8. T. Skolem, "Ein Verfahren zur Behandlung gewisser exponentialler Gleichungen und diophantischer Gleichungen," *8de Skand. Mat. Kongress Stockholm* (1934), pp. 163-188.
9. J. V. Uspensky & M. A. Heaslet, *Elementary Number Theory* (New York: McGraw-Hill Book Company, 1939).

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GENERALIZED TWO-PILE FIBONACCI NIM

JIM FLANIGAN
University of California at Los Angeles, Los Angeles, CA 90024

1. INTRODUCTION

Consider a take-away game with one pile of chips. Two players alternately remove a positive number of chips from the pile. A player may remove from 1 to $f(t)$ chips on his move, t being the number removed by his opponent on the previous move. The last player able to move wins.

In 1963, Whinihan [3] revealed winning strategies for the case when $f(t) = 2t$, the so-called *Fibonacci Nim*. In 1970, Schwenk [2] solved all games for f nondecreasing and $f(t) \geq t\ \forall t$. In 1977, Epp & Ferguson [1] extended the solution to the class where f is nondecreasing and $f(1) \geq 1$.

Recently, Ferguson solved a *two-pile analogue of Fibonacci Nim*. This motivated the author to investigate take-away games with more than one pile of chips. In this paper, winning strategies are presented for a class of two-pile take-away games which *generalize* two-pile Fibonacci Nim.

2. THE TWO-PILE GAME

Play begins with two piles containing m and m' chips and a positive integer w. Player I selects a pile and removes from 1 to w chips. Suppose t chips are taken. Player II responds by taking from 1 to $f(t)$ chips from one of the piles. We assume *f is nondecreasing* and $f(t) \geq t\ \forall t$. The two players alternate moves in this fashion. The player who leaves both piles empty is the winner. If $m = m'$, Player II is assured a win.

Set $d = m' - m$. For $d \geq 1$, define $L(m, d)$ to be the least value of w for which Player I can win. Set $L(m, 0) = \infty\ \forall m \geq 0$. One can systematically generate a *tableau* of values for $L(m, d)$. Given the position (m, d, w), the player about to move can win iff he can:

(1) take t chips, $1 \leq t \leq w$, from the large pile, leaving the next player in position $\big(m, d - t, f(t)\big)$ with $f(t) < L(m, d - t)$; or
(2) take t chips, $1 \leq t \leq w$, from the small pile, leaving the next player in position $\big(m - t, d + t, f(t)\big)$ with $f(t) < L(m - t, d + t)$.

(See Fig. 2.1.) Consequently, the tableau is governed by the functional equation

$$L(m,\ d) = \min\Big\{t > 0 \,\Big|\, f(t) < L(m,\ d - t) \ \text{ or } f(t) < L(m - t,\ d + t)\Big\}$$

subject to $L(m,\ 0) = +\infty\ \forall m \geq 0$. Note that $L(m,\ d) \leq d\ \forall d \geq 1$. Dr. Ferguson has written a computer program which can quickly furnish the players with a 60×40 tableau. As an illustration, Figure 2.2 gives a tableau for the two-pile game with $f(t) = 2t$, two-pile Fibonacci Nim.

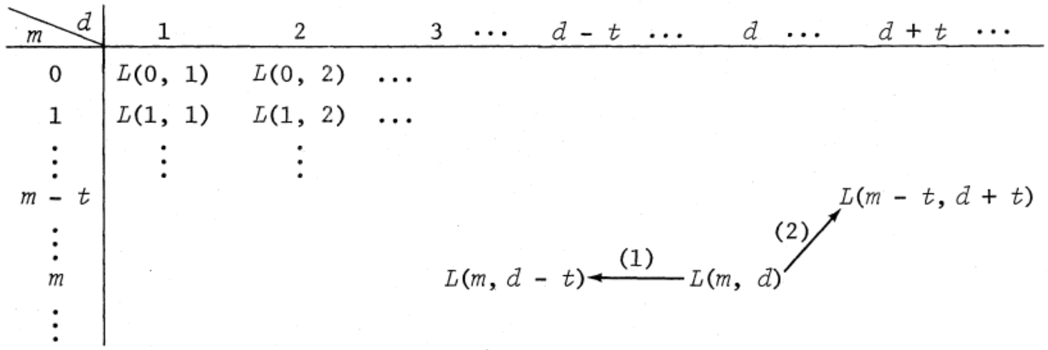

Fig. 2.1 The Tableau

Given f, one can construct a strictly-increasing infinite sequence $\langle H_k \rangle_1^\infty$ as follows: $H_1 = 1$ and for $k \geq 1$, $H_{k+1} = H_k + H_j$ where j is the least integer such that $f(H_j) \geq H_k$. For example, $\langle H_k \rangle_1^\infty$ is the Fibonacci sequence when $f(t) = 2t$, and $H_k = 2^{k-1}$, $k \geq 1$ when $f(t) = t$. Schwenk [2] showed that each positive integer d can be represented as a *unique sum* of the H_k's

$$(2.1) \qquad d = \sum_{i=1}^{s} H_{n_i} \text{ such that } f\left(H_{n_i}\right) < H_{n_i+1} \text{ for } i = 1, 2, \ldots, s - 1.$$

Moreover, for the take-away game with a single pile of d $(= m' - 0)$ chips, Player I can win iff he can remove H_{n_1} chips from the pile (i.e., iff $H_{n_1} \leq w$). So for the two-pile game with one pile exhausted,

$$(2.2) \qquad L(0, d) = H_{n_1}.$$

For the one-pile game with $d = H_{n_1} + \cdots + H_{n_s}$, $s \geq 1$, chips, H_{n_1} is the key term. It turns out that for the two-pile game where $d = m' - m = H_{n_1} + H_{n_2} + \cdots + H_{n_s}$, $s \geq 1$, H_{n_2} *(when it exists) as well as* H_{n_1} *plays a decisive role*. Denote $n_1 = n$ and n_2 (when it exists) $= n + r$. Thus, we shall write

$$d = H_n + H_{n+r} + \cdots + H_{n_s}, \; s \geq 1.$$

For each positive integer k, define $\ell(k)$ to be the greatest integer such that

$$(2.3) \qquad f\left(H_{k-\ell(k)}\right) \geq H_k.$$

Note that $\ell(1) = 0$, $\ell(k) \geq 0$, and $H_{k+1} = H_k + H_{k-\ell(k)}$ $\forall k \geq 1$.

In the sequel, we present winning strategies for the class of two-pile games for which $\ell(k) \in \{0, 1\}$ $\forall k$. We refer to such games as *generalized two-pile Fibonacci Nim*.

It would be nice if one could find some NASC on f such that $\ell(k) \in \{0, 1\}$ $\forall k$. The following partial results have been obtained:

(1) If $f(t) < (5/2)t$ $\forall t$, then $\ell(k) \in \{0, 1\}$ $\forall k$.
 In particular, for $f(t) = ct$,
 (a) if $1 \leq c < 2$, then $\ell(k) = 0$ $\forall k \geq 1$;
 (b) if $2 \leq c < 5/2$, then $\ell(k) = 1$ $\forall k \geq 2$;
 (c) if $c \geq 5/2$, then $\ell(3) = 2$ or $\ell(4) = 2$.

(2) If $\ell(k) \in \{0, 1\}$ $\forall k$, then $f(t) < 6t$ $\forall t$.

(3) A NASC such that $\ell(k) = 0$ $\forall k$ is $f(2^k) < 2^{k+1}$ $\forall k \geq 0$.

(4) A NASC such that $\ell(k) = 1\ \forall k \geq 2$ is $F_k \leq f(F_{k-1}) < F_{k+1}\ \forall k \geq 2$, where $\langle F_k \rangle_1^\infty$ is the Fibonacci sequence 1, 2, 3, 5, 8, 13, ...

m \ d	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34
0	1	2	3	1	5	1	2	8	1	2	3	1	13	1	2	3	1	5	1	2	21	1	2	3	1	5	1	2	8	1	2	3	1	34
1	1	1	3	1	5	1	1	8	1	1	3	1	13	1	1	3	1	5	1	1	21	1	1	3	1	5	1	1	8	1	1	3	1	34
2	1	1	2	1	5	1	1	8	1	1	2	1	13	1	1	2	1	5	1	1	21	1	1	2	1	5	1	1	8	1	1	2	1	34
3	1	2	2	1	3	1	1	8	1	2	2	1	13	1	2	2	1	3	1	1	21	1	2	2	1	3	1	1	8	1	2	2	1	34
4	1	2	2	1	3	1	1	8	1	2	2	1	13	1	2	2	1	3	1	1	21	1	2	2	1	3	1	1	8	1	2	2	1	34
5	1	2	3	1	3	1	1	5	1	2	2	1	13	1	2	3	1	3	1	1	21	1	2	3	1	3	1	1	5	1	2	2	1	34
6	1	1	3	1	3	1	1	5	1	2	2	1	13	1	1	3	1	3	1	1	21	1	1	3	1	3	1	1	5	1	2	2	1	34
7	1	1	3	1	3	1	1	5	1	2	2	1	13	1	1	3	1	3	1	1	21	1	1	3	1	3	1	1	5	1	2	2	1	34
8	1	1	3	1	5	1	1	5	1	2	2	1	8	1	1	3	1	3	1	1	21	1	1	3	1	5	1	1	5	1	2	2	1	34
9	1	1	3	1	5	1	1	5	1	2	2	1	8	1	1	3	1	3	1	1	21	1	1	3	1	5	1	1	5	1	2	2	1	34
10	1	1	2	1	5	1	1	5	1	2	2	1	8	1	1	3	1	3	1	1	21	1	1	2	1	5	1	1	5	1	2	2	1	34
11	1	2	2	1	5	1	1	5	1	2	2	1	8	1	1	3	1	3	1	1	21	1	2	2	1	5	1	1	5	1	2	2	1	34
12	1	2	2	1	5	1	1	5	1	2	2	1	8	1	1	3	1	3	1	1	21	1	2	2	1	5	1	1	5	1	2	2	1	34
13	1	2	2	1	5	1	1	8	1	2	2	1	8	1	1	3	1	3	1	1	13	1	2	2	1	5	1	1	5	1	2	2	1	34
14	1	2	2	1	5	1	1	8	1	2	2	1	8	1	1	3	1	3	1	1	13	1	2	2	1	5	1	1	5	1	2	2	1	34
15	1	2	2	1	5	1	1	8	1	2	2	1	8	1	1	3	1	3	1	1	13	1	2	2	1	5	1	1	5	1	2	2	1	34
16	1	2	2	1	3	1	1	8	1	2	2	1	8	1	1	3	1	3	1	1	13	1	2	2	1	5	1	1	5	1	2	2	1	34
17	1	2	2	1	3	1	1	8	1	2	2	1	8	1	1	3	1	3	1	1	13	1	2	2	1	5	1	1	5	1	2	2	1	34
18	1	2	3	1	3	1	1	8	1	2	2	1	8	1	1	3	1	3	1	1	13	1	2	2	1	5	1	1	5	1	2	2	1	34
19	1	1	3	1	3	1	1	8	1	2	2	1	8	1	1	3	1	3	1	1	13	1	2	2	1	5	1	1	5	1	2	2	1	34
20	1	1	3	1	3	1	1	8	1	2	2	1	8	1	1	3	1	3	1	1	13	1	2	2	1	5	1	1	5	1	2	2	1	34
21	1	1	3	1	3	1	1	8	1	2	2	1	13	1	1	3	1	3	1	1	13	1	2	2	1	5	1	1	5	1	2	2	1	21
22	1	1	3	1	3	1	1	8	1	2	2	1	13	1	1	3	1	3	1	1	13	1	2	2	1	5	1	1	5	1	2	2	1	21
23	1	1	3	1	3	1	1	8	1	2	2	1	13	1	1	3	1	3	1	1	13	1	2	2	1	5	1	1	5	1	2	2	1	21
24	1	1	3	1	3	1	1	8	1	2	2	1	13	1	1	3	1	3	1	1	13	1	2	2	1	5	1	1	5	1	2	2	1	21
25	1	1	3	1	3	1	1	8	1	2	2	1	13	1	1	3	1	3	1	1	13	1	2	2	1	5	1	1	5	1	2	2	1	21
26	1	1	3	1	3	1	1	5	1	2	2	1	13	1	1	3	1	3	1	1	13	1	2	2	1	5	1	1	5	1	2	2	1	21
27	1	1	3	1	3	1	1	5	1	2	2	1	13	1	1	3	1	3	1	1	13	1	2	2	1	5	1	1	5	1	2	2	1	21
28	1	1	3	1	3	1	1	5	1	2	2	1	13	1	1	3	1	3	1	1	13	1	2	2	1	5	1	1	5	1	2	2	1	21
29	1	1	3	1	5	1	1	5	1	2	2	1	13	1	1	3	1	3	1	1	13	1	2	2	1	5	1	1	5	1	2	2	1	21
30	1	1	3	1	5	1	1	5	1	2	2	1	13	1	1	3	1	3	1	1	13	1	2	2	1	5	1	1	5	1	2	2	1	21
31	1	1	2	1	5	1	1	5	1	2	2	1	13	1	1	3	1	3	1	1	13	1	2	2	1	5	1	1	5	1	2	2	1	21
32	1	2	2	1	5	1	1	5	1	2	2	1	13	1	1	3	1	3	1	1	13	1	2	2	1	5	1	1	5	1	2	2	1	21
33	1	2	2	1	5	1	1	5	1	2	2	1	13	1	1	3	1	3	1	1	13	1	2	2	1	5	1	1	5	1	2	2	1	21
34	1	2	2	1	5	1	1	5	1	2	2	1	13	1	1	3	1	3	1	1	21	1	2	2	1	5	1	1	5	1	2	2	1	21
35	1	2	2	1	5	1	1	5	1	2	2	1	13	1	1	3	1	3	1	1	21	1	2	2	1	5	1	1	5	1	2	2	1	21
36	1	2	2	1	5	1	1	5	1	2	2	1	13	1	1	3	1	3	1	1	21	1	2	2	1	5	1	1	5	1	2	2	1	21
37	1	2	2	1	5	1	1	5	1	2	2	1	13	1	1	3	1	3	1	1	21	1	2	2	1	5	1	1	5	1	2	2	1	21
38	1	2	2	1	5	1	1	5	1	2	2	1	13	1	1	3	1	3	1	1	21	1	2	2	1	5	1	1	5	1	2	2	1	21
39	1	2	2	1	5	1	1	5	1	2	2	1	13	1	1	3	1	3	1	1	21	1	2	2	1	5	1	1	5	1	2	2	1	21
40	1	2	2	1	5	1	1	5	1	2	2	1	13	1	1	3	1	3	1	1	21	1	2	2	1	5	1	1	5	1	2	2	1	21
41	1	2	2	1	5	1	1	5	1	2	2	1	13	1	1	3	1	3	1	1	21	1	2	2	1	5	1	1	5	1	2	2	1	21
42	1	2	2	1	5	1	1	5	1	2	2	1	8	1	1	3	1	3	1	1	21	1	2	2	1	5	1	1	5	1	2	2	1	21

Fig. 2.2 A Tableau for Two-Pile Fibonacci Nim $\left(f(t) = 2t\right)$*

*Computer program supplied by T. S. Ferguson.

3. SOME GOOD AND BAD MOVES

Lemma: For the position (m, d), $d = H_n + \cdots + H_{n_s}$, $s \geq 1$, it is *never* a winning move to take

(1) t chips from the large pile if $0 < t < H_n$.

(2) t chips from the small pile if $0 < t < H_n$, $t \neq H_{n-\ell(n)}$.

It is *always* a winning move to take

(3) H_n chips from the large pile with the possible exception of the *special case*: $d = H_n + H_{n+2} + \cdots + H_{n_s}$, $s \geq 2$, $\ell(n+1) = \ell(n+2) = 1$, $m \geq H_{n+1}$.

(4) $H_{n-\ell(n)}$ chips from the small pile when $d = H_n + H_{n+2} + \cdots + H_{n_s}$, $s \geq 2$, $\ell(n+1) = \ell(n+2) = 1$, $m \geq H_{n-\ell(n)}$. (This contains the special case.)

Proof: The statements (1)-(4) imply that $L(m, d) \in \{H_n, H_{n-\ell(n)}\}$ $\forall m \geq 0$. We shall use this observation and double induction in our argument.

Schwenk [2] proved the assertions for the positions $(0, d)$, $\forall d \geq 1$ (see equation 2.2). Suppose they hold for the positions (m, d) $\forall m \leq M - 1$, $\forall d \geq 1$ for some $M \geq 1$. We must show that (1)-(4) hold for the positions (M, d) $\forall d \geq 1$.

The claim is trivial for position $(M, 1)$. Suppose it is true for (M, d) $\forall d \leq D - 1$ for some $D \geq 2$. Consider the two types of moves which can be made from position (M, D), $D = H_n + H_{n+r} + \cdots + H_{n_s}$, $s \geq 1$.

A. *Taking from the big pile*:

Take t chips, $0 < t < H_n$, from the big pile. Then $D - t = H_k + \cdots + H_{n_s}$ where $k < n$. $t \geq H_{k-1}$ if $\ell(k) = 1$, and $t \geq H_k$ if $\ell(k) = 0$. By the inductive assumption $L(M, D - t) \leq H_k$. Hence,

$$f(t) \geq f(H_{k-1}) \geq H_k \geq L(M, D - t) \text{ if } \ell(k) = 1,$$

and

$$f(t) \geq f(H_k) \geq H_k \geq L(M, D - t) \text{ if } \ell(k) = 0.$$

Statement (1) follows.

Suppose you take $t = H_n$ chips from the big pile. Consider the following cases.

(1) $D = H_n$. Taking H_n chips from the large pile is obviously a winning move.

(2) $D > H_n$. Write $D - H_n = H_{n+r} + \cdots + H_s$.

(a) $r = 1$. Necessarily, $\ell(n + 1) = 0$. By the inductive assumption, $L(M, D - H_n) = H_{n+1}$. Thus, $f(H_n) < L(M, D - H_n)$ and it is a good move to take H_n chips from the large pile.

(b) $r \geq 3$. By the inductive assumption, $L(M, D - H_n) \geq H_{n+2}$. Thus, $f(H_n) < H_{n+2} \leq L(M, D - H_n)$ and it is a good move to take H_n chips from the large pile.

(c) $r = 2$.

(i) $\ell(n + 1) = 0$. $f(H_n) < H_{n+1}$ and, by the inductive assumption, $L(M, D - H_n) \geq H_{n+1}$. A good move is to take H_n chips from the big pile.

(ii) $\ell(n + 1) = 1$ and $\ell(n + 2) = 0$. By the second equation and the inductive assumption, $L(M, D - H_n) = H_{n+2}$. Thus, $f(H_n) < H_{n+2} \leq L(M, D - H_n)$, so taking H_n chips from the large pile wins.

(iii) $\ell(n + 1) = 1$ and $\ell(n + 2) = 1$. Here $f(H_n) \geq H_{n+1}$. By the inductive assumption, it is possible that $L(M, D - H_n) = H_{n+1}$. If $L(M, D - H_n) = H_{n+1}$, then $M \geq H_{n+1}$ follows from (1) of the Lemma. The possibility of $f(H_n) \geq L(M, D - H_n)$ signifies that taking H_n chips from the large pile might be a bad move. Thus, (3) holds.

B. Taking from the small pile:

If t chips, $0 < t < H_n$, $t \neq H_{n-\ell(n)}$, are removed from the small pile, the resulting position is $(M - t, D + t)$, $D + t = H_{n-k} + \cdots + H_{n_s}$ for some $k \geq 1$ and $t \geq H_{n-k}$. But $L(M - t, D + t) \leq H_{n-k}$ by assumption. Since

$$f(t) \geq t \geq H_{n-k} \geq L(M - t, D + t),$$

this is a bad move. Thus, (2) holds.

C. Case A2.c.(iii) *revisited*:

Here $D = H_n + H_{n+2} + \cdots + H_{n_s}$, $\ell(n + 1) = \ell(n + 2) = 1$. Suppose taking H_n chips from the large pile is *not* a good move. Then, $L(M, D - H_n) = H_{n+1}$.

For position (M, D), $M \geq H_{n-\ell(n)}$, take $H_{n-\ell(n)}$ chips from the small pile to get $\left(M - H_{n-\ell(n)}, D + H_{n-\ell(n)}\right)$. $D + H_{n-\ell(n)} = H_{n-\ell(n)} + H_n + H_{n+2} + \cdots + H_{n_s} = H_{n+1} + H_{n+2} + \cdots + H_{n_s'} = H_{n+k} + \cdots + H_{n_{s'}}$, for some $k \geq 3$ and $n_{s'} \geq n_s$, since $\ell(n + 2) = 1$. By the inductive assumption, $L\left(M - H_{n-\ell(n)}, D + H_{n-\ell(n)}\right) \geq H_{n+2}$. But $f\left(H_{n-\ell(n)}\right) \leq f(H_n) < H_{n+2}$. Thus,

$$f\left(H_{n-\ell(n)}\right) < L\left(M - H_{n-\ell(n)}, D + H_{n-\ell(n)}\right).$$

Taking $H_{n-\ell(n)}$ chips from the small pile is a good move, so (4) holds.

In A, B, and C we established that (1)-(4) hold for the position (M, D), which completes the induction on d. Hence, they hold for (M, d) $\forall d \geq 1$. This in turn completes the induction on m. Thus, (1)-(4) hold for (m, d) $\forall m \geq 0$, $\forall d \geq 1$. Q.E.D.

Corollary 1: $L(m, d) \in \left\{H_n, H_{n-\ell(n)}\right\}$ $\forall m \geq 0$.

Observe that if $\ell(n) = 0$, then $L(m, d) = H_n$ $\forall m \geq 0$. But when $\ell(n) = 1$, there are two possible values $L(m, d)$ might assume. However, if $m < H_{n-1}$, then $L(m, d) = H_n$.

Corollary 2—How to win (if you can) when you know $L(m, d)$:

(1) If $L(m, d) = H_{n-1}$, take H_{n-1} chips from the small pile to win.

(2) If $L(m, d) = H_n$, a winning move is to take H_n chips from the large pile, except possibly for the special case cited in the Lemma. In the special case, take H_n chips from the small pile to win.

4. HOW TO WIN IF YOU CAN

Knowing $L(m, d)$ at the beginning of play reveals whether Player I has a winning strategy. Compare $L(m, d)$ and w. If Player I knows the value of $L(m, d)$ and $w \geq L(m, d)$, he can use Corollary 2 to determine a winning move.

Which of the two possible values $L(m, d)$ assumes is not obvious under certain circumstances. The position (m, d, w) defies immediate classification when $L(m, d)$ is unknown and $H_{n-1} \leq w < H_n$.

Fortunately, not knowing whether one can win at the beginning of play does not prevent one from describing a winning strategy, provided such a strategy exists. A strategy of play, constructed from the Corollaries, is presented in Table 4.1. This table tells how to move optimally in all situations in which there exists a possibility of winning. An $N(P)$ represents a position for which there exists a winning move for Player I (II).

The only case in which the status of a position is now known at the start of play arises in 2(b) of the table. There, the player about to move is an *optimist* and *pretends* $L(m, d) = H_{n-1}$. This dictates taking H_{n-1} chips from the small pile. The outcome of the game will reveal the value of $L(m, d)$ depending on who wins.

Table 4.1. How To Win (If You Can) Without Knowing $L(m, d)$

(1) If $\ell(n) = 0$ [so necessarily $L(m, d) = H_n$] and

(a) $d = H_n + H_{n+2} + \cdots + H_{n_s}$, $s \geq 2$, $\ell(n+2) = \ell(n+1) = 1$

	$m \geq H_n$	$m < H_n$
$w \geq H_n$	N, Take H_n from s.p.	N, Take H_n from l.p.
$w < H_n$	P	P

(b) not as in (a)

	$m \geq H_n$	$m < H_n$
$w \geq H_n$	N, Take H_n from l.p.	N, Take H_n from l.p.
$w < H_n$	P	P

(2) If $\ell(n) = 1$ and

(a) $d = H_n + H_{n+2} + \cdots + H_{n_s}$, $s \geq 2$, $\ell(n+2) = \ell(n+1) = 1$

	$m \geq H_{n-1}\left(L(m, d) = H_{n-1}\right)$	$m < H_{n-1}\left(L(m, d) = H_n\right)$
$w \geq H_n$	N, Take H_{n-1} from s.p.	N, Take H_n from l.p.
$H_n > w \geq H_{n-1}$	N, Take H_{n-1} from s.p.	P
$w < H_{n-1}$	P	P

(b) not as in (a)

	$m \geq H_{n-1}\left(L(m, d) = ??\right)$	$m < H_{n-1}\left(L(m, d) = H_n\right)$
$w \geq H_n$	N, Take H_n from l.p.	N, Take H_n from l.p.
$H_n > w \geq H_{n-1}$	??, Take H_{n-1} from s.p.	P
$w < H_{n-1}$	P	P

(Note: s.p. = small pile; l.p. = large pile.)

As an illustration, consider two-pile Fibonacci Nim. It was first solved by Ferguson in the form of Table 4.1. For $f(t) = 2t$, the sequence $\langle H_k \rangle_1^\infty$ is the Fibonacci sequence. The first few values are

k	1	2	3	4	5	6	7	8	9	10
H_k	1	2	3	5	8	13	21	34	55	89

$\ell(1) = 0$ and $\ell(k) = 1$ $\forall k \geq 2$, since $H_{k+1} = H_k + H_{k-1}$ $\forall k \geq 2$. What is the status of position $m = 20$, $d = 42$, $w = 6$? $d = 34 + 8 = H_8 + H_5$. Player I is an optimist and *assumes* that $L(20, 42) = 5$, not 8. 2(b) in the table tells him to take 5 chips from the small pile.

Player II is left in position $m = 20 - 5 = 15$, $d = 42 + 5 = 47$, $w = f(5) = 10$. $d = 34 + 13 = H_8 + H_6$, $\ell(6) = 1$, $r = 2$, $\ell(8) = \ell(7) = 1$. $H_5 = 8 \leq w < H_6 = 13$. By 2(a) of the above table, this is a winning position ($L(15, 47) = 8$). Player II takes 8 chips from the small pile to win. We conclude that Player I has no winning strategy for the position (20, 42, 6). Consequently, $L(20, 42) = 8$, not 5.

Only after playing the game for a while were we able to determine who could win.

5. ELIMINATING SUSPENSE

It turns out that the suspense which can arise when $L(m, d)$ is unknown can be eliminated. The Theorem of this section presents a *simple method for computing* $L(m, d)$. If $d = H_n + \cdots + H_{n_s}$, then the entries in the dth column of the tableau can assume only the values H_n and H_{n-1}. We say that the dth column of the tableau makes k *flips*, $0 \leq k \leq \infty$, if it has the form in Figure 5.1. If $k < \infty$, the kth flip is followed by an infinite string of

$$\begin{cases} H_n\text{'s} & \text{if } k \text{ is even} \\ H_{n-1}\text{'s} & \text{if } k \text{ is odd.} \end{cases}$$

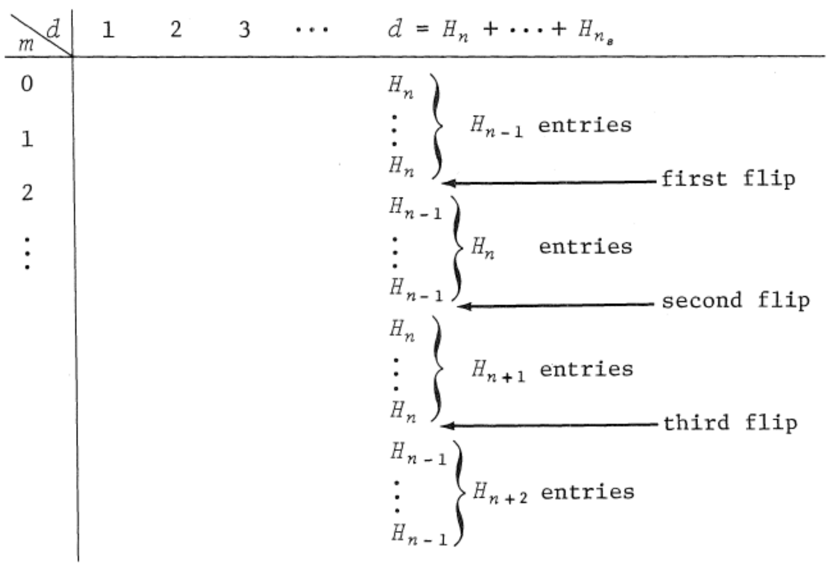

Fig. 5.1 The dth Column Makes k "flips"

Theorem: For $n \geq 1$, set $A_n = \{z \mid z \geq 0,\ \ell(n + z) = 0\}$. Then:

A. *Simple Case*: $d = H_n$. The dth column makes k flips, where $k = \min A_n$. (Convention: $\min \emptyset = \infty$.)

B. *Compound Case*: $d = H_n + H_{n+r} + \cdots + H_{n_s}$, $s \geq 2$. The dth column makes k flips, where

$$k = \begin{cases} r - 1 & \text{if } \min A_n > r, \\ \min A_n & \text{if } \min A_n \leq r. \end{cases}$$

Proof:

A. *Simple Case*:

(1) $A_n \neq \emptyset$. We proceed by induction on $k = \min A_n$. For $k = 0$, $L(m, H_n) = H_n \ \forall m \geq 0$, since $\ell(n) = 0$. There are zero flips.

Suppose the result holds $\forall k \leq K - 1$ for some $K \geq 1$. (That is, if $d = H_m$ and $\min A_m \leq K - 1$, then the column for $d = H_m$ makes $\min A_m$ flips.)

By the Lemma, each entry of the column $d = H_n$ is H_n, unless a good move can be made by taking H_{n-1} from the small pile. Removing H_{n-1} chips from the small pile is a winning move for position (M, H_n), $M \geq H_{n-1}$ iff $f(H_{n-1}) < L(M - H_{n-1}, H_n + H_{n-1})$. Since $\ell(n) = 1$, $H_n + H_{n-1} = H_{n+1}$ and $L(M - H_{n-1}, H_{n+1}) = H_{n+1}$ or H_n. Moreover, $H_{n+1} > f(H_{n-1}) \geq H_n$. This can be a good move iff $L(M - H_{n-1}, H_{n+1}) = H_{n+1}$. The column $d = H_{n+1}$ makes $K - 1$ flips. Thus, the column $d = H_n$ makes K flips. (See Fig. 5.2). This completes the induction on k.

(2) $A_n = \emptyset$, $\ell(n + k) = 1$ and $A_{n+k} = \emptyset \ \forall k \geq 0$. We show that each column $d = H_{n+k}$, $k \geq 0$, makes infinitely many flips. Let us proceed by induction on m.

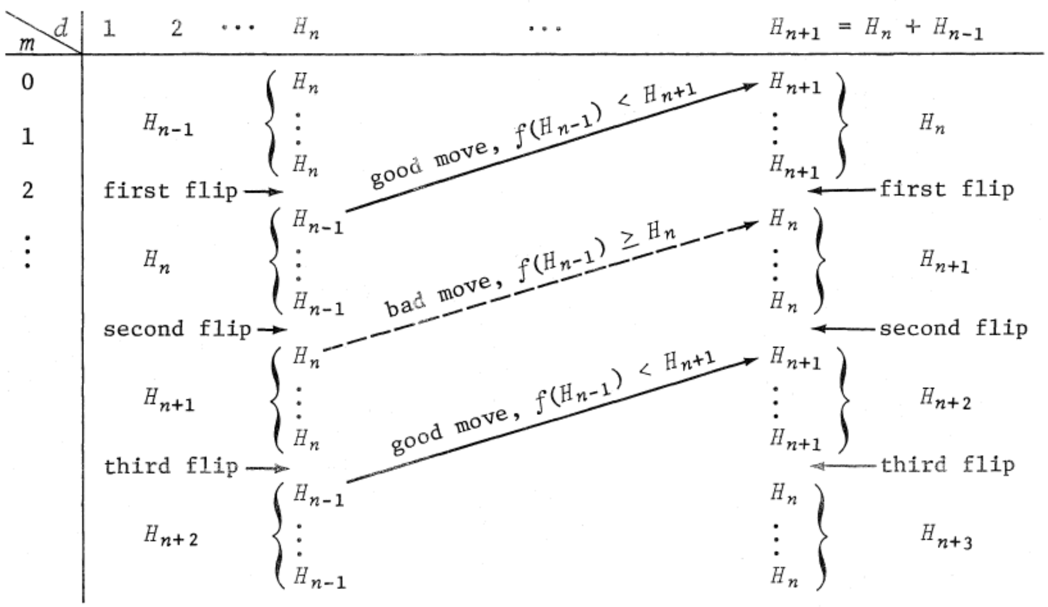

Fig. 5.2 Case A(1)

By the remark to Corollary 1, $L(m, H_{n+k}) = H_{n+k} \ \forall m < H_{n-1}$, $\forall k \geq 0$. The tableau has the desired values for the first H_{n-1} entries in columns $d = H_{n+k}$, $k \geq 0$.

Suppose that the tableau assumes the desired values in the entries $m = 0, 1, \ldots, M - 1$ in the columns $d = H_{n+k}$, $k \geq 0$, for some $M \geq H_{n-1}$. One can find $k_0 \geq 0$ such that

$$H_{n-1} + H_n + \cdots + H_{n+k_0} - 1 \geq M > H_{n-1} + H_n + \cdots + H_{n+k_0-1} - 1.$$

Equivalently, $H_n + \cdots + H_{n+k_0} - 1 \geq M - H_{n-1} > \begin{cases} -1 & \text{if } k_0 = 0, \\ H_n + \cdots + H_{n+k_0-1} - 1 & \text{if } k_0 \geq 1. \end{cases}$

By the inductive assumption,

$$L(M - H_{n-1},\ H_{n+1}) = \begin{cases} H_{n+1} & \text{if } k_0 \text{ is even,} \\ H_n & \text{if } k_0 \text{ is odd.} \end{cases}$$

(See Fig. 5.3.) Thus, for the position (M, H_n),

(a) if k_0 is even, taking H_{n-1} chips from the small pile is a good move since $f(H_{n-1}) < H_{n+1} = L(M - H_{n-1}, H_{n+1})$;

(b) if k_0 is odd, taking H_{n-1} chips from the small pile is a bad move since $f(H_{n-1}) \geq H_n = L(M - H_{n-1}, H_{n+1})$.

As desired, we conclude

$$L(M, H_n) = \begin{cases} H_n & \text{if } k_0 \text{ is odd,} \\ H_{n-1} & \text{if } k_0 \text{ is even.} \end{cases}$$

An identical argument reveals that the entries $L(M, H_{n+k})$, $k > 0$, have the desired values. Thus, the row $m = M$ assumes the desired values in the entries corresponding to columns $d = H_{n+k}$, $k \geq 0$. This completes the induction on m.

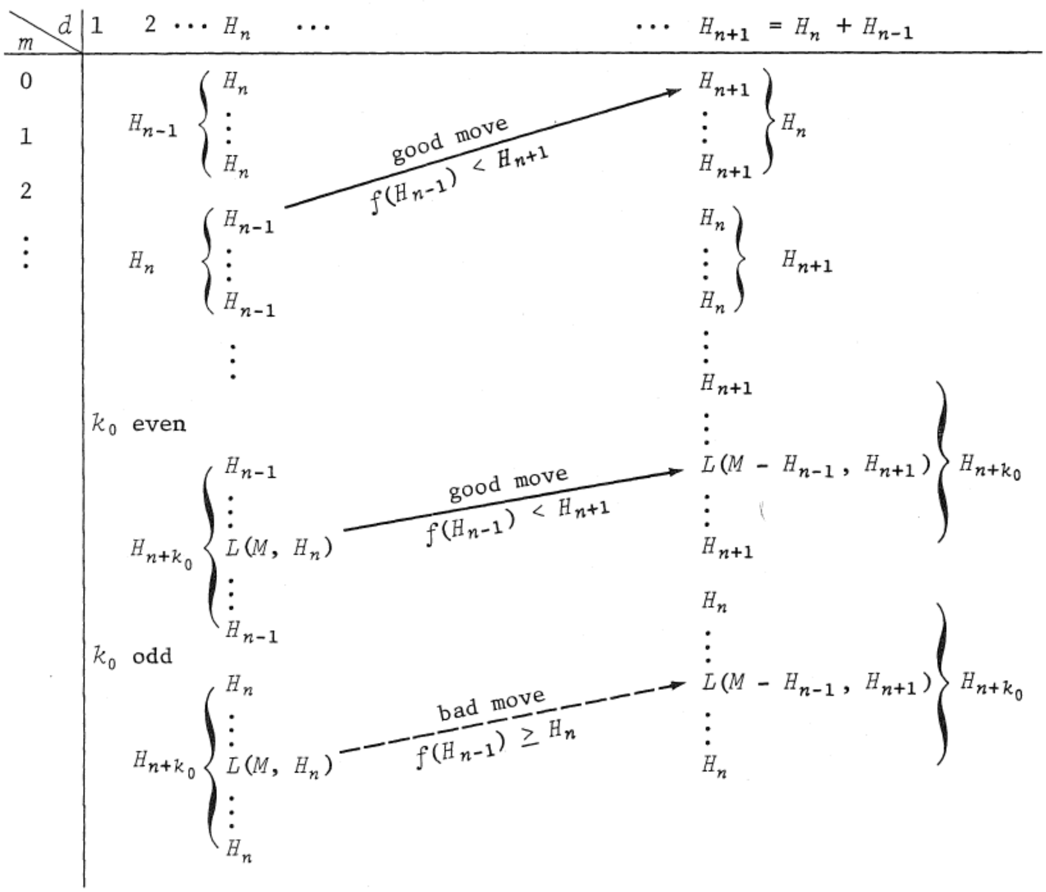

Fig. 5.3 Case A(2)

B. *Compound Case*:

Suppose $\underline{\ell(n) = 0}$. Then $L(m, d) = H_n\ \forall m \geq 0$. There are no flips in the dth column. Note that $\min A_n = 0$.

If $\underline{\ell(n) = 1}$, we consider two cases:

(1) $k = \min A_n \leq r$. By Corollary 1, $L(m, d - H_n + H_{n+k}) \geq H_{n+k}\ \forall m \geq 0$. The tableau from column $d - H_n + 1$ to column $d - H_n + H_{n+k} - 1$, inclusive, is a copy of the tableau from column 1 to column $H_{n+k} - 1$, inclusive. The dth column is identical to the H_nth column. By Part A, the latter column makes k flips, $k = \min A_n$.

(2) $\min A_n > r$. Here $\ell(n) = \ell(n + 1) = \cdots = \ell(n + r) = 1$. Necessarily, $r > 1$. Let $d' = d - H_n + H_{n+r-1}$. Since $\ell(n + r) = 1$, d' has the form $d' = H_{n+r+u} + \cdots + H_{n_{s'}}$, for some $u \geq 1$ and $n_{s'} \geq n_s$. By Corollary 1, $L(m, d') \geq H_{n+r+u-1}$. Consider the position (m, d''), $m \geq H_{n+r-3}$, where $d'' = d - H_n + H_{n+r-2}$. Note that $d'' + H_{n+r-3} = d'$. It is a good move to take H_{n+r-3} chips from the small pile, since $f(H_{n+r-3}) < H_{n+r-1} < L(m, d')$. Thus, $L(m, d'') = H_{n+r-3}\ \forall m \geq H_{n+r-3}$. The column $d'' = d - H_n + H_{n+r-2}$ makes one flip. Since the column d'' makes one flip, argue as in Part A(1) of the proof that column $d''' = d - H_n + H_{n+r-3}$ makes two flips. Similarly, column $d^{iv} = d - H_n + H_{n+r-4}$ makes three flips. Continue and argue that column $d = d - H_n + H_n$ makes $r - 1$ flips. (See Fig. 5.4). Q.E.D.

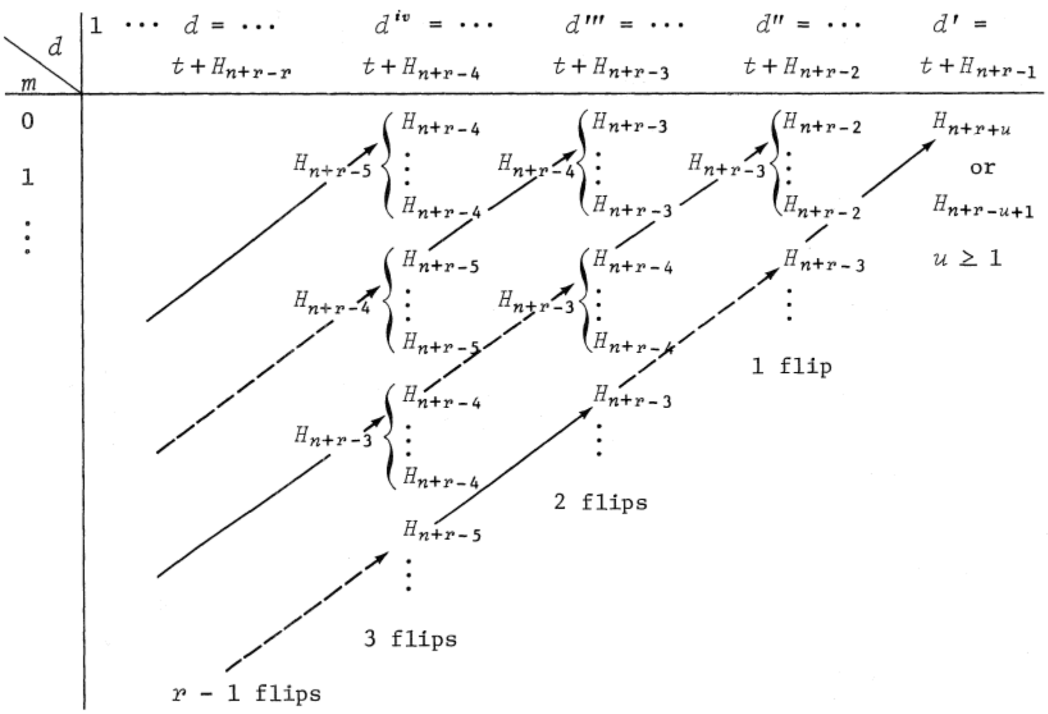

Fig. 5.4 Case B(2)

Notation: $t = d - H_n$.

$\longrightarrow$ = a good move.

$- - - \rightarrow$ = a bad move.

6. TWO-PILE FIBONACCI NIM REVISITED

Ferguson's solution for two-pile Fibonacci Nim was in the form of Table 4.1. His solution does not necessarily reveal which player can win at the beginning of play, because $L(m, d)$ might not be known then. The Theorem tells us the value of $L(m, d)$ be revealing the behavior of the columns of the tableau. Knowing $L(m, d)$ at the start of play leaves no uncertainty as to who can win. As an illustration of the Theorem, we compute $L(m, d)$ for two-pile Fibonacci Nim.

Suppose $d = H_n$ for some n. If $d = H_1$, then $L(m, d) = H_1 = 1\ \forall m \geq 0$, since $\ell(1) = 0$. If $n \geq 2$, the dth column makes infinitely many flips, since $A_n = \emptyset$. For a particular value of m, find the least integer $k_0 \geq -1$ such that $H_{n-1} + H_n + \cdots + H_{n+k_0} - 1 \geq m$. Then,

$$L(m, d) = \begin{cases} H_n & \text{if } k_0 \text{ is odd,} \\ H_{n-1} & \text{if } k_0 \text{ is even.} \end{cases}$$

Suppose d has compound form $d = H_n + H_{n+r} + \cdots + H_{n_s}$, $s \geq 2$. Note that $r > 1$. If $n = 1$, the dth column of the tableau has each entry equal to 1. If $n \geq 2$, the dth column makes $r - 1$ flips. If k_0 is the least integer such that $k_0 \geq -1$ and $H_{n-1} + H_n + \cdots + H_{n+k_0} - 1 \geq m$, then

$$L(m, d) = \begin{cases} H_n & \text{if } k_0 \text{ is odd and } k_0 \leq r - 2, \text{ or} \\ & \quad r \text{ is odd and } k_0 > r - 2. \\ H_{n-1} & \text{if } k_0 \text{ is even and } k_0 \leq r - 2, \text{ or} \\ & \quad r \text{ is even and } k_0 > r - 2. \end{cases}$$

7. CONCLUSION

The function $\ell(k)$ was defined by (2.3). In Table 4.1, a winning strategy (provided one exists) is given for the class of two-pile take-away games in which $\ell(k) \in \{0, 1\}\ \forall k \geq 1$. By revealing $L(m, d)$, the Theorem enables us to determine at the beginning of play whether such a strategy exists for the player about to move.

The author has considered several particular two-pile take-away games in which $\ell(k)$ assumes values other than 0 and 1. For example, when $f(t) = 3t$, then $\ell(k) = 3\ \forall k \geq 5$. I have found no general solution for any such game. Can we find solutions for the general class of games which impose no restrictions on $\ell(k)$? Can we extend to games beginning with arbitrarily many piles of chips? Let me know if you can.

REFERENCES

1. R. J. Epp & T. S. Ferguson, "A Note on Take-Away Games," paper contributed to Special Session on Combinatorial Games, AMS 83rd Annual Meeting, Jan. 1977.
2. A. J. Schwenk, "Take-Away Games," *The Fibonacci Quarterly*, Vol. 8, No. 3 (1970), pp. 225-234.
3. M. J. Whinihan, "Fibonacci Nim," *The Fibonacci Quarterly*, Vol. 1, No. 4 (1963), pp. 9-13.

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