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Abstract

It has been shown that the uniformly minimum variance unbiased (UMVU) estimator of the generalized variance always exists for any natural exponential family. In practice, however, this estimator is often difficult to obtain. This paper explicitly identifies the results in complete bivariate and symmetric multivariate gamma models, which are diagonal quadratic exponential families. For the non-independent multivariate gamma models, it is then pointed out that the UMVU and the maximum likelihood estimators are not proportional as conjectured for models belonging in certain quadratic exponential families.

AMS 2000 subject classification: 62F10; 62H12; 62H99

Keywords: Determinant; Diagonal variance function; Maximum likelihood estimator; Natural exponential family; UMVU estimator.

1 Introduction

Generalized variance estimators have been, for a long time, based on the determinant of the sample covariance matrix. Generally biased, some of properties of the sample generalized variance are known, in particular, under the normal distribution hypothesis. See e.g. [11] and [20] and references therein.

In the context of natural exponential families (NEFs) on $\mathbb{R}^d$ which include many usual distributions (Kotz et al. [16], Chap. 54), a common estimator

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of the generalized variance is obtained by considering the key result in Kokonendji and Seshadri [15] which we recall in the following proposition; see also [13] and [14] for this complete version. Let $\mathcal{M}(\mathbb{R}^d)$ denotes the set of σ-finite positive measures μ on $\mathbb{R}^d$ not concentrated on an affine subspace of $\mathbb{R}^d$, with the Laplace transform of μ given by

$$L_\mu(\boldsymbol{\theta}) = \int_{\mathbb{R}^d} \exp(\boldsymbol{\theta}^T\mathbf{x})\,\mu(d\mathbf{x})$$

and such that the interior $\Theta(\mu)$ of the domain $\{\boldsymbol{\theta} \in \mathbb{R}^d; L_\mu(\boldsymbol{\theta}) < \infty\}$ is non-empty. Defining the cumulant function as $K_\mu(\boldsymbol{\theta}) = \log L_\mu(\boldsymbol{\theta})$, the NEF generated by $\mu \in \mathcal{M}(\mathbb{R}^d)$, denoted by $F = F(\mu)$, is the family of probability measures $\{\mathcal{P}_{\boldsymbol{\theta},\mu}(d\mathbf{x}) = \exp[\boldsymbol{\theta}^\intercal\mathbf{x} - K_\mu(\boldsymbol{\theta})]\,\mu(d\mathbf{x}); \boldsymbol{\theta} \in \Theta(\mu)\}$.

Proposition 1 *Let $\mu \in \mathcal{M}(\mathbb{R}^d)$. Then, for all integers $n \geq d+1$, there exists a positive measure $\nu_n = \nu_n(\mu)$ on $\mathbb{R}^d$ satisfying the three following statements:*
(i) the measure ν_n is the image of

$$\frac{1}{(d+1)!}\left(\det\begin{bmatrix} 1 & 1 & \cdots & 1 \\ \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_{d+1}\end{bmatrix}\right)^2 \mu(d\mathbf{x}_1)...\mu(d\mathbf{x}_n)$$

by the map $(\mathbf{x}_1, ..., \mathbf{x}_n) \longmapsto \mathbf{x}_1 + ... + \mathbf{x}_n$;
(ii) the Laplace transform of ν_n is given by

$$L_{\nu_n}(\boldsymbol{\theta}) = (L_\mu(\boldsymbol{\theta}))^n \det K''_\mu(\boldsymbol{\theta}), \quad \boldsymbol{\theta} \in \Theta(\mu) \tag{1}$$

where $K''_\mu(\boldsymbol{\theta}) = \partial^2 K_\mu(\boldsymbol{\theta})/(\partial\boldsymbol{\theta}^T\partial\boldsymbol{\theta})$ is the Hessian matrix of $K_\mu(\boldsymbol{\theta})$;
(iii) there exists $C_n : \mathbb{R}^d \longrightarrow \mathbb{R}$ such that

$$\nu_n(d\mathbf{x}) = C_n(\mathbf{x})\,\mu^{*n}(d\mathbf{x}), \tag{2}$$

where μ^{*n} denotes the n-th convolution power of μ.

We also recall that any NEF can be reparametrized in terms of the mean $\mathbf{m}$ such that

$$\mathbf{m} = \mathbf{m}(\boldsymbol{\theta}) = \mathbb{E}_{\boldsymbol{\theta}}(\mathbf{X}) = \frac{\partial K_\mu(\boldsymbol{\theta})}{\partial\boldsymbol{\theta}} = K'_\mu(\boldsymbol{\theta}),$$

where $\mathbf{X}$ is a random vector distributed according to a $\mathcal{P}_{\boldsymbol{\theta},\mu}$ in F. The mean domain $M_F = K'_\mu(\Theta(\mu))$ depends only on F, and not on the choice of the generating measure μ of F; so we can write $F = \{\mathcal{P}(\mathbf{m}, F); \mathbf{m} \in M_F\}$. The function

$$V_F(\mathbf{m}) = Var_{\boldsymbol{\theta}(\mathbf{m})}(\mathbf{X}) = K''_\mu(\boldsymbol{\theta}(\mathbf{m})), \quad \mathbf{m} \in M_F$$

is called the *variance function* of the family F. Here $\boldsymbol{\theta}(\cdot)$ denotes the inverse of the mapping $\mathbf{m}(\boldsymbol{\theta}) = K'_\mu(\boldsymbol{\theta})$. The pair $(V_F(\cdot), M_F)$ characterizes F within the class of all NEFs.

Thus, the authors ([13], [14] and [15]) of Proposition 1 have shown that

$$C_n\left(n\overline{\mathbf{X}}_n\right) = C_n(\mathbf{X}_1 + ... + \mathbf{X}_n) \tag{3}$$

is the uniformly minimum variance unbiased (UMVU) estimator of the generalized variance $\det V_F(\mathbf{m}) = \det K''_\mu(\boldsymbol{\theta})$ based on $n \geq d+1$ observations $\mathbf{X}_1, ..., \mathbf{X}_n$ of $\mathcal{P}(\mathbf{m}, F)$. Obviously, the crucial problem of this estimator (3) is to exhibit $C_n(\cdot)$ defined in (2). In the previous papers we only find the explicit expressions of $C_n(\cdot)$ for NEFs having homogeneous and simple quadratic variance functions of Casalis ([5] and [6]). Pommeret [19] provides another construction of the generalized variance UMVU estimator which is limited to the simple quadratic NEFs. Moreover, in order to compare the UMVU estimator $C_n(n\overline{\mathbf{X}}_n)$ to the maximum likelihood (ML) estimator $\det V_F(\overline{\mathbf{X}}_n)$ of $\det V_F(\mathbf{m})$, Kokonendji and Pommeret [14] have conjectured the following proportionality which holds for all homogeneous and simple quadratic NEFs: there exists $\beta_n > 0$ such that $C_n(n\overline{\mathbf{X}}_n) = \beta_n \det V_F(\overline{\mathbf{X}}_n)$ if and only if there exists $(a, \mathbf{b}, c) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}$ such that the canonical generalized variance is

$$\det K''_\mu(\boldsymbol{\theta}) = \exp\left\{aK_\mu(\boldsymbol{\theta}) + \mathbf{b}^T\boldsymbol{\theta} + c\right\}, \quad \boldsymbol{\theta} \in \Theta(\mu). \tag{4}$$

The condition (4) is used by Consonni et al. [7] for references prior analysis of the simple quadratic NEFs and by Kokonendji and Masmoudi [12] (with $a = 0$) for starting the characterization of the corresponding NEFs. We note in passing that if μ or $F = F(\mu)$ is infinitely divisible then there exists a positive measure $\rho = \rho(\mu)$ such that $L_\rho(\boldsymbol{\theta}) = \det K''_\mu(\boldsymbol{\theta})$, for all $\boldsymbol{\theta} \in \Theta(\mu)$ [9].

Motivated by the recent result of Bernardoff [4] "Which multivariate gamma distributions are infinitely divisible?" and the use of a multivariate gamma NEF for mixing Poisson distribution by Ferrari et al. [8], this paper is devoted to the UMVU and ML estimators of some generalized variances under the multivariate gamma hypothesis. Considered by the previous authors to be the natural multivariate extension of the real gamma NEF, the following multivariate gamma models belong to the diagonal quadratic NEFs in the sense of Bar-Lev et al. [2] (see Proposition 2) and, however, does not hold the condition (4) of proportionality between UMVU and ML estimators of the generalized variance for certain quadratic NEFs. The present paper is structured as follows. In Section 2, definition and variance function of d-dimensional gamma NEFs are given. In Section 3, particular cases of the generalized variance for a bivariate and a symmetric multivariate gamma families are presented. In Section 4, the UMVU estimator of the generalized variance in the bivariate case is pointed out and compared to the corresponding ML estimator. We shall show that these two estimators are not proportional. In Section 5, symmetric multivariate gamma models are considered. We shall observe that its corresponding bivariate is trivially a particular case of the one of the previous section. Finally, Section 6 concludes on independent multivariate case. To make easy the reading of the results all proofs are collected in the Appendix.

2 Multivariate gamma NEFs

For $d = 1$, for λ and $a > 0$ the real gamma distribution with shape parameter λ and scale parameter a is

$$\boldsymbol{\gamma}_{\lambda,a}(dx) = \frac{x^{\lambda-1}e^{-x/a}}{a^{\lambda}\Gamma(\lambda)}1_{(0,\infty)}(x)dx.$$

This is an element of the univariate gamma NEF $F = F(\mu_\lambda)$ generated by $\mu_\lambda(dx) = (\Gamma(\lambda))^{-1}x^{\lambda-1}1_{(0,\infty)}(x)dx$, which is characterized by its variance function $V_F(m) = m^2/\lambda$, $m \in (0,\infty) = M_F$ (see Morris [17]). We also note $F = F(\boldsymbol{\gamma}_{\lambda,a})$. The Laplace transform of $\boldsymbol{\gamma}_{\lambda,a}$ is $L_{\boldsymbol{\gamma}_{\lambda,a}}(\theta) = (1 - a\theta)^{-\lambda}$ for a suitable θ.

For $d > 1$, we consider the multivariate gamma distribution defined by its Laplace transform $(P(-\boldsymbol{\theta}))^{-\lambda}$, $\lambda > 0$, where

$$P(\boldsymbol{\theta}) = \sum_{S\subseteq\{1,...,d\}} a_S \prod_{i\in S}\theta_i \tag{5}$$

is an affine polynomial in $\boldsymbol{\theta} = (\theta_1, ..., \theta_d)$ (i.e., $\partial^2 P/\partial\theta_i^2 = 0$ for $i = 1, ..., d$) with suitable $a_S \in \mathbb{R}$ and $a_\varnothing = 1$. We denote this distribution by $\boldsymbol{\gamma}_{\lambda,P}$. For simplicity, if $S = \{i_1, ..., i_k\}$ then we shall write $a_{\{i_1,...,i_k\}} = a_{i_1...i_k}$. The associated multivariate gamma NEF $F = F(\boldsymbol{\gamma}_{\lambda,P})$ is such that $\boldsymbol{\gamma}_{\lambda,P}$ must belong to $\mathcal{M}(\mathbb{R}^d)$. This study on the pair (λ, P) of $\boldsymbol{\gamma}_{\lambda,P}$ is a difficult problem and only sufficient or necessary conditions are known. In Bernardoff [4] the necessary and sufficient conditions for existence and infinite divisibility of $\boldsymbol{\gamma}_{1,P}$ are found, with the restriction to $a_{1...d} \neq 0$ in (5).

Let us present three important examples:
- *A bivariate* case ($d = 2$) which is infinitely divisible is defined with $\lambda > 0$ and

$$P(\theta_1, \theta_2) = 1 + a_1\theta_1 + a_2\theta_2 + a_{12}\theta_1\theta_2 \tag{6}$$

for a_1, a_2 and $a_{12} > 0$ such that $a_1a_2 - a_{12} \geq 0$. If $a_{12} = a_1a_2$ (i.e. $P(\theta_1, \theta_2) = (1 + a_1\theta_1)(1 + a_2\theta_2)$) then the corresponding $\boldsymbol{\gamma}_{\lambda,P}$ is the distribution of the random variable $X = (Y_1, Y_2)$ where Y_1 and Y_2 are independent real random variables with respective distribution $\boldsymbol{\gamma}_{\lambda,a_i}$ for $i = 1, 2$.
- *A symmetric multivariate* case, also infinitely divisible, is defined with $\lambda > 0$ and

$$P(\boldsymbol{\theta}) = 1 - \frac{1}{a} + \frac{1}{a}\prod_{i=1}^{d}(1 + a\theta_i), \quad a \in (0, 1) \tag{7}$$

for $\theta_i < 1/a$, $i = 1, ..., d$ and $\prod_{i=1}^{d}(1 - a\theta_i) > 1 - a$ [4].

- *A line multivariate* case with $\lambda > 0$ and

$$P(\boldsymbol{\theta}) = 1 + \sum_{i=1}^{d} a_i \theta_i, \quad a_i > 0$$

is the distribution $\boldsymbol{\gamma}_{\lambda,P}$ of the random variable $X = (a_1 Y, ..., a_d Y)$ where Y is a real random variable with distribution $\boldsymbol{\gamma}_{\lambda,1}$ [8].

The following preliminary result shows that all multivariate gamma NEFs have a diagonal quadratic variance function (see also Bar-Lev et al.[2]).

Proposition 2 *Let P be an affine polynomial (5) in d variables and $\lambda > 0$. If F is a multivariate gamma NEF associated with (λ, P) then its variance function $V_F(\mathbf{m}) = (V_{i,j})_{i,j=1,...,d}$, $\mathbf{m} = (m_1, ..., m_d) \in M_F$ satisfies $V_{i,i} = m_i^2/\lambda$ and*

$$\left(P(-\boldsymbol{\theta}) \frac{\partial^2 P(-\boldsymbol{\theta})}{\partial\theta_i \partial\theta_j} - \frac{\partial P(-\boldsymbol{\theta})}{\partial\theta_i} \frac{\partial P(-\boldsymbol{\theta})}{\partial\theta_j}\right) \left(V_{i,j} - \frac{m_i m_j}{\lambda}\right)^2 + \lambda \left(\frac{\partial^2 P(-\boldsymbol{\theta})}{\partial\theta_i \partial\theta_j}\right)^2 V_{i,j} = 0, \quad i \neq j \qquad (8)$$

which does not depend on θ_i and θ_j.

In the sequel, we only investigate the generalized variance $\det V_F(\mathbf{m})$ in the infinitely divisible cases of the bivariate and symmetric multivariate gamma NEFs. Since the off-diagonal elements $V_{i,j}$ of $V_F(\mathbf{m})$ are difficult to exhibit via equation (8) for some affine polynomials P given in (5), these particular cases (6) and (7) shall suffice for instance to illustrate the problem of UMVU and ML estimators presented in the Introduction.

3 Generalized variance for some multivariate gamma NEFs

We here show two results of the generalized variance in the multivariate gamma NEFs for which we investigate their estimators in the next sections. The first concerns the bivariate case.

Proposition 3 *Let $P(\theta_1, \theta_2) = 1 + a_1\theta_1 + a_2\theta_2 + a_{12}\theta_1\theta_2$ be the associated affine polynomial (6) of the bivariate gamma NEF $F = F(\boldsymbol{\gamma}_{\lambda,P})$ with $\lambda > 0$. We denote $b_{12} = (a_1 a_2 - a_{12})/a_{12}^2 \geq 0$. Then, for $\mathbf{m} = (m_1, m_2) \in M_F = (0, \infty)^2$, if $b_{12} = 0$ we have $\det V_F(\mathbf{m}) = m_1^2 m_2^2/\lambda^2$ and, if $b_{12} > 0$ the gener-*

alized variance is

$$\det V_F(\mathbf{m}) = \frac{\lambda^2}{2b_{12}^2}\left[\left(1+\frac{2b_{12}m_1m_2}{\lambda^2}\right)\left(1+\frac{4b_{12}m_1m_2}{\lambda^2}\right)^{1/2} - \left(1+\frac{4b_{12}m_1m_2}{\lambda^2}\right)\right]$$

and tends to $m_1^2m_2^2/\lambda^2$ *when* b_{12} *tends to* 0.

The second result is devoted to the symmetric multivariate gamma NEFs.

Proposition 4 *Let* $P(\boldsymbol{\theta}) = 1-1/a+(1/a)(1+a\theta_1)...(1+a\theta_d)$ *be the associated affine polynomial (7) of the symmetric multivariate gamma NEF* $F = F(\boldsymbol{\gamma}_{\lambda,P})$ *with* $\lambda > 0$. *Then, for* $\mathbf{m} = (m_1, ..., m_d) \in M_F$,

$$\det V_F(\mathbf{m}) = \frac{a^{d-1}(ds - a\lambda(d-1))}{\lambda s^d}(m_1 \dots m_d)^2$$

where $s = m_i(1-a\theta_i)$, $i = 1, \dots, d$ *is the unique real non-negative solution of the equation of degree* d

$$s^d - \lambda a s^{d-1} - (1-a)m_1 \dots m_d = 0. \tag{9}$$

Writing $s = ((1-a)m_1 \dots m_d)^{1/d} y$ and $u = \lambda a((1-a)m_1 \dots m_d)^{-1/d}$ the equation (9) becomes

$$y^d - uy^{d-1} - 1 = 0,$$

and following Hochstadt [10] (p. 77) its unique real non-negative solution can be expressed as

$$\begin{aligned} y(u) &= \frac{1}{d}\sum_{r=0}^{\infty}(-1)^r \frac{\Gamma\left(\frac{r(d-1)+1}{d}\right)}{\Gamma\left(\frac{r(d-1)+1}{d}+1-r\right)}\frac{(-u)^r}{r!} \\ &= \frac{1}{d}\sum_{r=0}^{\infty}\frac{\Gamma\left(\left(1-\frac{r-1}{d}\right)+(r-1)\right)}{\Gamma\left(1-\frac{r-1}{d}\right)}\frac{u^r}{r!} \\ &= 1+\frac{u}{d}+\frac{1}{d}\sum_{r=2}^{\infty}\left[\prod_{k=1}^{r-1}\left(\frac{1-r}{d}+k\right)\right]\frac{u^r}{r!}, \quad |u| < d\,(d-1)^{(1-d)/d}. \end{aligned}$$

For example, when $d = 2$ we have $y(u) = u/2 + (1+u^2/4)^{1/2}$ and the Taylor expansion provides the corresponding above result.

4 Generalized variance estimators for bivariate gamma NEF

Following Bernardoff [3] and with the notations of Proposition 3 the density of the bivariate gamma distribution $\boldsymbol{\gamma}_{\lambda,P}$ can be written, for $\mathbf{x}=(x_1,x_2)$, as

$$\boldsymbol{\gamma}_{\lambda,P}(d\mathbf{x})=\frac{(x_1x_2)^{\lambda-1}}{a_{12}^{\lambda}(\Gamma(\lambda))^2}\exp\left(-\frac{a_2x_1+a_1x_2}{a_{12}}\right){}_0F_1(\lambda;b_{12}x_1x_2)\,1_{(0,\infty)^2}(\mathbf{x})\,d\mathbf{x} \tag{10}$$

where ${}_0F_q(b_1,\ldots,b_q;z)$ is the generalized hypergeometric function defined by

$${}_0F_q(b_1,\ldots,b_q;z)=\sum_{k=0}^{\infty}\frac{\Gamma(b_1)\ldots\Gamma(b_q)}{\Gamma(b_1+k)\ldots\Gamma(b_q+k)}\frac{z^k}{k!}.$$

We now show its UMVU generalized variance estimator.

Theorem 5 *Let $\mathbf{X}_1,\ldots,\mathbf{X}_n$ be i.i.d. sample from the bivariate gamma distribution $\boldsymbol{\gamma}_{\lambda,P}$ of (10) for fixed $\lambda>0$. With the notations of Proposition 3, we assume $n\geq 3$ and $b_{12}>0$, then the UMVU estimator $C_n\left(n\overline{\mathbf{X}}_n\right)=C_n(\mathbf{X}_1+\ldots+\mathbf{X}_n)$ of $\det V_F(\mathbf{m})$ is such that, for $(x_1,x_2)\in(0,\infty)^2$,*

$$\begin{aligned}C_n(x_1,x_2)&=\frac{(x_1x_2)^2}{n^2(\lambda n+1)^2}\times\frac{1}{{}_0F_1(\lambda n;b_{12}x_1x_2)}\\&\quad\times\left\{{}_0F_1(\lambda n+2;b_{12}x_1x_2)+\frac{2b_{12}x_1x_2}{(\lambda n+2)^2}\,{}_0F_1(\lambda n+3;b_{12}x_1x_2)\right\}\\&=\frac{\lambda x_1x_2}{n(\lambda n+1)b_{12}}\times\frac{1}{I_{\lambda n-1}[2(b_{12}x_1x_2)^{1/2}]}\\&\quad\times\left\{I_{\lambda n+1}\left[2(b_{12}x_1x_2)^{1/2}\right]+\frac{2(b_{12}x_1x_2)^{1/2}}{\lambda n+2}I_{\lambda n+2}\left[2(b_{12}x_1x_2)^{1/2}\right]\right\},\end{aligned}$$

where I_α is the modified Bessel function with index α such that

$$\sum_{k=0}^{\infty}\frac{1}{\Gamma(\alpha+k)}\frac{z^k}{k!}=\frac{1}{\Gamma(\alpha)}\,{}_0F_1(\alpha;z)=z^{(1-\alpha)/2}I_{\alpha-1}\left(2z^{1/2}\right). \tag{11}$$

From Proposition 3 the ML estimator of $\det V_F(\mathbf{m})$ is $\det V_F\left(\overline{\mathbf{X}}_n\right)$. The following proposition gives the ratio of the previous two estimators.

Proposition 6 *Under the assumptions of Theorem 5 the ratio of the estimators $C_n\left(n\overline{\mathbf{X}}_n\right)$ and $\det V_F\left(\overline{\mathbf{X}}_n\right)$ of $\det V_F(\mathbf{m})$ is, for fixed $\lambda>0$ and $n\geq 3$,*

$$\frac{C_n\left(n\overline{\mathbf{X}}_n\right)}{\det V_F\left(\overline{\mathbf{X}}_n\right)} = \frac{2p^3\left(t^2+2p^2\right)}{(p+2)(p+1)t^3}\left[\left(1+\frac{t^2}{2p^2}\right)\left(1+\frac{t^2}{p^2}\right)^{-1/2}+1\right] \times \left(\frac{I_p(t)}{I_{p-1}(t)} - \frac{pt}{t^2+2p^2}\right)$$

where $p = n\lambda$ *and* $t = 2n(b_{12}\overline{X}_1\overline{X}_2)^{1/2}$ *with* $\overline{\mathbf{X}}_n = (\overline{X}_1, \overline{X}_2)$.

We observe that the ratio $C_n\left(n\overline{\mathbf{X}}_n\right) / \det V_F\left(\overline{\mathbf{X}}_n\right)$ depends on the components of the sample mean $\overline{\mathbf{X}}_n = (\overline{X}_1, \overline{X}_2)$. Graphically it is shown by Maple that $C_n\left(n\overline{\mathbf{X}}_n\right) / \det V_F\left(\overline{\mathbf{X}}_n\right) < 1$ (Fig. 1). Also, it is pointed out in the proof of Theorem 5 that the canonical generalized variance is a sum of two exponential terms. This means that the equation (4) does not hold for non-independent bivariate gamma distributions ($b_{12} > 0$). Thus, the conjecture of proportionality is not contradicted in this case.

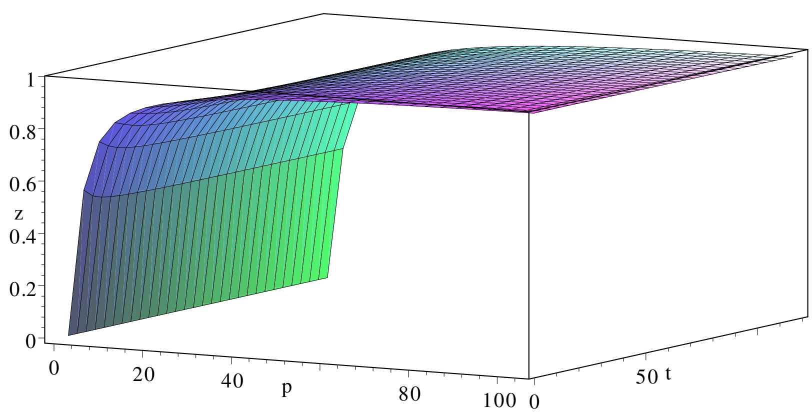

Fig. 1: Graphic of the ratio $z = C_n\left(n\overline{\mathbf{X}}_n\right) / \det V_F\left(\overline{\mathbf{X}}_n\right) = z(t;p)$ with $p = n\lambda$ and $t = 2n(b_{12}\overline{X}_1\overline{X}_2)^{1/2}$ of Proposition 6.

5 Generalized variance estimators in a symmetric multivariate case

Bernardoff [4] has defined the density of the corresponding symmetric multivariate gamma distribution $\boldsymbol{\gamma}_{\lambda,P}$ of Proposition 4 as

$$\boldsymbol{\gamma}_{\lambda,P}(d\mathbf{x}) = \frac{(x_1\ldots x_d)^{\lambda-1}}{a^{(d-1)\lambda}\left(\Gamma(\lambda)\right)^d}\exp\left(-\frac{x_1+\ldots+x_d}{a}\right) \times {}_0F_{d-1}\left(\lambda,\ldots,\lambda;(1-a)a^{-d}x_1\ldots x_d\right)1_{(0,\infty)^d}(\mathbf{x})\,(d\mathbf{x}), \quad (12)$$

for $\mathbf{x} = (x_1, ..., x_d)$. The following theorem states its UMVU generalized variance estimator.

Theorem 7 *Let $\mathbf{X}_1, ..., \mathbf{X}_n$ be i.i.d. sample from the symmetric multivariate gamma distribution $\boldsymbol{\gamma}_{\lambda,P}$ of (12) for fixed $\lambda > 0$. With the notations of Proposition 4, we assume $n \geq 3$, then the UMVU estimator $C_n\left(n\overline{\mathbf{X}}_n\right) = C_n(\mathbf{X}_1 + ... + \mathbf{X}_n)$ of $\det V_F(\mathbf{m})$ is such that*

$$C_n(\mathbf{x}) = \frac{\lambda^d (x_1 \dots x_d)^2}{{}_0F_{d-1}(n\lambda, \dots, n\lambda; (1-a)a^{-d}x_1 \dots x_d)} \times \sum_{k=0}^{d-1} \binom{d-1}{k} (k+1) \left[\frac{\Gamma(n\lambda)}{\Gamma(n\lambda + 2 + k)}\right]^d \left[(1-a)a^{-d}x_1 \dots x_d\right]^k \times {}_0F_{d-1}\left(n\lambda + 2 + k, \dots, n\lambda + 2 + k; (1-a)a^{-d}x_1 \dots x_d\right),$$

for $\mathbf{x} = (x_1, ..., x_d) \in (0, \infty)^d$.

For $d = 2$ the formula becomes

$$C_n(\mathbf{x}) = \frac{(\lambda x_1 x_2)^2}{n^2 (\lambda n + 1)^2} \times \frac{1}{{}_0F_1(n\lambda; (1-a)a^{-2}x_1x_2)} \times \left\{ {}_0F_1\left(n\lambda + 2; \frac{x_1x_2}{a^2/(1-a)}\right) + \frac{2(1-a)x_1x_2}{a^2(\lambda n + 2)^2} \, {}_0F_1\left(n\lambda + 3; \frac{x_1x_2}{a^2/(1-a)}\right)\right\},$$

which is the particular case of the bivariate gamma models (6) with $a_1 = a_2 = 1$, $a_{12} = a$ and then $b_{12} = (1-a)/a^2$. These symmetric multivariate gamma models are non-independent.

The ML estimator of $\det V_F(\mathbf{m})$ is $\det V_F\left(\overline{\mathbf{X}}_n\right)$ by using Proposition 4. However, the comparison study of these two estimators may be just more complex to write.

6 Concluding remarks

A very standard case of the generalized variance estimators is to consider the univariate ($d = 1$) situation where the sample size is $n = d + 1$. See e.g. Antoniadis et al. [1] for the practical use in wavelet shrinkage.

The standard multivariate case ($d > 1$) for the gamma models can be seen through the independent multivariate gamma NEF $F = F(\boldsymbol{\gamma}_{\lambda,P})$ with $\lambda > 0$

and

$$P(\boldsymbol{\theta}) = \prod_{i=1}^{d}(1 - a_i\theta_i), \quad a_i > 0.$$

Recall that the density of $\boldsymbol{\gamma}_{\lambda,P}$ can be written as

$$\boldsymbol{\gamma}_{\lambda,P}(d\mathbf{x}) = \frac{(x_1 \ldots x_d)^{\lambda-1}}{(a_1 \ldots a_d)^{\lambda}(\Gamma(\lambda))^d} \exp\left\{-\left(\frac{x_1}{a_1} + \ldots + \frac{x_d}{a_d}\right)\right\} 1_{(0,\infty)^d}(\mathbf{x})(d\mathbf{x}).$$

It is easy to check the corresponding ingredients: the generalized variance is

$$\det V_F(\mathbf{m}) = \det diag\left(\frac{m_1^2}{\lambda}, \ldots, \frac{m_d^2}{\lambda}\right) = \frac{m_1^2 \ldots m_d^2}{\lambda^d}, \quad \mathbf{m} \in (0,\infty)^d,$$

the UMVU function (2) is $C_n(\mathbf{x}) = (x_1 \ldots x_d)^2 n^{-d}(n\lambda+1)^{-d}$, and the ratio $C_n(n\overline{\mathbf{X}})/\det V_F(\overline{\mathbf{X}})$ is $\beta_n = [n\lambda/(n\lambda+1)]^d < 1$. Thus the conjecture of proportionality between UMVU and ML estimators holds for the independent multivariate gamma models, because the canonical generalized variance verifies (4) with $a = 2/\lambda$, $\mathbf{b} = \mathbf{0}$, $c = \log(\lambda^d a_1^2 \ldots a_d^2)$. For fixed n and λ we think that this coefficient β_n is the upper-bound for any sample of the multivariate gamma models.

In general, for non-independent multivariate gamma NEFs, the canonical generalized variance $\det K''_{\boldsymbol{\gamma}_{\lambda,P}}(\boldsymbol{\theta})$ is on one hand a sum of exponential terms as in the right side member of (4) and, on the other hand, the UMVU and the ML estimators are not proportional. The bivariate and the symmetric multivariate gamma NEFs treated in this paper cover a wide spectrum of different situations of gamma distributions (5), which include many other multivariate gamma distributions proposed in the literature (e.g. Kotz et al. [16], Chap. 48 and references therein). Finally, the conjecture of proportionality always holds for canonical generalized variance satisfying (4).

Appendix: Proofs

Proof of Proposition 2. Let $K_\mu(\boldsymbol{\theta}) = -\lambda \log P(-\boldsymbol{\theta})$. The means $m_i = \partial K_\mu(\boldsymbol{\theta})/\partial\theta_i$ are

$$m_i = \frac{\lambda}{P(-\boldsymbol{\theta})} \times \frac{\partial P(-\boldsymbol{\theta})}{\partial\theta_i}, \quad i = 1, \ldots d.$$

For $i = j$ the diagonal elements $V_{i,i} = \partial^2 K_\mu(\boldsymbol{\theta})/\partial\theta_i^2$ of $V_F(\mathbf{m})$ are

$$V_{i,i} = \frac{\lambda}{(P(-\boldsymbol{\theta}))^2}\left(\frac{\partial P(-\boldsymbol{\theta})}{\partial\theta_i}\right)^2 = \frac{m_i^2}{\lambda}, \quad i = 1, \ldots, d,$$

because P is an affine polynomial with $\partial^2 P/\partial\theta_i^2 = 0$ for all $i = 1, ..., d$.

For $i \neq j$ the off-diagonal elements $V_{i,j} = \partial^2 K_\mu(\boldsymbol{\theta})/(\partial\theta_i\partial\theta_j)$ of $V_F(\mathbf{m})$ are, successively,

$$V_{i,j} = \frac{\lambda}{(P(-\boldsymbol{\theta}))^2}\left[-P(-\boldsymbol{\theta})\frac{\partial^2 P(-\boldsymbol{\theta})}{\partial\theta_i\partial\theta_j} + \frac{\partial P(-\boldsymbol{\theta})}{\partial\theta_i}\frac{\partial P(-\boldsymbol{\theta})}{\partial\theta j}\right] \tag{13}$$

$$= \frac{-\lambda}{P(-\boldsymbol{\theta})} \times \frac{\partial^2 P(-\boldsymbol{\theta})}{\partial\theta_i\partial\theta_j} + \frac{m_i m_j}{\lambda}. \tag{14}$$

Note that

$$\frac{\partial}{\partial\theta_i}\frac{\partial^2 P(-\boldsymbol{\theta})}{\partial\theta_i\partial\theta_j} = \frac{\partial}{\partial\theta_j}\frac{\partial^2 P(-\boldsymbol{\theta})}{\partial\theta_i\partial\theta_j} = 0$$

and

$$0 = \frac{\partial}{\partial\theta_i}\left[(P(-\boldsymbol{\theta}))^2 V_{i,j}\right] = \lambda\left[\frac{\partial P(-\boldsymbol{\theta})}{\partial\theta_i}\frac{\partial^2 P(-\boldsymbol{\theta})}{\partial\theta_i\partial\theta_j} - \frac{\partial P(-\boldsymbol{\theta})}{\partial\theta_i}\frac{\partial^2 P(-\boldsymbol{\theta})}{\partial\theta_i\partial\theta_j}\right]$$

$$= \frac{\partial}{\partial\theta_j}\left[(P(-\boldsymbol{\theta}))^2 V_{i,j}\right].$$

Thus $P(-\boldsymbol{\theta})(V_{i,j} - m_i m_j/\lambda)$ and $(P(-\boldsymbol{\theta}))^2 V_{i,j}$ do not depend on the variables θ_i and θ_j. From (14) we can write

$$V_{i,j} - \frac{m_i m_j}{\lambda} = \frac{-\lambda}{P(-\boldsymbol{\theta})}\frac{\partial^2 P(-\boldsymbol{\theta})}{\partial\theta_i\partial\theta_j}$$

and then

$$\left(V_{i,j} - \frac{m_i m_j}{\lambda}\right)^2 = \frac{\lambda^2}{(P(-\boldsymbol{\theta}))^2}\left(\frac{\partial^2 P(-\boldsymbol{\theta})}{\partial\theta_i\partial\theta_j}\right)^2.$$

Replacing the denominator $(P(-\boldsymbol{\theta}))^2$ of the last equality by its expression from (13), we therefore obtain

$$\left(P(-\boldsymbol{\theta})\frac{\partial^2 P(-\boldsymbol{\theta})}{\partial\theta_i\partial\theta_j} - \frac{\partial P(-\boldsymbol{\theta})}{\partial\theta_i}\frac{\partial P(-\boldsymbol{\theta})}{\partial\theta j}\right)\left(V_{i,j} - \frac{m_i m_j}{\lambda}\right)^2 = -\lambda\left(\frac{\partial^2 P(-\boldsymbol{\theta})}{\partial\theta_i\partial\theta_j}\right)^2 V_{i,j}$$

which is equivalent to (8). It also follows that the above equation does not depend on θ_i and θ_j. □

Proof of Proposition 3. We first apply Proposition 2 with $P(\theta_1, \theta_2) = 1 + a_1\theta_1 + a_2\theta_2 + a_{12}\theta_1\theta_2$ to get $V_F(\mathbf{m}) = (V_{i,j})_{i,j=1,2}$, where $V_{i,i} = m_i^2/\lambda$ and $V_{1,2} = V_{2,1}$ satisfies the corresponding equation (8):

$$b_{12}\left(V_{1,2} - \frac{m_1 m_2}{\lambda}\right)^2 - \lambda V_{1,2} = 0 \tag{15}$$

for $m_1 = \lambda(a_1 + a_{12}\theta_2)/P(-\theta_1, -\theta_2)$ and $m_2 = \lambda(a_2 + a_{12}\theta_1)/P(-\theta_1, -\theta_2) > 0$.

For $b_{12} = 0$ in (15) we obviously have $V_{1,2} = 0$ and, then, $\det V_F(\mathbf{m}) = m_1^2 m_2^2/\lambda^2$.

For $b_{12} > 0$, and since $\lambda^2 + 4b_{12}m_1m_2 > 0$, the equation (15) has for solutions

$$V_{1,2}(\varepsilon) = \frac{m_1m_2}{\lambda} + \frac{\lambda}{2b_{12}}\left[1 + \varepsilon\left(1 + \frac{4b_{12}m_1m_2}{\lambda^2}\right)^{1/2}\right], \quad \varepsilon = \pm 1.$$

The solution $V_{1,2} = V_{1,2}(+1)$ is associated to the negative definite matrix $V_F(\mathbf{m}) = (V_{i,j})_{i,j=1,2}$ because its corresponding determinant is clearly

$$\det V_F(\mathbf{m}) = \left(\frac{m_1m_2}{\lambda}\right)^2 - \left\{\frac{m_1m_2}{\lambda} + \frac{\lambda}{2b_{12}}\left[1 + \left(1 + \frac{4b_{12}m_1m_2}{\lambda^2}\right)^{1/2}\right]\right\}^2 < 0.$$

Thus, the positive definite matrix $V_F(\mathbf{m}) = (V_{i,j})_{i,j=1,2}$ which is associated to the adequate solution of (15) is obtained with $V_{1,2} = V_{1,2}(-1)$. From this it may be deduced the result as follows:

$$\begin{aligned}
\det V_F(\mathbf{m}) &= \left(\frac{m_1m_2}{\lambda}\right)^2 - \left\{\frac{m_1m_2}{\lambda} + \frac{\lambda}{2b_{12}}\left[1 - \left(1 + \frac{4b_{12}m_1m_2}{\lambda^2}\right)^{1/2}\right]\right\}^2 \\
&= \left[\frac{-\lambda}{2b_{12}} + \frac{\lambda}{2b_{12}}\left(1 + \frac{4b_{12}m_1m_2}{\lambda^2}\right)^{1/2}\right] \\
&\quad \times \left[\frac{2m_1m_2}{\lambda} + \frac{\lambda}{2b_{12}} - \frac{\lambda}{2b_{12}}\left(1 + \frac{4b_{12}m_1m_2}{\lambda^2}\right)^{1/2}\right] \\
&= \left(\frac{\lambda^2}{2b_{12}^2} + \frac{m_1m_2}{b_{12}}\right)\left(1 + \frac{4b_{12}m_1m_2}{\lambda^2}\right)^{1/2} - \left(\frac{\lambda^2}{2b_{12}^2} + \frac{2m_1m_2}{b_{12}}\right).
\end{aligned}$$

Moreover, since $(1 + 4b_{12}m_1m_2/\lambda^2)^{1/2} = 1 + 2b_{12}m_1m_2/\lambda^2 - 2b_{12}^2m_1^2m_2^2/\lambda^4 + o(b_{12}^2)$ we can write $\det V_F(\mathbf{m}) = m_1^2m_2^2/\lambda^2 + o(1)$, and the proposition is then proven. □

Proof of Proposition 4. By a direct calculation of $V_F(\mathbf{m}) = (V_{i,j})_{i,j=1,\ldots,d}$ we first express the means $m_i = \partial K_{\boldsymbol{\gamma}_{\lambda,P}}(\boldsymbol{\theta})/\partial\theta_i = \partial[-\lambda \log P(-\boldsymbol{\theta})]/\partial\theta_i$ with $P(\boldsymbol{\theta}) = 1 - 1/a + (1/a)(1 + a\theta_1)...(1 + a\theta_d)$, $\boldsymbol{\theta} \in \Theta(\boldsymbol{\gamma}_{\lambda,P})$, as

$$m_i = \frac{\lambda}{P(-\boldsymbol{\theta})}\prod_{k=1,k\neq i}^{d}(1 - a\theta_k) = \frac{-\lambda a\prod_{k=1,k\neq i}^{d}(1 - a\theta_k)}{1 - a - \prod_{k=1}^{d}(1 - a\theta_k)}, \quad i = 1, \ldots, d.$$

To define $\boldsymbol{\theta}(\cdot)$ in terms of $\mathbf{m}$ we can solve the previous system of equations $m_i = m_i(\boldsymbol{\theta}), i = 1, ..., d$ as follows: letting $m_1(1 - a\theta_1) = m_i(1 - a\theta_i)$, $i =$

$2, \ldots, d$, we obtain

$$m_d \prod_{k=1}^{d} (1 - a\theta_k) - \lambda a \prod_{k=1}^{d-1} (1 - a\theta_k) - (1 - a) m_d = 0,$$

which is equivalent to

$$\prod_{k=1}^{d} m_k (1 - a\theta_k) - \lambda a \prod_{k=1}^{d-1} m_k (1 - a\theta_k) - (1 - a) \prod_{k=1}^{d} m_k = 0$$

and, finally,

$$(m_1(1 - a\theta_1))^d - \lambda a \left(m_1(1 - a\theta_1)\right)^{d-1} - (1 - a) \prod_{k=1}^{d} m_k = 0.$$

This proves the condition (9) resolved by Hochstadt [10] (p. 77).

Then, the diagonal elements $V_{i,i} = \partial^2 K_{\boldsymbol{\gamma}_{\lambda,P}}(\boldsymbol{\theta}) / \partial\theta_i^2, i = 1, ..., d$ are

$$V_{i,i} = \frac{\partial}{\partial\theta_i} \left[\frac{\lambda}{P(-\boldsymbol{\theta})} \prod_{k=1, k\neq i}^{d} (1 - a\theta_k) \right] = \frac{\lambda}{(P(-\boldsymbol{\theta}))^2} \left[\prod_{k=1, k\neq i}^{d} (1 - a\theta_k) \right]^2 = \frac{m_i^2}{\lambda},$$

and the off-diagonal elements $V_{i,j} = \partial^2 K_{\boldsymbol{\gamma}_{\lambda,P}}(\boldsymbol{\theta}) / (\partial\theta_i \partial\theta_j), i \neq j$ are

$$\begin{aligned}
V_{i,j} &= \frac{\partial}{\partial\theta_j} \left[\frac{\lambda}{P(-\boldsymbol{\theta})} \prod_{k=1, k\neq i}^{d} (1 - a\theta_k) \right] \\
&= \frac{-\lambda a}{P(-\boldsymbol{\theta})(1 - a\theta_i)} \prod_{k=1, k\neq j}^{d} (1 - a\theta_k) + \frac{\lambda}{(P(-\boldsymbol{\theta}))^2} \prod_{k=1, k\neq i}^{d} (1 - a\theta_k) \prod_{k=1, k\neq j}^{d} (1 - a\theta_k) \\
&= \frac{-a m_j}{1 - a\theta_i} + \frac{m_i m_j}{\lambda} \\
&= \left(\frac{1}{\lambda} - \frac{a}{s} \right) m_i m_j, \quad i \neq j,
\end{aligned}$$

where $s > 0$ is the solution of (9).

Now, letting $\mathbf{1}_d = (1, ..., 1)$ the d-unit vector of $\mathbb{R}^d$ and $\boldsymbol{I}_d = diag(1, ..., 1)$ the $(d \times d)$ unit matrix. Using the standard rules of the determinant calculus (e.g.

Muir [18]) we successively obtain the desired result

$$\begin{aligned}\det V_F(\mathbf{m}) &= (m_1 \dots m_d)^2 \det\left[\frac{1}{\lambda}\mathbf{1}_d^T\mathbf{1}_d + \frac{a}{s}\left(\boldsymbol{I}_d - \mathbf{1}_d^T\mathbf{1}_d\right)\right] \\ &= (m_1 \dots m_d)^2 \left(\frac{1}{\lambda} + (d-1)\left(\frac{1}{\lambda} - \frac{a}{s}\right)\right)\left(\frac{1}{\lambda} - \left(\frac{1}{\lambda} - \frac{a}{s}\right)\right)^{d-1} \\ &= (m_1 \dots m_d)^2 \frac{a^{d-1}\{s + (d-1)(s - a\lambda)\}}{\lambda s^d} \\ &= (m_1 \dots m_d)^2 \frac{a^{d-1}(ds - a\lambda(d-1))}{\lambda s^d}. \ \square\end{aligned}$$

Proof of Theorem 5. We apply Proposition 1 (ii) and (iii) with $\mu = \boldsymbol{\gamma}_{\lambda,P}$, $\lambda > 0$ and $P(\theta_1, \theta_2) = 1 + a_1\theta_1 + a_2\theta_2 + a_{12}\theta_1\theta_2$. Since $\partial^2 K_{\boldsymbol{\gamma}_{\lambda,P}}(\theta_1, \theta_2)/\partial\theta_i^2 = \partial^2[-\lambda \log P(-\theta_1, -\theta_2)]/\partial\theta_i^2 = \lambda(-a_i + a_{12}\theta_{i^c})/(P(-\theta_1, -\theta_2))^2$, $i = 1 = i^c - 1$, $i = 2 = i^c + 1$ and $\partial^2 K_{\boldsymbol{\gamma}_{\lambda,P}}(\theta_1, \theta_2)/(\partial\theta_1\partial\theta_2) = \lambda(a_1a_2 + a_{12})/(P(-\theta_1, -\theta_2))^2$, we successively have

$$\begin{aligned}\det K''_{\boldsymbol{\gamma}_{\lambda,P}}(\theta_1, \theta_2) &= \det\left(\partial^2 K_{\boldsymbol{\gamma}_{\lambda,P}}(\theta_1, \theta_2)/(\partial\theta_i\partial\theta_j)\right)_{i,j=1,2} \\ &= \lambda^2 a_{12}(a_{12}^2\theta_1\theta_2 - a_{12}a_1\theta_1 - a_{12}a_2\theta_2 - a_{12} + 2a_1a_2)(P(-\theta_1, -\theta_2))^{-3} \\ &= \lambda^2\left\{a_{12}^2(P(-\theta_1, -\theta_2))^{-2} + 2a_{12}^3 b_{12}(P(-\theta_1, -\theta_2))^{-3}\right\} \\ &= \exp\left\{\frac{2}{\lambda}K_{\boldsymbol{\gamma}_{\lambda,P}}(\theta_1, \theta_2) + 2\log a_{12} + 2\log\lambda\right\} \\ &\quad + \exp\left\{\frac{3}{\lambda}K_{\boldsymbol{\gamma}_{\lambda,P}}(\theta_1, \theta_2) + \log(2b_{12}) + 3\log a_{12} + 2\log\lambda\right\} \\ &= L_{\lambda^2 a_{12}^2\boldsymbol{\gamma}_{2,P} + 2\lambda^2 a_{12}^3 b_{12}\boldsymbol{\gamma}_{3,P}}(\theta_1, \theta_2).\end{aligned}$$

Then, the associated measure $\nu_n = \nu_n(\boldsymbol{\gamma}_{\lambda,P})$ defined by its Laplace transform $L_{\nu_n}(\theta_1, \theta_2) = (L_{\boldsymbol{\gamma}_{\lambda,P}}(\theta_1, \theta_2))^n \det K''_{\boldsymbol{\gamma}_{\lambda,P}}(\theta_1, \theta_2)$ can be written $\nu_n = \lambda^2 a_{12}^2\boldsymbol{\gamma}_{\lambda n+2,P} + 2\lambda^2 a_{12}^3 b_{12}\boldsymbol{\gamma}_{\lambda n+3,P}$. From (10) we write

$$\begin{aligned}\nu_n(d\mathbf{x}) &= \lambda^2\left(a_{12}^2\boldsymbol{\gamma}_{\lambda n+2,P} + 2a_{12}^3 b_{12}\boldsymbol{\gamma}_{\lambda n+3,P}\right)(d\mathbf{x}) \\ &= \frac{\lambda^2(x_1x_2)^{\lambda n+1}}{a_{12}^{\lambda n}[\Gamma(\lambda n+2)]^2}\exp\left(-\frac{a_2x_1 + a_1x_2}{a_{12}}\right) \\ &\quad \times\left\{{}_0F_1(\lambda n+2; b_{12}x_1x_2) + \frac{2b_{12}x_1x_2}{(\lambda n+2)^2}\,{}_0F_1(\lambda n+3; b_{12}x_1x_2)\right\} \\ &\quad \times 1_{(0,\infty)^2}(\mathbf{x})(d\mathbf{x}).\end{aligned}$$

and

$$\begin{aligned}\boldsymbol{\gamma}_{\lambda,P}^{*n}(d\mathbf{x}) &= \boldsymbol{\gamma}_{\lambda n,P}(d\mathbf{x}) \\ &= \frac{(x_1x_2)^{\lambda n-1}}{a_{12}^{\lambda n}[\Gamma(\lambda n)]^2}\exp\left(-\frac{a_2x_1 + a_1x_2}{a_{12}}\right){}_0F_1(\lambda n; b_{12}x_1x_2)1_{(0,\infty)^2}(\mathbf{x})\,d\mathbf{x},\end{aligned}$$

which easily lead to the first expression of $C_n(\cdot)$ by using $C_n(\mathbf{x}) = \nu_n(d\mathbf{x}) / \boldsymbol{\gamma}^{*n}_{\lambda,P}(d\mathbf{x})$. The second expression of $C_n(\cdot)$ in terms of the modified Bessel function I_α is therefore deduce from the first one by using (11) (see Watson [21]). □

Proof of Proposition 6. From Theorem 5 and Proposition 3 we write the ratio as follows:

$$\begin{aligned}
\frac{C_n\left(n\overline{\mathbf{X}}_n\right)}{\det V_F\left(\overline{\mathbf{X}}_n\right)} &= \frac{n\lambda \overline{X}_1\overline{X}_2}{(n\lambda+1)b_{12}} \times \frac{1}{I_{\lambda n-1}\left[2n(b_{12}\overline{X}_1\overline{X}_2)^{1/2}\right]} \\
&\quad \times \left\{ I_{\lambda n+1}\left[2n(b_{12}\overline{X}_1\overline{X}_2)^{1/2}\right] + \frac{2n(b_{12}\overline{X}_1\overline{X}_2)^{1/2}}{\lambda n+2} I_{\lambda n+2}\left[2n(b_{12}\overline{X}_1\overline{X}_2)^{1/2}\right]\right\} \\
&\quad \times \left\{ \frac{\lambda^2}{2b_{12}^2}\left[\left(1+\frac{2b_{12}\overline{X}_1\overline{X}_2}{\lambda^2}\right)\left(1+\frac{4b_{12}\overline{X}_1\overline{X}_2}{\lambda^2}\right)^{1/2} - \left(1+\frac{4b_{12}\overline{X}_1\overline{X}_2}{\lambda^2}\right)\right]\right\}^{-1} \\
&= \frac{2n\lambda}{(\lambda n+1)}\left(\frac{2(b_{12}\overline{X}_1\overline{X}_2)^{1/2}}{2\lambda}\right)^2 \frac{1}{I_{\lambda n-1}\left[2n(b_{12}\overline{X}_1\overline{X}_2)^{1/2}\right]} \\
&\quad \times \left\{ I_{\lambda n+1}\left[2n(b_{12}\overline{X}_1\overline{X}_2)^{1/2}\right] + \frac{2n(b_{12}\overline{X}_1\overline{X}_2)^{1/2}}{\lambda n+2} I_{\lambda n+2}\left[2n(b_{12}\overline{X}_1\overline{X}_2)^{1/2}\right]\right\} \\
&\quad \times \left[\left(1+\frac{2b_{12}\overline{X}_1\overline{X}_2}{\lambda^2}\right)\left(1+\frac{4b_{12}\overline{X}_1\overline{X}_2}{\lambda^2}\right)^{1/2} - \left(1+\frac{4b_{12}\overline{X}_1\overline{X}_2}{\lambda^2}\right)\right]^{-1}.
\end{aligned}$$

Denoting $p = n\lambda$ and $t = 2n(b_{12}\overline{X}_1\overline{X}_2)^{1/2}$ we rewrite this ratio as:

$$\begin{aligned}
\frac{C_n\left(n\overline{\mathbf{X}}_n\right)}{\det V_F\left(\overline{\mathbf{X}}_n\right)} &= \frac{2p}{(p+1)(p+2)}\left(\frac{t}{2p}\right)^2 \frac{(p+2)\, I_{p+1}(t) + tI_{p+2}(t)}{I_{p-1}(t)} \\
&\quad \times \left[\left(1+\frac{t^2}{2p^2}\right)\left(1+\frac{t^2}{p^2}\right)^{1/2} - \left(1+\frac{t^2}{p^2}\right)\right]^{-1} \\
&= \frac{2p^3}{(p+1)(p+2)\,t^2}\left[\left(1+\frac{t^2}{2p^2}\right)\left(1+\frac{t^2}{p^2}\right)^{-1/2} + 1\right] \\
&\quad \times \frac{(p+2)\, I_{p+1}(t) + tI_{p+2}(t)}{I_{p-1}(t)}.
\end{aligned}$$

To obtain the desired result we simplify the expression

$$\frac{(p+2)\, I_{p+1}(t) + tI_{p+2}(t)}{I_{p-1}(t)}$$

by using the following identity:

$$I_{p-1}(t) - I_{p+1}(t) = \frac{2p}{t} I_p(t) \qquad (16)$$

(e.g. Watson [21], p. 79). Indeed, substituting $p+1$ to p and multiplying by t in (16) we have

$$tI_p(t) = 2(p+1)I_{p+1}(t) + tI_{p+2}(t),$$

that is

$$(p+2)I_{p+1}(t) + tI_{p+2}(t) = tI_p(t) - pI_{p+1}(t).$$

Thus we successively obtain

$$\begin{aligned}\frac{(p+2)I_{p+1}(t) + tI_{p+2}(t)}{I_{p-1}(t)} &= \frac{tI_p(t) - pI_{p+1}(t)}{I_{p-1}(t)} \\ &= \frac{tI_p(t) - p\left[I_{p-1}(t) - 2pI_p(t)/t\right]}{I_{p-1}(t)} \\ &= \frac{1}{t}\left[\left(t^2+2p^2\right)\frac{I_p(t)}{I_{p-1}(t)} - pt\right] \\ &= \frac{t^2+2p^2}{t}\left(\frac{I_p(t)}{I_{p-1}(t)} - \frac{pt}{t^2+2p^2}\right),\end{aligned}$$

and the proposition is finally deduced. □

Proof of Theorem 7. We use Proposition 1 (ii) and (iii) with $\mu = \boldsymbol{\gamma}_{\lambda,P}$, $\lambda > 0$ and $P(\boldsymbol{\theta}) = 1 - 1/a + (1/a)(1+a\theta_1)...(1+a\theta_d)$, $\boldsymbol{\theta} \in \Theta(\boldsymbol{\gamma}_{\lambda,P})$. Denoting $P(-\boldsymbol{\theta}) = \mathbf{P}$ and $\prod_{k=1}^{d}(1-a\theta_k) = \prod_{k=1}^{d}\alpha_k = \boldsymbol{\alpha}$ we have the relation $\boldsymbol{\alpha} = a\mathbf{P} + 1 - a$. As in the proof of Proposition 4 we consider $\mathbf{1}_d = (1,...,1)$ and $\boldsymbol{I}_d = diag(1,...,1)$. Then, we successively write

$$\begin{aligned}\det K''_{\boldsymbol{\gamma}_{\lambda,P}}(\boldsymbol{\theta}) &= \det\left(\partial^2 K_{\boldsymbol{\gamma}_{\lambda,P}}(\boldsymbol{\theta})/(\partial\theta_i\partial\theta_j)\right)_{i,j=1,...,d} \\ &= \lambda^d \mathbf{P}^{-2d} \det\begin{pmatrix} \left(\frac{\boldsymbol{\alpha}}{\alpha_1}\right)^2 & \frac{(1-a)\boldsymbol{\alpha}}{\alpha_1\alpha_j} & \frac{(1-a)\boldsymbol{\alpha}}{\alpha_1\alpha d} \\ & \ddots & \\ \frac{(1-a)\boldsymbol{\alpha}}{\alpha_1\alpha_j} & \left(\frac{\boldsymbol{\alpha}}{\alpha_j}\right)^2 & \frac{(1-a)\boldsymbol{\alpha}}{\alpha_j\alpha_d} \\ & & \ddots \\ \frac{(1-a)\boldsymbol{\alpha}}{\alpha_1\alpha_d} & \frac{(1-a)\boldsymbol{\alpha}}{\alpha_j\alpha_d} & \left(\frac{\boldsymbol{\alpha}}{\alpha_d}\right)^2 \end{pmatrix} \\ &= \lambda^d \mathbf{P}^{-2d} \boldsymbol{\alpha}^{d-2} \det\left[\boldsymbol{\alpha}\boldsymbol{I}_d + (1-a)\left(\mathbf{1}_d^T\mathbf{1}_d - \boldsymbol{I}_d\right)\right].\end{aligned}$$

Using $\boldsymbol{\alpha} = a\mathbf{P} + 1 - a$ its expansion can be expressed as

$$\begin{aligned}
\det K''_{\boldsymbol{\gamma}_{\lambda,P}}(\boldsymbol{\theta}) &= \lambda^d \mathbf{P}^{-2d} \boldsymbol{\alpha}^{d-2} \left\{ (d-1)(1-a) + \boldsymbol{\alpha} \right\} (\boldsymbol{\alpha} - 1 + a)^{d-1} \\
&= \lambda^d a^{d-1} \mathbf{P}^{-(d+1)} (a\mathbf{P} + 1 - a)^{d-2} (a\mathbf{P} + d(1-a)) \\
&= \lambda^d a^{d-1} \mathbf{P}^{-(d+1)} \left[(a\mathbf{P} + 1 - a)^{d-1} + (d-1)(1-a)(a\mathbf{P} + 1 - a)^{d-2} \right] \\
&= \lambda^d a^{d-1} \mathbf{P}^{-(d+1)} \\
&\times \left[\sum_{k=0}^{d-1} \binom{d-1}{k} a^k (1-a)^{d-1-k} \mathbf{P}^k + (d-1)(1-a) \sum_{k=0}^{d-2} \binom{d-2}{k} a^k (1-a)^{d-2-k} \mathbf{P}^k \right] \\
&= \lambda^d a^{d-1} \mathbf{P}^{-(d+1)} \left\{ a^{d-1} \mathbf{P}^{d-1} + \sum_{k=0}^{d-2} \left[\binom{d-1}{k} + (d-1) \binom{d-2}{k} \right] a^k (1-a)^{d-1-k} \mathbf{P}^k \right\} \\
&= \lambda^d \left[a^{2(d-1)} \mathbf{P}^{-2} + \sum_{k=1}^{d-1} (k+1) \binom{d-1}{k} a^{2(d-1)-k} (1-a)^k \mathbf{P}^{-(k+2)} \right] \\
&= \lambda^d \sum_{k=0}^{d-1} (k+1) \binom{d-1}{k} a^{2(d-1)-k} (1-a)^k \mathbf{P}^{-(k+2)} \\
&= L_{\lambda^d \sum_{k=0}^{d-1} (k+1) \binom{d-1}{k} p^{2(d-1)-k} (1-a)^k \boldsymbol{\gamma}_{2+k,P}} (\boldsymbol{\theta}).
\end{aligned}$$

Thus, the associated measure $\nu_n = \nu_n(\boldsymbol{\gamma}_{\lambda,P})$ defined by its Laplace transform $L_{\nu_n}(\boldsymbol{\theta}) = (L_{\boldsymbol{\gamma}_{\lambda,P}}(\boldsymbol{\theta}))^n \det K''_{\boldsymbol{\gamma}_{\lambda,P}}(\boldsymbol{\theta})$ is $\nu_n = \lambda^d \sum_{k=0}^{d-1} (k+1) \binom{d-1}{k} a^{2(d-1)-k} (1-a)^k \boldsymbol{\gamma}_{n\lambda+2+k,P}$. Using (12) we deduce

$$\begin{aligned}
\nu_n(d\mathbf{x}) &= \lambda^d \sum_{k=0}^{d-1} (k+1) \binom{d-1}{k} a^{2(d-1)-k} (1-a)^k \boldsymbol{\gamma}_{n\lambda+2+k,P}(d\mathbf{x}) \\
&= \{ \lambda^d \sum_{k=0}^{d-1} (k+1) \binom{d-1}{k} a^{2(d-1)-k} \frac{(1-a)^k a^{(1-d)(n\lambda+2+k)}}{(\Gamma(n\lambda+2+k))^d} \exp\left(-\frac{x_1 + \ldots + x_d}{a} \right) \\
&\quad \times (x_1 \ldots x_d)^{n\lambda+1+k} \, {}_0F_{d-1}\left(n\lambda+2+k, \ldots, n\lambda+2+k; (1-a)a^{-d} x_1 \ldots x_d \right) \} \\
&\quad \times 1_{(0,\infty)^d}(\mathbf{x})(d\mathbf{x}) \\
&= \{ (x_1 \ldots x_d)^{n\lambda+1} a^{-(d-1)n\lambda} \lambda^d \exp\left(-\frac{x_1 + \ldots + x_d}{a} \right) \\
&\quad \times \sum_{k=0}^{d-1} \binom{d-1}{k} \frac{(k+1)}{(\Gamma(n\lambda+2+k))^d} \left((1-a)a^{-d} x_1 \ldots x_d \right)^k \\
&\quad \times \, {}_0F_{d-1}\left(n\lambda+2+k, \ldots, n\lambda+2+k; (1-a)a^{-d} x_1 \ldots x_d \right) \} 1_{(0,\infty)^d}(\mathbf{x})(d\mathbf{x})
\end{aligned}$$

and

$$\begin{aligned}
\boldsymbol{\gamma}^{*n}_{\lambda,P}(d\mathbf{x}) &= \boldsymbol{\gamma}_{n\lambda,P}(d\mathbf{x}) \\
&= \frac{(x_1 \ldots x_d)^{n\lambda-1}}{a^{(d-1)n\lambda} (\Gamma(n\lambda))^d} \exp\left(-\frac{x_1 + \ldots + x_d}{a} \right) \\
&\quad \times {}_0F_{d-1}\left(n\lambda, \ldots, n\lambda; (1-a)a^{-d} x_1 \ldots x_d \right) 1_{(0,\infty)^d}(\mathbf{x})(d\mathbf{x}).
\end{aligned}$$

Finally, the expression of $C_n(\cdot)$ is obtained by $C_n(\mathbf{x}) = \nu_n(d\mathbf{x}) / \boldsymbol{\gamma}^{*n}_{\lambda,P}(d\mathbf{x})$. $\square$

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