mu,\epsilon,\delta}$ approaches $\infty$ as $\delta \to 0$ or $\epsilon \to 0$. When $\mu = 0$, the constant depends on $\epsilon$ only.

We shall also make use of the following lemma that bounds the $L^2$-norm of the logarithm of function.

LEMMA 3.3. *Let $\mu \geq 0$ be a real number. Then, for any $\delta > 0$,*

$$\|(\ln(I-\Delta))^{\mu} f\|_{L^2} \leq C_{\mu,\delta}\|f\|_{L^2}\left(\ln\left(1+\frac{\|f\|_{\dot{H}^\delta}}{\|f\|_{L^2}}\right)\right)^{\mu}. \tag{3.1}$$

*where $C_{\mu,\delta}$ is a constant depending on $\mu$ and $\delta$ only.*

In the rest of this section, we first prove Theorem 1.3, then Proposition 3.1, and finally Lemma 3.3.

PROOF OF THEOREM 1.3. The proof obtains a local a priori bound for $\|\theta\|_{H^4}$. Once the local bound is at our disposal, a standard approach such as successive approximation can be employed to provide a complete proof for the local existence and uniqueness. Since this portion involves no essential difficulties, the details will be omitted.

To establish the local $H^4$-bound, we start with the $L^2$-bound. By $\nabla \cdot u = 0$,

$$\frac{1}{2}\frac{d}{dt}\|\theta\|_{L^2}^2 + \kappa\|\Lambda^\alpha\theta\|_{L^2}^2 = 0 \quad \text{or} \quad \|\theta(\,\cdot\,,t)\|_{L^2} \leq \|\theta_0\|_{L^2}.$$

Now let $\sigma$ be a multi-index with $|\sigma| = 4$. Then,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|D^\sigma\theta\|_{L^2}^2 + \kappa\|\Lambda^\alpha D^\sigma\theta\|_{L^2}^2 &= -\int D^\sigma\theta\, D^\sigma(u\cdot\nabla\theta)dx \\ &= J_1 + J_2 + J_3 + J_4 + J_5,\end{aligned} \tag{3.2}$$

where

$$J_1 = -\int D^\sigma\theta\, D^\sigma u\cdot\nabla\theta\, dx,$$

$$J_2 = -\sum_{\substack{|\sigma_1|=3\\ \sigma_1+\sigma_2=\sigma}}\int D^\sigma\theta\, D^{\sigma_1}u\cdot D^{\sigma_2}\nabla\theta\, dx,$$

$$J_3 = -\sum_{\substack{|\sigma_1|=2\\ \sigma_1+\sigma_2=\sigma}}\int D^\sigma\theta\, D^{\sigma_1}u\cdot D^{\sigma_2}\nabla\theta\, dx,$$

$$J_4 = -\sum_{\substack{|\sigma_1|=1\\ \sigma_1+\sigma_2=\sigma}}\int D^\sigma\theta\, D^{\sigma_1}u\cdot D^{\sigma_2}\nabla\theta\, dx,$$

$$J_5 = \int D^\sigma\theta\, u\cdot\nabla D^\sigma\theta\, dx.$$

By $\nabla\cdot u = 0$, $J_5 = 0$. To bound $J_1$, we write it as a commutator integral. Applying (2.7) with $A := \nabla^{\perp}(\log(I-\Delta))^{\mu}$, $f := D^{\sigma}\theta$, and $g := \nabla\theta$, we have

$$J_1 = \frac{1}{2}\int D^{\sigma}\theta[(\log(I-\Delta))^{\mu}\nabla^{\perp}\cdot, \nabla\theta]D^{\sigma}\theta\, dx.$$

By Hölder's inequality and Proposition 3.1,

$$\begin{aligned}|J_1| &\le C\|D^{\sigma}\theta\|_{L^2}\,\|[(\log(I-\Delta))^{\mu}\nabla^{\perp}\cdot, \nabla\theta]D^{\sigma}\theta\|_{L^2}\\ &\le C\|D^{\sigma}\theta\|_{L^2}^2\,\|\nabla\theta\|_{H^{2+\epsilon}}(1+(\ln(1+\|D^{\sigma}\theta\|_{H^{\delta}}))^{\mu})\\ &\le C_{\epsilon}\,\|D^{\sigma}\theta\|_{L^2}^2\,\|\theta\|_{H^{3+\epsilon}}\,(\ln(1+\|\theta\|_{H^{4+\delta}}))^{\mu}.\end{aligned}$$

Applying Hölder's inequality, Lemma 3.3, and the Sobolev embedding

$$H^{1+\epsilon}(\mathbb{R}^2) \hookrightarrow L^{\infty}(\mathbb{R}^2), \quad \epsilon > 0, \tag{3.3}$$

we obtain

$$\begin{aligned}|J_2| &\le C \sum_{\substack{|\sigma_1|=3\\ \sigma_1+\sigma_2=4}} \|D^{\sigma}\theta\|_{L^2}\,\|D^{\sigma_1}u\|_{L^2}\,\|D^{\sigma_2}\nabla\theta\|_{L^{\infty}}\\ &\le C_{\epsilon}\,\|D^{\sigma}\theta\|_{L^2}^2(\ln(1+\|\theta\|_{H^{4+\delta}}))^{\mu}\|\theta\|_{H^{3+\epsilon}}.\end{aligned}$$

To bound $J_3$, we first apply Hölder's inequality to obtain

$$|J_3| \le C \sum_{|\sigma_1|=2,\sigma_1+\sigma_2=4} \|D^{\sigma}\theta\|_{L^2}\,\|D^{\sigma_1}u\|_{L^4}\,\|D^{\sigma_2}\nabla\theta\|_{L^4}.$$

By the Sobolev inequality

$$\|f\|_{L^4(\mathbb{R}^2)} \le C\|f\|_{L^2(\mathbb{R}^2)}^{1/2}\,\|\nabla f\|_{L^2(\mathbb{R}^2)}^{1/2}$$

and applying Lemma 3.3, we have

$$\begin{aligned}|J_3| &\le C \sum_{|\sigma_1|=2,\sigma_1+\sigma_2=4} \Big(\|D^{\sigma}\theta\|_{L^2}\,\|D^{\sigma_1}u\|_{L^2}^{1/2}\,\|\nabla D^{\sigma_1}u\|_{L^2}^{1/2}\\ &\qquad\qquad \times\|D^{\sigma_2}\nabla\theta\|_{L^2}^{1/2}\,\|\nabla D^{\sigma_2}\nabla\theta\|_{L^2}^{1/2}\Big)\\ &\le C\|D^{\sigma}\theta\|_{L^2}\,\|\theta\|_{H^4}^2(\ln(1+\|\theta\|_{H^{4+\delta}}))^{\mu}.\end{aligned}$$

By Hölder's inequality, (3.3), and Lemma 3.3,

$$\begin{aligned}|J_4| &\le C \sum_{\substack{|\sigma_1|=1\\ \sigma_1+\sigma_2=4}} \|D^{\sigma}\theta\|_{L^2}\,\|D^{\sigma_1}u\|_{L^{\infty}}\,\|D^{\sigma_2}\nabla\theta\|_{L^2}\\ &\le C \sum_{|\sigma_1|=1,\sigma_1+\sigma_2=4} \|D^{\sigma}\theta\|_{L^2}\,\|D^{\sigma_1}u\|_{H^{1+\epsilon}}\,\|D^{\sigma_2}\nabla\theta\|_{L^2}\\ &\le C\|D^{\sigma}\theta\|_{L^2}\|\theta\|_{H^4}\,\|\theta\|_{H^{3+\epsilon}}(\ln(1+\|\theta\|_{H^{3+\epsilon+\delta}}))^{\mu}.\end{aligned}$$

Let $0 < \epsilon \le 1$ and $0 < \delta < \alpha$. The estimates above on the right-hand side of (3.2) then imply that

$$\frac{1}{2}\frac{d}{dt}\|D^\sigma\theta\|_{L^2}^2 + \kappa\|\Lambda^\alpha D^\sigma\theta\|_{L^2}^2 \le C\|\theta\|_{H^4}^3(\ln(1+\|\theta\|_{H^{4+\alpha}}))^\mu.$$

This inequality is obtained for $|\sigma| = 4$. Obviously, for $|\sigma| = 1, 2, 3$, the bound on the right remains valid. Therefore, if we sum the inequalities for $|\alpha| = 1, 2, 3, 4$ we have

$$\frac{1}{2}\frac{d}{dt}\|\theta\|_{H^4}^2 + \kappa\|\theta\|_{H^{4+\alpha}}^2 \le C\|\theta\|_{H^4}^3(\ln(1+\|\theta\|_{H^{4+\alpha}}))^\mu.$$

The local (in time) a priori bound for $\|\theta\|_{H^4}$ then follows if we notice the simple inequality $(\ln(1+a))^\mu \le a$ for large $a > 0$. This completes the proof of Theorem 1.3. □

We now present the proof of Proposition 3.1.

PROOF OF PROPOSITION 3.1. This proof employs Besov spaces and related concepts such as the Fourier localization operator $\Delta_j$ for $j = -1, 0, 1, \dots$, and the operator $S_j$. These tools are now standard and can be found in several books, say [4, 8, 13]. A self-contained quick introduction to the notation used in this proof can be found in [2].

We start by identifying $L^2$ with the inhomogeneous Besov space $B^0_{2,2}$, namely,

$$\|f\|_{L^2}^2 = \sum_{j=-1}^{\infty}\|\Delta_j f\|_{L^2}^2.$$

Let $N \ge 1$ be an integer to be determined later. We write

$$\|[(\ln(I-\Delta))^\mu\partial_x, g]f\|_{L^2}^2 = K_1 + K_2, \tag{3.4}$$

where

$$K_1 = \sum_{j=-1}^{N-1}\|\Delta_j[(\ln(I-\Delta))^\mu\partial_x, g]f\|_{L^2}^2, \tag{3.5}$$

$$K_2 = \sum_{j=N}^{\infty}\|\Delta_j[(\ln(I-\Delta))^\mu\partial_x, g]f\|_{L^2}^2. \tag{3.6}$$

Following Bony's notion of paraproducts,

$$FG = \sum_k S_{k-1}F\Delta_k G + \sum_k \Delta_k F S_{k-1}G + \sum_k \Delta_k F\widetilde{\Delta}_k G$$

with $\widetilde{\Delta}_k = \Delta_{k-1} + \Delta_k + \Delta_{k+1}$, we have the decomposition

$$\begin{aligned} f &= (\ln(I-\Delta))^\mu\partial_x(fg) - ((\ln(I-\Delta))^\mu\partial_x f)g \\ &= L_1 + L_2 + L_3, \end{aligned} \tag{3.7}$$

where

$$L_1 = \sum_k (\ln(I-\Delta))^\mu \, \partial_x(S_{k-1}f \, \Delta_k g) - S_{k-1}((\ln(I-\Delta))^\mu \partial_x f)\Delta_k g,$$

$$L_2 = \sum_k (\ln(I-\Delta))^\mu \, \partial_x(\Delta_k f \, S_{k-1} g) - \Delta_k((\ln(I-\Delta))^\mu \partial_x f) S_{k-1} g,$$

$$L_3 = \sum_k (\ln(I-\Delta))^\mu \, \partial_x(\Delta_k f \, \widetilde{\Delta}_k g) - \Delta_k((\ln(I-\Delta))^\mu \partial_x f)\widetilde{\Delta}_k g.$$

Inserting the decomposition (3.7) into (3.5) and (3.6) yields the following corresponding decompositions in $K_1$ and $K_2$:

$$K_1 \le K_{11} + K_{12} + K_{13}, \quad K_2 \le K_{21} + K_{22} + K_{23},$$

with

$$K_{11} = \sum_{j=-1}^{N-1} \|\Delta_j L_1\|_{L^2}^2, \quad K_{12} = \sum_{j=-1}^{N-1} \|\Delta_j L_2\|_{L^2}^2, \quad K_{13} = \sum_{j=-1}^{N-1} \|\Delta_j L_3\|_{L^2}^2,$$

$$K_{21} = \sum_{j=N}^{\infty} \|\Delta_j L_1\|_{L^2}^2, \quad K_{22} = \sum_{j=N}^{\infty} \|\Delta_j L_2\|_{L^2}^2, \quad K_{23} = \sum_{j=N}^{\infty} \|\Delta_j L_3\|_{L^2}^2.$$

Attention is now focused on bounding these terms; we start with $K_{11}$. When $\Delta_j$ is applied to $L_1$, the summation over $k$ in $L_1$ becomes a finite summation for $k$ satisfying $|k-j| \le 3$, namely,

$$\Delta_j L_1 = \sum_{|k-j|\le 3} \Delta_j \big((\ln(I-\Delta))^\mu \partial_x(S_{k-1} f \Delta_k g) - S_{k-1}((\ln(I-\Delta))^\mu \partial_x f)\Delta_k g\big).$$

For the sake of brevity, we shall just estimate the representative term with $k=j$ in $\Delta_j L_1$. The treatment of the rest of the terms satisfying $|k-j| \le 3$ is similar and yields the same bound. Therefore,

$$\|\Delta_j L_1\|_{L^2} \le C \big\| \Delta_j \big((\ln(I-\Delta))^\mu \partial_x(S_{j-1} f \Delta_j g) - S_{j-1}((\ln(I-\Delta))^\mu \partial_x f)\Delta_j g\big)\big\|_{L^2}.$$

Without loss of generality, we set $\partial_x = \partial_{x_1}$. By Plancherel's theorem,

$$\|\Delta_j L_1\|_{L^2}^2 \le C \left\| \Phi_j(\xi) \int_{\mathbb{R}^2} (H(\xi) - H(\xi-\eta)) \widehat{S_{j-1} f}(\xi-\eta) \widehat{\Delta_j g}(\eta) \, d\eta \right\|_{L^2}^2,$$

where $\Phi_j$ denotes the symbol of $\Delta_j$, namely $\widehat{\Delta_j f}(\xi) = \Phi_j(\xi)\hat{f}(\xi)$, and

$$H(\xi) = (\ln(1+|\xi|^2))^\mu \xi_1.$$

To further the estimate, we first invoke the inequality

$$|H(\xi)-H(\xi-\eta)| \le |\eta|\big((\ln(1+\max\{|\xi|^2,|\xi-\eta|^2\}))^{\mu} + \mu(\ln(1+\max\{|\xi|^2,|\xi-\eta|^2\}))^{\mu-1}\big).$$

Clearly, the first term on the right-hand side dominates. We assume, without loss of generality, that

$$|H(\xi)-H(\xi-\eta)| \le C|\eta|(\ln(1+\max\{|\xi|^2,|\xi-\eta|^2\}))^{\mu}. \tag{3.8}$$

Noticing that

$$\operatorname{supp}\Phi_j,\ \operatorname{supp}\widehat{\Delta_j g} \subset \{\xi\in\mathbb{R}^2 : 2^{j-1}\le|\xi|<2^{j+1}\},$$

we have, for $-1\le j\le N-1$,

$$\begin{aligned}\|\Delta_j L_1\|_{L^2}^2 &\le C\Big\|\Phi_j(\xi)\int_{\mathbb{R}^2}(\ln(1+\max\{|\xi|^2,|\xi-\eta|^2\}))^{\mu} |\widehat{S_{j-1}f}(\xi-\eta)|\,|\eta\widehat{\Delta_j g}(\eta)|d\eta\Big\|_{L^2}^2\\ &\le C(\ln(1+2^{2N}))^{2\mu}\Big\|\Phi_j(\xi)\int_{\mathbb{R}^2}|\widehat{S_{j-1}f}(\xi-\eta)|\,|\eta\widehat{\Delta_j g}(\eta)|d\eta\Big\|_{L^2}^2\\ &\le C(\ln(1+2^{2N}))^{2\mu}\Big\|\int_{\mathbb{R}^2}|\widehat{S_{j-1}f}(\xi-\eta)|\,|\eta\widehat{\Delta_j g}(\eta)|d\eta\Big\|_{L^2}^2.\end{aligned} \tag{3.9}$$

By Young's inequality for convolution,

$$\|\Delta_j L_1\|_{L^2}^2 \le C(\ln(1+2^{2N}))^{2\mu}\|\widehat{S_{j-1}f}\|_{L^2}^2\,\|\eta\widehat{\Delta_j g}(\eta)\|_{L^1}^2.$$

By Plancherel's theorem and Hölder's inequality, for any $\epsilon>0$,

$$\|\widehat{S_{j-1}f}\|_{L^2} = \|S_{j-1}f\|_{L^2} \le \|f\|_{L^2}, \quad \|\eta\widehat{\Delta_j g}(\eta)\|_{L^1} \le C_\epsilon\|\Lambda^{2+\epsilon}\Delta_j g\|_{L^2}.$$

Therefore,

$$\begin{aligned}K_{11} &\le C_\epsilon(\ln(1+2^{2N}))^{2\mu}\|f\|_{L^2}^2\sum_{j=-1}^{N-1}\|\Lambda^{2+\epsilon}\Delta_j g\|_{L^2}\\ &\le C_\epsilon(\ln(1+2^{2N}))^{2\mu}\|f\|_{L^2}^2\,\|g\|_{H^{2+\epsilon}}^2.\end{aligned} \tag{3.10}$$

We now estimate $K_{12}$. As in $\Delta_j L_1$, we have

$$\Delta_j L_2 = \sum_{|k-j|\le 3}\Delta_j\big((\ln(I-\Delta))^{\mu}\partial_x(\Delta_k f S_{k-1}g) - \Delta_k((\ln(I-\Delta))^{\mu}\partial_x f)S_{k-1}g\big).$$

It suffices to estimate the representative term with $k = j$. As in the estimate of $\Delta_j L_1$, we have

$$\begin{aligned}\|\Delta_j L_2\|_{L^2}^2 &\le C(\ln(1+2^{2N}))^{2\mu}\left\|\int_{\mathbb{R}^2} |\widehat{\Delta_j f}(\xi-\eta)|\,|\eta\widehat{S_{j-1}g}(\eta)|d\eta\right\|_{L^2}^2\\ &\le C(\ln(1+2^{2N}))^{2\mu}\|\widehat{\Delta_j f}\|_{L^2}^2\,\|\eta\widehat{S_{j-1}g}(\eta)\|_{L^1}^2\\ &\le C(\ln(1+2^{2N}))^{2\mu}\|\Delta_j f\|_{L^2}^2\,\|g\|_{H^{2+\epsilon}}^2.\end{aligned}$$

Therefore,

$$\begin{aligned}K_{12} &\le C(\ln(1+2^{2N}))^{2\mu}\sum_{j=-1}^{N-1}\|\Delta_j f\|_{L^2}^2\,\|g\|_{H^{2+\epsilon}}^2\\ &\le C(\ln(1+2^{2N}))^{2\mu}\|f\|_{L^2}^2\,\|g\|_{H^{2+\epsilon}}^2.\end{aligned} \tag{3.11}$$

$K_{13}$ involves the interaction between high frequencies of $f$ and $g$, and the estimate is slightly more complicated. First we notice that

$$\Delta_j L_3 = \sum_{k\ge j-1}\Delta_j\big((\ln(I-\Delta))^{\mu}\partial_x(\Delta_k f\,\widetilde{\Delta}_k g) - \Delta_k\big((\ln(I-\Delta))^{\mu}\partial_x f\big)\widetilde{\Delta}_k g\big).$$

Applying Plancherel's theorem and invoking (3.8), we find

$$\begin{aligned}\|\Delta_j L_3\|_{L^2}^2 &\le \sum_{k\ge j-1}\big\|\Delta_j\big((\ln(I-\Delta))^{\mu}\partial_x(\Delta_k f\,\widetilde{\Delta}_k g)\\ &\qquad\qquad - \Delta_k\big((\ln(I-\Delta))^{\mu}\partial_x f\big)\widetilde{\Delta}_k g\big)\big\|_{L^2}^2\\ &\le C\sum_{k\ge j-1}\left\|\Phi_j(\xi)\int_{\mathbb{R}^2}(\ln(1+\max\{|\xi|^2,|\xi-\eta|^2\}))^{\mu}\right.\\ &\qquad\qquad\left.\times|\widehat{\Delta_k f}(\xi-\eta)|\,|\eta\widehat{\widetilde{\Delta}_k g}(\eta)|d\eta\right\|_{L^2}^2.\end{aligned} \tag{3.12}$$

Since $\Phi_j$ is supported on $\{\xi\in\mathbb{R}^2 : 2^{j-1}\le|\xi|<2^{j+1}\}$ and $\widehat{\Delta_k f}$ is on $\{\xi\in\mathbb{R}^2 : 2^{k-1}\le|\xi|<2^{k+1}\}$, we have, for $k\ge j-1$,

$$\begin{aligned}(\ln(1+\max\{|\xi|^2,|\xi-\eta|^2\}))^{\mu} &\le (\ln(1+\max\{2^{2j+2},2^{2(k+1)}\})^{\mu}\\ &\le (\ln(1+2^{2k+4}))^{\mu}.\end{aligned}$$

Therefore,

$$\|\Delta_j L_3\|_{L^2}^2 \le C \sum_{k\ge j-1} (\ln(1+2^{2k+4}))^{2\mu} \times \left\| \Phi_j(\xi) \int_{\mathbb{R}^2} |\widehat{\Delta_k f}(\xi-\eta)|\, |\eta \widehat{\widetilde{\Delta}_k g}(\eta)| d\eta \right\|_{L^2}^2.$$

When $\eta$ is in the support of $\widehat{\widetilde{\Delta}_k g}$, $|\eta|$ is comparable to $2^k$ and $|\eta|^{2\epsilon} \sim 2^{2\epsilon k}$. Using this fact and Young's inequality for convolution, we have

$$\begin{aligned}\|\Delta_j L_3\|_{L^2}^2 &\le C \sum_{k\ge j-1} (\ln(1+2^{2k+4}))^{2\mu} 2^{-2\epsilon k} \times \left\| \int_{\mathbb{R}^2} |\widehat{\Delta_k f}(\xi-\eta)|\, \left||\eta|^{1+2\epsilon} \widehat{\widetilde{\Delta}_k g}(\eta)\right| d\eta \right\|_{L^2}^2 \\ &\le C \sum_{k\ge j-1} (\ln(1+2^{2k+4}))^{2\mu} 2^{-2\epsilon k} \|\widehat{\Delta_k f}\|_{L^2}^2 \left\| |\eta|^{1+2\epsilon} \widehat{\widetilde{\Delta}_k g}(\eta)\right\|_{L^1}^2.\end{aligned}$$

Using the fact that

$$(\ln(1+2^{2k+4}))^{2\mu} 2^{-\epsilon k} \le C_\epsilon, \quad \left\| |\eta|^{1+2\epsilon} \widehat{\widetilde{\Delta}_k g}(\eta)\right\|_{L^1} \le C_\epsilon \|g\|_{H^{2+3\epsilon}},$$

we obtain

$$\|\Delta_j L_3\|_{L^2}^2 \le C_\epsilon \|g\|_{H^{2+3\epsilon}}^2 \sum_{k\ge j-1} 2^{-\epsilon k} \|\Delta_k f\|_{L^2}^2.$$

Therefore,

$$\begin{aligned}K_{13} &= \sum_{j=-1}^{N-1} \|\Delta_j L_3\|_{L^2}^2 \\ &\le C_\epsilon \|g\|_{H^{2+3\epsilon}}^2 \sum_{j=-1}^{N-1} 2^{-\epsilon j} \sum_{k\ge j-1} 2^{-\epsilon(k-j)} \|\Delta_k f\|_{L^2}^2 \\ &\le C_\epsilon \|g\|_{H^{2+3\epsilon}}^2 \|f\|_{L^2}^2. \qquad (3.13)\end{aligned}$$

We now turn to $K_{21}$. $\Delta_j L_1$ is bounded differently. As in (3.9), we have

$$\|\Delta_j L_1\|_{L^2}^2 \le C \left\| \Phi_j(\xi) \int_{\mathbb{R}^2} (\ln(1+\max\{|\xi|^2, |\xi-\eta|^2\}))^\mu \times |\widehat{S_{j-1} f}(\xi-\eta)|\, |\eta \widehat{\Delta_j g}(\eta)| d\eta \right\|_{L^2}^2.$$

Since $\operatorname{supp}\Phi_j, \operatorname{supp}\widehat{\Delta_j g} \subset \{\xi \in \mathbb{R}^2 : 2^{j-1} \le |\xi| < 2^{j+1}\}$, we have

$$(\ln(1+\max\{|\xi|^2, |\xi-\eta|^2\}))^\mu \le C(\ln(1+2^{2j}))^\mu,$$

and $\eta \in \operatorname{supp}\widehat{\Delta_j g}$ indicates that $|\eta|$ is comparable to $2^j$. Therefore,

$$\begin{aligned}\|\Delta_j L_1\|_{L^2}^2 &\le C(\ln(1+2^{2j}))^{2\mu}2^{-2\epsilon j}\left\|\int_{\mathbb{R}^2}|\widehat{S_{j-1}f}(\xi-\eta)|\,\big||\eta|^{1+\epsilon}\widehat{\Delta_j g}(\eta)\big|d\eta\right\|_{L^2}^2 \\ &\le C(\ln(1+2^{2j}))^{2\mu}2^{-2\epsilon j}\|\widehat{S_{j-1}f}\|_{L^2}^2\,\||\eta|^{1+\epsilon}\widehat{\Delta_j g}(\eta)\|_{L^1}^2 \\ &\le C(\ln(1+2^{2j}))^{2\mu}2^{-2\epsilon j}\|f\|_{L^2}^2\,\|\Lambda^{2+2\epsilon}\Delta_j g\|_{L^2}^2.\end{aligned}$$

Therefore,

$$\begin{aligned}K_{21} &= \sum_{j=N}^{\infty}\|\Delta_j L_1\|_{L^2}^2 \\ &\le C\|f\|_{L^2}^2\sum_{j=N}^{\infty}(\ln(1+2^{2j}))^{2\mu}2^{-2\epsilon j}\|\Lambda^{2+2\epsilon}\Delta_j g\|_{L^2}^2 \\ &\le C\|f\|_{L^2}^2(\ln(1+2^{2N}))^{2\mu}2^{-2\epsilon N}\|g\|_{H^{2+2\epsilon}}^2 \\ &\le C\|f\|_{L^2}^2\,\|g\|_{H^{2+2\epsilon}}^2.\end{aligned} \tag{3.14}$$

We now bound $K_{22}$. $\Delta_j L_2$ admits the following bound:

$$\begin{aligned}\|\Delta_j L_2\|_{L^2}^2 \le C\Bigg\|\Phi_j(\xi)\int_{\mathbb{R}^2}&(\ln(1+\max\{|\xi|^2,|\xi-\eta|^2\}))^\mu \\ &\times|\widehat{\Delta_j f}(\xi-\eta)|\,|\eta\widehat{S_{j-1}g}(\eta)|d\eta\Bigg\|_{L^2}^2.\end{aligned}$$

Since $\operatorname{supp}\Phi_j \subset \{\xi \in \mathbb{R}^2 : 2^{j-1} \le |\xi| < 2^{j+1}\}$ and $\operatorname{supp}\widehat{S_{j-1}g} \subset \{\xi \in \mathbb{R}^2 : |\xi| < 2^j\}$, we still have

$$(\ln(1+\max\{|\xi|^2,|\xi-\eta|^2\}))^\mu \le C(\ln(1+2^{2j}))^\mu.$$

In contrast to the previous estimate on $\Delta_j L_1$, $\eta \in \widehat{S_{j-1}g}$ no longer implies that $|\eta|$ is comparable to $2^j$. However, any $\xi \in \operatorname{supp}\widehat{\Delta_j f}$ must have $|\xi|$ comparable to $2^j$.

Therefore, for any $\delta > 0$,

$$\begin{aligned}
&\|\Delta_j L_2\|_{L^2}^2 \\
&\quad\le C(\ln(1+2^{2j}))^{2\mu}2^{-2\delta j}\left\| \int_{\mathbb{R}^2} \big||\xi-\eta|^\delta \widehat{\Delta_j f}(\xi-\eta)\big|\, |\eta\widehat{S_{j-1}g}(\eta)|\,d\eta \right\|_{L^2}^2 \\
&\quad\le C(\ln(1+2^{2j}))^{2\mu}2^{-2\delta j}\, \||\xi-\eta|^\delta \widehat{\Delta_j f}(\xi-\eta)\|_{L^2}^2\, \|\eta\widehat{S_{j-1}g}(\eta)\|_{L^1}^2 \\
&\quad\le C(\ln(1+2^{2j}))^{2\mu}2^{-2\delta j}\, \|\Delta_j\Lambda^\delta f\|_{L^2}^2\, \|g\|_{H^{2+\epsilon}}^2.
\end{aligned}$$

Thus,

$$\begin{aligned}
K_{22} &\le C\sum_{j=N}^{\infty}(\ln(1+2^{2j}))^{2\mu}2^{-2\delta j}\,\|\Delta_j\Lambda^\delta f\|_{L^2}^2\,\|g\|_{H^{2+\epsilon}}^2 \\
&\le C(\ln(1+2^{2N}))^{2\mu}2^{-2\delta N}\,\|g\|_{H^{2+\epsilon}}^2\sum_{j=N}^{\infty}\|\Delta_j\Lambda^\delta f\|_{L^2}^2 \\
&\le C(\ln(1+2^{2N}))^{2\mu}2^{-2\delta N}\,\|g\|_{H^{2+\epsilon}}^2\,\|f\|_{H^\delta}^2. \qquad (3.15)
\end{aligned}$$

The last term $K_{23}$ can be dealt with exactly as $K_{13}$. The bound for $K_{23}$ is

$$K_{23} \le C_\epsilon \|g\|_{H^{2+3\epsilon}}^2\,\|f\|_{L^2}^2. \qquad (3.16)$$

Collecting the estimates in (3.10), (3.11), (3.13), (3.14), (3.15), and (3.16), and inserting them into (3.4), we obtain, for any integer $N > 1$,

$$\begin{aligned}
\|[(\ln(I-\Delta))^\mu\partial_x, g]f\|_{L^2}^2 \le{}& C_\epsilon(\ln(1+2^{2N}))^{2\mu}\,\|f\|_{L^2}^2\,\|g\|_{H^{2+\epsilon}}^2 \\
&+ C_\epsilon\|f\|_{L^2}^2\,\|g\|_{H^{2+3\epsilon}}^2 \\
&+ C_\epsilon(\ln(1+2^{2N}))^{2\mu}2^{-2\delta N}\,\|f\|_{H^\delta}^2\,\|g\|_{H^{2+\epsilon}}^2.
\end{aligned}$$

We now choose $N$ such that $2^{-2\delta N}\,\|f\|_{H^\delta}^2 \le C\,\|f\|_{L^2}^2$. In fact, we can choose

$$N = \left[\frac{1}{\delta}\log_2\frac{\|f\|_{H^\delta}}{\|f\|_{L^2}}\right]. \qquad (3.17)$$

It then follows that

$$\|[(\ln(I-\Delta))^\mu\partial_x, g]f\|_{L^2} \le C_{\mu,\epsilon,\delta}\left(1+\left(\ln\left(1+\frac{\|f\|_{H^\delta}}{\|f\|_{L^2}}\right)\right)^\mu\right)\|f\|_{L^2}\,\|g\|_{H^{2+3\epsilon}},$$

where $C_{\mu,\epsilon,\delta}$ is a constant depending on $\mu$, $\epsilon$, and $\delta$ only. It is easy to see that the inhomogeneous Sobolev norm $\|f\|_{H^\delta}$ can be replaced by the homogeneous norm $\|f\|_{\dot H^\delta}$. This completes the proof of Proposition 3.1. □

Finally we prove Lemma 3.3.

PROOF OF LEMMA 3.3. Let $N \geq 1$ be an integer to be specified later. We write

$$\|(\ln(I-\Delta))^{\mu} f\|_{L^2}^2 = L_1 + L_2$$

where

$$L_1 = \sum_{j=-1}^{N-1} \|\Delta_j (\ln(I-\Delta))^{\mu} f\|_{L^2}^2, \quad L_2 = \sum_{j=N}^{\infty} \|\Delta_j (\ln(I-\Delta))^{\mu} f\|_{L^2}^2.$$

According to theorem 1.2 in [2], we have, for $j \geq 0$,

$$\|\Delta_j (\ln(I-\Delta))^{\mu} f\|_{L^2} \leq C(\ln(1+2^{2j}))^{\mu} \|\Delta_j f\|_{L^2}.$$

Clearly, for $j = -1$,

$$\|\Delta_{-1} (\ln(I-\Delta))^{\mu} f\|_{L^2} \leq C \|\Delta_{-1} f\|_{L^2}.$$

Therefore,

$$L_1 \leq C(\ln(1+2^{2N}))^{2\mu} \sum_{j=-1}^{N-1} \|\Delta_j f\|_{L^2}^2 \leq C(\ln(1+2^{2N}))^{2\mu} \|f\|_{L^2}^2.$$

For any $\delta > 0$,

$$\begin{aligned} L_2 &\leq \sum_{j=N}^{\infty} (\ln(1+2^{2j}))^{2\mu} 2^{-2\delta j}\, 2^{2\delta j} \|\Delta_j f\|_{L^2}^2 \\ &\leq (\ln(1+2^{2N}))^{2\mu} 2^{-2\delta N} \|f\|_{H^\delta}^2. \end{aligned}$$

Therefore,

$$\begin{aligned} \|(\ln(I-\Delta))^{\mu} f\|_{L^2}^2 \leq\ & C(\ln(1+2^{2N}))^{2\mu} \|f\|_{L^2}^2 \\ & + (\ln(1+2^{2N}))^{2\mu} 2^{-2\delta N} \|f\|_{H^\delta}^2. \end{aligned}$$

If we choose $N$ in a similar fashion as in (3.17), we obtain the desired inequality (3.1). This completes the proof of Lemma 3.3. □

## 4 Global Weak Solutions

This section establishes the global existence of weak solutions to (1.2), namely Theorem 1.6. The following commutator estimate will be used:

LEMMA 4.1. *Let $s \geq 0$. Let $j = 1$ or 2. Then, for any $\epsilon > 0$, there exists a constant $C$ depending on $s$ and $\epsilon$ such that*

$$\|[\Lambda^s \partial_{x_j}, g]h\|_{L^2(\mathbb{T}^2)} \leq C(\|h\|_{L^2} \|g\|_{H^{2+s+\epsilon}} + \|\Lambda^s h\|_{L^2} \|g\|_{H^{2+\epsilon}}). \tag{4.1}$$

Although the lemma is for the periodic setting, it can be proven in a similar manner as Proposition 2.1 and we thus omit its proof.

PROOF OF THEOREM 1.6. The proof follows a standard approach, the Galerkin approximation. Let $n > 0$ be an integer and let $K_n$ denote the subspace of $L^2(\mathbb{T}^2)$,

$$K_n = \{e^{im\cdot x} : m \neq 0 \text{ and } |m| \leq n\}.$$

Let $\mathbb{P}_n$ be the projection onto $K_n$. For each fixed $n$, we consider the solution of the projected equation,

$$\begin{aligned}
&\partial_t \theta_n + \mathbb{P}_n(u_n \cdot \nabla \theta_n) = 0,\\
&u_n = \nabla^\perp \Lambda^{-2+\beta} \theta_n,\\
&\theta_n(x,0) = \mathbb{P}_n \theta_0(x).
\end{aligned}$$

This equation has a unique global solution $\theta_n$. Clearly, $\theta_n$ obeys the $L^2$ global bound

$$\|\theta_n(\,\cdot\,,t)\|_{L^2} = \|\mathbb{P}_n\theta_0\|_{L^2} \leq \|\theta_0\|_{L^2}. \tag{4.2}$$

In addition, let $\psi_n$ be the corresponding stream function, namely $\Delta\psi_n = \Lambda^\beta \theta_n$. Then we have

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\Lambda^{1-\frac{\beta}{2}}\psi_n\|_{L^2}^2 &= -\int \psi_n \mathbb{P}_n(u_n\cdot\nabla\theta_n)dx\\
&= -\int \psi_n\, u_n \cdot \nabla\theta_n\, dx.
\end{aligned}$$

Noticing that $u_n = \nabla^\perp \psi_n$, we integrate by parts in the last term to obtain

$$-\int \psi_n u_n \cdot \nabla\theta_n\, dx = \int \psi_n u_n \cdot \nabla \theta_n\, dx.$$

Therefore,

$$\frac{d}{dt}\|\Lambda^{1-\frac{\beta}{2}}\psi_n\|_{L^2}^2 = 0 \quad \text{or} \quad \|\Lambda^{1-\frac{\beta}{2}}\psi_n\|_{L^2} \leq \|\Lambda^{1-\frac{\beta}{2}}\psi_0\|_{L^2}. \tag{4.3}$$

Furthermore, for any $\phi \in H^{3+\epsilon}$ with $\epsilon > 0$, we have

$$\int \partial_t\theta_n(x,t)\phi(x)dx = -\int (u_n\cdot\nabla\theta_n)\mathbb{P}_n\phi\, dx = \int \theta_n u_n \cdot \nabla \mathbb{P}_n \phi\, dx. \tag{4.4}$$

On the one hand, $\theta_n = \Lambda^{2-\beta}\psi_n$ and

$$\begin{aligned}
\int \theta_n u_n \cdot \nabla\mathbb{P}_n\phi\, dx &= \int \psi_n\, \Lambda^{2-\beta}(u_n\cdot\nabla\mathbb{P}_n\phi)dx\\
&= \int \psi_n\, \Lambda^{2-\beta}(\nabla^\perp\psi_n\cdot\nabla\mathbb{P}_n\phi)dx.
\end{aligned}$$

On the other hand, $u_n = \nabla^\perp\psi_n$ and

$$\begin{aligned}
\int \theta_n u_n \cdot \nabla\mathbb{P}_n\phi\, dx &= \int \theta_n \nabla^\perp\cdot(\psi_n\, \nabla\mathbb{P}_n\phi)dx\\
&= -\int \psi_n \nabla^\perp \Lambda^{2-\beta}\psi_n \cdot \nabla\mathbb{P}_n\phi\, dx.
\end{aligned}$$

Thus,

$$\int \theta_n u_n \cdot \nabla \mathbb{P}_n \phi \, dx = \frac{1}{2} \int \psi_n [\Lambda^{2-\beta} \nabla^\perp \cdot, \nabla \mathbb{P}_n \phi] \psi_n \, dx.$$

It then follows from Hölder's inequality and Lemma 4.1 that

$$\begin{aligned} \left| \int \theta_n u_n \cdot \nabla \mathbb{P}_n \phi \, dx \right| &\le C \|\psi_n\|_{L^2} \|\psi_n\|_{H^{2-\beta}} \|\mathbb{P}_n \phi\|_{H^{3+\epsilon}} \\ &\le C \|\Lambda^{-2+\beta} \theta_n\|_{L^2} \|\theta_n\|_{L^2} \|\phi\|_{H^{3+\epsilon}} \\ &\le C \|\theta_0\|_{L^2}^2 \|\phi\|_{H^{3+\epsilon}} \end{aligned} \tag{4.5}$$

where the fact that mean-zero functions in $L^2(\mathbb{T}^2)$ are also in $H^{-2+\beta}(\mathbb{T}^2)$ has been used. Therefore, by (4.4),

$$\|\partial_t \theta_n\|_{H^{-3-\epsilon}} \le C \|\theta_0\|_{L^2}^2. \tag{4.6}$$

The bounds in (4.2), (4.3), and (4.6), together with the compact embedding relation $L^2(\mathbb{T}^2) \hookrightarrow H^{-2+\beta}(\mathbb{T}^2)$ for $1 < \beta < 2$, imply that there exists $\theta \in C([0,T]; L^2(\mathbb{T}^2))$ such that

$$\theta_n \rightharpoonup \theta \quad \text{in } L^2, \qquad \psi_n \to \psi \quad \text{in } L^2. \tag{4.7}$$

In addition, because of the uniform boundedness of $\|\theta_n\|_{L^2}$ and the embedding $L^2(\mathbb{T}^2) \hookrightarrow H^{-3-\epsilon}(\mathbb{T}^2)$, the Arzelà-Ascoli theorem implies

$$\lim_{n\to\infty} \sup_{t\in[0,T]} \left| \int (\theta_n(x,t) - \theta(x,t)) \phi(x) dx \right| \to 0, \tag{4.8}$$

where $\phi \in H^{3+\epsilon}(\mathbb{T}^2)$.

The convergence in (4.7) and (4.8) allows us to prove that $\theta$ satisfies (1.4). Clearly, $\theta_n$ satisfies the integral equation

$$\int_0^T \int_{\mathbb{T}^2} \theta_n (\partial_t \phi + u_n \cdot \nabla \mathbb{P}_n \phi) dx \, dt = \int_{\mathbb{T}^2} \mathbb{P}_n \theta_0(x) \phi(x,0) dx.$$

It is easy to check that

$$\int_{\mathbb{T}^2} \mathbb{P}_n \theta_0(x) \phi(x,0) dx \to \int_{\mathbb{T}^2} \theta_0(x) \, \phi(x,0) dx,$$

and (4.8) implies that, as $n \to \infty$,

$$\int_0^T \int_{\mathbb{T}^2} \theta_n \, \partial_t \phi \, dx \, dt \to \int_0^T \int_{\mathbb{T}^2} \theta \, \partial_t \phi \, dx \, dt.$$

To show the convergence in the nonlinear term, we write

$$
\begin{aligned}
\int_0^T \int_{\mathbb{T}^2} \theta_n u_n \cdot \nabla \mathbb{P}_n \phi \, dx \, dt - \int_0^T \int_{\mathbb{T}^2} \theta \, u \cdot \nabla \phi \, dx \, dt \\
&= \frac{1}{2} \int_0^T \int_{\mathbb{T}^2} \psi_n [\Lambda^{2-\beta} \nabla^{\perp} \cdot, \nabla \mathbb{P}_n \phi] \psi_n \, dx \, dt \\
&\quad - \frac{1}{2} \int_0^T \int_{\mathbb{T}^2} \psi [\Lambda^{2-\beta} \nabla^{\perp} \cdot, \nabla \phi] \psi \, dx \, dt \\
&= \frac{1}{2} \int_0^T \int_{\mathbb{T}^2} \psi_n [\Lambda^{2-\beta} \nabla^{\perp} \cdot, \nabla (\mathbb{P}_n \phi - \phi)] \psi_n \, dx \, dt \\
&\quad + \frac{1}{2} \int_0^T \int_{\mathbb{T}^2} (\psi_n - \psi) [\Lambda^{2-\beta} \nabla^{\perp} \cdot, \nabla \phi] \psi_n \, dx \, dt \\
&\quad + \frac{1}{2} \int_0^T \int_{\mathbb{T}^2} \psi [\Lambda^{2-\beta} \nabla^{\perp} \cdot, \nabla \phi] (\psi_n - \psi) dx \, dt.
\end{aligned}
$$

In order to get the convergence for the first two terms above, we appeal to Lemma 4.1 and the strong convergence of $\psi_n$ in $L^2$. Let us point out that in the last term for $\Lambda^{2-\beta} \psi_n$ we only have weak convergence in $L^2$ so we have to proceed in a different manner. We consider the following integral:

$$
\begin{aligned}
Q_n(t) &= \int_{\mathbb{T}^2} \psi [\Lambda^{2-\beta} \nabla^{\perp} \cdot, \nabla \phi] (\psi_n - \psi) dx \\
&= \sum_{k \neq 0} \widehat{\psi}(-k) ([\Lambda^{2-\beta} \nabla^{\perp} \cdot, \nabla \phi] (\psi_n - \psi))\widehat{\ }(k),
\end{aligned}
$$

which is bounded by

$$
\begin{aligned}
|Q_n(t)| \leq \Big( \sum_{k \neq 0} ||k|^{2-\beta} \widehat{\psi}(-k)|^2 \Big)^{1/2} \\
\times \Big( \sum_{k \neq 0} \big| |k|^{\beta-2} ([\Lambda^{2-\beta} \nabla^{\perp} \cdot, \nabla \phi] (\psi_n - \psi))\widehat{\ }(k) \big|^2 \Big)^{1/2}.
\end{aligned}
$$

The first sum above is controlled by $\|\theta_0\|_{L^2}$. Using a similar notation as before, the coefficients in the second sum have the form

$$
|k|^{\beta-2} ([\Lambda^{2-\beta} \partial_x, \varphi] (\psi_n - \psi))\widehat{\ }(k)
$$

where $\partial_x$ is either $\partial_{x_1}$ or $\partial_{x_2}$ and $\varphi$ is $\partial_x \phi$. Since

$$([\Lambda^{2-\beta}\partial_x, \varphi](\psi_n - \psi))\widehat{\ }(k) = \sum_j i(k_a|k|^{2-\beta} - (k-j)_a|k-j|^{2-\beta})(\psi_n - \psi)\widehat{\ }(k-j)\widehat{\varphi}(j)$$

for $a = 1, 2$, following the bounds in Section 2 we obtain

$$\begin{aligned} &|([\Lambda^{2-\beta}\partial_x, \varphi](\psi_n - \psi))\widehat{\ }(k)| \\ &\quad \le C\sum_j (|k|^{2-\beta} + |k-j|^{2-\beta})|(\psi_n - \psi)\widehat{\ }(k-j)|\,|j|\,|\widehat{\varphi}(j)| \\ &\quad \le C\sum_j (|k|^{2-\beta} + |j|^{2-\beta})|(\psi_n - \psi)\widehat{\ }(k-j)|\,|j|\,|\widehat{\varphi}(j)|. \end{aligned}$$

For $|k| \neq 0$, it yields

$$|k|^{\beta-2}|([\Lambda^{2-\beta}\partial_x, \varphi](\psi_n - \psi))\widehat{\ }(k)| \le C\sum_j |(\psi_n - \psi)\widehat{\ }(k-j)|\,|j|(1 + |j|^{2-\beta})|\widehat{\varphi}(j)|.$$

The above bound provides

$$|Q_n(t)| \le C_\epsilon \|\theta_0\|_{L^2}\, \|\phi\|_{H^{5-\beta+\epsilon}}\, \|\psi_n - \psi\|_{L^2}$$

for any $\epsilon > 0$. It then follows from (4.7) that $\lim_{n\to\infty} Q_n(t) = 0$. The dominated convergence theorem then leads to the desired convergence of the third term. Therefore, $\theta$ is a weak solution of 1.2 in the sense of Definition 1.5. This completes the proof of Theorem 1.6. □

## 5 Local Existence for Smooth Patches

This section is devoted to proving Theorem 1.7.

PROOF OF THEOREM 1.7. Since $\beta = 2$ corresponds to the trivial steady-state solution, it suffices to consider the case when $1 < \beta < 2$. The major efforts are devoted to establishing a priori local (in time) bound for $\|x(\cdot, t)\|_{H^4} + \|F(x)\|_{L^\infty}(t)$ for $x$ satisfying the contour dynamics equation (1.10) and $F(x)(\gamma, \eta, t)$ defined in (1.7).

This proof follows the ideas in Gancedo [6]. The difference here is that the kernel in (1.10) is more singular but the function space concerned here is $H^4(\mathbb{T})$, which is more regular than in [6] and compensates for the singularity of the kernel.

For notational convenience, we shall omit the coefficient $C_\beta(\theta_1 - \theta_2)$ in the contour dynamics equation (1.10). In addition, the $t$-variable will sometimes be suppressed. We start with the $L^2$-norm. Dotting (1.10) by $x(\gamma,t)$ and integrating over $\mathbb{T}$, we have

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}} |x(\gamma,t)|^2\,dx = I_1 + I_2,$$

where

$$I_1 = \int_{\mathbb{T}}\int_{\mathbb{T}} x(\gamma,t)\cdot\frac{\partial_\gamma x(\gamma,t)-\partial_\gamma x(\gamma-\eta,t)}{|x(\gamma,t)-x(\gamma-\eta,t)|^\beta}\,d\eta\,d\gamma,$$

$$I_2 = \int_{\mathbb{T}} \lambda(\gamma)\,x(\gamma,t)\cdot\partial_\gamma x(\gamma,t)\,d\gamma.$$

$I_1$ is actually 0. In fact, by the symmetrizing process,

$$\begin{aligned} I_1 &= \frac{1}{2}\int_{\mathbb{T}}\int_{\mathbb{T}} \frac{(x(\gamma)-x(\gamma-\eta))\cdot(\partial_\gamma x(\gamma)-\partial_\gamma x(\gamma-\eta))}{|x(\gamma)-x(\gamma-\eta)|^\beta}\,d\eta\,d\gamma \\ &= \frac{1}{2(2-\beta)}\int_{\mathbb{T}}\int_{\mathbb{T}} \partial_\gamma(|x(\gamma)-x(\gamma-\eta)|^{2-\beta})d\gamma\,d\eta \\ &= 0. \end{aligned}$$

To bound $I_2$, we first apply Hölder's inequality to obtain

$$|I_2| \le \|\lambda\|_{L^\infty}\,\|x\|_{L^2}\,\|\partial_\gamma x\|_{L^2}.$$

By the representation of $\lambda$ in (1.11) and using the fact that

$$\frac{1}{|\partial_\gamma x|^2} \le \|F(x)\|^2_{L^\infty}(t),$$

we have

$$\begin{aligned} \|\lambda\|_{L^\infty} &\le C\|F(x)\|^2_{L^\infty}(t)\int_{\mathbb{T}} |\partial_\gamma x|\left|\partial_\gamma\int_{\mathbb{T}} \frac{\partial_\gamma x(\gamma)-\partial_\gamma x(\gamma-\eta)}{|x(\gamma)-x(\gamma-\eta)|^\beta}\,d\eta\right|d\gamma \\ &= C\|F(x)\|^2_{L^\infty}(t)(I_{21}+I_{22}), \end{aligned}$$

where

$$I_{21} = \int_{\mathbb{T}} |\partial_\gamma x|\int_{\mathbb{T}} \frac{|\partial_\gamma^2 x(\gamma)-\partial_\gamma^2 x(\gamma-\eta)|}{|x(\gamma)-x(\gamma-\eta)|^\beta}\,d\eta\,d\gamma,$$

$$I_{22} = \int_{\mathbb{T}} |\partial_\gamma x|\int_{\mathbb{T}} \frac{|\partial_\gamma x(\gamma)-\partial_\gamma x(\gamma-\eta)|^2}{|x(\gamma)-x(\gamma-\eta)|^{\beta+1}}\,d\eta\,d\gamma.$$

It is not hard to see that $I_{21}$ and $I_{22}$ can be bounded as follows:

$$I_{21} \le C\|F(x)\|^\beta_{L^\infty}(t)\,\|\partial_\gamma x\|_{L^2}\,\|\partial_\gamma^3 x\|_{L^2},$$

$$I_{22} \le C\|F(x)\|^{1+\beta}_{L^\infty}(t)\,\|\partial_\gamma^2 x\|^2_{L^2}\,\|\partial_\gamma x\|_{L^2}.$$

Therefore,

$$\frac{d}{dt}\|x\|_{L^2}^2 \le C\|F(x)\|_{L^\infty}^{3+\beta}(t)\|x\|_{H^3}^5.$$

We now estimate $\|\partial_\gamma^4 x\|_{L^2}$:

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}} |\partial_\gamma^4 x|^2\, d\gamma = I_3 + I_4,$$

where

$$I_3 = C\int_{\mathbb{T}} \partial_\gamma^4 x(\gamma)\cdot \partial_\gamma^4 \int_{\mathbb{T}} \frac{(\partial_\gamma x(\gamma) - \partial_\gamma x(\gamma-\eta))}{|x(\gamma)-x(\gamma-\eta)|^\beta}\, d\eta\, d\gamma,$$

$$I_4 = \int_{\mathbb{T}} \partial_\gamma^4 x(\gamma)\cdot \partial_\gamma^4(\lambda\partial_\gamma x)(\gamma) d\gamma.$$

$I_3$ can be further decomposed into five terms, namely $I_3 = I_{31} + I_{32} + I_{33} + I_{34} + I_{35}$, where

$$I_{31} = \int_{\mathbb{T}}\int_{\mathbb{T}} \partial_\gamma^4 x(\gamma)\cdot \frac{(\partial_\gamma^5 x(\gamma) - \partial_\gamma^5 x(\gamma-\eta))}{|x(\gamma)-x(\gamma-\eta)|^\beta}\, d\eta\, d\gamma,$$

$$I_{32} = 4\int_{\mathbb{T}}\int_{\mathbb{T}} \partial_\gamma^4 x(\gamma)\cdot (\partial_\gamma^4 x(\gamma) - \partial_\gamma^4 x(\gamma-\eta))\partial_\gamma(|x(\gamma)-x(\gamma-\eta)|^{-\beta}) d\eta\, d\gamma,$$

$$I_{33} = 6\int_{\mathbb{T}}\int_{\mathbb{T}} \partial_\gamma^4 x(\gamma)\cdot (\partial_\gamma^3 x(\gamma) - \partial_\gamma^3 x(\gamma-\eta))\partial_\gamma^2(|x(\gamma)-x(\gamma-\eta)|^{-\beta}) d\eta\, d\gamma,$$

$$I_{34} = 4\int_{\mathbb{T}}\int_{\mathbb{T}} \partial_\gamma^4 x(\gamma)\cdot (\partial_\gamma^2 x(\gamma) - \partial_\gamma^2 x(\gamma-\eta))\partial_\gamma^3(|x(\gamma)-x(\gamma-\eta)|^{-\beta}) d\eta\, d\gamma,$$

$$I_{35} = \int_{\mathbb{T}}\int_{\mathbb{T}} \partial_\gamma^4 x(\gamma)\cdot (\partial_\gamma x(\gamma) - \partial_\gamma x(\gamma-\eta))\partial_\gamma^4(|x(\gamma)-x(\gamma-\eta)|^{-\beta}) d\eta\, d\gamma.$$

By symmetrizing, $I_{31}$ can be written as

$$\begin{aligned}
I_{31} &= \frac{1}{2}\int_{\mathbb{T}}\int_{\mathbb{T}} \frac{(\partial_\gamma^4 x(\gamma) - \partial_\gamma^4 x(\gamma-\eta))\cdot(\partial_\gamma^5 x(\gamma) - \partial_\gamma^5 x(\gamma-\eta))}{|x(\gamma)-x(\gamma-\eta)|^\beta}\, d\eta\, d\gamma \\
&= \frac{1}{4}\int_{\mathbb{T}}\int_{\mathbb{T}} \frac{\partial_\gamma(|\partial_\gamma^4 x(\gamma) - \partial_\gamma^4 x(\gamma-\eta)|^2)}{|x(\gamma)-x(\gamma-\eta)|^\beta}\, d\eta\, d\gamma \\
&= \frac{\beta}{4}\int_{\mathbb{T}}\int_{\mathbb{T}} \frac{|\partial_\gamma^4 x(\gamma) - \partial_\gamma^4 x(\gamma-\eta)|^2(x(\gamma)-x(\gamma-\eta))\cdot(\partial_\gamma x(\gamma) - \partial_\gamma x(\gamma-\eta))}{|x(\gamma)-x(\gamma-\eta)|^{\beta+2}}\, d\eta\, d\gamma.
\end{aligned}$$

Setting

$$B(\gamma,\eta) = (x(\gamma) - x(\gamma-\eta)) \cdot (\partial_\gamma x(\gamma) - \partial_\gamma x(\gamma-\eta))$$

and using the fact that $\partial_\gamma x(\gamma)\cdot\partial_\gamma^2 x(\gamma) = 0$, we have

$$|I_{31}| \le C\|F(x)\|^{2+\beta}_{L^\infty}(t) \int_{\mathbb{T}}\int_{\mathbb{T}} \left|\partial_\gamma^4 x(\gamma) - \partial_\gamma^4 x(\gamma-\eta)\right|^2 \times \frac{B(\gamma,\eta)\eta^{-2} - \partial_\gamma x(\gamma)\cdot\partial_\gamma^2 x(\gamma)}{|\eta|^\beta}\, d\eta\, d\gamma.$$

Using the bound that

$$\left|B(\gamma,\eta)\eta^{-2} - \partial_\gamma x(\gamma)\cdot\partial_\gamma^2 x(\gamma)\right| \le C\|x\|^2_{C^3}|\eta|,$$

we obtain

$$|I_{31}| \le C\|F(x)\|^{2+\beta}_{L^\infty}(t)\, \|x\|^2_{C^3}\, \|x\|^2_{H^4}.$$

To estimate $I_{32}$, we realize that, after computing $\partial_\gamma(|x(\gamma)-x(\gamma-\eta)|^{-\beta})$, $I_{32}$ can be bounded in the same fashion as $I_{31}$. That is,

$$|I_{32}| \le C\|F(x)\|^{2+\beta}_{L^\infty}(t)\, \|x\|^4_{H^4}.$$

In order to estimate $I_{33}$, we further decompose it into three terms, $I_{33} = I_{331} + I_{332} + I_{333}$, where

$$I_{331} = C\int_{\mathbb{T}}\int_{\mathbb{T}} \partial_\gamma^4 x(\gamma)\cdot\left(\partial_\gamma^3 x(\gamma) - \partial_\gamma^3 x(\gamma-\eta)\right)\frac{D(\gamma,\eta)}{|x(\gamma)-x(\gamma-\eta)|^{2+\beta}}\, d\eta\, d\gamma,$$

$$I_{332} = C\int_{\mathbb{T}}\int_{\mathbb{T}} \partial_\gamma^4 x(\gamma)\cdot\left(\partial_\gamma^3 x(\gamma) - \partial_\gamma^3 x(\gamma-\eta)\right)\frac{|\partial_\gamma x(\gamma)-\partial_\gamma x(\gamma-\eta)|^2}{|x(\gamma)-x(\gamma-\eta)|^{2+\beta}}\, d\eta\, d\gamma,$$

$$I_{333} = C\int_{\mathbb{T}}\int_{\mathbb{T}} \partial_\gamma^4 x(\gamma)\cdot\left(\partial_\gamma^3 x(\gamma) - \partial_\gamma^3 x(\gamma-\eta)\right)\frac{B^2(\gamma,\eta)}{|x(\gamma)-x(\gamma-\eta)|^{4+\beta}}\, d\eta\, d\gamma$$

with

$$D(\gamma,\eta) = (x(\gamma)-x(\gamma-\eta))\cdot\left(\partial_\gamma^2 x(\gamma) - \partial_\gamma^2 x(\gamma-\eta)\right).$$

It is not very difficult to see that

$$|I_{331}|, |I_{332}|, |I_{333}| \le C\|F(x)\|^{2+\beta}_{L^\infty}(t)\|x\|^4_{H^4}.$$

$I_{34}$ also admits a similar bound. In $I_{35}$ one has to use identity the

$$\partial_\gamma x(\gamma)\cdot\partial_\gamma^4 x(\gamma) = 3\partial_\gamma^2 x(\gamma)\cdot\partial_\gamma^3 x(\gamma)$$

to find the same control. We shall not provide the detailed estimates since they can be obtained by modifying the lines in [6]. We also need to deal with $I_4$. To do so, we use the representation formula (1.11) and obtain

$$|I_4| \le C\|F(x)\|^{4+\beta}_{L^\infty}(t)\|x\|^5_{H^4}$$

In summary, we have

$$\frac{d}{dt}\|x\|_{H^4}^2 \le C\|F(x)\|_{L^\infty}^{4+\beta}(t)\|x\|_{H^4}^5. \tag{5.1}$$

We now derive the estimate for $\|F(x)\|_{L^\infty}(t)$. For any $p > 2$, we have

$$\frac{d}{dt}\|F(x)\|_{L^p}^p(t) \le p\int_{\mathbb{T}}\int_{\mathbb{T}}\left(\frac{|\eta|}{|x(\gamma)-x(\gamma-\eta)|}\right)^{p+1} \frac{|x_t(\gamma,t)-x_t(\gamma-\eta,t)|}{|\eta|}\,d\eta\,d\gamma. \tag{5.2}$$

Invoking the contour dynamics equation (1.10), we have

$$\begin{aligned} x_t(\gamma)-x_t(\gamma-\eta) &= I_5+I_6+I_7+I_8 \\ &\equiv \int_{\mathbb{T}}\left(\frac{\partial_\gamma x(\gamma)-\partial_\gamma x(\gamma-\xi)}{|x(\gamma)-x(\gamma-\xi)|^\beta} - \frac{\partial_\gamma x(\gamma)-\partial_\gamma x(\gamma-\xi)}{|x(\gamma-\eta)-x(\gamma-\eta-\xi)|^\beta}\right)d\xi \\ &\quad + \int_{\mathbb{T}} \frac{\partial_\gamma x(\gamma)-\partial_\gamma x(\gamma-\eta)+\partial_\gamma x(\gamma-\eta-\xi)-\partial_\gamma x(\gamma-\xi)}{|x(\gamma-\eta)-x(\gamma-\eta-\xi)|^\beta}\,d\xi \\ &\quad + (\lambda(\gamma)-\lambda(\gamma-\eta))\partial_\gamma x(\gamma) + \lambda(\gamma-\eta)(\partial_\gamma x(\gamma)-\partial_\gamma x(\gamma-\eta)). \end{aligned}$$

Following the argument as in [6], we have

$$\begin{aligned} |I_5| &\le C\|F(x)\|_{L^\infty}^{2\beta}(t)\,\|x\|_{C^2}^{1+\beta}\,|\eta|, \\ |I_6| &\le C\|F(x)\|_{L^\infty}^{\beta}(t)\,\|x\|_{C^3}\,|\eta|, \\ |I_7| &\le C\|F(x)\|_{L^\infty}^{3+\beta}(t)\,\|x\|_{H^4}^4\,|\eta|, \\ |I_8| &\le C\|F(x)\|_{L^\infty}^{3+\beta}(t)\,\|x\|_{H^4}^4\,|\eta|. \end{aligned}$$

Inserting these estimates into (5.2), we find

$$\frac{d}{dt}\|F(x)\|_{L^p}(t) \le C\|x\|_{H^4}^4\,\|F(x)\|_{L^\infty}^{4+\beta}(t)\,\|F(x)\|_{L^p}(t).$$

After integrating in time and taking the limit as $p\to\infty$, we obtain

$$\frac{d}{dt}\|F(x)\|_{L^\infty}(t) \le C\|x\|_{H^4}^4\,\|F(x)\|_{L^\infty}^{5+\beta}(t).$$

Combining the above with (5.1), we obtain

$$\frac{d}{dt}(\|x\|_{H^4} + \|F(x)\|_{L^\infty}(t)) \le C\|x\|_{H^4}^4\,\|F(x)\|_{L^\infty}^{5+\beta}(t).$$

This inequality would allow us to deduce a local (in time) bound for $\|x\|_{H^4}$. This completes the proof of Theorem 1.7. □

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DONGHO CHAE
Department of Mathematics
Chung-Ang University
Dongjak-gu Heukseok-ro 84
Seoul 156-756
KOREA
E-mail: dchae@cau.ac.kr

PETER CONSTANTIN
Department of Mathematics
Program in Applied and Computational Mathematics
Princeton University
Fine Hall, Washington Road
Princeton, NJ 08544-1000
E-mail: const@math.princeton.edu

DIEGO CÓRDOBA
Instituto de Ciencias Matemáticas
Consejo Superior de Investigaciones Científicas
C. Nicolás Cabrera, 13-15
Campus Cantoblanco UAM, 28049
SPAIN
E-mail: dcg@icmat.es

FRANCISCO GANCEDO
Departamento de Análisis Matemático
Universidad de Sevilla
41012 Sevilla
SPAIN
E-mail: fgancedo@us.es

JIAHONG WU
Department of Mathematics
Oklahoma State University
401 Mathematical Sciences
Stillwater, OK 74078
E-mail: jiahong@math.okstate.edu

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