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Generalized Typically Real Functions

Stanisława Kanas[a], Anna Tatarczak[b]

[a]*University of Rzeszów*
[b]*Maria Curie-Sklodowska University in Lublin*

Abstract. Let $f(z) = z + a_2z^2 + \cdots$ be regular in the unit disk and real valued if and only if z is real and $|z| < 1$. Then $f(z)$ is said to be typically real function. Rogosinski found the necessary and sufficient condition for a regular function to be typically-real. The main purpose of the paper is a consideration of the generalized typically-real functions defined via the generating function of the generalized Chebyshev polynomials of the second kind

$$\Psi_{p,q}(e^{i\theta}; z) = \frac{1}{(1 - pze^{i\theta})(1 - qze^{-i\theta})} = \sum_{n=0}^{\infty} U_n(p, q; e^{i\theta})z^n,$$

where $-1 \le p, q \le 1,\ \theta \in \langle 0, 2\pi\rangle,\ |z| < 1$.

1. Background and Fundamental Definitions

By $\mathcal{H}$ we denote the class of function f, analytic in the unit disk $\mathbb{D} = \{z : |z| < 1\}$ with the standard normalization $f(0) = 0 = f'(0) - 1$. Next, by $\mathcal{S}$ we denote the subclass of $\mathcal{H}$ consisting of univalent functions, and by $\mathcal{ST}$ (resp. $\mathcal{CV}$) its subclass of *starlike* (resp. *convex*) functions. It is well known that $f \in \mathcal{H}$ is starlike (resp. convex) if and only if $\mathrm{Re}[zf'(z)/f(z)] > 0$ (resp. $\mathrm{Re}[1 + zf''(z)/f'(z)] > 0$).

Let $\mathcal{P}$ be the class of functions of the form $\varphi(z) = 1 + p_1z + \cdots$, holomorphic, univalent and with positive real part in the unit disk $\mathbb{D}$; a class of such functions is called the Carathéodory class. $\mathcal{P}$ has interesting properties and many useful applications, particularly in the study of special classes of univalent functions. Any function $\varphi \in \mathcal{P}$ has useful Herglotz representation

$$\varphi(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1 + ze^{-i\theta}}{1 - ze^{-i\theta}} d\mu(\theta) = \frac{1}{2\pi}\int_0^{2\pi} L(ze^{-i\theta}) d\mu(\theta),$$

where $\mu(\theta)$ is a probability measure on $\langle 0, 2\pi\rangle$ and $L(z) = (1 + z)/(1 - z)$ is the Möbius function. A significant subclass of the class $\mathcal{P}$ is a class of all functions which are real on $(-1, 1)$; we denote it here by $\mathcal{P}_{\mathbb{R}}$. Since $\varphi \in \mathcal{P}_{\mathbb{R}}$ has real coefficients then $\varphi(z) = \overline{\varphi(\bar{z})}$. Therefore [11]

$$\varphi(z) = \frac{\varphi(z) + \overline{\varphi(\bar{z})}}{2} = \frac{1}{2\pi}\int_0^{2\pi} \frac{1 - z^2}{1 - 2z\cos\theta + z^2} d\mu(\theta). \tag{1}$$

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Email addresses: skanas@ur.edu.pl (Stanisława Kanas), anna.tatarczak@poczta.umcs.lublin.pl (Anna Tatarczak)

Successively, the new subclasses of $\mathcal{P}$ and $\mathcal{P}_{\mathbb{R}}$ consisting of functions φ such that $\varphi(\mathbb{D})$ is a proper subdomain of a right halfplane appeared in the literature (see, for example [4]).

Now, we construct a natural extension of the class $\mathcal{P}_{\mathbb{R}}$ as follows. For $-1 \le p, q \le 1$ let

$$\varphi_1(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1+e^{i\theta}pz}{1-e^{i\theta}pz} d\mu(\theta) = \frac{1}{2\pi}\int_0^{2\pi} L(pze^{-i\theta}) d\mu(\theta),$$

$$\varphi_2(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1+e^{-i\theta}qz}{1-e^{-i\theta}qz} d\mu(\theta) = \frac{1}{2\pi}\int_0^{2\pi} \overline{L(q\bar{z}e^{-i\theta})} d\mu(\theta).$$

Then $\varphi_1, \varphi_2 \in \mathcal{P}$. Hence

$$\frac{1}{2}\left(\varphi_1(z) + \varphi_2(z)\right) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-pqz^2}{(1-e^{i\theta}pz)(1-e^{-i\theta}qz)} d\mu(\theta)$$

is also in $\mathcal{P}$, since $\mathcal{P}$ is convex. Therefore, we define the class $\mathcal{P}^{p,q}$ as the class of functions $\varphi \in \mathcal{P}$ that are of the form

$$\varphi(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-pqz^2}{(1-e^{i\theta}pz)(1-e^{-i\theta}qz)} d\mu(\theta), \tag{2}$$

where $\mu(\theta)$ is the unique probability measure on the interval $\langle 0, 2\pi\rangle$. We note that setting $p = q = 1$ the class $\mathcal{P}^{p,q}$ becomes $\mathcal{P}_{\mathbb{R}}$.

Let now $p = 1, q = -1$ or $p = -1, q = 1$. Then (2) reduces to

$$\varphi(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1+z^2}{1+2zi\sin\theta - z^2} d\mu(\theta), \tag{3}$$

and the classes $\mathcal{P}^{1,-1} = \mathcal{P}^{-1,1}$ consists of all functions φ, satisfying $\varphi(z) = \overline{\varphi(-\bar{z})}$, that is functions symmetric with respect to the imaginary axis.

We note that for $\varphi \in \mathcal{P}^{p,q}$ we immediately obtain

Proposition 1.1. *Let $-1 \le p, q \le 1$, and let $\varphi \in \mathcal{P}^{p,q}$. Then for $z = re^{it}$, $0 < r < 1$, it holds*

$$|\varphi(z)| \le \frac{1+|p||q|r^2}{(1-|p|r)(1-|q|r)}.$$

By the integral representation (2) we have that if $\varphi(z) = 1 + p_1 z + \cdots$, then

$$p_n = \frac{1}{2\pi}\int_0^{2\pi} (e^{in\theta}p^n + e^{-in\theta}q^n) d\mu(\theta).$$

Therefore, the immediate consequence follows, below.

Proposition 1.2. *Let $-1 \le p, q \le 1$, and let $\varphi \in \mathcal{P}^{p,q}$ is of the form $\varphi(z) = 1 + p_1 z + \cdots$, then*

$$|p_n| \le |p|^n + |q|^n. \tag{4}$$

We observe that the above bounds become known estimates in the class $\mathcal{P}$ in the case, when $|p| = |q| = 1$.

For the considered above range of parameters $-1 \le p, q \le 1$, let the generalized Chebyshev polynomials of the second kind be defined by the following generating function

$$\Psi_{p,q}(e^{i\theta}; z) = \frac{1}{(1 - pze^{i\theta})(1 - qze^{-i\theta})} = \sum_{n=0}^{\infty} U_n(p, q; e^{i\theta}) z^n \ (\theta \in \langle 0, 2\pi \rangle, \ |z| < 1). \tag{5}$$

The classical Chebyshev polynomials of the second kind are the well known; the above generalization was defined in [8]. From (5) we have

$$U_0(p, q; e^{i\theta}) = 1, \quad U_1(p, q; e^{i\theta}) = pe^{i\theta} + qe^{-i\theta},$$

and

$$U_{n+2}(p, q; e^{i\theta}) - (pe^{i\theta} + qe^{-i\theta}) U_{n+1}(p, q; e^{i\theta}) + pq U_n(p, q; e^{i\theta}) = 0 \ (n = 0, 1, \ldots).$$

We note that there is close relation between $\mathcal{P}^{p,q}$ and the generalized Chebyshev polynomials of the second kind, namely if $\varphi \in \mathcal{P}^{p,q}$, then

$$\varphi(z) = \frac{1}{2\pi} \int_0^{2\pi} (1 - pqz^2) \Psi_{p,q}(e^{i\theta}; z) d\mu(\theta).$$

2. Generalized Koebe Function

In this section we study the function that is strictly related to the generalized Chebyshev polynomials of the second kind as well as to the class $\mathcal{P}^{p,q}$, defined in the previous section. We describe a geometric properties of that function and find some its specific bounds.

Let $-1 \le p, q \le 1$. Consider the generalized Koebe function

$$k_{p,q}(z) = \frac{z}{(1 - pz)(1 - qz)} = \frac{1}{2(p + q)} \left[\frac{(1 + pz)(1 + qz)}{(1 - pz)(1 - qz)} - 1 \right] \ (z \in \mathbb{D}). \tag{6}$$

The case $p = q = 1$ leads to the famous standard Koebe function therefore, we will understand the function $k_{p,q}$ as its generalization. The function $k_{p,q}$ maps the unit disk onto a domain symmetric with respect to real axis. Indeed, the cases $p = q = -1$, as well as $p = q = 1$, are well known; such functions map the unit disk onto the complex plane without the slit $(-\infty, -1/4\rangle$ along the real axis, and slit $\langle 1/4, \infty)$, respectively. When p or q are zero, we get the function $k_{p,0}(z) = z/(1 - pz)$ that maps the unit disk onto a disk, that becomes the halfplane in the limiting cases $p = \pm 1$. The case $q = p$ leads to the function $k_{p,p}(z) = z/(1 - pz)^2$ that maps the unit disk onto the ordinary Pascal snail (scaling in the direction of the imaginary axis by some factor) which starts from the disk (small p), through dimpled cardioid that becomes a plane with the single slit in the limiting case $p \to \pm 1$. The other special case $k_{p,-p}(z) = z/(1 - p^2 z^2)$ maps the unit disk onto the Cassini oval which in the limiting case $p \to \pm 1$ reduces to the plane with two disjunctive slits from $i/2$, and $-i/2$ to infinity.

Consider now the cases $p, q \ne \pm 1$. Let

$$h_{p,q}(z) = \frac{(1 - pz)(1 - qz)}{z}.$$

It is easy to find that $h_{p,q}$ maps the unit disk onto the interior of the ellipse

$$\mathcal{E} = \left\{ w = u + iv : \frac{(u + (p + q))^2}{(1 + pq)^2} + \frac{v^2}{(1 - pq)^2} = 1 \right\},$$

that has a center at $(-(p+q),0)$, eccentricity $\varepsilon = 2\sqrt{|pq|}/(1+pq)$ $(pq \neq -1)$, and intersects the real axis at the points $((1-p)(1-q),0), (-(1+p)(1+q),0)$. The inverse transformation $T(w) = 1/w$ maps the ellipse onto the curve

$$\mathcal{E}' = \left\{u+iv : \frac{(u+(p+q)(u^2+v^2))^2}{(1+pq)^2} + \frac{v^2}{(1-pq)^2} = (u^2+v^2)^2\right\},$$

known as *Pascal snail (limaçon of Pascal)*.

In the case when one of the parameter p or q is zero, say $q = 0$, the curve $\mathcal{E}'$ is the circle with the center at $S = (p/(1-p^2),0)$ and the radius $R = 1/(1-p^2)$. In the other special case, when $p+q = 0$, we obtain the hippopede $(u^2+v^2)^2 = cu^2 + dv^2$, with $c = 1/(1+pq)^2, d = 1/(1-pq)^2$ that is the bicircular rational algebraic curve of degree 4, symmetric with respect to both axes. When $d > 0$ (that is our case) such a curve is known as an elliptic leminiscate of Booth. For the case $|p+q| = 1$ the Pascal snail is the conchoid of the circle, known also as a cardioid. For no case the hippopede is the Bernoulli leminiscate because it corresponds the case $d = -c$.

The equation of Pascal snail can be also transformed onto a form

$$\left[u^2+v^2-\frac{p+q}{(1-p^2)(1-q^2)}u\right]^2 = \frac{1}{(1-p^2)(1-q^2)}\left(u^2+\frac{(1+pq)^2}{(1-pq)^2}v^2\right).$$

The intersection points of the $\mathcal{E}'$ with the real axis are

$$\left(\frac{1}{(1-p)(1-q)},0\right) \text{ and } \left(-\frac{1}{(1+p)(1+q)},0\right).$$

Below, we present some images of the unit disk by $k_{p,q}$ for some special choice of the parameters.

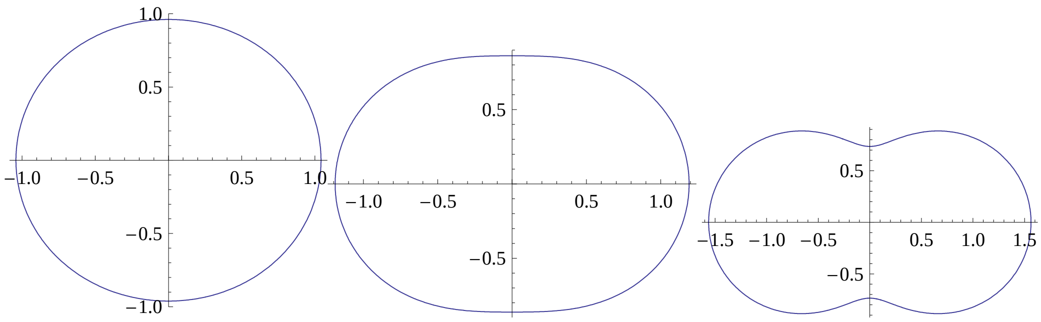

Fig. 1. The images of the unit disk by $k_{p,q}$ for $p+q = 0$.

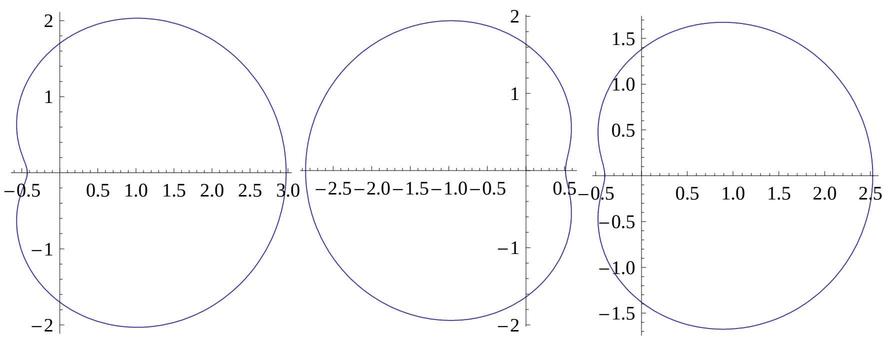

Fig. 2. The images of the unit disk by $k_{p,q}$ for $p+q = 1; p+q < 1; p+q > 1$.

The special case of the Pascal snail, it means an elliptic lemniscate (named later Booth lemniscate), was a topic of investigation by Booth ([1], [2] apparently given this name by Loria [6]). It has many interesting applications, for example, in mechanical linkages [14] and fluid physics [10]. It also appears in a solid geometry as an intersection of a plane with a spindle torus, or with Fresnel's elasticity surface [13].

We remain also, that in the case $q = 1/p$ and $p \in (0,1)$, the function $k_{p,q}$ maps the unit disk onto a complement of the interval $\langle -p/(1-p)^2, -p/(1+p)^2 \rangle$ on the negative axis. In the family $S(p)$ of meromorphic univalent functions with the normalization $f(0) = 0, f'(0) = 1$, and $f(p) = \infty$ the function

$$k_{p,\frac{1}{p}} = \frac{z}{\left(1 - \frac{1}{p}z\right)(1 - pz)}$$

plays role of the Koebe functions. For example, Jenkins [5] showed that, if $f \in S(p)$, and

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \text{ and } k_{p,1/p}(z) = z + \sum_{n=2}^{\infty} A_n z^n \quad (|z| < p),$$

then $a_n \leq A_n$ for any $n \geq 2$. Fenchel [3] solved the minimum modulus problem and showed that

$$\min_{|z| \to r} |f(z)| \geq |k_{p,1/p}(-r)| \quad \text{for all } f \in S(p).$$

However, when $q = 1/p$ and $p \in (0,1)$ we get $q > 1$ so that, this case is outside the considered interval.

Now, we discuss some particular properties of $k_{p,q}$. By the properties of the ellipse $\mathcal{E}$ it is easy to see that the points of the intersection with the real axis are $\left(-\frac{1}{(1+p)(1+q)}, 0\right)$ and $\left(\frac{1}{(1-p)(1-q)}, 0\right)$. Therefore we may expect that they are realize min and max of the real part. Although, as we can see on Fig. 1. and Fig. 2. this is not a case.

Proposition 2.1. *Let* $-1 < p, q < 1$. *The values of* $\max_{0 \leq t \leq 2\pi} \operatorname{Re} k_{p,q}(e^{it})$, $\min_{0 \leq t \leq 2\pi} \operatorname{Re} k_{p,q}(e^{it})$, *are the following*

$$\max_{0 \leq t \leq 2\pi} \operatorname{Re} k_{p,q}(e^{it}) = \begin{cases} \frac{1}{1-p} & for \quad q = 0, \\ \frac{1}{1-q} & for \quad p = 0, \\ \frac{1}{(1-p)(1-q)} & for \quad pq < 0, \\ \frac{1}{(1-p)^2} & for \quad q = p, \\ \frac{1}{(1-p)(1-q)} & for \quad p, q \in (0,1),\ p \neq q, \\ \frac{-(1+pq)^2}{2(1-pq)[(p+q)(1-pq)+2\sqrt{pq(1-p^2)(1-q^2)}]} & for \quad p, q \in (-1,0),\ p \neq q. \end{cases}$$

$$\min_{0 \leq t \leq 2\pi} \operatorname{Re} k_{p,q}(e^{it}) = \begin{cases} \frac{-1}{1+p} & for \quad q = 0, \\ \frac{-1}{1+q} & for \quad p = 0, \\ \frac{-1}{(1+p)(1+q)} & for \quad pq < 0, \\ \frac{-1}{(1+p)^2} & for \quad q = p, \\ \frac{(1+pq)^2}{2(1-pq)[2\sqrt{pq(1-p^2)(1-q^2)}-(p+q)(1-pq)]} & for \quad p, q \in (0,1),\ p \neq q, \\ \frac{-1}{(1+p)(1+q)} & for \quad p, q \in (-1,0),\ p \neq q. \end{cases}$$

Proof. Observe that

$$\operatorname{Re} k_{p,q}(e^{it}) = \frac{-(p+q) + (1+pq)\cos t}{(1 - 2p\cos t + p^2)(1 - 2q\cos t + q^2)} =: R(t),$$

so that the extremes of $\operatorname{Re} k_{p,q}(e^{it})$ are located at the critical points of the derivative of R with respect to t. The equation $R'(t) = 0$ is equivalent to the equation

$$\sin t \left\{4pq(1+pq)\cos^2 t - 8pq(p+q)\cos t + (1+pq)\left[(p+q)^2 - (1-pq)^2\right]\right\} = 0,$$

therefore the obvious critical points are $t = 0, t = \pi$, and the solutions of the equation

$$4pq(1+pq)\cos^2 t - 8pq(p+q)\cos t + (1+pq)\left[(p+q)^2 - (1-pq)^2\right] = 0.$$

If p or q are zero, say $q = 0$, then the only critical points remain $t = 0$ or $t = \pi$. For such t we have $\operatorname{Re} k_{p,q}(e^{it}) = 1/(1-p)$ and $-1/(1+p)$, respectively.

Let $x = \cos\, t$, then the above equality may be rewritten into the form

$$4pq(1+pq)x^2 - 8pq(p+q)x + (1+pq)\left[(p+q)^2 - (1-pq)^2\right] = 0.$$

Denote the above polynomial by $V(x)$.

Assume now, $pq \neq 0$. Then V is a nondegenerate quadratic polynomial with the discriminant $\Delta = 16pq(1-p^2)(1-q^2)(1-pq)^2$ which is negative for $pq < 0$, and positive when $pq > 0$. In the first case ($pq < 0$) we are going back to the critical points $t = 0$ or $t = \pi$, and for such t we obtain $\operatorname{Re} k_{p,q}(e^{it}) = 1/((1-p)(1-q))$, and $-1/((1+p)(1+q))$, respectively. In the second case (when $pq > 0$), there are two roots of V. We have to check that they satisfy the condition $|x| \leq 1$, because of the trigonometric substitution.

Note that the vertex of the parabola $V(x)$ has the coordinates

$$x_v = \frac{p+q}{1+pq}, \text{ and } y_v = -\frac{(1-pq)^2(1-p^2)(1-q^2)}{1+pq}.$$

It is easy to see that $|x_v| < 1$ and $y_v < 0$. Moreover,

$$V(-1) = (1+pq)(1-p^2)(1-q^2) + 8pq(1+p)(1+q) > 0$$

and

$$V(1) = (1+pq)(1-p^2)(1-q^2) + 8pq(1-p)(1-q) > 0.$$

From this follows that the roots are situated in the interval $(-1, 1)$, and its location depend on the value $V(0)$. The polynomial V possesses the roots of the form:

$$x_{1,2} = \frac{p+q}{1+pq} \pm \frac{1-pq}{2(1+pq)}\sqrt{\frac{(1-p^2)(1-q^2)}{pq}},$$

and assume $x_1 < x_2$. Let $\cos t_1 = x_1$, and $\cos t_2 = x_2$. The values of $\operatorname{Re} k_{p,q}(e^{it})$ at t_1, t_2 are

$$R_1 = R(t_1) = \frac{-(1+pq)^2}{2(1-pq)[2\sqrt{pq(1-p^2)(1-q^2)} + (p+q)(1-pq)]},$$

and

$$R_2 = R(t_2) = \frac{(1+pq)^2}{2(1-pq)[2\sqrt{pq(1-p^2)(1-q^2)} - (p+q)(1-pq)]}.$$

Moreover $R_2 < R_1 < 0$ for $p, q \in (0, 1)$, and $R_1 > R_2 > 0$ for $p, q \in (-1, 0)$.

Set also

$$R_\pi = R(\pi) = \frac{-1}{(1+p)(1+q)}, \; R_0 = R(0) = \frac{1}{(1-p)(1-q)}.$$

Consider first the case $p, q \in (0, 1)$, $p \neq q$. Then, we have $R_\pi, R_2, R_1 < 0$, $R_0 > 0$, and $R_2 < R_1$. Therefore, the maximum value of $\operatorname{Re} k_{p,q}(e^{it})$ is R_0, and minimum is $\min\{R_\pi, R_2\} = R_2$.

Let now $p, q \in (-1, 0), p \neq q$. Then $R_\pi < 0$, and $R_0, R_2, R_1 > 0$, hence $\min \operatorname{Re} k_{p,q}(e^{it}) = R_\pi$, and $\max \operatorname{Re} k_{p,q}(e^{it}) = \max\{R_0, R_1\} = R_1$. In conclusion we obtain the thesis. □

Proposition 2.2. *Let $-1 < p, q < 1$. The values of $\max_{0\le t\le 2\pi} |k_{p,q}(e^{it})|$, are the following*

$$\max_{0\le t\le 2\pi} |k_{p,q}(e^{it})| = \begin{cases} \frac{1}{(1-p)(1-q)} & for \quad q \ge -p, \\ \frac{1}{(1+p)(1+q)} & for \quad q \le -p. \end{cases}$$

Proof. We note that

$$|k_{p,q}(e^{it})|^2 = \frac{1}{(1+p^2-2p\cos t)(1+q^2-2q\cos t)} =: M(t).$$

Then

$$M'(t) = -\frac{2[(p+q)(1+pq)-4pq\cos t]\sin t}{(1+p^2-2p\cos t)^2(1+q^2-2q\cos t)^2}.$$

Hence $M'(t) = 0$ if and only if $[(p+q)(1+pq)-4pq\cos t]\sin t = 0$, for $t \in \langle 0, 2\pi\rangle$. The roots of the last equation are $t = 0$ or $t = \pi$, and the solutions of the equation

$$\cos t = \frac{(p+q)(1+pq)}{4pq},$$

if and only if the expression of the right hand side lies in the interval $\langle -1, 1\rangle$. Let t_0 be the solution of the last equation. We have

$$M(0) = \frac{1}{(1-p)^2(1-q)^2}, \quad M(\pi) = \frac{1}{(1+p)^2(1+q)^2},$$

and

$$M(t_0) = \frac{-4pq}{(q-p)^2(1-pq)^2} \quad for \quad pq < 0.$$

Comparing all values we conclude that $M(t_0) < M(0)$ and $M(t_0) < M(\pi)$, therefore the thesis holds. □

Proposition 2.3. *Let $-1 < p, q < 1$. The values of $\max_{0\le t\le 2\pi} |\mathrm{Im} k_{p,q}(e^{it})|$ are the following*

$$\max_{0\le t\le 2\pi} \left|\mathrm{Im}\, k_{p,q}(e^{it})\right| = \begin{cases} \frac{1}{1-p^2} & for & q = 0, \\ \frac{1}{1-q^2} & for & p = 0, \\ \frac{1-p^2}{(1+p^2)^2} & for & p \in (1-\sqrt{2}, \sqrt{2}-1) \setminus \{0\}, q = -p, \\ \frac{1}{4|p|} & for & p \in (-1, 1-\sqrt{2}) \cup (\sqrt{2}-1, 1), q = -p, \\ \Phi(t_i) & for & the\ remaining\ cases, \end{cases}$$

where $\Phi(t)$ is a function given by the equality

$$\Phi(t) = \frac{(1+pq)\sin t}{(1-2p\cos t+p^2)(1-2q\cos t+q^2)},$$

and t_i $(i = 1, 2)$ is a solution of the equation

$$-4pqx^3 + [8pq + (1+p^2)(1+q^2)]x - 2(p+q)(1+pq) = 0,$$

from the interval $\langle 0, \pi/2)$, and $\langle \pi/2, \pi)$ (with $x = \cos t$), respectively.

Proof. We note that $\mathrm{Im}\, k_{p,q}(e^{it})$ equals

$$\mathrm{Im}\, k_{p,q}(e^{it}) = \frac{(1+pq)\sin t}{(1-2p\cos t+p^2)(1-2q\cos t+q^2)} =: \Phi(t),$$

and attains its maximum at the point in which

$$-4pqx^3 + [8pq + (1 + p^2)(1 + q^2)]x - 2(p + q)(1 + pq) = 0, \tag{7}$$

where $x = \cos\, t$.

In the case when $pq = 0$, say $q = 0$, we immediately have

$$\max_{0\le t\le 2\pi} \left|\operatorname{Im} k_{p,q}(e^{it})\right| = \frac{1}{1-p^2}.$$

Assume now $pq \neq 0$ and let $a = p + q$, $b = pq$ (hence $b \neq 0$). Then the equation (7) may be rewritten into the form

$$-4bx^3 + [a^2 + (b + 3)^2 - 8]x - 2a(b + 1) = 0, \tag{8}$$

where $-2 < a < 2$, $-1 < b < 1$, $b \neq 0$. Denote the above polynomial by $W(x)$. The derivative $W'(x) = -12bx^2 + [a^2 + (b + 3)^2 - 8]$ has the roots satisfying $x^2 \le 1$ only if $a^2 + (b - 3)^2 \le 8$ that is equivalent to $(p - q)^2 + (1 - pq)^2 \le 0$, which is impossible by the assertions. Therefore the sign of W' is constant on $(-1, 1)$. Since

$$W(-1) = -(a + b + 1)^2 = -(1 + p)^2(1 + q)^2 < 0,$$
$$W(1) = (a - b + 1)^2 = (p + q - pq + 1)^2 > 0,$$

and the sign of W' is constant in $(-1, 1)$ we conclude that W increases in the entire interval. Because of the fact that $W(0) = -2a(b + 1)$ the single root is located at $(-1, 0)$ or $(0, 1)$. Next two roots, if there exist, are outside the interval $(-1, 1)$.

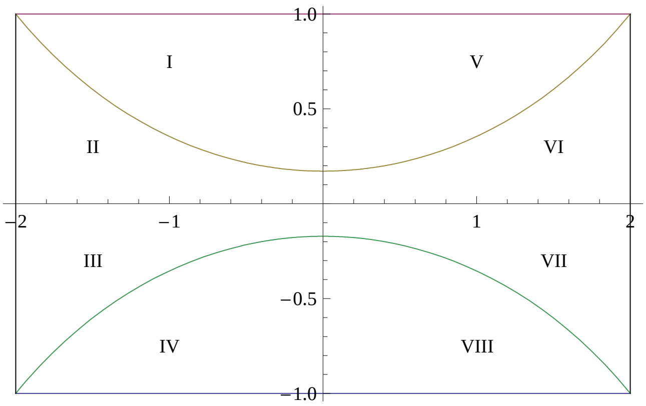

Fig. 3. The range of the parameters a, b.

Consider now the location of the roots of the polynomial W.

1. $a = 0$, then $q = -p$. This assertion imposes the second one, namely $pq < 0$, that is $b < 0$. The equation (8) now reduces to the following

$$-4bx^3 + [(b + 1)^2 + 4b]x = 0,$$

from that $x = 0$ or $x^2 = \frac{(b+1)^2+4b}{4b}$. By the trigonometric substitution $x = \cos\, t$, it must hold

$$\frac{(b + 1)^2 + 4b}{4b} = 1 + \frac{(b + 1)^2}{4b} \le 1 \quad \text{and} \quad 1 + \frac{(b + 1)^2}{4b} \ge 0,$$

that is satisfied only when $b \in (-1, 2\sqrt{2} - 3)$. Thus, for $b \in (-1, 2\sqrt{2} - 3)$ the polynomial W has three roots, and a single root when $b \notin (-1, 2\sqrt{2} - 3)$.

2. $a < 0, b > 0$, and $a^2 + (b-3)^2 < 8$ (a domain I). Since $a^2 + (b-3)^2 < 8$ is equivalent to $(p-q)^2 + (1-pq)^2 < 0$ the case leads to the contradiction.
3. $a < 0, b > 0$, and $a^2 + (b-3)^2 > 8$ (a domain II). Since then also $a^2 + (b+3)^2 > 8$ the sign change of the coefficients for $W(x)$ is $(-, +, +)$ and $(+, -, +)$ for $W(-x)$. Hence, by the Descartes-Harriot rule, there are one positive and two negative roots. By the fact that $W(-1) < 0, W(0) > 0$ the root is located in $(-1, 0)$. Next two zeros are located in $(-\infty, -1)$ and $(1, \infty)$.
4. $a < 0$, $b < 0$, and $a^2 + (b+3)^2 > 8$ (a domain III). The sign change of the coefficients is $(+, +, +)$ for $W(x)$ and $(-, -, +)$ for $W(-x)$. Therefore W has no positive roots, and one negative zero, that belongs to $(-1, 0)$.
5. $a < 0$, $b < 0$, and $a^2 + (b+3)^2 < 8$ (a domain IV). Then the sign change of the coefficients of $W(x)$ is $(+, -, +)$ and $(-, +, +)$ for $W(-x)$. Hence there are two positive and one negative root. By the fact that $W(-1) < 0, W(0) > 0$ the root is located in $(-1, 0)$. Next two zeros, if there exist, are located in $(1, \infty)$.
6. $a > 0$, $b > 0$, and $a^2 + (b-3)^2 < 8$ (a domain V). This case leads to the contradiction by the similar argument as in the case (2).
7. $a > 0$, $b > 0$, and $a^2 + (b-3)^2 > 8$, then $a^2 + (b+3)^2 > 8$ (a domain VI). Then the sign change of the coefficients is $(-, +, -)$ for $W(x)$ and $(+, -, -)$ for $W(-x)$. Hence there are two positive and one negative root. By the fact that $W(0) < 0, W(1) > 0$, one root is located in $(0, 1)$. Next two zeros are located in $(-\infty, -1)$ and $(1, \infty)$.
8. $a > 0$, $b < 0$, and $a^2 + (b+3)^2 > 8$ (a domain VII). Then the sign change of coefficients is $(+, +, -)$ for $W(x)$ and $(-, -, -)$ for $W(-x)$. Hence there are one positive and no negative root. By the fact that $W(0) < 0, W(1) > 0$ the root is located in $(0, 1)$.
9. $a > 0, b < 0, a^2 + (b+3)^2 < 8$, then also $a^2 + (b-3)^2 > 8$. (a domain VIII). Thus the sign change of the coefficients $(+, -, -)$ for $W(x)$ and $(-, +, -)$ for $W(-x)$. Hence there are one positive and two negative roots. By the fact that $W(0) < 0, W(1) > 0$ the root is located in $(0, 1)$. Next two zeros, if there exist, are located in $(-\infty, -1)$.
10. $a^2 + (b+3)^2 = 8$, so that $b < 0$. In this case a root is given by the equation

$$x^3 = -\frac{a(b+1)}{2b} \in (-1, 1) \setminus \{0\},$$

and this condition is satisfied for $2 - \frac{2}{b+1} < a < -2 + \frac{2}{b+1}$, with $b < 0$.

In terms of parameters p and q the domains I - VIII are represented in a Fig. 4. Since the cases (2) and (6) lead to the contradiction (domains I and V), we study only the remaining cases, and domains I, V are not marked on a Fig. 4.

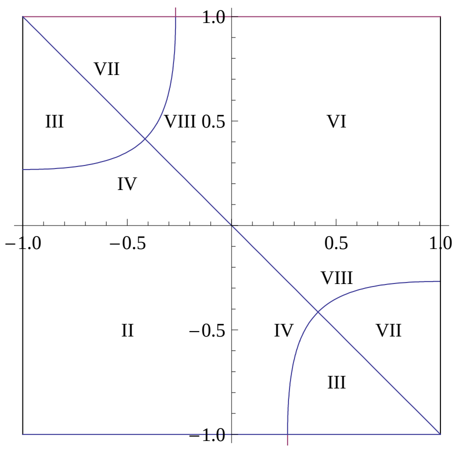

Fig. 4. The range of the parameters p, q that correspond to the range of a, b.

We note that each root of the polynomial $W(x)$ corresponds to two roots of the polynomial $W(\cos t)$ in $(0, 2\pi)$ but, by the symmetry, we may consider only the interval $(0, \pi)$. Observe next, that the function $W(\cos t)$ is decreasing in the interval $(0, \pi)$, as a function of the variable t, therefore the root of the polynomial $W(x)$ located in the interval $(-1, 1)$ is the local maximum of $W(\cos t)$.

Now, we analyze the polynomial $W(\cos\ t)$ as regards to the variable t.

In the case (1) we obtain two subcases. The first correspond to $p \in (1-\sqrt{2}, \sqrt{2}-1) \setminus \{0\}$. In this case there is the unique critical point $x = 0$, and $\max\limits_{0\le t\le 2\pi} \left|\operatorname{Im} k_{p,q}(e^{it})\right| = \frac{1-p^2}{(1+p^2)^2}$. The second subcase corresponds to $p \in (-1, 1-\sqrt{2}) \cup (\sqrt{2}-1, 1)$, and $q = -p$, and then there are three critical points: $x = 0$ and $x = \pm\sqrt{\frac{(b+1)^2+4b}{4b}}$ from which $\max |\operatorname{Im} k_{p,q}|$ is attained at the point t corresponding to $x = \pm\sqrt{\frac{(b+1)^2+4b}{4b}}$ and the maximum then equals $\frac{1}{4|p|}$.

The case (3) (set II, Fig. 4.) corresponds to the set

$$\{(p, q) : q < -p, pq > 0, (p + q)^2 + (pq - 3)^2 > 8\} = \{(p, q) : p \in (-1, 0), q \in (-1, 0)\}.$$

In this instance there is a single solution of (8), that corresponds to a root $t_0 \in (\pi/2, \pi)$, where $\max |\operatorname{Im} k_{p,q}(e^{it})|$ is attained.

The case (4) (domain III, Fig. 4.) corresponds to the set

$$\begin{aligned} &\{(p, q) : q < -p, pq < 0, (p + q)^2 + (pq + 3)^2 > 8\} \\ = &\left\{(p, q) : p \in (-1, \sqrt{3}-2), \frac{-4p-\sqrt{16p^2-(p^2+1)^2}}{1+p^2} \le q < -p\right\}. \end{aligned}$$

In this instance there is single solution of (8), that is a single zero in $t_0 \in (\pi/2, \pi)$, where $\max |\operatorname{Im} k_{p,q}(e^{it})|$ is attained.

The case (5) (set IV, Fig. 4.) corresponds to

$$\begin{aligned} &\{(p, q) : q < -p, pq < 0, (p + q)^2 + (pq + 3)^2 < 8\} \\ = &\left\{(p, q) : p \in (-1, 1-\sqrt{2}), 0 < q < \frac{-4p-\sqrt{16p^2-(p^2+1)^2}}{1+p^2}\right\} \\ &\cup\{(p, q) : p \in (1-\sqrt{2}, 0), 0 < q < -p\}. \end{aligned}$$

In this case there is single zero of (8), and a single $t_0 \in (\pi/2, \pi)$, where $\max |\operatorname{Im} k_{p,q}(e^{it})|$ is attained.

The case (9) (domain VIII, Fig. 4.) corresponds to a set

$$\begin{aligned} &\{(p, q) : q > -p, pq < 0, (p + q)^2 + (pq + 3)^2 < 8\} \\ = &\left\{(p, q) : p \in (1-\sqrt{2}, \sqrt{3}-2), -p < q < \frac{-4p-\sqrt{16p^2-(p^2+1)^2}}{1+p^2}\right\} \\ &\cup\{(p, q) : p \in (\sqrt{3}-2, 0), -p < q < 1\}, \end{aligned}$$

and a root of W is in the interval $(0, \pi/2)$.

The case (7) (set VI, Fig. 4.) is symmetric to the case (3) and corresponds to the set $\{(p, q) : q > -p, pq > 0\}$.

The case (8) (domain VII, Fig. 4.) corresponds to the set

$$\{(p, q) : q > -p, pq < 0, (p + q)^2 + (pq + 3)^2 > 8\},$$

and a root of W lies in the interval $(0, \pi/2)$.

For the domains II - IV there is only one zero in $(\pi/2, \pi)$, say t_2, and for the domains VI - VIII there is only one zero in $(0, \pi/2)$, say t_1. Then $\max |\operatorname{Im} k_{p,q}(e^{it})| = \Phi(t_i)$, where t_i is the only zero in the respective interval $(i = 1, 2)$.

Reassuming, we obtain the thesis. □

It is easy to find the coefficients of $k_{p,q}$ since

$$k_{p,q}(z) = \frac{1}{p-q}\left(\frac{1}{1-pz} - \frac{1}{1-qz}\right),$$

so that for $n = 1, 2, \cdots$

$$a_n = \begin{cases} \dfrac{p^n - q^n}{p-q} & for \quad p \neq q, \\ np^{n-1} & for \quad q = p. \end{cases}$$

Also, we note that

$$k_{p,q}(z) = \frac{z}{(1-pz)(1-qz)} = z\Psi_{p,q}(1; z) = z\sum_{n=0}^{\infty} U_n(p, q; 1)z^n,$$

so that there is a close relationship between generalized Koebe function and generalized Czebyshev polynomials.

As we can see the domain bounded by the ellipse $\mathcal{E}'$ is convex or convex in the direction of real axis for some cases of the parameters p and q. The same concerns the starlikeness of $k_{p,q}$. Indeed, in [7] it was proved that the function $k_{p,q}$ is α - starlike in $\mathbb{D}$ with

$$\alpha = \alpha(p, q) = \frac{1}{2}\left(\frac{1-|p|}{1+|p|} + \frac{1-|q|}{1+|q|}\right),$$

and convex in the disk $|z| < r_c(p, q)$, where

$$r_c(p, q) = \frac{2}{t + \sqrt{t^2 - 4|p||q|}} \quad and \quad t = \frac{|p| + |q| + \sqrt{|p|^2 + |q|^2 + 34|p||q|}}{2}.$$

Lemma 2.4. *Let $-1 \leq p, q \leq 1$, $|pq| \neq 1$. The function $k_{p,q}$ is convex of order α in $\mathbb{D}$ for*

$$\alpha(p, q) = \frac{2(1-|pq|)}{(1+|p|)(1+|q|)} - \frac{1+|pq|}{1-|pq|}. \tag{9}$$

Proof. We note that

$$1 + \frac{zk''_{p,q}(z)}{k'_{p,q}(z)} = \frac{1+pz}{1-pz} + \frac{1+qz}{1-qz} - \frac{1+pqz^2}{1-pqz^2},$$

hence, applying

$$\frac{1-r}{1+r} \leq \mathrm{Re}\frac{1+z}{1-z} \leq \frac{1+r}{1-r}, \ |z| = r < 1,$$

we find that

$$\mathrm{Re}\left(1 + \frac{zk''_{p,q}(z)}{k'_{p,q}(z)}\right) \geq \frac{1-|p|}{1+|p|} + \frac{1-|q|}{1+|q|} - \frac{1+|p||q|}{1-|p||q|}.$$

The function is convex of order α, if the inequality

$$\frac{1-|p|}{1+|p|} + \frac{1-|q|}{1+|q|} - \frac{1+|p||q|}{1-|p||q|} > \alpha$$

holds, it means if $\alpha < \alpha(p, q)$, where

$$\alpha < \frac{2(1-|pq|)}{(1+|p|)(1+|q|)} - \frac{1+|pq|}{1-|pq|}.$$

Since the right hand side is less than 1, the assertion follows. □

3. Generalized Typically Real Functions

Let $\mathcal{T}_{\mathbb{R}}$ denote the class of functions that are analytic in $\mathbb{D} = \{z : |z| < 1\}$, and satisfying the condition $\operatorname{Im}\{f(z)\}\operatorname{Im}\{z\} \geq 0$ in $\mathbb{D}$. Thus $\mathcal{T}_{\mathbb{R}}$ is the class of typically real functions introduced by Rogosinski [12]. Robertson [11] has shown that, if $f \in \mathcal{T}_{\mathbb{R}}$, then

$$f(z) = \int_{-1}^{1} \frac{z d\mu(t)}{1 - 2tz + z^2} \qquad (z \in \mathbb{D}),$$

where $\mu(t)$ is the unique probability measure on the segment $\langle -1, 1\rangle$. By the Rogosinski's result [12] we know that a (normalized) typically real function f can be written as

$$f(z) = \frac{z}{1 - z^2}\varphi(z)$$

for some $\varphi \in \mathcal{P}_{\mathbb{R}}$, and that the set of extreme points of the class $\mathcal{T}_{\mathbb{R}}$ consists of the functions

$$\varphi_t(z) = \frac{z}{1 - 2tz + z^2} \qquad (t \in \langle -1, 1\rangle).$$

The class $\mathcal{T}_{\mathbb{R}}$ was intensively studied because of the Bieberbach conjecture, and the relation $\mathcal{T}_{\mathbb{R}} = \overline{co}\,\mathcal{S}_{\mathbb{R}}$, where $\mathcal{S}_{\mathbb{R}}$ denotes the class of univalent functions in $\mathbb{D}$ with real coefficients, and $\overline{co}\,\mathcal{S}_{\mathbb{R}}$ denotes the closed convex hull of $\mathcal{S}_{\mathbb{R}}$.

Let $\mathcal{T}^{p,q}$ be the class of *generalized typically-real functions* [7], that is the class of all functions analytic in the unit disk, and given by

$$f(z) = \frac{1}{2\pi}\int_{0}^{2\pi} \frac{z d\mu(\theta)}{(1 - e^{i\theta}pz)(1 - e^{-i\theta}qz)} \qquad (z \in \mathbb{D}), \tag{10}$$

where $\mu(\theta)$ is the probability measure on $\langle 0, 2\pi\rangle$.

We note that there is a close connection between $\mathcal{P}^{p,q}$ and $\mathcal{T}^{p,q}$, below.

Theorem 3.1. *Let $-1 \leq p, q \leq 1$. If $g \in \mathcal{P}^{p,q}$ then*

$$f(z) = \frac{z}{1 - pqz^2}g(z) \in \mathcal{T}^{p,q}.$$

Conversely, if $f \in \mathcal{T}^{p,q}$, then

$$g(z) = \frac{1 - pqz^2}{z}f(z) \in \mathcal{P}^{p,q}. \tag{11}$$

Proof. Let $f \in \mathcal{T}^{p,q}$. Then the function

$$g(z) = \frac{1 - pqz^2}{z}f(z) = 1 + a_2 z + (a_3 - pq)z^2 + \cdots$$

is regular in $\mathbb{D}$ since the pole at $z = 0$ is canceled by the zero of f at the same point. Further $g(0) = 1$. Also, we note that

$$g(z) = \frac{1 - pqz^2}{z}\frac{1}{2\pi}\int_{0}^{2\pi} \frac{z d\mu(\theta)}{(1 - e^{i\theta}pz)(1 - e^{-i\theta}qz)} = \frac{1}{2\pi}\int_0^{2\pi} \frac{1 - pqz^2}{(1 - e^{i\theta}pz)(1 - e^{-i\theta}qz)}d\mu(\theta).$$

Then $g \in \mathcal{P}^{p,q}$.

Conversely suppose $g \in \mathcal{P}^{p,q}$ and $f(z) = zg(z)/(1 - pqz^2)$. The function

$$f(z) = \frac{z}{1 - pqz^2} g(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{z}{(1 - e^{i\theta}pz)(1 - e^{-i\theta}qz)} d\mu(\theta)$$

has the standard normalization $f(0) = 0 = f'(0) - 1$ and is univalent in $\mathbb{D}$. □

We observe that $\mathcal{T}^{1,1} = \mathcal{T}_{\mathbb{R}}$ and $\mathcal{T}^{1,0} = \mathcal{T}^{0,1} = \overline{co}\,\mathcal{CV}$.

Remark 3.1. Let $z = e^{it}$, and $f \in \mathcal{T}^{p,q}$. Then, we have

$$\begin{aligned} \mathrm{Re}\frac{1 - pqz^2}{z} f(z) &= \mathrm{Re}\left(e^{-it} - pqe^{it}\right) f(e^{it}) \\ &= (1 - pq)\cos\, t\,\mathrm{Re} f(e^{it}) + (1 + pq)\sin\, t\,\mathrm{Im} f(e^{it}) \end{aligned}$$

so that $\mathrm{Re}\frac{1 - pqz^2}{z} f(z) > 0$, if and only if (extending the inequality to the whole $\mathbb{D}$)

$$(1 - pq)\mathrm{Re}\{z\}\mathrm{Re}\{f(z)\}) + (1 + pq)\mathrm{Im}\{f(z)\}\,\mathrm{Im}\{z\} > 0. \tag{12}$$

We see that if $pq = 1$ the above condition reduces to the following

$$\mathrm{Im}\{f(z)\}\,\mathrm{Im}\{z\} \geq 0 \quad (z \in \mathbb{D}), \tag{13}$$

that is a geometric interpretation of a class $\mathcal{T}_{\mathbb{R}}$. Also, when $pq = -1$, we obtain

$$\mathrm{Re}\{z\}\mathrm{Re}\{f(z)\} > 0 \quad (z \in \mathbb{D}), \tag{14}$$

that means that the image of the unit disk under f is symmetric with respect to the imaginary axis.
In the remaining cases the condition (12) may be rewritten in the form

$$\frac{1 - pq}{2}\mathrm{Re}\{z\}\mathrm{Re}\{f(z)\} + \frac{1 + pq}{2}\mathrm{Im}\{f(z)\}\,\mathrm{Im}\{z\} > 0. \tag{15}$$

We observe that (15) constitute an arithmetic means of (13) and (14), thus the class $\mathcal{T}^{p,q}$ provides an arithmetic bridge between the classes of normalized functions symmetric with respect to the imaginary and real axis.

We note that, directly from an integral representation (10) and Propositions from the Section 2 we may solve several extremal problems in the class $\mathcal{T}^{p,q}$, below.

Theorem 3.2. *Let $-1 \leq p, q \leq 1$, $pq \neq 0$. If $f \in \mathcal{TR}^{p,q}$, then*

$$|f(re^{i\theta})| \leq \frac{r}{(1 - |p|r)(1 - |q|r)}. \tag{16}$$

The result is sharp; the equality is attained for $k_{p,q}(r)$ for $q \leq -p$ and $-k_{p,q}(-r)$ for $q \geq -p$. Moreover

$$|f(z)| \leq \begin{cases} \frac{1}{(1-p)(1-q)} & for \quad p \in (-1, 1),\ q \geq -p, \\ \frac{1}{(1+p)(1+q)} & for \quad p \in (-1, 1),\ q \leq -p, \end{cases} \tag{17}$$

$$\mathrm{Im}\, f(z) \leq \max_{0 \leq t \leq 2\pi} \left|\mathrm{Im}\, k_{p,q}(e^{it})\right|, \tag{18}$$

and

$$\min_{0 \leq t \leq 2\pi} \mathrm{Re}\, k_{p,q}(e^{it}) \leq \mathrm{Re}\, f(z) \leq \max_{0 \leq t \leq 2\pi} \mathrm{Re}\, k_{p,q}(e^{it}), \tag{19}$$

where $\max_{0 \leq t \leq 2\pi} \mathrm{Re}\, k_{p,q}(e^{it})$, $\min_{0 \leq t \leq 2\pi} \mathrm{Re}\, k_{p,q}(e^{it})$ are described in the Proposition 2.1, and $\max_{0 \leq t \leq 2\pi} \left|\mathrm{Im}\, k_{p,q}(e^{it})\right|$ is given in the Proposition 2.3.

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