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GENERALIZED SYMMETRY CONDITIONS AT A CORE POINT*

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ABSTRACT

Previous analyses have shown that if a point x is to be a core of a majority rule voting game in Euclidean space, when preferences are smooth, then the utility gradients must satisfy certain restrictive symmetry conditions. In this paper these results are generalized to the case of an arbitrary voting rule, and necessary and sufficient conditions, expressed in terms of "pivotal" coalitions, are obtained.

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1. INTRODUCTION

It is now well known that if the set of alternatives, W, can be represented as a subset of Euclidean space, and individual preferences are smooth, then the individual utility gradients at a point in the majority core must satisfy strong symmetry conditions [9]. The necessity that these symmetry conditions be satisfied can be used to prove the generic non-existence of core points in certain situations [7]. The same symmetry conditions can be used to show that if the majority rule core is empty, then it will generally be the case that voting trajectories can be constructed throughout the space.

This paper generalizes the Plott symmetry conditions to deal with arbitrary social choice functions, obtaining restrictions on the gradients at a point which are necessary and sufficient for that point to be in the core. The generalized gradient restrictions that we identify show the central role of what we term the "pivotal" collections of coalitions in determining when core points exist. Specifically, we define a coalition, M, to be <u>pivotal</u> in a subset L of the voters, if it is the case that whenever we partition L-M into two subsets, at least one of these subsets, together with the members of M, constitutes a decisive coalition. Our symmetry conditions specify that for x to be a core point, the utility gradients of the members of any subset, L, of voters must satisfy the following condition: For every pivotal coalition M in L, the set of voters, M*, whose gradients lie in the subspace spanned by those in M, must positively span 0 (the zero vector). Taking L to be the set of non-satiated voters, it is easily shown that the Plott symmetry conditions for the existence of a majority core point are implied by this condition. The pivotal gradient condition can also be applied to get necessary conditions for a point to be in the constrained core, and hence for a point to be in the cycle set.

Because of the effect of constituencies or party systems, political decision processes cannot in general be identified with simple majority rule. Early results [9, 16] on the analysis of the core of voting games have focused on majority rule, and as a consequence, these characterizations have not been applicable to a wide class of political phenomena. The generalized symmetry conditions which we present here give a technique for analyzing a much broader class of voting mechanisms. See [15] for example.

2. DEFINITION AND NOTATION

We let $W \subset \mathbb{R}^w$ represent the set of <u>alternatives</u>. Let $N = \{1,2,\dots,n\}$ be a finite set indexing <u>voters</u>. Let U denote the set of smooth, real valued functions on W, and let $u = (u_1,\dots,u_n) \in U^n$, with u_i representing the <u>utility function</u> for voter i. Throughout this paper, we consider only a fixed $u \in U^n$, and call such a $u \in U^n$ a smooth profile.

For any binary relation $Q \subseteq W \times W$, we use the standard notation $xQy \Leftrightarrow (x,y) \in Q$. We write P_i for the binary relation on W defined by $xP_iy \Leftrightarrow u_i(x) > u_i(y)$, and for any $C \subseteq N$, write $P_C = \bigcap_{i \in C} P_i$.

We are given a set $\mathbb{D}$ of subsets $C \subseteq N$, called the set of decisive coalitions, which is assumed to satisfy

(a) $C \in \mathbb{D}$ and $C \subseteq C' \Rightarrow C' \in \mathbb{D}$ ($\mathbb{D}$ is monotonic).

(b) $C \in \mathbb{D} \Rightarrow N - C \notin \mathbb{D}$ ($\mathbb{D}$ is proper).

We can then define the social order $P \subseteq W \times W$ by

$$xPy \Leftrightarrow xP_Cy \text{ for some } C \in \mathbb{D}.$$

For any binary relation, $Q \subseteq W \times W$, and $x \in W$, define $Q(x) = \{y \in X: yQx\}$, and write $Q^1(x) = Q(x)$. For any integer $j \geq 1$, define $Q^j(x) = \{y \in W: yQz \text{ for some } z \in Q^{j-1}(x)\}$. Then define $Q^*(x) = \bigcup_{j=1}^{\infty} Q^j(x)$. Also, for any $V \subseteq W$, and $Q \subseteq W \times W$, define $Q|_V = Q \cap (V \times V)$ to be the binary relation Q, restricted to V. We can then define the core, or global optima set to be the set of socially unbeaten alterantives in V:

$$GO(V:\mathbb{D}) = \{x \in V: (P|_V)(x) = \emptyset\},$$

and the local optima set on V by

$$LO(V:\mathbb{D}) = \{x \in V: x \in GO(V':\mathbb{D}) \text{ for some neighborhood } V' \text{ of } x \text{ in } V\}.$$

We define the global cycle set to be the set of points which are

elements of a cycle in V, under the social order:

$$GC(V:\mathbb{D}) = \{x \in V: x \in (P|_V)^*(x)\},$$

and the local cycle set by

$$LC(V:\mathbb{D}) = \{x \in V: x \in GC(V':\mathbb{D}) \text{ for all neighborhoods, } V' \text{ of } x \text{ in } V\}.$$

When there is no fear of ambiguity we write GO(V), LO(V) etc. for these sets. We will also write GO = GO(W), LO = LO(W), etc. and call these the global or local optima sets with respect to $\mathbb{D}$. Clearly,

$$GO \subseteq LO \text{ and } LC \subseteq GC.$$

3. CONSTRAINTS ON GRADIENTS AT A CORE POINT

In this section, we define the critical optimal set, $IO(W:\mathbb{D})$, give its relation to the global and local optimal sets, and characterize this set in terms of conditions on the utility gradients of members of decisive coalitions.

For any $x \in W$, and $i \in N$, let $p_i(x) = \nabla u_i(x) \in \mathbb{R}^W$ represent voter i's utility gradient at the point x. For $C \subseteq N$, let $p_C(x) = \{y \in \mathbb{R}^W: y = \sum_{i \in C} \alpha_i p_i(x),\ \alpha_i \geq 0\ \forall i \in C \text{ and } \exists i \in C \text{ st } \alpha_i \neq 0\}$ be the semi-positive cone generated by $\{p_i(x) | i \in C\}$, and let $sp_C(x) = \{y \in \mathbb{R}^W: y = \sum_{i \in C} \alpha_i p_i(x) \text{ with } \alpha_i \in \mathbb{R}\}$ be the subspace spanned by $\{p_i(x): i \in C\}$.

We use the notation Int W to refer to the interior of W in the standard topology on $\mathbb{R}^W$, and write $\partial W = W \backslash \text{Int } W$ for the boundary of

W. We also make the assumption that $W \subset$ clos Int W where clos means the closure in the topology on $\mathbb{R}^W$. This eliminates the possibility that W includes isolated points. Define the preference cone of coalition $C \subseteq N$ at x by

$$H_C^+(x) = \{y \in W : p_i(x) \cdot (y - x) > 0 \ \forall i \in C\}$$

Define the <u>infinitesimal</u> (or critical) optima set on $V \subseteq W$ with respect to $\mathbb{D}$ by

$$IO(V:\mathbb{D}) = \{x \in V : V \cap H_C^+(x) = \emptyset \ \forall C \in \mathbb{D}\}$$

The critical optima set for $\mathbb{D}$ may be thought of as the analogue, for a social order, of the set of critical points of a smooth function. It is the set of points which, on the basis of "first derivative" information are candidates for global optima. Thus the critical optima set contains the global optima set, but may also contain other points. We shall obtain necessary and sufficient conditions on the utility gradients at x for x to belong to $IO(W:\mathbb{D})$. Consequently these conditions will be necessary for a point to belong to the core. Under some conditions the critical and global optima sets coincide, and in this case, our conditions are necessary and sufficient for a point to belong to the core.

Say the smooth profile $(u_1,\ldots,u_n)$ is <u>strictly pseudo-concave</u> iff $\forall i \in N$, any $x,y \in W$ it is the case that $u_i(y) \geq u_i(x)$ implies that $p_i(x)(y - x) > 0$. More generally say the preference profile is <u>semi-convex</u> iff $\forall i \in N$, any $x \in W$

$$\{y \in W : yP_i x\} \subseteq H_{\{i\}}^+(x).$$

It is an easy matter to show that if the profile is strictly pseudo-concave then it is semi-convex in the above sense, and then $GO(W) = IO(W:\mathbb{D})$.

LEMMA 1: (i) $GO(W) \subset LO(W) \subset IO(W:\mathbb{D})$. Moreover if preferences are semi-convex then these sets are identical.

(ii) If $x \in$ Int W then a necessary and sufficient condition for $x \in IO(W:\mathbb{D})$ is that

$$0 \in \bigcap_{C \in \mathbb{D}} p_C(x).$$

<u>Proof</u>: Using Taylor's Theorem, it is a simple matter to prove that if $H_C^+(x) \neq \Phi$, for some $C \in \mathbb{D}$, then in any neighborhood V of x, $\exists y \in V$ such that $yP_C x$ (see [13], Lemma 4.19 for example). Thus $x \notin IO(W:\mathbb{D})$ implies $x \notin LO(W)$ and hence $x \notin GO(W)$. When preferences are semi-convex, then for any $C \subseteq N$,

$$\{y \in W : yP_C x\} \subseteq H_C^+(x)$$

Thus

$$x \notin GO(W) \Rightarrow H_C^+(x) \neq \emptyset \text{ for some } C \in \mathbb{D}$$
$$\Rightarrow x \notin IO(W:\mathbb{D})$$

(ii) From a standard argument (see for example [10], [12]) if $x \in$ Int W then for any $C \subseteq N$,

$$H_C^+(x) = \phi \text{ iff } 0 \in p_C(x)$$

Q.E.D.

Thus a necessary condition for $x \in \text{Int } W \cap GO(W)$ is that $0 \in p_C(x)$ $\forall C \in \mathbb{D}$. We now show that this latter condition is equivalent to a condition on pivotal rather than decisive coalitions.

4. SYMMETRY CONDITIONS FOR A CORE

In this section we define the notion of "pivotal" coalitions and use this notion to develop symmetry conditions, similar to the Plott symmetry condition for majority rule, which characterize $IO(W:\mathbb{D})$ for a fixed smooth profile, u.

DEFINITION 1: Given any family $\mathbb{D}$ of subsets of N and any $L \subseteq N$, we define the set of pivotal coalitions for $\mathbb{D}$ in L, written $\mathbb{E}_L(\mathbb{D})$, as the set of all coalitions $M \subseteq L$ such that for every binary partition $\{C,D\}$ of L-M, either $M \cup C \in \mathbb{D}$ or $M \cup D \in \mathbb{D}$. We write $\mathbb{E}(\mathbb{D}) = \mathbb{E}_N(\mathbb{D})$. We also sometimes write $\mathbb{E}_L$ for $\mathbb{E}_L(\mathbb{D})$ when there is no danger of confusion.

It is easy to see that since $\mathbb{D}$ is monotonic so is $\mathbb{E}_L$. I.e., any superset of a pivotal coalition is also pivotal.

DEFINITION 2: Let $x \in W$. We say x satisfies the pivotal gradient restrictions (PGR) with respect to $\mathbb{D}$ iff, for every $L \subseteq N$ and every $M \in \mathbb{E}_L(\mathbb{D})$,

$$0 \in p_{M*}(x)$$

where $M^* = \{i \in L: p_i(x) \in sp_M(x)\}$.

We offer a loose interpretation of the above definition: Say that the pivotal coalition, $M \in \mathbb{E}_L$ is "blocked" if $0 \in p_{M*}(x)$. If M is blocked, then there are other members of L, whose g radients lie in the same subspace as those of M, but not in the same half space. See Figure 1. Thus, the members of M^* cannot agree on any common direction to move. The PGR condition, then, simply specifies that every pivotal coalition, in every subset L of N, must be blocked in the above sense.

[Insert Figure 1 here]

THEOREM 1: If $x \in \text{Int } W$ then a necessary and sufficient condition for $x \in IO(W:\mathbb{D})$ is that x satisfies PGR with respect to $\mathbb{D}$.

Proof: (i) Let $L \subseteq N$ and suppose, for some $M \in \mathbb{E}_L$, that $0 \notin p_{M*}(x)$. Suppose that $\dim[sp(M)] = w$. Then $M^* = L$. But since $M \in \mathbb{E}_L$, then L contains some decisive coalition, C say. But then $0 \notin p_{M*}(x)$ implies $0 \notin p_C(x)$, a contradiction. Suppose that $\dim[sp(M)] < w$. Then $\exists \beta \in \mathbb{R}^w$ with $\beta \cdot p_i(x) = 0$ for all $i \in M^*$, and $\beta \cdot p_i(x) \neq 0$ for all $i \in L - M^*$. Let $A = \{i \in L: \beta \cdot p_i(x) > 0\}$ and $B = \{i \in L: \beta \cdot p_i(x) < 0\}$. But since $M \in \mathbb{E}_L$, and $M^* \supseteq M$, we have $M^* \in \mathbb{E}_L$. Hence $M^* \cup A \in \mathbb{D}$ or $M^* \cup B \in \mathbb{D}$. W.l.o.g., assume $M^* \cup A \in \mathbb{D}$. Now if $0 \notin p_{M*}(x)$, then by the separating hyperplane theorem, $\exists \alpha \in sp_{M*}(x) = sp_M(x)$ with $\alpha \cdot p_i(x) > 0$ for all $i \in M^*$. Now pick $\delta \in \mathbb{R}^+$ with $(\beta + \delta\alpha)p_i(x) > 0$ for all $i \in A$ and set

$\gamma = \beta + \delta\alpha$. Then $\gamma \cdot p_i(x) > 0$ for all $i \in M^* \cup A$. But then $0 \notin p_C(x)$ where $C = M^* \cup A \in \mathbb{D}$. By Lemma 1, $x \notin IO(\text{Int } W:\mathbb{D})$. Hence PGR is a necessary condition.

(ii) To prove sufficiency, note that for any $M \in \mathbb{D}$ if we set $L = M$ then $M \in \mathbb{E}_L$ and $M^* = M$. Hence PGR implies that $0 \in p_M(x)$ $\forall M \in \mathbb{D}$.

Q.E.D.

COROLLARY 1: PGR is a necessary condition for an interior point of W to belong to GO(W). Moreover, with semi-convex preferences the condition is also sufficient.

Proof: This follows directly from Theorem 1 together with Lemma 1.

Q.E.D.

5. APPLICATIONS TO GENERAL RULES

We now show how the PGR conditions can be applied to particular social choice functions, and how for majority rule, the conditions imply the Plott symmetry conditions.

Note that the PGR conditions specify symmetry conditions that must hold for every $L \subseteq N$. However, if $p_i(x) = 0$ for some $i \in L$, then the PGR symmetry conditions are trivially satisfied for that L. Hence the most useful gradient restrictions are obtained by setting $L \subseteq \{i \in N : p_i(x) \neq 0\}$. In particular, a necessary condition for x to be a core point is that the PGR symmetry conditions be met for the set $L = \{i \in N : p_i(x) \neq 0\}$.

With these preliminaries, we now show how the Plott [9] symmetry conditions for the existence of a majority rule core obtain as a special case of the above theorem. Specifically, the Plott conditions deal with the case when n is odd and when no two voters have common satiation points. The conditions specify

(PO): $p_j(x) = 0$ for some $j \in N$, and for all $i \in N - \{j\}$,
$\exists k \in N - \{i,k\}$ with $p_i(x) = -\alpha_k p_k(x)$ for some $\alpha_k > 0$.

However, if $P_{\mathbb{D}}$ is majority rule, with n odd, and if no two voters have common satiation points, then it is easily verified that the pivotal gradient restrictions imply condition PO. To see this, set $L = \{i \in N : p_i(x) \neq 0\}$, and note:

(a) If $|L| = n$, then $\phi \in \mathbb{E}_L$, and $\phi^* = \phi$, implying $0 \in sp_\phi(x)$, a contradiction. So $x \notin IO(W:\mathbb{D})$.

(b) If $|L| = n - 1$, then $L = N - \{j\}$ for some $j \in N$ (i.e., $p_j(x) = 0$), and $\mathbb{E}_L = \{C \subseteq N - \{j\} : |C| \geq 1\}$. Hence, for all $i \in N - \{j\}$, $\{i\} \in \mathbb{E}_L$. Hence $0 \in sp_{i^*}(x)$, which implies that $\exists k \in N - \{i,j\}$ with $p_i(x) = -\alpha_k p_k(x)$ for some $\alpha_k > 0$.

This gives Plott's theorem as an immediate corollary of Theorem 1.

COROLLARY 2: Let $P_{\mathbb{D}}$ be majority rule, with n odd, and assume $x \in W$ satisfies $|\{i \in N : p_i(x) = 0\}| \leq 1$. Then $x \in \text{Int } W \cap GO$ implies that condition PO is met.

Note that in the case with n odd, and under the assumption that at the point x it is the case that $|\{i \in N : p_i(x) = 0\}| \leq 1$ then x satisfies

PGR with respect to the class of majority coalitions <u>if and only if</u> the Plott condition is satisfied.

As a second application, consider a <u>q-rule</u>, whose decisive coalitions are given by $\mathbb{D} = \{C \subset N: |C| \geq q\}$. The q-rule contains majority rule (with n odd or even) as a special case. Supra-majority rules of this kind have been studied by a number of writers (e.g., [2], [3], [8], [16], [17]). To obtain the core symmetry conditions for such a rule, assume that $q < n$ and define $w(n,q) = 2q - n - 1$. Then it is easy to verify that

if $|L| = n$, then $\mathbb{E}_L = \{M \subseteq N: |M| \geq e(n,q)\}$

if $|L| = n - 1$, then $\mathbb{E}_L = \{M \subseteq L: |M| \geq e(n,q) + 1\}$.

Thus, as above, we set $L = \{i \in N: p_i(x) \neq 0\}$, and obtain necessary conditions for a point x to be a core point when no more than one person is satiated at x: Either no one is satiated at x, and all coalitions of size w(n,q) are blocked, or one person is satiated at x, and all coalitions of size w(n,q) + 1 (among the remaining individuals) are blocked. (Compare to Slutsky [17].)

For a general social order it is useful to introduce the notion of a <u>structurally stable core</u>. We say that the set $IO(W:\mathbb{D})$ is <u>structurally stable</u> at the profile u if and only if $IO(W:\mathbb{D})$ is non-empty at u and there exists a neighborhood U of u in the Whitney topology on smooth profiles [see 11, 18] such that $IO(W:\mathbb{D})$ is non-empty to all u' in U. Conversely the set $IO(W:\mathbb{D})$ is structurally unstable at u if an arbitrary smally smooth perturbation of u is

sufficient to render $IO(W:\mathbb{D})$ empty. If $IO(W:\mathbb{D})$ is structurally unstable then say $GO(W:\mathbb{D})$ is also structurally unstable. If $IO(W:\mathbb{D})$ is structurally stable at a semi-convex profile u then we shall say the core, $GO(W:\mathbb{D})$, is structurally stable (see also [15]). As an application of this notion, observe that if $\{x\} = GO(W:\mathbb{D})$ and $|\{i \in N: p_i(x) = 0\}| \geq 2$ then $GO(W:\mathbb{D})$ must be structurally unstable. Since we are interested in the existence of a structurally stable core we may apply the pivotal gradient restrictions in the cases where $L = \{i \in N: p_i(x) \neq 0\}$ is of cardinality n or n - 1. For example, as we have shown elsewhere [7], the core for a q-rule can never be structurally stable if the dimension w satisfies $w \geq w(n,q) + 3 (= 2q - n + 2)$.

To show how Theorem 1 may be used in the general case, we let n = 5 and consider a social choice rule with the following decisive coalitions (we only list the minimal decisive sets): $\mathbb{D} = \{\{1,2,5\},\{1,3,5\},\{2,3,4\},\{2,3,5\},\{4,5\}\}$. Then we compute the pivotal sets for $|L| \geq 4$ as follows (we only list the minimal pivotal sets): Let $L_i = N - \{i\}$, and write $\mathbb{E}_{L_i} = \mathbb{E}_i$

L	Pivotal Sets	w_i^* (instability) dimension)
N	$\mathbb{E}_N = \{\{1\},\{2\},\{3\},\{4\},\{5\}\}$	2
L_1	$\mathbb{E}_1 = \{\{2\},\{3\},\{4\},\{5\}\}$	2
L_2	$\mathbb{E}_2 = \{\{1,3\},\{4\},\{5\}\}$	2
L_3	$\mathbb{E}_3 = \{\{1,2\},\{4\},\{5\}\}$	2
L_4	$\mathbb{E}_4 = \{\{1,2\},\{1,3\},\{5\}\}$	2
L_5	$\mathbb{E}_5 = \{\{2,4\},\{3,4\},\{2,3\}\}$	3

Thus, as above, setting $L = \{i \in N : p_i(x) \neq 0\}$, we obtain necessary conditions for x to be a core point if no more than one individual is satiated at x: Either no individual is satiated, and all coalitions in $\mathbb{E}_N$ are blocked, or individual i is satiated and all coalitions in $\mathbb{E}_i$ are blocked. The instability dimension, w_i^*, gives the lowest dimension in which these symmetry conditions are structurally unstable. For $w < w_i^*$, there is an open set of profiles for which the condition can be met, whereas for $w \geq w_i^*$, the conditions can be met only on a nowhere dense set of profiles. The rule as a whole, therefore, can have a structurally stable core (at player 5's most preferred point) in two dimensions, but only a structurally unstable core if $w \geq 3$. Figure 2 illustrates how a structurally stable core can occur in two dimensions, and Figure 3 illustrates how a structurally unstable core can occur in three dimensions. In these figures, we assume, for ease of illustration, that each player has a "Type I" or Euclidean preference of the type $u_i(x) = -1/2 \, ||x - x_i^*||^2$ on $\mathbb{R}^3$ where $|| \ ||$ is the standard Euclidean norm. Here x_i^* is the "bliss point" of player i, where $p_i(x_i^*) = 0$.

To illustrate, in Figure 3, if x_5^* is the core, then this point must belong to the set A, defined to be the convex hull of $\{x_2^*, x_3^*, x_4^*\}$. Transversality arguments [7] show that for an open dense set of profiles, the objects $\{x_5^*\}$ and A are respectively zero- and two-dimensional and do not intersect in $\mathbb{R}^3$. Thus the core in Figure 3 is structurally unstable. On the other hand, in the two dimensional case of Figure 2, the pivotal gradient restrictions are robust under small

perturbations and so the core is structurally stable.

[Insert Figures 2 and 3 here]

The results of section 4 and these examples above are valid when the set of alternatives is unconstrained. Political institutions frequently impose feasibility constraints on social choice, and in the following section we show how there can be incorporated in more general pivotal gradient restrictions.

6. SYMMETRY CONDITIONS AT A CONSTRAINED CORE

In this section we use Theorem 1 to characterize points in a constrained core. We fix $x \in W$. Then for any $v \in \mathbb{R}^W$, define the v restriction on W by

$$W_v = \{y \in W : y \cdot v \geq x \cdot v\}.$$

Say that x is a <u>v constrained core</u>, whenever $x \in GO(W_v)$. I.e., x is a core in the constrained set W_v.

Another way of thinking of a constrained core is that we introduce another voter, say voter "n + 1", who has utility gradient v, and who must be included in any winning coalition. Using this motivation, we define a new set $N_v = N \cup \{n + 1\}$ of voters, and the corresponding set, $\mathbb{D}_v$ of decisive coalitions by

$$\mathbb{D}_v = \{C \subseteq N_v : n + 1 \in C \text{ and } C - \{n + 1\} \in \mathbb{D}\}.$$

Given $\mathbb{D}_v$, and any $L \subseteq N_v$, then as before $E_L(\mathbb{D}_v)$ is the set of pivotal coalitions for $\mathbb{D}_v$ in L. Then Theorem 1 immediately gives the following corollary.

COROLLARY 3: If $x \in \text{Int } W$ then $x \in IO(W_v:\mathbb{D})$ iff x satisfies PGR with respect to $\mathbb{D}_v$.

Proof: From Theorem 1 it follows that $x \in IO(W:\mathbb{D}_v)$ iff x satisfies PGR w.r.t. $\mathbb{D}_v$. Hence we need only show that $IO(W_v:\mathbb{D}) = IO(W:\mathbb{D}_v)$. By definition $p_{n+1}(x) = v$. Hence $x \notin IO(W:\mathbb{D}_v) \iff \exists y \in W$ such that $y \in H_C^+(x)$ for some $C \in \mathbb{D}_v \iff \exists y \in W$ such that $(y - x)\cdot v > 0$ and $y \in H_B^+(x)$ for $B = C\backslash\{n + 1\} \in \mathbb{D} \Rightarrow \exists y \in W_v$ such that $H_B^+(x) \cap W_v \neq \emptyset$ for $B \in \mathbb{D} \Rightarrow x \notin IO(W_v:\mathbb{D})$. On the other hand if $x \notin IO(W_v:\mathbb{D})$ then $\exists y \in W_v$ such that $y \in H_B^+(x)$ for some $B \in \mathbb{D}$. Since utilities are smooth, $\exists y' \in H_B^+(x)$ such that $(y' - x)\cdot v > 0$. Hence $H_C^+(x) \neq \emptyset$ for some $C \in \mathbb{D}_v$, and so $x \notin IO(W:\mathbb{D}_v)$.

Q.E.D.

Lemma 1 then immediately gives the following corollary.

COROLLARY 4: If $x \in \text{Int } W$ then a necessary condition for x to be a v-constrained core is that x satisfies PGR w.r.t. $\mathbb{D}_v$. If preferences are semi-convex then the condition is sufficient.

Q.E.D.

To illustrate, consider the case of majority rule with n odd. As we have noted, $\emptyset \in \mathbb{E} = \mathbb{E}_N(\mathbb{D})$, so $\{n + 1\} \in \mathbb{E}_{N_v}(\mathbb{D}_v)$. Hence, it follows from the pivotal gradient restrictions that there exists $k \in N$ with $p_k(x) = -\lambda v$ for some $\lambda \in \mathbb{R}$ with $\lambda > 0$. Now let $L = N_v\backslash\{k\}$. Then it follows that any set of the form $M = \{j,n + 1\}$ is pivotal if

$j \notin \{k,n + 1\}$. It follows again, from the pivotal gradient conditions, that $0 \in p_{M*}(x)$, where $M* = \{i \in L:p_i(x) \in sp_M(x)\}$. In particular, it follows that $\exists \ell \in N\backslash\{j,k\}$ with $p_\ell(x) \in sp_M(x)$. But, if all gradients are non-zero, this is exactly the "joint symmetry" condition given by McKelvey [7]. Unlike the situation for an unconstrained majority rule core there is no requirement that $p_i(x) = 0$ for some $i \in N$. Similar symmetry properties for the constrained majority rule core with n even can also be obtained.

The symmetry properties for a majority rule constrained core which can be deduced from Corollary 4 are identical to those mentioned by Plott [9]. The corollary also shows how to obtain necessary symmetry properties at a constrained core for an arbitrary social order. Note also that Corollary 3 can be easily extended to the case where there exists a family of constraints at the point.

7. SYMMETRY CONDITIONS AND THE CYCLE SET

The notion of a constrained core is also helpful in characterizing the cycle set LC(W).

First of all define the critical cycle set [8] written IC(W) by $x \in IC(W)$ iff

(i) $0 \notin p_C(x)$ for at least one $C \in \mathbb{D}$

and (ii) $\emptyset = \bigcap_{C \in \mathbb{D}(x)} p_C(x)$ where $\mathbb{D}(x) = \{C \in \mathbb{D} : 0 \notin p_C(x)\}$.

Note that if $x \in W\backslash IC(W)$ then either

(i) $0 \in \bigcap_{C \in \mathbb{D}} p_C(x)$

or (ii) $\exists$ a vector $v_x \in \mathbb{R}^W\backslash\{0\}$ such that $v_x \in \bigcap_{C \in \mathbb{D}(x)} p_C(x)$.

The critical cycle set bears the same relation to the local and global cycle sets as the critical optima bear to the local and global are points. It is the set of points which, on the basis of "first derivative information," are candidates for the cycle sets. Earlier results have shown that IC(W) is open in W and

$$IC(W) \subset LC(W) \subset \text{clos } IC(W)$$

where clos IC(W) is the closure of IC(W) in W ([10], [14]).

THEOREM 2: $x \in \text{Int } W\backslash IC(W)$ iff there exists a vector $v_x \in \mathbb{R}^W\backslash\{0\}$ such that x satisfies PGR with respect to $\mathbb{D}_{v_x}$.

<u>Proof</u>: By definition there are two cases to consider.

(i) $0 \in \bigcap_{C \in \mathbb{D}} p_C(x)$.

But then $x \in IO(W:\mathbb{D})$. Moreover, for any $v \in \mathbb{R}^W\backslash\{0\}$ it is the case that $IO(W:\mathbb{D}) \subset IO(W_v:\mathbb{D})$. By Corollary 2, x must satisfy PGR w.r.t. $\mathbb{D}_v$.

(ii) $\exists B \in \mathbb{D}$ such that $0 \notin p_B(x)$. In this case, since $x \notin IC(W)$ there exists $v_x \in \mathbb{R}^W\backslash\{0\}$ such that, for all $B \in \mathbb{D}$ either $0 \in p_B(x)$ or $-v_x \in p_B(x)$. As before define $p_{n+1}(x) = v_x$, let $N_{v_x} = N \cup \{n + 1\}$ and let

$$\mathbb{D}_{v_x} = \{C \subseteq N_{v_x} : n + 1 \in C \text{ and } C\backslash\{n + 1\} \in \mathbb{D}\} .$$

Now for any $B \subseteq N$, $B \in \mathbb{D} \iff B' = B \cup \{n + 1\} \in \mathbb{D}_{v_x}$.

Hence $0 \in p_B(x)$ or $-v_x \in p_B(x)$, $\forall B \in \mathbb{D}$

$\iff 0 \in p_{B'}(x)$ $\forall B' \in \mathbb{D}_{v_x}$

$\iff$ x satisfies PGR w.r.t. $\mathbb{D}_{v_x}$, by Theorem 1.

Q.E.D.

COROLLARY 5: If LC(W) is empty then at every point in the interior of W, there exists a vector $v_x \in \mathbb{R}^W$ such that x satisfies PGR w.r.t. $\mathbb{D}_{v_x}$.

<u>Proof</u>: By previous results LC(W) is empty iff IC(W) is empty which implies that Int W $\cap$ IC(W) is empty. The result follows by Theorem 2.

Q.E.D.

Early results by Cohen and Matthews [1], Matthews ([4], [5]), McKelvey [6] and Schofield [12] were only valid for the analysis of the cycle set for majority rule. Theorem 2, together with Corollary 5 and the comments following Corollary 4, give the symmetry conditions which are necessary if a point is to lie outside the cycle set not just for majority rule but for an arbitrary social order.

Notice that Plott [9, p. 793] in his analysis of majority rule observed that a constraint could be represented by an "invisible" veto player. For an arbitrary social order the new player (n + 1) introduced in the proof of Corollary 3 and Theorem 2 has precisely the same function. This means that LC = $\emptyset$ effectively if and only if it is the case that at each point x, there exists an "invisible" veto player i_x who in fact "represents" the social order.

8. EXTENSIONS TO MANIFOLDS

The analysis of the previous sections can immediately be extended to the case where W is a smooth manifold of dimension (dim(W)) equal to w.

In this case the tangent space T_xW at a point x in the interior of W is linearly isomorphic to $\mathbb{R}^w$. At a point on the boundary ∂W the tangent space is isomorphic to

$$(\mathbb{R}^w)_v = \{y \in \mathbb{R}^w : y \cdot v \geq 0\}$$

for some $v \in \mathbb{R}^w \backslash \{0\}$. Here v may be thought of as normal to ∂W. The preference cone of C at x is now defined by

$$H_C^+(x) = \{v \in T_xW : p_i(x) \cdot v > 0 \ \forall i \in C\}$$

where $p_i(x)$ is again the utility gradient of i. IO(W:$\mathbb{D}$) can then be defined in the obvious fashion. With this modification the proofs of Theorems 1 and 2 are also valid. Corollary 2 may then be applied to the case of a point on the boundary.

COROLLARY 6: $x \in$ IO(∂W:$\mathbb{D}$) iff x satisfies PGR with respect to $\mathbb{D}_v$, where $v \in \mathbb{R}^w$ is "normal" to ∂W.

Finally, if $x \in \partial W \backslash$IC(W) then again one may define utility gradients restricted to the tangent space to ∂W, and show that there must satisfy appropriate pivotal gradient restrictions.

9. CONCLUSION

A social choice rule defines a family of decisive coalitions, and is parametrized by the smooth profile u. The space of all smooth profiles on a smooth manifold has a natural topology, the Whitney topology ([11], [18]). A property of the social order which is true for a residual, and thus dense, set of smooth profiles in this topology is called <u>generic</u>. The two theorems presented here can be applied to provide a generic classification of general voting rules.

For example, consider a q-rule of the kind discussed in Section 5. As observed there, a coalition M of size at least w(n,q) is pivotal. Remember w(n,q) = 2q − n − 1. It can be shown [7] that if dim(W) $\geq$ w(n,q) + 3 then it is generically the case that the pivotal gradient restrictions can apply at no point in W. Indeed if W has an empty boundary then the same is true if dim(W) $\geq$ w(n,q) + 2. That is to say the core must be structurally unstable when these dimensionality conditions are satisfied. It is also shown that if dim(W) $\geq$ w(n,q) + 3 then it is generically the case that there can be an invisible veto player only on nowhere-dense submanifolds of W. This immediately implies that IC(W) must be open-dense generically. Note that the "instability dimension," w(n,q) + 2, is either two or three in the case of majority rule, depending on whether n is odd or even. A later paper will attempt to compute the instability dimension for an arbitrary social order.

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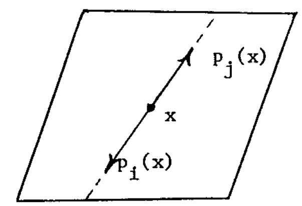
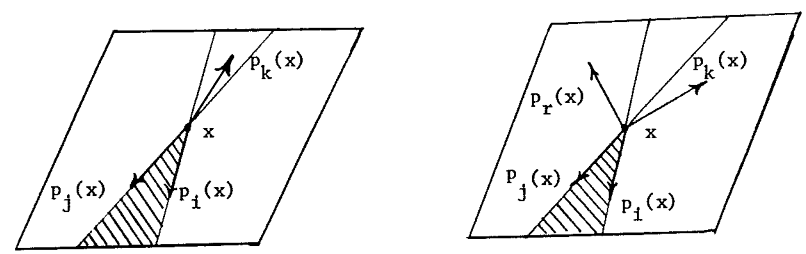


(a)
{i,j} blocked internally



(b)
{i,j}blocked by other players

Figure 1

Examples of ways in which {i,j} can be blocked

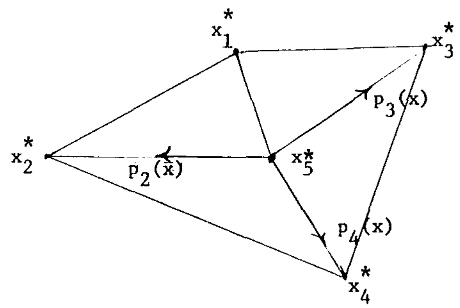


Figure 2
Example of Structurally Stable Core
for $\mathbb{D}$ in two dimensions

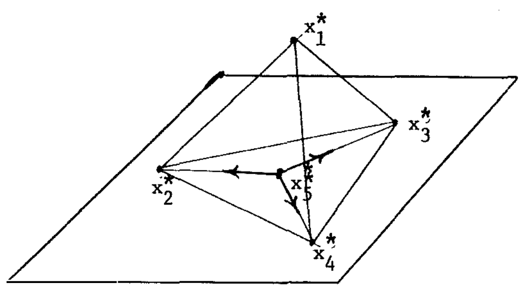


Figure 3*

Example of Structurally Unstable Core
for $\mathbb{D}$ in three dimensions

* The ideal points x_2^*, x_3^*, x_4^*, and x_5^*, are all in the subspace represented by the plane drawn. The ideal point x_1^* may be off the plane