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GENERALIZED THERMOELASTICITY FOR ANISOTROPIC MEDIA*

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Abstract. The equations of generalized thermoelasticity for an anisotropic medium are derived. Also, a uniqueness theorem for these equations is proved. A variational principle for the equations of motion is obtained.

1. Introduction. The theory of linear coupled thermoelasticity has been studied by many authors. Biot [1] has presented a unified treatment of the subject and Weiner [2] has proved a uniqueness theorem for the isotropic case.

Two generalizations of the equations of coupled thermoelasticity have arisen in the last decade. These generalizations eliminate the paradox of infinite speed of propagation of heat and elastic disturbances inherent in the coupled thermoelasticity theory.

The first generalization, due to Lord and Shulman [3] and Fox [4], modifies the well-known Fourier law of heat conduction but was until now restricted to isotropic homogeneous media.

The second generalization, due to Green and Lindsay [5], does not violate Fourier's law of heat conduction when the body under consideration has a center of symmetry, and was derived for both isotropic and anisotropic media. Green [6] supplemented this theory by proving a uniqueness theorem for a body which has a center of symmetry.

2. Derivation of the fundamental equations. We shall use the following notation:

V arbitrary material volume bounded by a closed and bounded surface S
q_i heat conduction vector
U internal energy per unit mass
η entropy per unit mass
T absolute temperature $= T_0 + \theta$
T_0 initial temperature
θ small temperature increment
σ_{ij} components of stress tensor
e_{ij} components of strain tensor
u_i components of displacement vector
v_i components of velocity vector
ρ density assumed independent of time
F_i external forces per unit mass

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n_i components of unit outward normal vector to the surface
k_{ij} thermal conductivity tensor
t time

We use the summation convention throughout. A superposed dot denotes differentiation with respect to time while a comma is used for material derivatives.

In the most general anisotropic medium, the equations of state relating stress, deformation and temperature are written as

$$\sigma_{ij} = c_{ijkl} e_{kl} - \beta_{ij} \theta. \tag{2.1}$$

The first law of thermodynamics takes the form

$$\frac{d}{dt} \int_V [\tfrac{1}{2} v_i v_i + U] \rho \, dV = \int_V \rho F_i v_i \, dV + \int_S (\sigma_{ji} v_i - q_j) n_j \, dS. \tag{2.2}$$

Using the divergence theorem and the equations of motion

$$\sigma_{ji,j} + \rho F_i = \rho \dot{v}_i, \tag{2.3}$$

$$\sigma_{ij} = \sigma_{ji}, \tag{2.4}$$

we get the pointwise form of (2.2)

$$-q_{i,i} = \rho \dot{U} - \sigma_{ij} \dot{e}_{ij}. \tag{2.5}$$

Using the entropy equation

$$q_{i,i} = -\rho T \dot{\eta}, \tag{2.6}$$

we get

$$\rho \, d\eta = \frac{\rho}{T} dU - \frac{1}{T} \sigma_{ij} \, de_{ij}.$$

This can be written as

$$\rho \, d\eta = \frac{\rho}{T} \frac{\partial U}{\partial T} dT + \frac{1}{T} \left(\frac{\partial U}{\partial e_{ij}} - \sigma_{ij} \right) de_{ij}. \tag{2.7}$$

The second law of thermodynamics requires that $d\eta$ be an exact differential in T and e_{ij}; therefore

$$\rho \frac{\partial \eta}{\partial T} = \frac{1}{T} \frac{\partial U}{\partial T}, \tag{2.8}$$

$$\rho \frac{\partial \eta}{\partial e_{ij}} = \frac{1}{T} \left(\frac{\partial U}{\partial e_{ij}} - \sigma_{ij} \right). \tag{2.9}$$

Using the identity

$$\frac{\partial^2 \eta}{\partial T \, \partial e_{ij}} = \frac{\partial^2 \eta}{\partial e_{ij} \, \partial T}$$

together with Eqs. (2.8) and (2.9), we get

$$\frac{1}{T} \left(\frac{\partial U}{\partial e_{ij}} - \sigma_{ij} \right) = \beta_{ij}. \tag{2.10}$$

Substituting from (2.10) in (2.7), we get

$$\rho \, d\eta = \frac{\rho}{T} \frac{\partial U}{\partial T} \, dT + \beta_{ij} \, de_{ij}. \tag{2.11}$$

Let

$$c_E = \frac{\partial U}{\partial T} \tag{2.12}$$

be the specific heat per unit mass in the absence of deformation (assumed independent of T in the neighborhood of the equilibrium state $T = T_0$). Substituting from (2.12) in (2.11), we obtain after integration

$$\rho\eta = \rho c_E \log T + \beta_{ij} e_{ij} + \text{constant}. \tag{2.13}$$

In (2.13) we choose the constant such that $\eta = 0$ when $T = T_0$ and $e_{ij} = 0$. Eq. (2.13), with this choice, takes the form

$$\rho\eta = \rho c_E \log\left(1 + \frac{\theta}{T_0}\right) + \beta_{ij} e_{ij}. \tag{2.14}$$

Expanding $\log(1 + (\theta/T_0))$ in a power series of θ/T_0 and neglecting higher orders of θ/T_0 than the first we get

$$\rho T_0 \eta = \rho c_E \theta + T_0 \beta_{ij} e_{ij}. \tag{2.15}$$

The linearized form of Eq. (2.6) is

$$q_{i,i} = -\rho T_0 \dot{\eta}. \tag{2.16}$$

By using Eq. (2.15) this reduces to

$$q_{i,i} = -\rho c_E \dot{\theta} - T_0 \beta_{ij} \dot{e}_{ij}. \tag{2.17}$$

We assume a generalized heat conduction equation of the form

$$q_i + \tau_0 \dot{q}_i = -k_{ij} \theta_{,j}. \tag{2.18}$$

Now, taking divergence of both sides of (2.18) and using Eq. (2.17) and its time derivative, we arrive at

$$\rho c_E(\dot{\theta} + \tau_0 \ddot{\theta}) + T_0 \beta_{ij}(\dot{e}_{ij} + \tau_0 \ddot{e}_{ij}) = \frac{\partial}{\partial x_i}(k_{ij}\theta_{,j}). \tag{2.19}$$

To get an equation satisfied by the displacements u_i we substitute from Eq. (2.1) in (2.3) and use the definition of strain

$$e_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}) \tag{2.20}$$

and the symmetry condition

$$c_{ijkl} = c_{klij}. \tag{2.21}$$

We get

$$\frac{\partial}{\partial x_j}(c_{ijkl} u_{k,l}) - \frac{\partial}{\partial x_j}(\beta_{ij}\theta) + \rho F_i = \rho \ddot{u}_i. \tag{2.22}$$

It is worth noting that for the case of isotropic material, Eqs. (2.19) and (2.22) reduce to the familiar form obtained by Lord and Shulman, namely

$$kT_{,ii} = \rho c_E(\dot{T} + \tau_0 \ddot{T}) + \beta T_0(e_{kk} + \tau_0 \ddot{e}_{kk})$$

and

$$\rho \ddot{u}_i = (\lambda + \mu)u_{j,\,ij} + \mu u_{i,\,jj} - \beta T_{,i} + \rho F_i,$$

where $\beta = (3\lambda + 2\mu)\alpha$, λ, μ being Lame's constants and α being the coefficient of linear expansion.

3. Uniqueness theorem. As usual, to prove uniqueness we assume there exist two sets of functions $\sigma_{ij}^{(1)}$ and $\sigma_{ij}^{(2)}$, $e_{ij}^{(1)}$ and $e_{ij}^{(2)}$, etc., and let

$$\sigma_{ij} = \sigma_{ij}^{(1)} - \sigma_{ij}^{(2)}, \qquad e_{ij} = e_{ij}^{(1)} - e_{ij}^{(2)}, \text{ etc.}$$

THEOREM. Given a regular region of space $V + S$ with boundary S then there exists at most one set of single-valued functions $\sigma_{ij}(x_{k,\,t})$ and $e_{ij}(x_{k,\,t})$ of class $C^{(1)}$, $u_i(x_{k,\,t})$ and $T(x_{k,\,t})$ of class $C^{(2)}$ in $V + S$, $t \geq 0$ which satisfy the following equations in V, $t > 0$:

$$\sigma_{ji,\,j} = \rho \ddot{u}_i, \tag{3.1}$$

$$\frac{\partial}{\partial x_i}(k_{ij}\theta_{,j}) = \rho c_E(\dot{\theta} + \tau_0 \ddot{\theta}) + T_0 \beta_{ij}(\dot{e}_{ij} + \tau_0 \ddot{e}_{ij}), \tag{3.2}$$

$$q_{i,\,i} = -\rho T_0 \dot{\eta}, \tag{3.3}$$

the following equations in $V + S$, $t \geq 0$:

$$\rho T_0 \eta = \rho c_E \theta + T_0 \beta_{ij} e_{ij}, \tag{3.4}$$

$$e_{ij} = \tfrac{1}{2}(u_{i,\,j} + u_{j,\,i}), \tag{3.5}$$

$$\sigma_{ij} = c_{ijkl} e_{kl} - \beta_{ij}\theta; \tag{3.6}$$

the following equations on the boundary S, $t > 0$:

$$T = 0, \qquad \text{on } S, \tag{3.7}$$

$$\begin{aligned} &u_i = 0, \qquad \text{on } S_1, \\ &p_i = \sigma_{ji} n_j = 0, \qquad \text{on } S - S_1; \end{aligned} \tag{3.8}$$

and the following equations in V, $t = 0$:

$$T = 0, \qquad u_i = 0, \qquad \dot{u}_i = 0, \tag{3.9–3.11}$$

where we assume that the elasticities satisfy the symmetry condition

$$c_{ijkl} = c_{klij} \tag{3.12}$$

and where the positive definiteness condition

$$c_{ijkl}\xi_{ij}\xi_{kl} \geq c\xi_{ij}\xi_{ij} \tag{3.13}$$

is satisfied for all non-zero tensors ξ_{ij} and for some positive constant c. We also assume that

$$\operatorname*{essinf}_{x_k \in V+S} \rho(x_k) > 0, \qquad T_0 > 0, \qquad c_E > 0. \tag{3.14}$$

Proof. Consider the integral

$$\int_V \sigma_{ij}\dot{e}_{ij}\,dV = \int_V \sigma_{ij}\dot{u}_{i,j}\,dV. \tag{3.15}$$

Using integration by parts and (3.7) we get

$$\int_V \sigma_{ij}\dot{e}_{ij}\,dV = -\int_V \sigma_{ij,j}\dot{u}_i\,dV. \tag{3.16}$$

Upon insertion of (3.1) in (3.16) the latter reduces to

$$\int_V [\sigma_{ij}\dot{e}_{ij} + \rho\dot{u}_i\ddot{u}_i]\,dV = 0. \tag{3.17}$$

Using Eqs. (3.6) and (3.12), Eq. (3.17) can be written in the form

$$\frac{d}{dt}\int_V \{\tfrac{1}{2}\rho\dot{u}_i\dot{u}_i + \tfrac{1}{2}c_{ijkl}e_{kl}e_{ij}\}\,dV - \int_V \theta\beta_{ij}\dot{e}_{ij}\,dV = 0. \tag{3.18}$$

Substituting for $\beta_{ij}\dot{e}_{ij}$ from Eq. (3.2) in (3.18), we get, after integration by parts and using (3.7),

$$\frac{d}{dt}\int_V \left\{\tfrac{1}{2}\rho\dot{u}_i\dot{u}_i + \tfrac{1}{2}c_{ijkl}e_{ij}e_{kl} + \frac{\rho c_E}{2T_0}\theta^2\right\}dV + \frac{1}{T_0}\left\{\int_V k_{ij}\theta_{,i}\theta_{,j}\,dV + \rho c_E\tau_0\int_V \theta\ddot{\theta}\,dV + T_0\tau_0\int_V \beta_{ij}\ddot{e}_{ij}\,dV\right\} = 0. \tag{3.19}$$

We now use the second law of thermodynamics in the form [4]

$$-q_i\theta_{,i} \geq 0.$$

Integrating this inequality and using (3.7), we get

$$\int_V \theta q_{i,i}\,dV \geq 0. \tag{3.20}$$

Substituting for $q_{i,i}$ from Eq. (2.18) in (3.20) we get

$$\int_V \theta\left[\tau_0\dot{q}_{i,i} + \frac{\partial}{\partial x_i}(k_{ij}\theta_{,j})\right]dV \geq 0. \tag{3.21}$$

Using Eq. (2.17) in (3.21) we get, after integration by parts,

$$\rho\tau_0 c_E\int_V \theta\ddot{\theta}\,dV + \tau_0 T_0\int_V \beta_{ij}\ddot{e}_{ij}\,dV + \int_V k_{ij}\theta_{,i}\theta_{,j}\,dV \geq 0. \tag{3.22}$$

From (3.22) and (3.19) we get

$$\frac{d}{dt}\int_V \left\{\tfrac{1}{2}\rho\dot{u}_i\dot{u}_i + \tfrac{1}{2}c_{ijkl}e_{ij}e_{kl} + \frac{\rho c_E}{2T_0}\theta^2\right\}dV \leq 0.$$

By using condition (3.13) this reduces to

$$\frac{d}{dt}\int_V \left\{\tfrac{1}{2}\rho\dot{u}_i\dot{u}_i + \tfrac{1}{2}c e_{ij}e_{ij} + \frac{\rho c_E}{2T_0}\theta^2\right\}dV \leq 0. \tag{3.23}$$

The integral on the left-hand side of (3.23) is initially zero, since the difference functions satisfy homogeneous initial conditions. By inequality (3.23), however, this integral either decreases (and therefore becomes negative) or remains equal to zero. Since the integral is the sum of squares, however, only the latter alternative is possible, that is

$$\int_V \left\{\rho \dot{u}_i \dot{u}_i + c e_{ij} e_{ij} + \frac{\rho c_E}{T_0} \theta^2\right\} dV = 0, \qquad t \geq 0. \tag{3.24}$$

It follows from (3.24) that the difference functions are identically zero throughout the body for all time. This completes the proof.

4. A variational principle. We introduce two invariants $\mathcal{V}$ and $\mathcal{D}$. The first invariant is the thermoelastic potential $\mathcal{V}$ defined by

$$\mathcal{V} = \int_V \left[W + \frac{\rho c_E}{2T_0} \theta^2\right] dV \tag{4.1}$$

where W is the isothermal mechanical energy given by

$$W = \tfrac{1}{2} c_{ijkl} e_{ij} e_{kl}. \tag{4.2}$$

In order to formulate the variational principle, the integrand $W + (\rho c_E / 2T_0)\theta^2$ must be expressed in terms of two vector fields. One is the displacement field

$$u_i = (u_x, u_y, u_z) \tag{4.3}$$

of the solid. The other is defined in terms of a vector S_i which represents the amount of heat flown in a given direction divided by the absolute temperature T_0. We call it the entropy flow after Biot [1]. S_i is given by

$$\dot{S}_i = q_i / T_0. \tag{4.4}$$

Eqs. (2.16) and (4.4) give

$$T_0 S_{i,i} = -\rho T_0 \eta. \tag{4.5}$$

Combining Eqs. (4.5) and (2.15), we get

$$T_0 S_{i,i} = -\rho c_E \theta - T_0 \beta_{ij} u_{i,j} \tag{4.6}$$

which gives

$$\theta = -\frac{T_0}{\rho c_E} (S_{i,i} + \beta_{ij} u_{i,j}). \tag{4.7}$$

In obtaining (4.6) we assume that $\beta_{ij} = \beta_{ji}$.

The second invariant is given by

$$\mathcal{D} = \frac{T_0}{2}\left(\frac{d}{dt} + \tau_0 \frac{d^2}{dt^2}\right) \int_V \lambda_{ij} S_i S_j \, dV \tag{4.8}$$

where λ_{ij}, the resistivity matrix, is the inverse of the thermal conductivity k_{ij}. It is convenient to modify somewhat the expression for $\mathcal{D}$ by writing, instead the operational expression,

$$\mathcal{D} = \frac{T_0}{2}(p + \tau_0 p^2) \int_V \lambda_{ij} S_i S_j \, dV \tag{4.9}$$

where $p = d/dt$. When calculating the variation, the operator p is treated as a constant and it is only in the final differential equations that it is replaced by an actual differential.

The variational principle is written as

$$\delta \mathcal{V} + \delta \mathcal{Q} = \int_S [p_i \, \delta u_i - \theta n_i \, \delta S_i] \, dS + \int_V \rho[F_i - \ddot{u}_i] \, \delta u_i \, dV. \tag{4.10}$$

The variation applied to all six components of the two vector fields and S_i, p_i is the boundary force per unit area in the x_i-direction given by

$$p_i = \sigma_{ji} n_j. \tag{4.11}$$

Indeed, we have

$$\delta \mathcal{V} + \delta \mathcal{Q} = \int_V \left\{ c_{ijkl} e_{kl} \, \delta e_{ij} + \frac{\rho c_E}{T_0} \theta \, \delta\theta + \frac{T_0}{2} (p + \tau_0 p^2) \lambda_{ij} S_j \, \delta S_i \right\} dV$$

$$= \int_V \{c_{ijkl} e_{kl} - \theta \beta_{ij}\} \, \delta u_{i,j} \, dV - \int_V \theta \, \delta S_{i,i} \, dV + T_0(p + \tau_0 p^2) \int_V \lambda_{ij} S_j \, \delta S_i \, dV.$$

Using integration by parts, we get

$$\delta \mathcal{V} + \delta \mathcal{Q} = \int_S [c_{ijkl} e_{kl} - \theta \beta_{ij}] n_j \, \delta u_i \, dS - \int_S \theta n_i \, \delta S_i \, dS$$

$$- \int_V [c_{ijkl} e_{kl} - \theta \beta_{ij}]_{,j} \, \delta u_i \, dV + \int_V [\theta_{,i} + T_0(p + \tau_0 p^2) \lambda_{ij} S_j] \, \delta S_i \, dV. \tag{4.12}$$

By substituting from (4.12) in the variational principle (4.10) the surface integrals cancel out and we are left with the condition that the surface integrals vanish identically. This implies two equations:

$$(c_{ijkl} e_{kl} - \theta \beta_{ij})_{,j} + \rho F_i = \rho \ddot{u}_i, \tag{4.13}$$

$$\theta_{,i} + T_0(p + \tau_0 p^2) \lambda_{ij} S_j = 0. \tag{4.14}$$

Eq. (4.14) can be written in the equivalent form

$$k_{ij} \theta_{,i} + T_0(p + \tau_0 p^2) S_j = 0$$

or

$$k_{ij} \theta_{,j} + T_0(p + \tau_0 p^2) S_i = 0. \tag{4.15}$$

In obtaining (4.15) we used the symmetry condition $k_{ij} = k_{ji}$. Differentiating (4.15) with respect to x_i, we get, after using Eq. (4.6),

$$\frac{\partial}{\partial x_i} (k_{ij} \theta_{,j}) = \rho c_E (\dot{\theta} + \tau_0 \ddot{\theta}) + T_0 \beta_{ij} (\dot{e}_{ij} + \tau_0 \ddot{e}_{ij}). \tag{4.16}$$

Eqs. (4.13) and (4.16) are the basic equations of motion for anisotropic generalized thermoelasticity.

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