

Generalized two-fluid theory of nonlinear magnetic structures

T. J. Schep

FOM Institute for Plasmaphysics, Rijnhuizen, Nieuwegein, The Netherlands

F. Pegoraro

Department of Theoretical Physics, Turin University, 10125 Turin, Italy

B. N. Kuvshinov

FOM Institute for Plasmaphysics, Rijnhuizen, Nieuwegein, The Netherlands

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A system of equations is introduced and discussed that describe the nonlinear dynamics of magnetic perturbations in a magnetized, high-temperature plasma. Diamagnetism, ion gyroradii effects, and finite electron mass are taken into account. These equations govern Alfvén as well as electrostatic waves and vortices and describe the nonlinear evolution of reconnecting modes. Electrons are treated in a fluid model. The equation for the ion response is new and is a nonlinear generalization to all orders in the thermal ion gyroradius of the nonlinear fluid model. This system of equations conserves two fluxes that are different from, but related to, the magnetic flux. Two-dimensional equilibrium solutions in the form of stationary propagating magnetic structures are obtained with the methods introduced in the theory of vector nonlinearities in electrostatic drift vortices. In the noncollisional regimes of interest the inertia of the electrons resolves the singularity in the current density that tends to develop at magnetic separatrices. The positions of the X points of the conserved fluxes are mirror symmetric and at a distance of the order of the electron skin depth from the resonant surface. The set of equations admits an energy integral and can be cast in noncanonical Hamiltonian form. The role of the Casimir invariants, that are functions of the conserved fluxes, is investigated and the connection with “reduced magnetohydrodynamics” is emphasized.

I. INTRODUCTION

Small-scale processes determine many aspects of the global behavior of magnetically confined toroidal plasmas. A well-known example is the so-called “internal disruption,” where reconnection of magnetic field lines in a narrow layer near the rational magnetic surface results in a global redistribution of the plasma density and temperature profiles. Another example is anomalous transport. According to current ideas the formation of small-scale coherent nonlinear structures, like magnetic islands, vortices, etc., may play an important role in the physics of enhanced heat and particle flows in a plasma.

Coherent nonlinear magnetic structures, such as magnetic islands and current sheets, have been studied in the zero frequency limit.¹ In this case the current density is constant on magnetic surfaces. In high-temperature plasmas, however, resistive modes, that lead to reconnection of magnetic field lines, are found to propagate with a finite velocity^{2,3} when diamagnetic and/or finite gyroradius effects are accounted for.

In this paper we present a set of equations that governs the linear and nonlinear evolution of plasma phenomena with frequencies below the ion cyclotron and the magnetosonic and above the ion-acoustic frequency. Diamagnetism, finite electron mass and full ion gyroradius effects are taken into account. The spatial scales of the phenomena may range from magnetohydrodynamic (MHD) scales down to the inertia electron skin depth. In a high-temperature plasma, this skin depth is smaller than the gyroradius of a thermal ion. Our set of equations is based on a fluid description of electrons and on a hybrid model for the ions. The equation that

describes the ion response is a generalization of both the nonlinear fluid ion response and of the linear response that is valid for all values of the thermal ion gyroradius.² The nonlinearities in the equations arise from the $E \times B$ advection and from the gradients of the electron pressure and current density along the total magnetic field.

In part of this paper we focus the discussion on two-dimensional (2-D) magnetic structures and extend the existing treatments^{4–8} to include finite electron mass and full ion gyroradius effects.⁹ These structures are characterized by a current density distribution that is not constant on magnetic surfaces and tend to develop large currents at magnetic separatrices. Electron mass effects arise from the electron inertia term and from a finite gyroradius contribution to the electron stress tensor.⁹ In the highly noncollisional regimes of interest finite electron mass effects resolve the current singularity at separatrices. Current density gradients, however, can still be large. The effect of electron inertia on reconnection and on linear reconnecting modes was examined in Refs. 10 and 11.

For stationary propagating modes and for arbitrary values of the ratio of the thermal ion gyroradius to the characteristic scale length of the structures, the electron equations can be integrated once following the methods introduced in the theory of vector nonlinearities in electrostatic drift vortices.¹² This leads to an equilibrium equation for the magnetic flux function that is a nonlinear eigenvalue equation for the propagation velocity. Although solutions to this equation are not known, a number of general properties, like the position of the X points, are derived.

The set of equations admits an energy integral that plays the role of the Hamiltonian functional, and the equations can

be cast in Hamiltonian form in terms of noncanonical Poisson brackets.^{13,14} The integral invariants (the conservation laws) determine which states are attainable by a system. One class of integral invariants consists of the Hamiltonian functional and of the functionals that are related to its symmetry properties. Another class, the Casimir invariants, arises from the algebraic properties of the Poisson brackets. The knowledge of a full set of integral invariants provides a general description of "equilibrium" solutions and gives the tools to investigate their nonlinear stability.¹⁵ It will be shown that in the three-dimensional (3-D) case, two electron and one ion Casimir functional exist. In the cold ion limit the ion Casimir is replaced by an infinite set. Similarly, if the problem is restricted to 2-D perturbations that are aligned with the background magnetic field, the two electron Casimirs are replaced by two infinite sets.

This paper is organized as follows. In Secs. II and III we introduce a set of three coupled nonlinear equations that are first order in time. In Sec. IV the linear limit of this set is briefly discussed since it serves as a boundary condition for the stationary propagating solutions discussed in Secs. V and VI. In Sec. V stationary propagating solutions are investigated in the limits of zero electron mass and the structure of the resulting singularity of the current density at the magnetic separatrices is analyzed in the limit of large and of small ion gyroradii. Electron inertia effects are reintroduced in Sec. VI. The nonlinear system of equations is integrated once in terms of two arbitrary functions. These depend on two linear combinations, denoted by f_{\pm} , of the generalized magnetic flux Ψ_e , which is the fluid analog of the generalized electron momentum, and of the logarithm of the electron density $\ln n$. The profile of the magnetic shear field and the linearized boundary conditions fix the two arbitrary functions in the regions outside the separatrices of f_{\pm} . A nonlinear differential equation is then obtained for the flux function Ψ . A number of properties that characterize the spatial structure of the solutions of this equation near the separatrices are discussed in different frequency intervals and the relationship between linear and nonlinear solutions is clarified. In particular the position of the X points of f_{\pm} is analyzed. In Sec. VII the equations are cast in a Hamiltonian functional form. It is shown that the two infinite sets of electron Casimirs are functionals of f_{\pm} , that are the conserved quantities of our dynamical system. Thus neither Ψ nor Ψ_e is conserved, and X points of f_{\pm} do not coincide with X points of Ψ or Ψ_e . The generalized flux Ψ_e is conserved only in the limit of zero electron temperature. We show that these conclusions hold irrespective of the ion response. Finally in Sec. VIII the conclusions are drawn and the validity of the fluid electron model adopted is discussed.

II. ELECTRON EQUATIONS

In this and in the following section we derive a set of equations that describe the time evolution of nonlinear electric and magnetic structures, such as vortices, islands, and current layers. Important characteristics of these equations are that they include the effects played by electron inertia in

limiting the electron current in regimes where the plasma collisionality is low and that they are valid to all orders in the thermal ion gyroradius.

We start from the electron momentum balance and the continuity equation¹⁶

$$m_e n \left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = -en \left(\mathbf{E} + \frac{1}{c} \mathbf{v} \times \mathbf{B} \right) - \nabla n T - \nabla \cdot \mathbf{\Pi} - m_e n \nu_e \mathbf{v} \quad (1)$$

and

$$\frac{\partial n}{\partial t} + \nabla \cdot n \mathbf{v} = 0, \quad (2)$$

where all symbols have their usual meaning; ν_e is the electron-ion collision frequency.

We will neglect magnetic curvature effects and approximate the geometry of a low- β toroidal configuration by a plane slab which is periodic in (y, z) and inhomogeneous along the x direction. The magnetic field is

$$\mathbf{B} = B_0 (\mathbf{e}_x + \mathbf{e}_z \times \nabla \Psi), \quad (3)$$

where B_0 is the constant field at the surface $x=0$. The flux Ψ corresponds to the helical flux function in a torus and is the sum of the shear flux $\Psi_0(x)$ of the background field and a fluctuating part $\tilde{\Psi}(\mathbf{x}, t)$. The electric field is

$$\mathbf{E} = -\nabla \phi + \frac{B_0}{c} \frac{\partial \Psi}{\partial t} \mathbf{e}_z. \quad (4)$$

Neglecting perpendicular inertia and resistivity, one obtains from (1) the velocity in the (x, y) plane,

$$\mathbf{v}_{\perp} = \frac{c}{B_0} \mathbf{e}_z \times \nabla \phi - \frac{c}{eB_0} \mathbf{e}_z \times \frac{\nabla n T}{n} + \nu_e \mathbf{e}_z \times \nabla \Psi. \quad (5)$$

The last term is supposed to be small with respect to the $\mathbf{E} \times \mathbf{B}$ drift.

The contribution from the stress tensor $\mathbf{\Pi}$ to the parallel momentum balance $(\nabla \cdot \mathbf{\Pi})_z = -(cm_e/eB_0) \nabla n T \times \mathbf{e}_z \cdot \nabla v_z$ cancels the pressure gradient contribution to $\mathbf{v}_{\perp} \cdot \nabla v_z$ in the inertia term.

The density can be written as $n(\mathbf{x}, t) = n_0(x) [1 + \tilde{n}(\mathbf{x}, t)]$, where $n_0(x)$ is the density of the background plasma and \tilde{n} represents the density fluctuations. Although \tilde{n} remains small, we take $\nabla \tilde{n} \sim \nabla \ln n_0(x)/n_0$. Further, we assume that the parallel ion velocity is much smaller than the electron velocity v_z . Thus, using Ampère's law, the velocity is related to the flux function according to $v_z \approx -J_z/en_0 \approx -(cB_0/4\pi en_0) \nabla_{\perp}^2 \Psi$, where $n_0 = n_0(0)$. In the low-collisionality regime under consideration, the electron fluid behaves isothermally along magnetic field lines. For simplicity we will take the electron temperature to be constant throughout the fluid.

After substitution of the perpendicular velocity (5), the parallel component of the momentum balance (1) and the continuity equation (2) become

$$\begin{aligned} \frac{\partial \Psi_e}{\partial t} + a[\Phi, \Psi_e] + a\left[\Psi, \ln \frac{n}{n_0}\right] \\ = a \frac{\partial \Phi}{\partial z} - a \frac{\partial}{\partial z} \ln \frac{n}{n_0} + \frac{\eta c^2}{4\pi} \nabla_{\perp}^2 \tilde{\Psi} \end{aligned} \quad (6)$$

and

$$\frac{\partial}{\partial t} \ln \frac{n}{n_0} + a\left[\Phi, \ln \frac{n}{n_0}\right] - \frac{a}{\beta_e} [\Psi, J] = \frac{a}{\beta_e} \frac{\partial J}{\partial z}, \quad (7)$$

respectively, where

$$\Psi_e = \Psi + v_z/\Omega_e = \Psi - d_e^2 \nabla_{\perp}^2 \Psi, \quad (8)$$

is the generalized flux function, $d_e = c/\omega_{pe}$ being the electron inertial skin depth, the brackets denote

$$[f, g] = \mathbf{e}_z \cdot \nabla f \times \nabla g \quad (9)$$

and $\Phi = e\phi/T$, $J = \nabla_{\perp}^2 \Psi$, $a = cT/(eB_0)$, and $\beta_e = 4\pi n_0 T/B_0^2$. We have included resistivity in Ohm's law (6). As will turn out, current gradients will be large so that electron viscosity should also be taken into account. However, we will consider ideal time scales on which dissipative processes do not play a role.

In part of this paper we will consider solutions that propagate with velocity u in the y direction at an angle with respect to the z axis determined by α , so that all quantities depend only on $\tau = t, x$ and $\lambda = y - ut + \alpha z$.

Stationary propagating modes, $\partial/\partial \tau = 0$, are solutions of

$$\left[\Phi - \frac{u}{a} x, \Psi_e + \alpha x\right] + \left[\Psi + \alpha x, \ln \frac{n}{n_0}\right] = 0 \quad (10)$$

and

$$\left[\Phi - \frac{u}{a} x, \ln \frac{n}{n_0}\right] - \frac{1}{\beta_e} [\Psi + \alpha x, \nabla_{\perp}^2 \Psi] = 0. \quad (11)$$

The electron density n is related to the ion density n_i through the quasineutrality condition.

III. ION RESPONSE

We assume that the nonlinear ion response to modes with parallel phase velocities above the ion thermal velocity is governed by

$$\begin{aligned} \frac{\partial}{\partial t} \left(\ln \frac{n}{n_0} + \tau_i (1 - \Gamma_0) \Phi \right) + a \left[\Phi, \ln \frac{n}{n_0} + \tau_i (1 - \Gamma_0) \Phi \right] \\ + a L_j \left[L_j \Phi, \ln \frac{n}{n_0} \right] = 0, \end{aligned} \quad (12)$$

where $\tau_i = T/T_i$, and Γ_0 is the integral operator $\Gamma_0 = \exp(\rho_i^2 \nabla_{\perp}^2) I_0(-\rho_i^2 \nabla_{\perp}^2)$, ρ_i being the thermal ion gyroradius and I_0 the Bessel function of the first kind. The operator L is defined as

$$L = \mathbf{e}_j L_j = \mathbf{e}_x L_x + \mathbf{e}_y L_y = \rho_i \nabla_{\perp} \left(\sum_{k=0}^{\infty} a_k \rho_i^{2k} \nabla_{\perp}^{2k} \right), \quad (13)$$

where the coefficients a_k have to be such that $L^2 = \Gamma_0 - 1$. In addition we require that $L_j \rightarrow i/2^{1/2}$ for $\rho_i \rightarrow \infty$. Summation over the index j in the (x, y) plane is assumed. The operators

in $1 - \Gamma_0$ and L act on fluctuating quantities and can be interpreted in terms of a Padé approximation² $\Gamma_0 \rightarrow 1/(1 - \rho_i^2 \nabla_{\perp}^2)$, $L \rightarrow \rho_i \nabla_{\perp} / (1 - \rho_i^2 \nabla_{\perp}^2)^{1/2}$.

The shear field coupling to the parallel ion motion has been neglected. We take Eq. (12) to be valid in a homogeneous magnetic field and in the limit of constant ion temperature. We expect that this model can be extended to finite ion temperature gradients as long as the ratio of the temperature and the density scale lengths is below a threshold value.¹⁷ The nonlinear equation (12) is defined for all values of the ion gyroradius. It is an ansatz and it is not rigorously derived from the Vlasov equation. However, it contains the well-known linear response and it leads to the correct nonlinear ion equations in the limits of small and large ion gyroradii. Note that the operator structure in Eq. (12) is analogous to the one in nonlinear gyrokinetic theory.¹⁸ In addition it is such that the system consisting of Eq. (12) and the electron equations (6) and (7) leads to an energy integral and as will be demonstrated in Sec. VII, that it can be written in Hamiltonian form.

Linearizing Eq. (12) and neglecting higher-order derivatives of the background density, it is easily seen that the linear Vlasov ion response is recovered. In the large- ρ_i limit, where $\Gamma_0 \rightarrow 0$ and $L_j \rightarrow i/2^{1/2}$, Eq. (12) yields the nonlinear Boltzmann response

$$\ln \frac{n}{n_0(x)} = -\tau_i \Phi. \quad (14)$$

In the small- ρ_i limit, Eq. (12) can be written in the form

$$\begin{aligned} \frac{\partial}{\partial t} \left(\ln \frac{n}{n_0} - \tau_i \rho_i^2 \nabla_{\perp}^2 \Phi \right) + a(1 + \rho_i^2 \nabla_{\perp}^2) \left[\Phi, \ln \frac{n}{n_0} \right] \\ - a \rho_i^2 \nabla_{\perp} \cdot \left[\Phi, \nabla_{\perp} \left(\ln \frac{n}{n_0} + \tau_i \Phi \right) \right] = 0. \end{aligned} \quad (15)$$

By multiplying with $1 - \rho_i^2 \nabla_{\perp}^2$ and by neglecting terms of order ρ_i^4 , Eq. (15) becomes

$$\frac{\partial}{\partial t} \left(\ln \frac{n}{n_0} - \rho_i^2 \nabla_{\perp}^2 h \right) + a \left[\Phi, \ln \frac{n}{n_0} \right] - a \rho_i^2 \nabla_{\perp} \cdot [\Phi, \nabla_{\perp} h] = 0, \quad (16)$$

where $h = \ln n/n_0 + \tau_i \Phi$. This expression is identical to the one that is obtained from fluid equations when the collisionless gyroviscosity contributions to the stress tensor are taken into account.¹⁶

For stationary propagating modes with velocity u in the y direction, Eq. (12) can be written as

$$\left[\Phi - \frac{u}{a} x, \ln \frac{n}{n_0} + \tau_i (1 - \Gamma_0) \Phi \right] + L_j \left[L_j \Phi, \ln \frac{n}{n_0} \right] = 0. \quad (17)$$

For small values of ρ_i , this equation is to leading order $[\Phi - ux/a, \ln n/n_0] = 0$, which has the solution $\ln n/n_0 = F(\Phi - ux/a)$ with arbitrary function F . It follows that to first order in ρ_i^2 , Eq. (17) can be written as

$$\left[\Phi - \frac{u}{a} x + \rho_i^2 \nabla_{\perp}^2 \Phi, \ln \frac{n}{n_0} - \tau_i \rho_i^2 \nabla_{\perp}^2 \Phi \right] = 0. \quad (18)$$

The set of our basic equations consists of the electron equations (6) and (7) and the ion equation (12). They describe the linear and nonlinear behavior of phenomena with frequencies below the ion cyclotron frequency and the magnetosonic frequency. Their spatial scales may be large or small as compared to the ion gyroradius and may range from MHD lengths to the electron inertia skin depth.

By multiplying Eq. (6) by $J = \nabla_{\perp}^2 \Psi$, Eq. (7) by $\beta_e \ln(n/n_0)$ and Eq. (12) by Φ , we obtain the energy integral

$$H = \frac{1}{2} \int d^3x \left[|\nabla_{\perp} \Psi|^2 + d_e^2 J^2 + \beta_e \ln^2 \frac{n}{n_0} + \beta_e \tau_i \Phi (1 - \Gamma_0) \Phi \right]. \quad (19)$$

The contributions to H represent the magnetic energy, the kinetic energy of the electron motion along field lines, the electron potential energy, and the ion energy, respectively. For nonlocalized modes that are periodic in λ we find

$$\frac{\partial H}{\partial t} = \left\langle \frac{1}{2} \Delta' \frac{\partial \Psi^2}{\partial t} \right\rangle, \quad (20)$$

where the angular brackets denote an average over λ at large values of x , and $\Delta' \equiv (2/\Psi)(\partial \Psi / \partial x)$ is the well-known logarithmic derivative at the boundary of the domain of integration. Here it is assumed that the perturbed current density, the particle density, and the potential vanish at the boundary faster than the perturbed flux function Ψ .

In the next three sections we will focus on stationary propagating modes. These modes are described by the electron equations (10) and (11) and by the ion equation (17). There, we will consider a background plasma with an exponential density profile and a linear shear field:

$$\ln \frac{n_0(x)}{n_0} = -\frac{x}{l_n} \quad \text{and} \quad \Psi_0 = -\frac{x^2}{2l_s}, \quad (21)$$

where l_n and l_s are the scale lengths of the density and of the shear field, respectively. As is well known, such a magnetic geometry is topologically unstable against perturbations Ψ that are even in x .

As far as the wave-number ratio α is concerned, there exist two limiting cases. For localized structures with radial scale length l_r such that $\alpha \gg l_r / (2l_s)$, the effect of the shear field is negligible and our set of equations describes Alfvén-type vortices. In the literature^{19,20} these vortices are treated in the small- ρ_i (MHD) limit. Our general ion response (12) will allow this theory to be extended to arbitrary spatial scale lengths with respect to the ion gyroradius. In this paper we will emphasize the opposite limit $\alpha \ll l_r / (2l_s)$, where magnetic shear is dominant. In this limit our equations describe the nonlinear evolution of reconnecting modes.

IV. LINEAR PERTURBATIONS

Linear perturbations of a background plasma with profiles given by Eq. (21) are of the form $\Phi(x, y) = \Phi(x) \exp(ik_y y - \omega t)$. The linearized forms of Eqs. (6), (7), and (12) are

$$\frac{l_n u_*}{l_s} \frac{u_*}{u} x (\ln \bar{n} - \tilde{\Phi}) + \left(1 - \frac{u_*}{u} \right) \tilde{\Psi} - \left(d_e^2 - \frac{\eta c^2}{4\pi i k_y u} \right) \nabla_{\perp}^2 \tilde{\Psi} = 0, \quad (22)$$

$$\ln \bar{n} - \frac{u_*}{u} \tilde{\Phi} = \frac{l_n u_*}{l_s \beta_e} \frac{u_*}{u} x \nabla_{\perp}^2 \tilde{\Psi}, \quad (23)$$

and

$$\ln \bar{n} - \frac{u_*}{u} \Gamma_0 \tilde{\Phi} = -\tau_i (1 - \Gamma_0) \tilde{\Phi}, \quad (24)$$

respectively, where $\omega \equiv k_y u$. Note that $u/u_* \equiv \omega/\omega_*$, where $\omega_* = k_y u_*$ is the electron drift frequency. The linear expressions (22)–(24) will be used in the next sections as boundary conditions on nonlinear solutions. Introducing the function $\tilde{\Theta} = (l_n/l_s)(\tau_i \tilde{\Phi} + \ln \bar{n})$ and using the Padé approximation to Γ_0 , we can write Eqs. (22)–(24) in the form

$$\rho_i^2 \nabla_{\perp}^2 \tilde{\Theta} = \frac{1}{\hat{\beta}} \frac{u_*}{u} x \nabla_{\perp}^2 \tilde{\Psi} \quad (25)$$

and

$$a_e x \tilde{\Theta} = a_i a_e \tilde{\Psi} + \left[\frac{1 + \tau_i}{\hat{\beta}} x^2 - \frac{u}{u_*} a_i \left(d_e^2 - \frac{\eta c^2}{4\pi i k_y u} \right) \right] \nabla_{\perp}^2 \tilde{\Psi}, \quad (26)$$

where $a_e = u/u_* - 1$, $a_i = 1 + \tau_i u/u_*$, and $\hat{\beta} = (l_s^2/l_n^2)\beta_e$. According to Eq. (14), the function $\tilde{\Theta}$ vanishes in the large- ρ_i limit so that Eqs. (25) and (26) become decoupled. For collisionless modes with real frequency such that $(u/u_*)a_i \equiv (\omega/\omega_*)(\tau_i \omega/\omega_* + 1) > 0$, the coefficient of the current density in (26) vanishes at

$$x_s^2 = \frac{u}{u_*} \frac{a_i}{1 + \tau_i} d_s^2, \quad (27)$$

where

$$d_s = \hat{\beta}^{1/2} d_e. \quad (28)$$

This corresponds to an infinite effective potential when Eq. (26) is written in Schrödinger form. This means that a regular perturbation must vanish at x_s . Hence, finite electron mass effects tend to shield the resonant surface from the perturbations. It will be shown that this property pertains in the nonlinear case.

In the absence of electron mass effects, Eqs. (25) and (26) are the equations in coordinate form corresponding to the equations discussed in Ref. 2. Several limiting cases of Eqs. (25) and (26) are treated in Ref. 10. In Ref. 21 it is shown that an MHD model which includes the Hall effect is equivalent to the approximate limit of a two-fluid model in which the Vlasov ion response is taken in the Padé approximation.

V. PROPAGATING NONLINEAR SOLUTIONS IN THE LIMIT OF ZERO ELECTRON INERTIA

In this and in the following section we will investigate the properties of stationary solutions that propagate stationary with velocity u in the y direction. The modes we consider are nonlocalized in the sense that at large values of x

they connect to the linear solutions discussed in the previous section. We start by neglecting electron inertia.

The limit of zero electron mass corresponds to the case where d_e is small compared to the other relevant scale length. In this limit Ψ_e and Ψ coincide. Then, the momentum balance (10) and the continuity equation (11) become

$$\left[\Phi - \frac{u}{a} x - \ln \frac{n}{n_0}, \Psi \right] = 0 \quad (29)$$

and

$$\left[\Phi - \frac{u}{a} x, \ln \frac{n}{n_0} \right] - \frac{1}{\beta_e} [\Psi, \nabla_{\perp}^2 \Psi] = 0, \quad (30)$$

respectively. These equations can be solved following the method of Ref. 12

$$\Phi - ux/a - \ln n/n_0 = F(\Psi), \quad (31)$$

where F is an arbitrary flux function. Inserting expression (31) into the continuity equation (30), we obtain

$$[\Psi, (\beta_e F' \ln n/n_0 - \nabla_{\perp}^2 \Psi)] = 0, \quad (32)$$

with solution

$$\nabla_{\perp}^2 \Psi = \beta_e F' \ln n/n_0 + H(\Psi), \quad (33)$$

where a prime denotes the derivative with respect to the argument. The flux functions F and H are determined^{4,6} by imposing that far from the rational surface at $x=0$, the expressions (31) and (33) reproduce the linearized equations (22) and (23). These boundary conditions yield

$$F(\Psi) = -\frac{a_e(2l_s)^{1/2}}{l_n} \sigma(\Psi_s - \Psi)^{1/2}, \quad H(\Psi) = \frac{a_e \hat{\beta}}{l_s}, \quad (34)$$

where Ψ_s is the reconnected flux and $\sigma = \text{sgn } x$. Inserting these expressions into the nonlinear solutions (31) and (33), we obtain

$$\Phi - \frac{a_e}{l_n} x - \ln \frac{n}{n_0(x)} = -\left(\frac{a_e}{l_n}\right) (2l_s)^{1/2} \sigma(\Psi_s - \Psi)^{1/2} \quad (35)$$

and

$$\bar{j} \equiv \nabla_{\perp}^2 \Psi = \frac{a_e \hat{\beta}}{l_s} \left(\frac{l_n}{(2l_s)^{1/2}} \sigma \frac{\ln n/n_0}{(\Psi_s - \Psi)^{1/2}} + 1 \right). \quad (36)$$

The boundary conditions (22) and (23) and the expressions (34) for F and H do not apply in the zero-frequency MHD limit which corresponds to $u, 1/l_n \rightarrow 0$. The density response in this limit arises from the ion polarization drift. If this ion inertia effect is neglected, (31) and (33) give that Φ and J are flux functions.

Equations (35) and (36) hold irrespective of the form of the density response. Because of quasineutrality, the density is determined by the ion response which depends on the ratio of the ion gyroradius to the characteristic radial scale of the propagating structures. This scale may be either large or small as compared to the thermal ion gyroradius ρ_i , as sketched in Fig. 1.

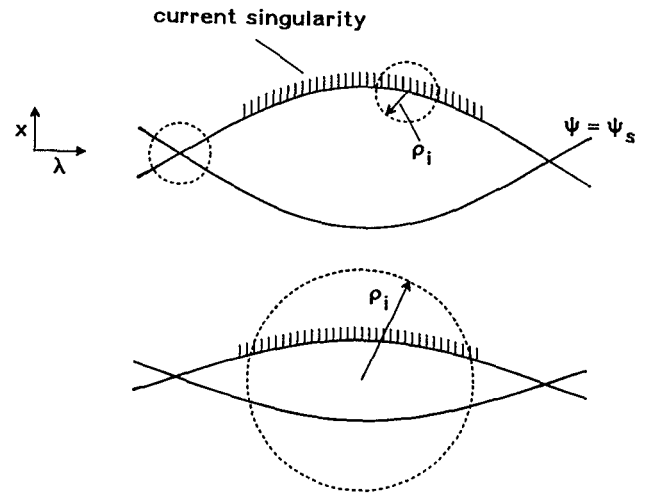


FIG. 1. Illustration of the relative sizes of an ion gyro-orbit, of a standard island, and of the singular current layer (shaded area).

In this section we consider the small- ρ_i limit. The ion response as obtained from (18) for an exponential density profile is,

$$\ln \frac{n}{n_0(x)} = \frac{u_*}{u} \Phi + \frac{u_*}{u} a_i \rho_i^2 \nabla_{\perp}^2 \Phi. \quad (37)$$

Upon combining this expression with (31) one obtains

$$\ln \frac{n}{n_0(x)} = \frac{x}{l_n} + \frac{F}{a_e} + \frac{u}{u_*} \frac{a_i}{a_e} \rho_i^2 \nabla_{\perp}^2 F. \quad (38)$$

Since in the small- ρ_i limit the last term is small, the density is almost a flux function with the exponential form

$$\frac{n(x, \lambda)}{n_0} \approx \exp\left(\frac{F}{a_e}\right) = \exp\left(\frac{-\sigma(2l_s)^{1/2}(\Psi_s - \Psi)^{1/2}}{l_n}\right). \quad (39)$$

This suggests that the excitation of ion-acoustic waves that would flatten the density gradient along field lines will play a minor role.²² Substituting this expression into (33) and using the expression (34) for F and H , leads to Ampère's law in the form

$$\begin{aligned} \nabla_{\perp}^2 \Psi &= \frac{\beta_e}{a_e} F' \left(F + \frac{u}{u_*} \frac{a_i}{a_e} \rho_i^2 \nabla_{\perp}^2 F \right) + H \\ &= -\frac{\hat{\beta}}{l_s} \frac{u}{u_*} a_i \rho_i^2 \frac{\nabla_{\perp}^2 (\Psi_s - \Psi)^{1/2}}{(\Psi_s - \Psi)^{1/2}}. \end{aligned} \quad (40)$$

We see that the current density is not a flux function. This is due to the fact that the small term in expression (38) for the density is dominant in Eq. (40).

Near the singular layer, the mode equation (40) can be solved in the limit $\partial/\partial x \gg \partial/\partial \lambda$. Recalling that $\Psi = -x^2/2l_s + \tilde{\Psi}$, Eq. (40) can now be brought to the energy form

$$\left(\frac{\partial f}{\partial \bar{x}}\right)^2 = \frac{g(\lambda) + f^2}{f^2 - \sigma_A}, \quad (41)$$

where $f = |A|^{-1/2}(\Psi_s - \Psi)^{1/2}$ is positive, $\bar{x} = (2l_s|A|)^{-1/2}x$, $A = (\hat{\beta}/2l_s)(u/u_*)a_i\rho_i^2$, $\sigma_A = \text{sgn } A$ and $g(\lambda)$ is an integration constant.

It is seen that a singularity occurs at $f=1$ for $\sigma_A=1$, i.e., for velocities such that $(u/u_*)(\tau_i u/u_* + 1) > 0$. The solution of Eq. (41) near the singularity $x=x_0(\lambda)$ is

$$\Psi_s - \Psi \equiv |A|f^2 \approx |A| + (6l_s|A|)^{2/3}(1+g)^{1/3} \times [x-x_0(\lambda)]^{2/3} \quad (x > x_0). \quad (42)$$

It follows that the current density, the magnetic field, and the density are singular.

Assume that inside the curve $x=x_0(\lambda)$ the flux Ψ is unperturbed, so that $\Psi_s=0$ and $\Psi=-x^2/2l_s$ for $x < x_0(\lambda)$. Then from Eq. (42) we obtain $x_0=(2l_s|A|)^{1/2}=(a_i u/u_*)^{1/2} \hat{\beta}^{1/2} \rho_i$ which is independent of λ . In the velocity interval $(u/u_*)(\tau_i u/u_* + 1) < 0$, no singularity occurs so that current layers cannot be formed and resistive instabilities do not exist. Near the singularity the small- ρ_i limit breaks down since the scale length x_0 is of the order ρ_i . In this region the large- ρ_i limit to the ion response is more appropriate. As will be pointed out in the next section, in the latter limit the current density is found to be singular along magnetic separatrices where $\Psi=\Psi_s$. This singularity is removed by the inertia of the electrons.

VI. ELECTRON INERTIA

When electron inertia is taken into account, stationary propagating modes are described by Eqs. (10) and (11). Taking $\alpha=0$, we first rewrite these equations in the following form:

$$\left[\Phi - \frac{u}{a} x - \ln \frac{n}{n_0}, \Psi_e \right] - d_e^2 \left[\ln \frac{n}{n_0}, \nabla_{\perp}^2 \Psi \right] = 0 \quad (43)$$

and

$$\left[\Phi - \frac{u}{a} x - \ln \frac{n}{n_0}, \ln \frac{n}{n_0} \right] - \frac{1}{\beta_e} [\Psi_e, \nabla_{\perp}^2 \Psi] = 0. \quad (44)$$

Upon multiplying (44) by $\beta_e^{1/2} d_e$, one obtains after adding and subtracting Eqs. (43) and (44)

$$\left[\Phi - \frac{u}{a} x - \ln \frac{n}{n_0} \pm \frac{d_e}{\beta_e^{1/2}} \nabla_{\perp}^2 \Psi, \Psi_e \pm \beta_e^{1/2} d_e \ln \frac{n}{n_0} \right] = 0. \quad (45)$$

The general solution is

$$\Psi_e \pm \beta_e^{1/2} d_e \ln \frac{n}{n_0} = F_{\pm} \left(\Phi - \frac{u}{a} x - \ln \frac{n}{n_0} \pm \frac{d_e}{\beta_e^{1/2}} \nabla_{\perp}^2 \Psi \right), \quad (46)$$

where the functions F_{\pm} are determined by the background profiles and the boundary conditions.

In the case of a background plasma with density profile and shear flux given by Eq. (21), the solutions that satisfy at large $|x|$ the linearized expressions (22) and (23) are

$$f_{\pm} = g_{\pm}^2, \quad (47)$$

where

$$f_{\pm} = 2l_s (\Psi_s - \Psi + d_e^2 \nabla_{\perp}^2 \tilde{\Psi}) + d_s^2 \pm 2d_s l_n \left(\frac{x}{l_n} - \ln \frac{n}{n_0(x)} \right) \quad (48)$$

and

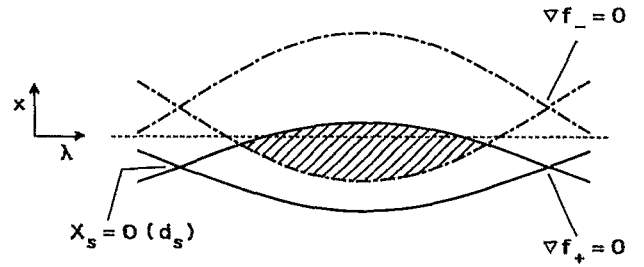


FIG. 2. Geometry of the f_{\pm} separatrices and X points. The quadratic solution for $f_{+}(f_{-})$ is valid outside the solid (dotted) lines. None of these solutions is valid in the shaded area.

$$g_{\pm} = -\frac{d_s}{a_e \hat{\beta}} (\nabla_{\perp}^2 \tilde{\Psi} l_s - a_e \hat{\beta}) \pm \frac{-l_n}{a_e} \left(\Phi - \ln \frac{n}{n_0(x)} - a_e \frac{x}{l_n} \right). \quad (49)$$

These expressions contain two length scales related with the electron mass: the inertial skin depth d_e and the inertial scale length d_s given by Eq. (28).

In the absence of perturbations $f_{\pm}=(x \pm d_s)^2$, and $g_{\pm}=d_s \pm x$ are the background values of f_{\pm} and g_{\pm} .

The functions g_{\pm} and f_{\pm} obey the "mirror" symmetry $g_{+}(x, \lambda) = g_{-}(-x, \lambda)$ and $f_{+}(x, \lambda) = f_{-}(-x, \lambda)$, which ensures that Ψ , Ψ_e , and $J = \nabla_{\perp}^2 \Psi$ are even functions of x , while Φ and $\ln n/n_0(x)$ are odd. The solutions (47) are valid outside regions where $f_{\pm} = \text{const}$ lines are confined to a finite part of the (x, λ) plane. The critical points of f_{\pm} are given by $\nabla f_{\pm} = 0$. These points are mirror symmetric for f_{+} and for f_{-} and, thus, do not coincide with the critical points of even functions like Ψ or Ψ_e . The geometry of f_{\pm} separatrices and X points are sketched in Fig. 2. Inside island structures the relationship (47) is not necessarily valid and different relationships $f_{\pm} = F(g_{\pm})$ may be assigned that have to satisfy certain matching conditions across separatrices.

Upon adding and subtracting Eqs. (47), it follows that

$$\Phi - \ln \frac{n}{n_0(x)} - a_e \frac{x}{l_n} = -\frac{a_e}{l_n} \sigma G \quad (50)$$

and

$$\ln \frac{n}{n_0(x)} - \frac{x}{l_n} = -\frac{1}{a_e^2 \hat{\beta}} \left(\Phi - \ln \frac{n}{n_0(x)} - a_e \frac{x}{l_n} \right) \times (l_s \nabla_{\perp}^2 \tilde{\Psi} - a_e \hat{\beta}), \quad (51)$$

where

$$G = (2l_s)^{1/2} \left[\Psi_s - \Psi + \frac{d_e^2}{a_e} \left(\frac{u}{u_*} - \frac{l_s}{2a_e \hat{\beta}} \nabla_{\perp}^2 \tilde{\Psi} \right) \nabla_{\perp}^2 \tilde{\Psi} \right]^{1/2}. \quad (52)$$

In the above derivation it comes out natural that Ψ and Ψ_s occur in the combination $\Psi - \Psi_s$, so that the linear perturbed flux is $\tilde{\Psi} = \Psi - \Psi_0 - \Psi_s$. This is related to the fact that the boundary conditions are satisfied before the square root is taken. This procedure resolves the ambiguity that could arise in the zero electron mass limit where the procedure of integrating the equations is such that the square root

turns up directly. This has led in the literature to the form⁴ $(-\Psi)^{1/2} - (-\Psi_s)^{1/2}$ in addition to the form $(-\Psi + \Psi_s)^{1/2}$. These two forms lead to quite different current distributions.

In the zero electron mass limit $d_e^2 \rightarrow 0$, Eqs. (50) and (51) are equivalent to Eqs. (35) and (36). Electron mass leads to three separate contributions arising from inertia and the electron stress tensor that add to the function under the square root in (52). Hence, the inclusion of electron mass effects is essential near the singular curve $\Psi = \Psi_s$.

Electron inertia only counts in a region with width of the order of the inertia skin depth $d_e = c/\omega_{pe}$. In a hot plasma d_e is smaller or at most of the order of the ion gyroradius. Therefore, we consider Eqs. (50) and (51) in the large- ρ_i limit where the density response is $\ln n/n_0(x) = -\tau_i \Phi$. The electric potential is

$$(1 + \tau_i)\Phi - a_e \frac{x}{l_n} = -\frac{a_e}{l_n} \sigma G. \quad (53)$$

The elimination of Φ and $\ln n/n_0$ from Eqs. (50) and (51) gives the mode equation (49) with the square root replaced by its finite inertia expression,

$$\nabla_{\perp}^2 \tilde{\Psi} = a_e a_i \frac{\hat{\beta}}{(1 + \tau_i) l_s} \left(1 - \frac{|x|}{G} \right), \quad (54)$$

with G given by (52). Equations (52) and (54) can be seen as a fourth-order polynomial in the current density with coefficients depending on x and the flux function Ψ .

While the explicit numerical solutions of Eq. (54) and of the related nonlinear eigenvalue problem are beyond the scope of the present paper, a number of qualitative properties of its solutions can be investigated analytically.

First, it is seen from (52) and (54) that in the limit of zero electron mass $G \rightarrow (2l_s)^{1/2} [\Psi_s - \Psi]^{1/2}$, so that the current density (54) is singular on surfaces $\Psi = \Psi_s$. The current density remains finite when inertia is included. It is seen that then G can only vanish at $x=0$ [if $x=0$ belongs to the domain where (47) is valid] where $G \propto x$.

Second, we may require that the current gradients remain finite. In contrast to the finiteness of the current, this condition is not built-in in the above equations, but can be imposed as a regularity condition. Using the relationship

$$g_{\pm} = \frac{d_s a_i}{(1 + \tau_i)} \frac{x}{G} - \frac{\tau_i}{1 + \tau_i} a_e d_s \pm G, \quad (55)$$

we see that at the critical points (λ_c, x_c) of g_{\pm} , where $\nabla g_{\pm} = 0$, these regularity conditions require that

$$G = \alpha_0 + \alpha_1(x - x_c) + \dots \quad (\alpha_0 > 0, \text{ if } x_c \neq 0), \quad (56)$$

where the constants $\alpha_0(\lambda_c)$ and $\alpha_1(\lambda_c)$ are related to the values of the current density and of its gradient at the X points

$$\begin{aligned} \bar{J}_c &= \frac{a_e a_i \hat{\beta}}{(1 + \tau_i) l_s} \left(1 - \frac{x_c}{\alpha_0} \right), \\ \frac{\partial \bar{J}}{\partial x_c} &= -\frac{a_e a_i \hat{\beta}}{(1 + \tau_i) l_s} \frac{1}{\alpha_0} \left(1 - \alpha_1 \frac{x_c}{\alpha_0} \right). \end{aligned} \quad (57)$$

The distance x_c of the X points from the rational surface $x=0$ is given by

$$\frac{a_i}{1 + \tau_i} \left(1 - \alpha_1 \frac{x_c}{\alpha_0} \right) \pm \frac{\alpha_0}{d_s} \alpha_1 = 0. \quad (58)$$

Assuming all terms to be of the same order, we obtain

$$x_c = O(d_s), \quad \alpha_0 = O(d_s), \quad \alpha_1 = O(d_s^0). \quad (59)$$

We see that the characteristic distance from the resonant surface $x=0$ of the X points of G_{\pm} is of order d_s . These X points do not necessarily coincide with the (possible) X points of Ψ or of Ψ_e . We conclude from Eq. (59) and from the linear dispersion equation (26), that it is not the inertia skin depth d_e but its β_e -modified version d_s that is the natural scale length on which electron inertia acts. For $d_s \rightarrow 0$, the current \bar{J}_c at the X points remains finite, but $\partial \bar{J} / \partial x_c \rightarrow \infty$, while the X points of g_{\pm} coincide with those of Ψ .

The general expression of the current gradient in the domain where (47) is valid if

$$\nabla \bar{J} = -\frac{a_e a_i \hat{\beta}}{(1 + \tau_i) l_s} \frac{N}{DN}, \quad (60)$$

where the numerator N and the denominator DN are given by

$$N = G^2 \nabla x + l_s x \nabla \Psi$$

and

$$DN = G^3 - x d_s^2 \frac{a_i}{1 + \tau_i} \left(\frac{u}{u_*} - \frac{l_s}{a_e \hat{\beta}} \bar{J} \right). \quad (61)$$

If there is a curve $x_s(\lambda)$ where DN vanishes within the domain of validity of (47), regularity of $\nabla \bar{J}$ requires that N vanishes on the same curve $x_s(\lambda)$. A class of solutions can then be constructed for which the resonant surface $x=0$ is shielded, i.e., the background configuration is not modified in the domain delimited by the curve $x_s(\lambda)$ inside which $\Psi = -x^2/2l_s$ and $\bar{J} = 0$. Requiring that $\nabla \Psi$ is continuous across the curve one obtains that the curve $x_s(\lambda)$ is the straight line given by Eq. (27). These shielded solutions can occur only if $(\tau_i u/u_* + 1)u/u_* > 0$ which is the nonlinear counterpart of the result obtained for regular perturbations in the linear limit [see Sec. IV below Eq. (16)].

It has been shown that in the frequency interval $(\tau_i u/u_* + 1)u/u_* < 0$, the linear equation (26) does not exhibit a singularity in the effective potential. In the same interval the nonlinear MHD solution remains regular and the large- ρ_i solutions are unshielded. It can be concluded that the quadratic relationship (47) is valid over all space up to the resonant surface $x=0$. This implies that islands and current sheets are not formed in this frequency interval. On the contrary, for frequencies such that $(\tau_i u/u_* + 1)u/u_* > 0$, both the linear effective potential and the nonlinear MHD solution are singular. In the large- ρ_i limit infinite current gradients tend to develop. For a limited class of solutions this singularity can be removed by shielding. However, in the general case, it implies a breakdown of the quadratic relationship (47). This means that islands and current sheets exist.

VII. HAMILTONIAN FORMULATION

In the remainder of this paper we will return to the nonlinear electron equations (6) and (7), and to the nonlinear ion equation (12) and we will investigate their Hamiltonian structure. First we show that our set of equations can be put in noncanonical Hamiltonian form. In order to do this we first rewrite our equations. We define the variables

$$\begin{aligned}\xi_1 &= \Psi_e, & \xi_2 &= d_e \beta_e^{1/2} \ln \frac{n}{n_0}, \\ \xi_3 &= -d_e \beta_e^{1/2} \left(\ln \frac{n}{n_0} + \tau_i (1 - \Gamma_0) \Phi \right).\end{aligned}\quad (62)$$

The functional derivatives of the energy integral (19) are

$$\begin{aligned}\frac{\delta H}{\delta \xi_1} &= -J = -\nabla_{\perp}^2 \Psi, & \frac{\delta H}{\delta \xi_2} &= \frac{\beta_e^{1/2}}{d_e} \left(\ln \frac{n}{n_0} - \Phi \right), \\ \frac{\delta H}{\delta \xi_3} &= -\frac{\beta_e^{1/2}}{d_e} \Phi.\end{aligned}\quad (63)$$

With these expressions, our set of equations can be written as

$$\frac{\beta_e^{1/2}}{ad_e} \frac{\partial \xi_1}{\partial t} = - \left[\xi_1, \frac{\delta H}{\delta \xi_2} \right] - \left[\xi_2, \frac{\delta H}{\delta \xi_1} \right] - \frac{\partial}{\partial z} \frac{\delta H}{\delta \xi_2}, \quad (64)$$

$$\frac{\beta_e^{1/2}}{ad_e} \frac{\partial \xi_2}{\partial t} = - \left[\xi_1, \frac{\delta H}{\delta \xi_1} \right] - \left[\xi_2, \frac{\delta H}{\delta \xi_2} \right] - \frac{\partial}{\partial z} \frac{\delta H}{\delta \xi_1}, \quad (65)$$

and

$$\frac{\beta_e^{1/2}}{ad_e} \frac{\partial \xi_3}{\partial t} = - \left[\xi_3, \frac{\delta H}{\delta \xi_3} \right] + L_k \left[\xi_2, L_k \frac{\delta H}{\delta \xi_3} \right]. \quad (66)$$

Noncanonical Poisson brackets are defined by

$$\begin{aligned}\{F, G\} &= \frac{ad_e}{\beta_e^{1/2}} \int d^3x W_{ij} \left[\frac{\delta F}{\delta \xi_i}, \frac{\delta G}{\delta \xi_j} \right] \\ &\quad - \frac{ad_e}{\beta_e^{1/2}} \int d^3x W_{ij}^{(z)} \frac{\delta F}{\delta \xi_i} \frac{\partial}{\partial z} \frac{\delta G}{\delta \xi_j} \\ &\quad + \frac{ad_e}{\beta_e^{1/2}} \int d^3x \xi_2 \left[L_k \frac{\delta F}{\delta \xi_3}, L_k \frac{\delta G}{\delta \xi_3} \right],\end{aligned}\quad (67)$$

where the symmetric matrices W_{ij} and $W_{ij}^{(z)}$ are defined by

$$W_{ij} = \begin{pmatrix} \xi_2 & \xi_1 & 0 \\ \xi_1 & \xi_2 & 0 \\ 0 & 0 & \xi_3 \end{pmatrix}, \quad W_{ij}^{(z)} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (68)$$

It can be verified directly that the noncanonical Poisson brackets (67) are antisymmetric and satisfy the Jacobi identity $\{F, \{G, H\}\} + \{G, \{H, F\}\} + \{H, \{F, G\}\} = 0$.

Equations (64)–(66) read in Hamiltonian form

$$\frac{\partial \xi_i}{\partial t} = \{\xi_i, H\}, \quad i = 1, 2, 3. \quad (69)$$

Apart from the ξ_2 contribution to the integrand in the last term in Eq. (67), we see from the form of the matrices (68) that the Poisson brackets do not contain any coupling be-

tween $\xi_{1,2}$ and ξ_3 . Coupling would arise when ion currents are no longer negligible and/or when charge neutrality is violated.

We will explore the constants of the motion that arise from the structure of the Poisson brackets and that do not depend on the specific form of the Hamiltonian.

In the general case, our Hamiltonian system admits three global constants of the motion

$$Q_i = \int d^3x \xi_i \quad (i = 1, 2, 3). \quad (70)$$

These constants are Casimirs since they commute with all functionals F , i.e., $\{Q_i, F\} = 0$. In the cold ion limit, $T_i \rightarrow 0$, the operators L_k in (67) vanish. Then the system contains an infinite number of Casimirs involving ξ_3 only

$$C_3 = \int d^3x h(\xi_3), \quad (71)$$

with h an arbitrary function. Note that in this limit $\xi_3 = -d_e \beta_e^{1/2} [\ln(n/n_0) - \rho_s^2 \nabla_{\perp}^2 \Phi]$. In the case of linear MHD where $\partial \xi_3 / \partial t = 0$, ξ_3 either vanishes or is a constant in functional space (ξ_3 is an explicit function of coordinates only). Then the Casimirs (71) become trivial.

In the remainder of this section we will restrict ourselves to the two-dimensional case, and consider all quantities to depend on the coordinates $\tau = t, x$, and $\lambda = y + \alpha z - ut$. All relevant equations are obtained in terms of these coordinates by taking $\partial / \partial z = 0$ and applying the following transformation to the expressions given in this section

$$\begin{aligned}t \rightarrow \tau, & \quad (x, y, z) \rightarrow (x, \lambda), \quad d^3x \rightarrow d^2x = dx d\lambda, \\ \Phi \rightarrow \Phi - \frac{u}{a} x, & \quad \Psi \rightarrow \Psi + \alpha x.\end{aligned}\quad (72)$$

In this 2-D case our Poisson brackets admit two additional infinite set of Casimirs,

$$C_{\pm} = \int d^2x h_{\pm}(\xi_1 \pm \xi_2) = \int d^2x h_{\pm} \left(\Psi_e \pm \frac{\beta_e^{1/2} d_e \ln n}{n_0} \right), \quad (73)$$

with h_{\pm} arbitrary functions. Note that the arguments $\xi_1 \pm \xi_2$ are just the functions f_{\pm} defined in Eq. (48) of the preceding section. This is quite natural since the first variation of the Hamiltonian (19), keeping all the Casimirs constant (isotopological variation¹⁵), yields the equilibrium equations (45). The Casimirs (73) exist for all values of the ion gyroradius. The dominant scale length involved in these Casimirs is $d_s = \beta_e^{1/2} d_e l_s / l_n$, where l_s and l_n are the characteristic scale lengths of the magnetic field and of the density.

The Casimirs (71) and (73) generalize in an elegantly symmetrical way different special limits. In the limit $m_e \rightarrow 0$, $u \rightarrow 0$, $\ln n/n_0 \rightarrow \rho_s^2 \nabla_{\perp}^2 \Phi = \Omega$, where Ω is the normalized vorticity, the Hamiltonian functional (19) becomes

$$H = \frac{1}{2} \int d^2x [(\nabla \Psi)^2 + \beta_e \Omega^2 - \beta_e \Phi \Omega], \quad (74)$$

with c_A the Alfvén velocity in the field B_0 . In this reduced system the Casimirs (71) become trivial, while the Casimirs (73) become those¹³ of reduced MHD²³

$$C_1 = \int d^2x F(\Psi) \quad \text{and} \quad C_2 = \int d^2x \Omega G(\Psi). \quad (75)$$

Another interesting limit corresponds to the case of cold electrons $T \rightarrow 0$. Then the Hamiltonian functional becomes

$$H = \frac{1}{2} \int d^2x [(\nabla \Psi)^2 + d_e^2 (\nabla_{\perp}^2 \Psi)^2 + \beta_e \tau_i \Phi (1 - \Gamma_0) \Phi], \quad (76)$$

and the Casimirs (73) become

$$C_1 = \int d^2x F(\Psi_e) \quad \text{and} \quad C_2 = \int d^2x G(\Psi_e) \ln \frac{n}{n_0}. \quad (77)$$

It is seen from Eq. (75), that in the reduced MHD model the magnetic flux is conserved, while according to Eq. (77), the generalized flux Ψ_e is conserved in the zero temperature limit. In the general case where electron inertia and finite temperature are accounted for, neither of these fluxes are conserved. Instead the more subtle conservation (73) is valid. This implies that magnetic reconnection in Ψ_e and/or Ψ can occur in the presence of an infinite set of conservation laws. This is due to the fact that inertia is particularly important in regions where the reconnection process can occur.

VIII. CONCLUSIONS

In this paper we have presented a set of nonlinear, generalized two-fluid equations that describe low-frequency phenomena in hot plasmas over a wide range of spatial scale lengths. The hybrid nonlinear model for the ion response, that is valid for all values of the ion thermal gyroradius, is a new ingredient of the theory. It is shown that two-dimensional equilibrium solutions in the form of stationary propagating modes tend to develop large currents at magnetic separatrices. The current density is prevented from becoming singular by electron inertia, but current gradients can still be large.

The set of equations has an energy integral that allows for a Hamiltonian representation in terms of noncanonical Poisson brackets. We have explored the algebraic invariance (Casimirs) of the Poisson brackets which represent the physical quantities that are conserved by the plasma dynamics. The number of Casimirs depends on the dimensionality of the problem and on the physical regime. The infinite set of ion Casimirs, that exist in the cold ion limit, is reduced to a single Casimir when finite ion gyroradius effects are taken into account. In 2-D problems, two infinite sets of electron Casimirs are found, $C_{\pm} = \int d^2x h(\Psi_e \pm \beta_e^{1/2} d_e \ln n/n_0)$. This implies that neither the magnetic flux Ψ , nor its generalized form Ψ_e , is conserved by the plasma motion. The conserved quantities are $f_{\pm} \equiv \Psi_e \pm \beta_e^{1/2} d_e \ln n/n_0$, and their X points do not coincide with the X points of Ψ or Ψ_e . We have shown in Sec. VI that the natural scale length associated with the X points of the conserved quantities is the β_e modified electron skin depth d_s defined by Eq. (28). We note, however, that at this scale length we find for propagating modes with phase velocity u of the order of the diamagnetic velocity u_* , $u \sim u_* \sim (d_s/l_s)v_{th e}$, so that Landau resonances become important and the fluid model breaks down.

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