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Generalized Transvectants-Rankin–Cohen Brackets

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Abstract. We introduce $o(p+1, q+1)$-invariant bilinear differential operators on the space of tensor densities on $\mathbb{R}^n$ generalizing the well-known bilinear sl_2-invariant differential operators in the one-dimensional case, called Transvectants or Rankin–Cohen brackets. We also consider already known linear $o(p+1, q+1)$-invariant differential operators given by powers of the Laplacian.

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1. Introduction

In the one-dimensional case, the problem of classification of SL_2-invariant (bi)linear differential operators was treated in the classical literature.

Consider the following action of $SL(2, \mathbb{R})$ on the space of (smooth) functions in one variable, for instance, on $\mathbb{R}, S^1$, or holomorphic functions on the upper half-plane:

$$f(x) \mapsto f\left(\frac{ax+b}{cx+d}\right)(cx+d)^{-2\lambda}, \tag{1.1}$$

where λ is a parameter $\lambda \in \mathbb{R}$ (or $\mathbb{C}$). This $SL(2, \mathbb{R})$-module of functions is called the space of λ-densities and denoted $\mathcal{F}_\lambda$.

The classification of $SL(2, \mathbb{R})$-invariant linear differential operators from $\mathcal{F}_\lambda$ to $\mathcal{F}_\mu$ (i.e. of the operators commuting with the action (1.1)) was obtained in classical works on projective differential geometry, namely, for every $k = 1, 2, \ldots$, there exists a unique (up to a constant) $SL(2, \mathbb{R})$-invariant linear differential operator of order k:

$$A_k : \mathcal{F}_{\frac{1-k}{2}} \to \mathcal{F}_{\frac{1+k}{2}}. \tag{1.2}$$

It is given by $A_k(f) = \mathrm{d}^k f/\mathrm{d}x^k$.

Linear $\mathrm{SL}(2,\mathbb{R})$-invariant differential operators from $\mathcal{F}_\lambda \otimes \mathcal{F}_\mu$ to $\mathcal{F}_\nu$ were already considered by Gordan [13]. For generic values of λ and μ, more precisely, for

$$\lambda, \mu \neq 0, -\tfrac{1}{2}, -1, \ldots$$

and for every $k = 0, 1, 2, \ldots,$ there exists a unique (up to a constant) $\mathrm{SL}(2,\mathbb{R})$-invariant bilinear differential operator of order k

$$B_k : \mathcal{F}_\lambda \otimes \mathcal{F}_\mu \to \mathcal{F}_{\lambda+\mu+k}.$$

This differential operator is given by the formula

$$B_k(f,g) = \sum_{i+j=k} (-1)^i \binom{2\mu + k - 1}{i} \binom{2\lambda + k - 1}{j} f^{(i)} g^{(j)} \tag{1.3}$$

and is called the Transvectant (cf. [16]). It is also known as the kth Rankin–Cohen bracket [5, 21].

The operators (1.3) play an important role in the theory of modular forms; they have been recently used in [6, 18, 19] to construct $\mathrm{SL}(2,\mathbb{R})$-invariant star-products on T^*S^1 and in the cohomology of Lie algebras of vector fields [2].

The purpose of this Letter is to extend the classical Gordan Transvectants to the multi-dimensional case. We will consider $\mathrm{o}(p+1, q+1)$-invariant bilinear differential operators on tensor densities on $\mathbb{R}^n$, where $n = p + q$. It should be stressed that there are other ways to generalize sl_2-symmetries in the multi-dimensional case. For instance, one considers the $\mathrm{sl}(n+1,\mathbb{R})$-action on $\mathbb{R}^n$; this leads to projective differential geometry.

Another approach to the problem of classification of conformally invariant differential operators can be found in the recent papers [7] and [4].

2. Conformally-Invariant Differential Operators

In the multi-dimensional case, one has to distinguish the conformally flat case that can be reduced to $\mathbb{R}^n$ endowed with the standard $\mathrm{o}(p+1, q+1)$-action, where $n = p + q$, and the 'curved' or generic case of an arbitrary pseudo-Riemannian manifold M. In this paper we will consider only the conformally flat case.

2.1. LIE ALGEBRA OF CONFORMAL SYMMETRIES

Denote g the standard quadratic form on $\mathbb{R}^n$ of signature $p - q$, where $p + q = n$. The Lie algebra of infinitesimal conformal transformations is generated by the vector fields

$$\begin{aligned} X_i &= \frac{\partial}{\partial x^i}, \qquad X_{ij} = x_i \frac{\partial}{\partial x^j} - x_j \frac{\partial}{\partial x^i}, \\ X_0 &= x^i \frac{\partial}{\partial x^i}, \qquad \bar{X}_i = x_j x^j \frac{\partial}{\partial x^i} - 2 x_i x^j \frac{\partial}{\partial x^j} \end{aligned} \tag{2.1}$$

where $(x^1, \ldots, x^n)$ are coordinates on $\mathbb{R}^n$ and $x_i = g_{ij}x^j$. Throughout this Letter, sum over repeated indices is understood. Let us also consider the Lie subalgebras

$$o(p,q) \subset e(p,q) \subset o(p+1,q+1) \subset \mathrm{Vect}(\mathbb{R}^n), \tag{2.2}$$

where $o(p,q)$ is generated by the X_{ij} and the Euclidean subalgebra $e(p,q)$ by X_i, X_{ij}.

It is worth noticing that the conformal Lie algebra $o(p+1,q+1)$ is maximal in the class of finite-dimensional subalgebras of $\mathrm{Vect}(\mathbb{R}^n)$, that is, any bigger subalgebra of $\mathrm{Vect}(\mathbb{R}^n)$ is infinite-dimensional (see [1]). This maximality property explains the uniqueness results.

Remark 2.1. All the results of this paper are valid for an arbitrary manifold M endowed with a conformally flat structure. Such a manifold is locally identified with $\mathbb{R}^n$ and the action (2.1) of the conformal Lie algebra is defined locally on M.

2.2. MODULES OF DIFFERENTIAL OPERATORS

Let us define the space $\mathcal{F}_\lambda$ of tensor densities of degree λ on $\mathbb{R}^n$. As a vector space, $\mathcal{F}_\lambda$ is isomorphic to $C^\infty(\mathbb{R}^n)$; the action of the Lie algebra, $\mathrm{Vect}(\mathbb{R}^n)$, of vector fields on $\mathbb{R}^n$ on $\mathcal{F}_\lambda$, is defined by

$$L_X^\lambda = X^i \frac{\partial}{\partial x^i} + \lambda\, \mathrm{Div}(X) \tag{2.3}$$

and depends on λ. Geometrically speaking, $\mathcal{F}_\lambda$ is the space of smooth sections of the line bundle $\Delta_\lambda(\mathbb{R}^n) = |\Lambda^n T^*\mathbb{R}^n|^{\otimes\lambda}$ over $\mathbb{R}^n$.

Consider the space $\mathcal{D}_{\lambda,\mu}$ of linear differential operators from $\mathcal{F}_\lambda$ to $\mathcal{F}_\mu$ and the space $\mathcal{D}_{\lambda,\mu;\nu}$ of bilinear differential operators from $\mathcal{F}_\lambda \otimes \mathcal{F}_\mu$ to $\mathcal{F}_\nu$. These spaces are naturally $\mathrm{Vect}(\mathbb{R}^n)$-modules. We will restrict these modules structures to the subalgebra $o(p+1,q+1) \subset \mathrm{Vect}(\mathbb{R}^n)$ and consider the spaces $\mathcal{D}_{\lambda,\mu}$ and $\mathcal{D}_{\lambda,\mu;\nu}$ as $o(p+1,q+1)$-modules. More precisely, we will be interested in the $o(p+1, q+1)$-invariant differential operators, that is, in the differential operators commuting with the $o(p+1,q+1)$-action.

Note that the $o(p+1,q+1)$-modules $\mathcal{D}_{\lambda,\mu}$ have been studied in a series of recent papers (see [8] and references therein).

2.3. POWERS OF THE LAPLACIAN

In the conformally flat case, the analogues of the operators (1.2) have been classified in [11]. The result is as follows.

THEOREM 2.2 ([11]). *For every $k = 1, 2, \ldots,$ there exists a unique (up to a constant) $o(p+1,q+1)$-invariant linear differential operator of order k:*

$$A_{2k} : \mathcal{F}_{\frac{n-2k}{2n}} \to \mathcal{F}_{\frac{n+2k}{2n}} \tag{2.4}$$

and there are no other $o(p+1, q+1)$*-invariant linear differential operators of order* $k \geqslant 1$ *from* $\mathcal{F}_\lambda$ *to* $\mathcal{F}_\mu$.

In the coordinate system (2.1), the explicit expressions of the operators A_{2k} are

$$A_{2k} = \Delta^k, \quad \text{where } \Delta = g^{ij} \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j}.$$

Remark 2.3. In the generic (curved) case, the situation is much more difficult, see [14, 15]. We also refer to [9, 10], see also [3], for a recent study of conformally invariant differential operators on tensor fields, and a complete list of references.

We will give a simple direct proof of Theorem 2.2.

2.4. MULTI-DIMENSIONAL TRANSVECTANTS

The following theorem is the main result of this Letter. It provides a classification of bilinear $o(p+1, q+1)$-invariant differential operators on tensor densities of generic degrees λ and μ.

THEOREM 2.4. (i) *For every* $k = 0, 1, 2, \ldots,$ *there exists a unique up to a constant* $o(p+1, q+1)$*-invariant bilinear differential operator of order* k

$$B_{2k} : \mathcal{F}_\lambda \otimes \mathcal{F}_\mu \to \mathcal{F}_{\lambda+\mu+\frac{2k}{n}} \tag{2.5}$$

provided neither of λ *and* μ *belongs to the set*

$$\left\{0, -\frac{1}{n}, -\frac{2}{n}, \ldots, \frac{2-2k}{n}\right\} \cup \left\{\frac{n-2}{2n}, \frac{n-4}{2n}, \ldots, \frac{n-2k}{2n}\right\}. \tag{2.6}$$

(ii) *There is no other bilinear* $o(p+1, q+1)$*-invariant differential operator on tensor densities, for generic values of* λ *and* μ.

The case, when λ and μ simultaneously belong to the set (2.6), is particular. There may be no $o(p+1, q+1)$-invariant operators in this case, as well as there may be no uniqueness. For instance, if $\lambda = \mu = (n-2k)/2n$, both operators $B(f, g) = A_{2k}(f)g$ and $B(f, g) = fA_{2k}(g)$, are $o(p+1, q+1)$-invariant. The classification in this case is much more complicated and will not be considered in this paper (cf. [12] for the one-dimensional case).

We will give explicit formulæ for the operators (2.5) in Section 4.3.

Remark 2.5. In this paper, we will not consider the curved case. We formulate here a problem of existence of bilinear conformally invariant differential operators for an arbitrary (not necessarily flat) conformal structure.

3. Proof of Theorem 2.2

We will start the proof with classical results of the theory of invariants and describe the differential operators invariant with respect to the action of the Lie algebra $\mathrm{e}(p,q)$. We refer [22] as a classical source and [8] for the description of the Euclidean invariants.

3.1. EUCLIDEAN INVARIANTS

Using the standard affine connection on $\mathbb{R}^n$, one identifies the space of linear differential operators on $\mathbb{R}^n$ with the corresponding space of symbols, i.e., with the space of smooth functions on $T^*\mathbb{R}^n \cong \mathbb{R}^n \oplus (\mathbb{R}^n)^*$ polynomial on $(\mathbb{R}^n)^*$. This identification is an isomorphism of modules over the algebra of affine transformations and allows us to apply the theory of invariants.

Moreover, choosing a (dense) subspace of symbols which are also polynomials on the first summand, one reduces the classification of $\mathrm{e}(p,q)$-invariant differential operators from $\mathcal{D}_{\lambda,\mu}$ to the classification of $\mathrm{e}(p,q)$-invariant polynomials in the space $\mathbb{C}[x^1,\ldots,x^n,\xi_1,\ldots,\xi_n]$, where $(\xi_1,\ldots,\xi_n)$ are the coordinates on $(\mathbb{R}^n)^*$ dual to $(x^1,\ldots,x^n)$.

Consider first invariants with respect to $\mathrm{o}(p,q)\subset \mathrm{e}(p,q)$. It is well-known (see [22]) that the algebra of $\mathrm{o}(p,q)$-invariant polynomials is generated by three elements

$$\mathrm{R}_{xx} = \mathrm{g}_{ij}\,x^i x^j, \quad \mathrm{R}_{x\xi} = x^i\xi_i, \quad \mathrm{R}_{\xi\xi} = \mathrm{g}^{ij}\,\xi_i\xi_j.$$

Second, taking into account the invariance with respect to translations in $\mathrm{e}(p,q)$, any $\mathrm{e}(p,q)$-invariant polynomial $P(x,\xi)$ satisfies $\partial P/\partial x^i = 0$. The only remaining generator is $\mathrm{R}_{\xi\xi}$ and, therefore, $\mathrm{e}(p,q)$-invariant linear differential operators from $\mathcal{F}_\lambda$ to $\mathcal{F}_\mu$ are linear combinations of operators (2.4).

Note that the obtained result is, of course, independent from λ and μ since the degree of tensor densities does not intervene in the $\mathrm{e}(p,q)$-action.

3.2. PROOF OF THEOREM 2.2

It remains to check for which values of λ and μ the operators (2.4) from $\mathcal{F}_\lambda$ to $\mathcal{F}_\mu$ are invariant with respect to the action of the full Lie algebra $\mathrm{o}(p+1,q+1)$.

By definition, the action of a vector field X on an element $A\in\mathcal{D}_{\lambda,\mu}$ is given by

$$L_X^{\lambda,\mu}(A) = L_X^{\mu}\circ A - A\circ L_X^{\lambda},$$

where L_X^λ is the operator of Lie derivative (2.3).

Consider the action of the generator X_0 in (2.1) on the operator $A=\sum_{k\geqslant 0} c_k\mathrm{R}_{\xi\xi}^k$. Using the preceding expressions, one readily gets

$$L_{X_0}^{\lambda,\mu}(A) = \sum_{i\geqslant 0}(n\delta - 2k)c_k\mathrm{R}_{\xi\xi}^k$$

where $\delta = \mu - \lambda$. Therefore, the invariance condition $L^{\lambda,\mu}_{X_0}(A) = 0$ is satisfied if and only if for each k in the above sum either $c_k = 0$ or $\delta = 2k/n$, and one obtains the values of the shift δ in accordance with (2.4).

Consider, at last, the action of the generators $\bar{X}_i$ (with $i = 1, \ldots, n$). After the identification of the differential operators with polynomials one has the following explicit expressions for the Lie derivative.

PROPOSITION 3.1. *The action of the generator $\bar{X}_i$ on $\mathcal{D}_{\lambda,\mu}$ is as follows*

$$L^{\lambda,\mu}_{\bar{X}_i} = L^{\delta}_{\bar{X}_i} - \xi_i T + 2(\mathcal{E} + n\lambda)\,\partial_{\xi^i}$$

where

$$L^{\delta}_{\bar{X}_i} = x_j x^j \partial_i - 2x_i x^j \partial_j - 2(\xi_i x_j - \xi_j x_i)\partial_{\xi_j} + 2\xi_j x^j \partial_{\xi^i} - 2n\delta x_i$$

is the cotangent lift, and where $T = \partial_{\xi^j}\partial_{\xi_j}$ is the trace and $\mathcal{E} = \xi_j \partial_{\xi_j}$ the Euler operator.

This formula has been obtained in [8], it can be also easily checked by a straightforward computation.

Applying $L^{\lambda,\mu}_{\bar{X}_i}$ to the operator $\mathrm{R}^k_{\xi\xi}$ one then obtains

$$L^{\lambda,\mu}_{\bar{X}_i}(\mathrm{R}^k_{\xi\xi}) = 2(2k - n\delta)x_i R^k_{\xi\xi} \; + 2k(n(2\lambda - 1) + 2k)\xi_i \mathrm{R}^{k-1}_{\xi\xi}.$$

The first term in this expression vanishes for $2k - n\delta = 0$, this condition is precisely the preceding one; the second term vanishes if and only if $\lambda = (n - 2k)/2n$. Theorem 2.2 is proved.

4. Proof of Theorem 2.4

In this section we will prove our main result and give an explicit formula for the operators (2.5).

4.1. EUCLIDEAN-INVARIANT BILINEAR OPERATORS

As in Section 3.1, let us first consider the operators invariant with respect to the Lie algebra $\mathrm{e}(p, q)$. Again, identifying the bilinear differential operators with their symbols, one is led to study the algebra of $\mathrm{e}(p, q)$-invariant polynomials in the space $\mathbb{C}[x^1, \ldots, x^n, \xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_n]$. The Weyl invariant theory just applied guarantees that there are three generators:

$$\mathrm{R}_{\xi\xi} = \mathrm{g}^{ij}\,\xi_i\xi_j, \quad \mathrm{R}_{\xi\eta} = \mathrm{g}^{ij}\,\xi_i\eta_j, \quad \mathrm{R}_{\eta\eta} = \mathrm{g}^{ij}\,\eta_i\eta_j.$$

Any $\mathrm{e}(p, q)$-invariant bilinear differential operator is then of the form

$$B = \sum_{r,s,t \geqslant 0} c_{rst} \mathrm{R}^{r,s,t}$$

where, to simplify the notations, we put $\mathrm{R}^{r,s,t} = \mathrm{R}^r_{\xi\xi}\mathrm{R}^s_{\xi\eta}\mathrm{R}^t_{\eta\eta}$.

4.2. INVARIANCE CONDITION

The action of a vector field X on a bilinear operator B: $\mathcal{F}_\lambda \otimes \mathcal{F}_\mu \to \mathcal{F}_\nu$ is defined as follows

$$(L_X^{\lambda,\mu;\nu} B)(f,g) = L_X^\nu B(f,g) - B(L_X^\lambda f, g) - B(f, L_X^\mu g).$$

Let us apply the generator X_0 of $\mathrm{o}(p+1,q+1)$ to the operator B, one has

$$L_{X_0}^{\lambda,\mu;\nu}(B) = \sum_{r,s,t \geqslant 0} (n\delta - 2(r+s+t)) c_{rst} \mathrm{R}^{r,s,t}$$

where $\delta = \nu - \mu - \lambda$. The equation $L_{X_0}^{\lambda,\mu}(B) = 0$ leads to the homogeneity condition

$$\delta = \frac{2(r+s+t)}{n}.$$

The general expression for B retains the form

$$B_{2k} = \sum_{r+s+t=k} c_{rst} \mathrm{R}^{r,s,t}. \tag{4.1}$$

The operator B is of order $2k$ and $\nu = \lambda + \mu + 2k/n$.

Now, to determine the coefficients c_{rst}, one has to apply the generators $\bar{X}_i$ of the Lie algebra $\mathrm{o}(p+1,q+1)$. A straightforward computation yields

PROPOSITION 4.1. *The action of the generator $\bar{X}_i$ on $\mathcal{D}_{\lambda,\mu;\nu}$ is given by*

$$L_{\bar{X}_i}^{\lambda,\mu;\nu} = L_{\bar{X}_i}^{\delta} - \xi_i T_\xi - \eta_i T_\eta + 2\big((\mathcal{E}_\xi + n\lambda)\,\partial_{\xi^i} + (\mathcal{E}_\eta + n\mu)\,\partial_{\eta^i}\big)$$

where

$$\begin{aligned} L_{\bar{X}_i}^{\delta} =& x_j x^j \partial_i - 2x_i x^j \partial_j - 2n\delta x_i - \\ & - 2\big((\xi_i x_j - \xi_j x_i)\partial_{\xi_j} + (\eta_i x_j - \eta_j x_i)\partial_{\eta_j}\big) + \\ & + 2\big(\xi_j x^j \partial_{\xi^i} + \eta_j x^j \partial_{\eta^i}\big) \end{aligned}$$

is just the natural lift of $\bar{X}_i$ to $T^*\mathbb{R}^n \oplus T^*\mathbb{R}^n$.

Applying $L_{\bar{X}_i}^{\lambda,\mu;\nu}$ to each monomial, $\mathrm{R}^{r,s,t} = \mathrm{R}^r_{\xi\xi}\mathrm{R}^s_{\xi\eta}\mathrm{R}^t_{\eta\eta}$, in the operator B_{2k}, one immediately gets

$$\begin{aligned} L_{\bar{X}_i}^{\lambda,\mu;\nu}(\mathrm{R}^{r,s,t}) =& 2(2k - n\delta)x_i \mathrm{R}^{r,s,t} + \\ & + \big(2r(2r + n(2\lambda - 1))\mathrm{R}^{r-1,s,t} - s(s-1)\mathrm{R}^{r,s-2,t+1} + \\ & + 2s(s + 2t + n\mu - 1)\mathrm{R}^{r,s-1,t}\big)\xi_i + \\ & + \big(2t(2t + n(2\mu - 1))\mathrm{R}^{r,s,t-1} - s(s-1)\mathrm{R}^{r+1,s-2,t} + \\ & + 2s(s + 2r + n\lambda - 1)\mathrm{R}^{r,s-1,t}\big)\eta_i. \end{aligned}$$

Finally, applying $L_{\tilde{X}_i}^{\lambda,\mu;\nu}$ to the operator B_{2k} written in the form (4.1) and collecting the terms, one readily gets the following recurrent system of two linear equations

$$\begin{aligned} &2(r+1)(2(r+1)+n(2\lambda-1))c_{r+1,s,t}- \\ &\quad-(s+2)(s+1)\,c_{r,s+2,t-1}+ \\ &\quad+2(s+1)(s+2t+n\mu)\,c_{r,s+1,t}=0 \end{aligned} \tag{4.2}$$

and

$$\begin{aligned} &2(t+1)(2(t+1)+n(2\mu-1))c_{r,s,t+1}- \\ &\quad-(s+2)(s+1)\,c_{r-1,s+2,t}+ \\ &\quad+2(s+1)(s+2r+n\lambda)\,c_{r,s+1,t}=0 \end{aligned} \tag{4.3}$$

for the coefficients. The system (4.2), (4.3) is a necessary and sufficient condition of $o(p+1,q+1)$-invariance for the operator (4.1).

4.3 EXPLICIT SOLUTION OF THE SYSTEM

Let us now show that the system (4.2), (4.3) is compatible and, moreover, give its explicit solution.

Equations (4.2) with $t=0$ readily give

$$c_{r,s,0}=\frac{(-1)^r}{2^r}\binom{r+s}{r}\frac{(s+n\mu)_r}{(n(\lambda-\frac{1}{2})+1)}c_{0,r+s,0},$$

where we use the standard notation $(a)_r := a(a+1)\cdots(a+r-1)$ for the Pochhammer symbol. Similarly, from Equations (4.3) with $r=0$, one obtains

$$c_{0,s,t}=\frac{(-1)^t}{2^t}\binom{s+t}{t}\frac{(s+n\lambda)_t}{(n(\mu-\frac{1}{2})+1)_t}c_{0,s+t,0}.$$

We now substitute the above expressions for $c_{0,s,t}$ to (4.2) to obtain the coefficients $c_{r,s,t}$ with $r\leqslant t$. The answer

$$\begin{aligned} c_{r,s,t}=&\frac{(-1)^{t-r}}{2^r r!}\binom{r+s+t}{t}\frac{(s+1)_r}{(n(\lambda-\frac{1}{2})+1)_r}\times \\ &\times\sum_{p=0}^{r}\frac{r!t!}{p!}\frac{(r+s-p+n\lambda)_{t-p}(s+2t+n\mu)_{r-p}}{(n(\mu-\frac{1}{2})+1)_{t-p}}\times \\ &\times c_{0,r+s+t,0} \end{aligned} \tag{4.4}$$

can be easily get by induction. Similarly, substituting the expressions for $c_{r,s}$ to (4.3), one obtains the coefficients $c_{r,s,t}$ with $r\geqslant t$. The explicit formula, in this case, coincides with (4.4) after the simultaneous exchange $(r\leftrightarrow t)$ and $(\lambda\leftrightarrow\mu)$.

Finally, to prove the compatibility of the system (4.2), (4.3), one checks by a straightforward computation, that the coefficients (4.4) satisfy the equations (4.3).

We proved that the system (4.2), (4.3) has a unique solution, provided neither of λ and μ belongs to the set of critical values (2.6). Theorem 2.4 is proved.

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