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GENERALIZED WEIGHTED COMPOSITION OPERATORS FROM AREA NEVANLINNA SPACES TO WEIGHTED-TYPE SPACES

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Abstract. Let $H(\mathbb{D})$ denote the class of all analytic functions on the open unit disk $\mathbb{D}$ of the complex plane $\mathbb{C}$. Let n be a nonnegative integer, φ be an analytic self-map of $\mathbb{D}$ and $u \in H(\mathbb{D})$. The generalized weighted composition operator is defined by

$$D^n_{\varphi,u} f = u f^{(n)} \circ \varphi, \ f \in H(\mathbb{D}).$$

The boundedness and compactness of the generalized weighted composition operator from area Nevanlinna spaces to weighted-type spaces and little weighted-type spaces are characterized in this paper.

1. Introduction

Let μ be a positive continuous function on $[0,1)$. We say that μ is normal if there exist $\delta \in [0,1)$ and two positive numbers a and b with $0 < a < b$, such that (see [21])

$$\frac{\mu(r)}{(1-r)^a} \text{ is decreasing on } [\delta,1), \lim_{r\to 1} \frac{\mu(r)}{(1-r)^a} = 0;$$
$$\frac{\mu(r)}{(1-r)^b} \text{ is increasing on } [\delta,1), \lim_{r\to 1} \frac{\mu(r)}{(1-r)^b} = \infty.$$

A non-negative function $\|\cdot\|$ on a vector space X (over the real or complex field K) is called an F-norm if the following properties are satisfied:

(i) $\|x\| = 0 \Leftrightarrow x = 0$;
(ii) $\|\lambda x\| \le \|x\|$ for all $\lambda \in K$ with $|\lambda| \le 1$;
(iii) $\|x+y\| \le \|x\| + \|y\|$ for all $x, y \in X$;
(iv) if $\lambda_m \to 0$ and $\lambda_m \in K$, then $\|\lambda_m x\| \to 0$.

An F-norm $\|\cdot\|$ induces a transitive invariant distance d by $d(x,y) = \|x-y\|$ for all $x, y \in X$. A vector space X with an F-norm $\|\cdot\|$ is said to be an F*-space. A complete F*-space is called an F-space.

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Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$. We denote by $H(\mathbb{D})$ the class of all holomorphic functions on $\mathbb{D}$. Let $p \in [1,\infty)$, $\alpha > -1$. An $f \in H(\mathbb{D})$ is said to belong to the area Nevanlinna space $\mathcal{N}_\alpha^p = \mathcal{N}_\alpha^p(\mathbb{D})$, if

$$\|f\|_{\mathcal{N}_\alpha^p}^p = \int_{\mathbb{D}} \big[\log(1+|f(z)|)\big]^p dA_\alpha(z) < \infty,$$

where $dA_\alpha(z) = (1-|z|^2)^\alpha dA(z)$. It is easy to see that

$$\|f+g\|_{\mathcal{N}_\alpha^p} \le \|f\|_{\mathcal{N}_\alpha^p} + \|g\|_{\mathcal{N}_\alpha^p} \tag{1}$$

for all $f, g \in \mathcal{N}_\alpha^p$. Consequently, $\mathcal{N}_\alpha^p$ becomes a metric space.

Also, we have by subharmonicity

$$\log(1+|f(z)|) \le C\frac{\|f\|_{\mathcal{N}_\alpha^p}}{(1-|z|^2)^{(2+\alpha)/p}},\ f \in \mathcal{N}_\alpha^p, \tag{2}$$

where C depends only on p and α (see, e.g. [2]).

From (1) and (2), it is easy to verify that $\|\cdot\|_{\mathcal{N}_\alpha^p}$ is an F-norm of the space $\mathcal{N}_\alpha^p$. Moreover, from (2), it follows that if $f_m \to f$ in $\mathcal{N}_\alpha^p$, then $f_m \to f$ locally uniformly. Here, locally uniform convergence means uniform convergence on every compact subset of $\mathbb{D}$. Therefore, under the F-norm, $\mathcal{N}_\alpha^p$ becomes an F-space, i.e., a translation-invariant complete metric space.

In this paper, a subset A of $\mathcal{N}_\alpha^p$ is called bounded if there exists a positive number r such that $A \subset \{f \in \mathcal{N}_\alpha^p : \|f\|_{\mathcal{N}_\alpha^p} < r\}$. Given a Banach space X, we say that a linear map $T : \mathcal{N}_\alpha^p \to X$ is bounded if $T(A) \subset X$ is bounded for every bounded subset A of $\mathcal{N}_\alpha^p$. We say that T is compact if $T(A) \subset X$ is relatively compact for every bounded subset $A \subset \mathcal{N}_\alpha^p$.

Suppose μ is a normal function on $[0,1)$. An analytic function f on $\mathbb{D}$ is said to belong to the weighted-type space $\mathcal{H}_\mu^\infty = \mathcal{H}_\mu^\infty(\mathbb{D})$, if

$$\|f\|_{\mathcal{H}_\mu^\infty} = \sup_{z\in\mathbb{D}} \mu(|z|)|f(z)| < \infty.$$

The little weighted-type space $\mathcal{H}_{\mu,0}^\infty = \mathcal{H}_{\mu,0}^\infty(\mathbb{D})$ is a subspace of $\mathcal{H}_\mu^\infty$ consisting of all $f \in H(\mathbb{D})$ such that

$$\lim_{|z|\to 1^-} \mu(|z|)|f(z)| = 0.$$

See [1] for more information on $\mathcal{H}_\mu^\infty$.

Let n be a nonnegative integer, φ be an analytic self-map of $\mathbb{D}$ and $u \in H(\mathbb{D})$. The generalized weighted composition operator $D_{\varphi,u}^n$ is defined by

$$D_{\varphi,u}^n f = u f^{(n)} \circ \varphi,\ f \in H(\mathbb{D}), \tag{3}$$

where $f^{(0)} = f$. The generalized weighted composition operator $D_{\varphi,u}^n$ can be regarded as a product of composition operator C_φ, multiplication operator M_u and the n-th differentiation operator D^n. The generalized weighted composition operator $D_{\varphi,u}^n$ was introduced in [38], and studied in [25, 31, 38, 39, 41, 43, 29, 30].

It is interesting to provide a function theoretic characterization of the conditions under which the generalized weighted composition operator $D^n_{\varphi,u}$ becomes a bounded or compact operator on various spaces of analytic functions. The books [4, 36] contain plenty of information on this topic for $D^n_{\varphi,u}$ in the case of $n = 0$ and $u(z) = 1$, i.e., for the composition operator C_φ.

In the case of $n = 0$, $D^n_{\varphi,u}$ is the weighted composition operator uC_φ. Weighted composition operators between analytic function spaces have been studied in [3, 5, 9, 10, 11, 12, 13, 16, 18, 20, 22, 23, 26, 31, 34, 37, 40, 42] (see also related references therein). Xiao studied the compact composition operator on the area Nevanlinna space in [32], and characterized the boundedness and compactness of the composition operator from the area Nevanlinna space to the Bloch space in [33]. Zhu studied the weighted composition operator from the area Nevanlinna space to the Bloch space in [42].

The case $n = 1$ and $u(z) = \varphi'(z)$, that is $D^n_{\varphi,u} = DC_\varphi$, was studied in [6, 8, 27, 35]. The case $n = 1$ and $u(z) = 1$, that is $D^n_{\varphi,u} = C_\varphi D$, was studied in [6, 19, 35, 28, 14, 27, 24, 15].

This paper focuses on the boundedness and compactness of the generalized weighted composition operator from the area Nevanlinna space to the weighted-type space and the little weighted-type space. Some sufficient and necessary conditions for the generalized weighted composition operator $D^n_{\varphi,u}$ to be bounded or compact are given.

Throughout this paper C denotes a positive constant which may be differ from one occurrence to another.

2. Main results and proofs

In this section, we will give our main results and proofs. In order to prove our main results, we need some auxiliary results which are incorporated in the following lemmas.

Lemma 1. *Let n be a nonnegative integer, $1 \le p < \infty$ and $\alpha > -1$. Then there exists some C such that for each $f \in \mathcal{N}^p_\alpha$ and $z \in \mathbb{D}$,*

$$|f^{(n)}(z)| \le \frac{1}{(1-|z|^2)^n} \exp\Big[\frac{C\|f\|_{\mathcal{N}^p_\alpha}}{(1-|z|^2)^{\frac{2+\alpha}{p}}}\Big]. \tag{4}$$

Proof. For $z \in \mathbb{D}$ and $\xi \in \partial\mathbb{D}$, we have

$$1 - \left|z + \frac{1-|z|}{2}\xi\right|^2 \ge 1 - \frac{(1+|z|)^2}{4} \ge \frac{1-|z|^2}{4}.$$

From this, the Cauchy's integral formula for derivatives and (2), we have

$$\begin{aligned}|f^{(n)}(z)| &= \left|\frac{n!}{2\pi i}\int_{\partial\mathbb{D}} \frac{f(\xi)}{(\xi - z)^{n+1}} d\xi\right| \\ &\le \frac{n! 2^n}{2\pi(1-|z|)^n}\int_{\partial\mathbb{D}} |f(z + \frac{1-|z|}{2}\xi)||d\xi|\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{2\pi(1-|z|^2)^n}\int_{\partial\mathbb{D}}(1+n!4^n|f(z+\frac{1-|z|}{2}\xi)|)|d\xi| \\
&\leq \frac{1}{2\pi(1-|z|^2)^n}\int_{\partial\mathbb{D}}\exp\Big[n!4^n\log(1+|f(z+\frac{1-|z|}{2}\xi)|)\Big]|d\xi| \\
&\leq \frac{1}{2\pi(1-|z|^2)^n}\int_{\partial\mathbb{D}}\exp\Big[\frac{C\|f\|_{\mathcal{N}_\alpha^p}}{(1-|z+\frac{1-|z|}{2}\xi|^2)^{\frac{2+\alpha}{p}}}\Big]|d\xi| \\
&\leq \frac{1}{2\pi(1-|z|^2)^n}\int_{\partial\mathbb{D}}\exp\Big[\frac{C\|f\|_{\mathcal{N}_\alpha^p}}{(\frac{1-|z|^2}{4})^{\frac{2+\alpha}{p}}}\Big]|d\xi| \\
&= \frac{1}{(1-|z|^2)^n}\exp\Big[\frac{C\|f\|_{\mathcal{N}_\alpha^p}}{(1-|z|^2)^{\frac{2+\alpha}{p}}}\Big],
\end{aligned}
$$

from which the result follows. □

The proof of the following lemma is similar to the proof of Lemma 1 in [17], so we omit it here.

Lemma 2. *Suppose μ is a normal function on $[0,1)$. A closed set K in $\mathcal{H}_{\mu,0}^\infty$ is compact if and only if it is bounded and satisfies*

$$\lim_{|z|\to 1}\sup_{f\in K}\mu(|z|)|f(z)| = 0.$$

The following criterion for compactness follows from arguments similar to those in Proposition 2.3 of [7].

Lemma 3. *Suppose that φ is an analytic self-map of $\mathbb{D}$, $u\in H(\mathbb{D})$, $1\leq p<\infty$, $\alpha>-1$ and μ is a normal function on $[0,1)$. Then the operator $D_{\varphi,u}^n:\mathcal{N}_\alpha^p\to\mathcal{H}_\mu^\infty$ is compact if and only if for each sequence $\{f_k\}_{k\in\mathbb{N}}$ which is bounded in $\mathcal{N}_\alpha^p$ and converges to zero locally uniformly on $\mathbb{D}$, we have $\|D_{\varphi,u}^n f_k\|_{\mathcal{H}_\mu^\infty}\to 0$ as $k\to\infty$.*

Now, we are ready to state and prove the main results of this paper.

Theorem 1. *Suppose that φ is an analytic self-map of $\mathbb{D}$, $u\in H(\mathbb{D})$, $1\leq p<\infty$, $\alpha>-1$ and μ is a normal function on $[0,1)$. Then for each nonnegative integer n, $D_{\varphi,u}^n:\mathcal{N}_\alpha^p\to\mathcal{H}_\mu^\infty$ is bounded if and only if for all $c>0$,*

$$M(c) =: \sup_{z\in\mathbb{D}}\frac{\mu(|z|)|u(z)|}{(1-|\varphi(z)|^2)^n}\exp\Big[\frac{c}{(1-|\varphi(z)|^2)^{\frac{2+\alpha}{p}}}\Big]<\infty. \tag{5}$$

Proof. Suppose that $D_{\varphi,u}^n:\mathcal{N}_\alpha^p\to\mathcal{H}_\mu^\infty$ is bounded. Let $f(z)=\frac{z^n}{n!}\in\mathcal{N}_\alpha^p$. Then we have

$$\|u\|_{\mathcal{H}_\mu^\infty}=\|D_{\varphi,u}^n f\|_{\mathcal{H}_\mu^\infty}\leq\|D_{\varphi,u}^n\|_{\mathcal{N}_\alpha^p\to\mathcal{H}_\mu^\infty}\|f\|_{\mathcal{N}_\alpha^p}<\infty.$$

Hence $u(z)\in\mathcal{H}_\mu^\infty$.

For any $c > 0$ and $z \in \mathbb{D}$, take

$$f_z(\omega) = \exp\Big\{c\Big[\frac{1-|\varphi(z)|^2}{(1-\overline{\varphi(z)}\omega)^2}\Big]^{\frac{2+\alpha}{p}}\Big\}.$$

By using the inequality $|e^u| \le e^{|u|}$, $u \in \mathbb{C}$, we have

$$\int_{\mathbb{D}} \big[\log(1+|f_z(\omega)|)\big]^p dA_\alpha(\omega) \le \int_{\mathbb{D}} c^p\Big|\frac{1-|\varphi(z)|^2}{(1-\overline{\varphi(z)}\omega)^2}\Big|^{2+\alpha} dA_\alpha(\omega) < \infty.$$

Hence $f_z \in \mathcal{N}_\alpha^p$ for all $z \in \mathbb{D}$.

On the other hand,

$$\begin{aligned} f_z^{(n)}(\omega) = \Big[&\frac{c_0 A^n \overline{(\varphi(z))^n}}{(1-\overline{\varphi(z)}w)^{\frac{2n(2+\alpha)}{p}+n}} + \frac{c_1\overline{(\varphi(z))^n}A^{n-1}}{(1-\overline{\varphi(z)}w)^{\frac{2(n-1)(2+\alpha)}{p}+n}} + \cdots \\ &+ \frac{c_{n-1}\overline{(\varphi(z))^n}A}{(1-\overline{\varphi(z)}w)^{\frac{2(2+\alpha)}{p}+n}}\Big] \exp\Big\{c\Big[\frac{1-|\varphi(z)|^2}{(1-\overline{\varphi(z)}\omega)^2}\Big]^{\frac{2+\alpha}{p}}\Big\},\ \omega \in \mathbb{D}, \end{aligned}$$

where $A = (1-|\varphi(z)|^2)^{\frac{2+\alpha}{p}}$ and $c_0, c_1, \ldots, c_{n-1}$ are positive numbers depending only on c, α, n and p, for example, $c_0 = (\frac{2c(2+\alpha)}{p})^n$. Then

$$\begin{aligned} &\|D_{\varphi,u}^n f_z\|_{\mathcal{H}_\mu^\infty} \\ &= \sup_{\omega\in\mathbb{D}} \mu(|\omega|)|u(\omega)f_z^{(n)}(\varphi(\omega))| \\ &\ge \mu(|z|)|u(z)f_z^{(n)}(\varphi(z))| \\ &= \mu(|z|)|u(z)| \exp\Big\{c\Big[\frac{1-|\varphi(z)|^2}{(1-|\varphi(z)|^2)^2}\Big]^{\frac{2+\alpha}{p}}\Big\} \times \Big|\frac{c_0A^n\overline{(\varphi(z))^n}}{(1-|\varphi(z)|^2)^{\frac{2n(2+\alpha)}{p}+n}} \\ &\quad + \frac{c_1\overline{(\varphi(z))^n}A^{n-1}}{(1-|\varphi(z)|^2)^{\frac{2(n-1)(2+\alpha)}{p}+n}} + \cdots + \frac{c_{n-1}\overline{(\varphi(z))^n}A}{(1-|\varphi(z)|^2)^{\frac{2(2+\alpha)}{p}+n}}\Big| \\ &= \mu(|z|)|u(z)||\varphi(z)|^n \exp\Big[\frac{c}{(1-|\varphi(z)|^2)^{\frac{2+\alpha}{p}}}\Big] \times \Big|\frac{c_0}{(1-|\varphi(z)|^2)^{\frac{n(2+\alpha)}{p}+n}} \\ &\quad + \frac{c_1}{(1-|\varphi(z)|^2)^{\frac{(n-1)(2+\alpha)}{p}+n}} + \cdots + \frac{c_{n-1}}{(1-|\varphi(z)|^2)^{\frac{2+\alpha}{p}+n}}\Big| \\ &\ge \frac{c_0\mu(|z|)|u(z)||\varphi(z)|^n}{(1-|\varphi(z)|^2)^{\frac{n(2+\alpha)}{p}+n}} \exp\Big[\frac{c}{(1-|\varphi(z)|^2)^{\frac{2+\alpha}{p}}}\Big], \end{aligned}$$

which implies that

$$\frac{\mu(|z|)|u(z)|}{(1-|\varphi(z)|^2)^n} \exp\Big[\frac{c}{(1-|\varphi(z)|^2)^{\frac{2+\alpha}{p}}}\Big] \le \frac{\|D_{\varphi,u}^n f_z\|_{\mathcal{H}_\mu^\infty}(1-|\varphi(z)|^2)^{\frac{n(2+\alpha)}{p}}}{c_0|\varphi(z)|^n}. \tag{6}$$

Applying (6), it is easy to see that

$$\sup_{\{z\in\mathbb{D}:|\varphi(z)|>\frac{1}{2}\}}\frac{\mu(|z|)|u(z)|}{(1-|\varphi(z)|^2)^n}\exp\Big[\frac{c}{(1-|\varphi(z)|^2)^{\frac{2+\alpha}{p}}}\Big]<\infty. \tag{7}$$

Moreover, from $u(z)\in\mathcal{H}_\mu^\infty$, we have

$$\begin{aligned}&\sup_{\{z\in\mathbb{D}:|\varphi(z)|\le\frac{1}{2}\}}\frac{\mu(|z|)|u(z)|}{(1-|\varphi(z)|^2)^n}\exp\Big[\frac{c}{(1-|\varphi(z)|^2)^{\frac{2+\alpha}{p}}}\Big]\\ \le\ &(\frac{4}{3})^n\exp\Big[c(\frac{4}{3})^{\frac{2+\alpha}{p}}\Big]\sup_{z\in\mathbb{D}}\mu(|z|)|u(z)|<\infty.\end{aligned} \tag{8}$$

From (7) and (8), it follows that condition (5) holds.

Conversely, suppose (5) holds. Let S be a bounded subset in $\mathcal{N}_\alpha^p$. Then there exists a positive number K such that $\|f\|_{\mathcal{N}_\alpha^p}\le K$ for all $f\in S$. By Lemma 1 and (3), we have

$$\begin{aligned}\|D_{\varphi,u}^n f\|_{\mathcal{H}_\mu^\infty}&=\sup_{z\in\mathbb{D}}\mu(|z|)|(D_{\varphi,u}^n f)(z)|\\&=\sup_{z\in\mathbb{D}}\mu(|z|)|u(z)||f^{(n)}(\varphi(z))|\\&\le\sup_{z\in\mathbb{D}}\frac{\mu(|z|)|u(z)|}{(1-|\varphi(z)|^2)^n}\exp\Big[\frac{C\|f\|_{\mathcal{N}_\alpha^p}}{(1-|\varphi(z)|^2)^{\frac{2+\alpha}{p}}}\Big]\\&=M(CK)<\infty\end{aligned}$$

for all $f\in S$. This implies $D_{\varphi,u}^n(S)$ is a bounded subset of $\mathcal{H}_\mu^\infty$, and then $D_{\varphi,u}^n:\mathcal{N}_\alpha^p\to\mathcal{H}_\mu^\infty$ is a bounded operator. The proof is completed. □

Theorem 2. *Suppose that φ is an analytic self-map of $\mathbb{D}$, $u\in H(\mathbb{D})$, $1\le p<\infty$, $\alpha>-1$ and μ is a normal function on $[0,1)$. Then for each nonnegative integer n, $D_{\varphi,u}^n:\mathcal{N}_\alpha^p\to\mathcal{H}_\mu^\infty$ is compact if and only if $u(z)\in\mathcal{H}_\mu^\infty$ and for all $c>0$,*

$$\lim_{|\varphi(z)|\to1}\frac{\mu(|z|)|u(z)|}{(1-|\varphi(z)|^2)^n}\exp\Big[\frac{c}{(1-|\varphi(z)|^2)^{\frac{2+\alpha}{p}}}\Big]=0. \tag{9}$$

Proof. Suppose that $D_{\varphi,u}^n:\mathcal{N}_\alpha^p\to\mathcal{H}_\mu^\infty$ is compact. Then it is clear that $D_{\varphi,u}^n:\mathcal{N}_\alpha^p\to\mathcal{H}_\mu^\infty$ is bounded. Now from the proof of Theorem 1, we can conclude $u(z)\in\mathcal{H}_\mu^\infty$

Let $\{z_k\}$ be a sequence such that $|\varphi(z_k)|\to1$ as $k\to\infty$. For any $c>0$, set

$$f_k(\omega)=\exp\Big\{c\Big[\frac{1-|\varphi(z_k)|^2}{(1-\overline{\varphi(z_k)}\omega)^2}\Big]^{\frac{2+\alpha}{p}}\Big\}-1.$$

It is easy to see that the sequence $\{f_k\}$ converges to zero local uniformly on any compact subsets of $\mathbb{D}$. Moreover, from the proof of Theorem 1, we can see $\{f_k\}$ is a bounded sequence in $\mathcal{N}_\alpha^p$.

Since $D^n_{\varphi,u}$ is compact, by Lemma 3 we have

$$\|D^n_{\varphi,u} f_k\|_{\mathcal{H}^\infty_\mu} = 0.$$

On the other hand, (6) implies

$$\frac{\mu(|z_k|)|u(z_k)|}{(1-|\varphi(z_k)|^2)^n} \exp\Big[\frac{c}{(1-|\varphi(z_k)|^2)^{\frac{2+\alpha}{p}}}\Big] \le \frac{\|D^n_{\varphi,u} f_k\|_{\mathcal{H}^\infty_\mu}(1-|\varphi(z_k)|^2)^{\frac{n(2+\alpha)}{p}}}{c_0|\varphi(z_k)|^n}.$$

Taking $k\to\infty$ on both sides of above inequality, we obtain

$$\lim_{k\to\infty} \frac{\mu(|z_k|)|u(z_k)|}{(1-|\varphi(z_k)|^2)^n} \exp\Big[\frac{c}{(1-|\varphi(z_k)|^2)^{\frac{2+\alpha}{p}}}\Big] = 0,$$

which implies that (9) holds.

Conversely, suppose $u(z)\in\mathcal{H}^\infty_\mu$ and (9) holds. Then for every $c,\varepsilon>0$, there is a $\delta\in(0,1)$ such that

$$\frac{\mu(|z|)|u(z)|}{(1-|\varphi(z)|^2)^n} \exp\Big[\frac{c}{(1-|\varphi(z)|^2)^{\frac{2+\alpha}{p}}}\Big] < \varepsilon, \tag{10}$$

whenever $\delta<|\varphi(z)|<1$.

Now let $\{f_k\}$ be a sequence in $\mathcal{N}^p_\alpha$ such that $f_k\to 0$ local uniformly on $\mathbb{D}$ and $\|f\|_{\mathcal{N}^p_\alpha}\le K$. Then $f_k^{(n)}\to 0$ local uniformly on $\mathbb{D}$. Therefore using (3) and (4), we have

$$\begin{aligned}
\|D^n_{\varphi,u} f_k\|_{\mathcal{H}^\infty_\mu} &= \sup_{z\in\mathbb{D}} \mu(|z|)|(D^n_{\varphi,u} f_k)(z)| \\
&= \sup_{z\in\mathbb{D}} \mu(|z|)|u(z)||f_k^{(n)}(\varphi(z))| \\
&\le \sup_{\{z\in\mathbb{D}:|\varphi(z)|\le\delta\}} \mu(|z|)|u(z)||f_k^{(n)}(\varphi(z))| \\
&\quad + \sup_{\{z\in\mathbb{D}:\delta<|\varphi(z)|<1\}} \mu(|z|)|u(z)||f_k^{(n)}(\varphi(z))| \\
&\le \|u\|_{\mathcal{H}^\infty_\mu} \sup_{\{w\in\mathbb{D}:|w|\le\delta\}} |f_k^{(n)}(w)| \\
&\quad + \sup_{\{z\in\mathbb{D}:\delta<|\varphi(z)|<1\}} \frac{\mu(|z|)|u(z)|}{(1-|\varphi(z)|^2)^n} \exp\Big[\frac{C\|f_k\|_{\mathcal{N}^p_\alpha}}{(1-|\varphi(z)|^2)^{\frac{2+\alpha}{p}}}\Big] \\
&\le \|u\|_{\mathcal{H}^\infty_\mu} \sup_{\{w\in\mathbb{D}:|w|\le\delta\}} |f_k^{(n)}(w)| \\
&\quad + \sup_{\{z\in\mathbb{D}:\delta<|\varphi(z)|<1\}} \frac{\mu(|z|)|u(z)|}{(1-|\varphi(z)|^2)^n} \exp\Big[\frac{CK}{(1-|\varphi(z)|^2)^{\frac{2+\alpha}{p}}}\Big] \\
&\le \|u\|_{\mathcal{H}^\infty_\mu} \sup_{\{w\in\mathbb{D}:|w|\le\delta\}} |f_k^{(n)}(w)| + \varepsilon.
\end{aligned}$$

Note that $u(z) \in \mathcal{H}_\mu^\infty$ and the positive number ε is arbitrary. This yields

$$\lim_{k\to\infty} \|D_{\varphi,u}^n f_k\|_{\mathcal{H}_\mu^\infty} = 0.$$

It follows from Lemma 3 that the operator $D_{\varphi,u}^n : \mathcal{N}_\alpha^p \to \mathcal{H}_\mu^\infty$ is compact. The proof is completed. □

From Theorems 1 and 2, we can obtain the following corollary.

Corollary 3. *Suppose that φ is an analytic self-map of $\mathbb{D}$, $u \in H(\mathbb{D})$, $1 \le p < \infty$, $\alpha > -1$ and μ is a normal function on $[0,1)$. Then for each nonnegative integer n, the following are equivalent*:
(i) $D_{\varphi,u}^n : \mathcal{N}_\alpha^p \to \mathcal{H}_\mu^\infty$ *is bounded*;
(ii) $D_{\varphi,u}^n : \mathcal{N}_\alpha^p \to \mathcal{H}_\mu^\infty$ *is compact*;
(iii) $u(z) \in \mathcal{H}_\mu^\infty$ *and for all $c > 0$ condition* (9) *hold.*

Proof. From the proof of Theorem 2, we can get (iii)⇒(ii), and (ii)⇒(i) is obvious.

(i)⇒(iii). Suppose that $D_{\varphi,u}^n : \mathcal{N}_\alpha^p \to \mathcal{H}_\mu^\infty$ is bounded. By Theorem 1, $u(z) \in \mathcal{H}_\mu^\infty$. Moreover, for all $\delta, c > 0$, we have $M(c) < \infty$ and $M(c+\delta) < \infty$. Using the equation

$$\lim_{|\varphi(z)|\to 1} \exp\Big[\frac{\delta}{(1-|\varphi(z)|^2)^{\frac{2+\alpha}{p}}}\Big] = \infty,$$

we conclude that the condition (9) holds for all $c > 0$. The proof is completed. □

Theorem 4. *Suppose that φ is an analytic self-map of $\mathbb{D}$, $u \in H(\mathbb{D})$, $1 \le p < \infty$, $\alpha > -1$ and μ is a normal function on $[0,1)$. Then for each nonnegative integer n, the following are equivalent*:
(i) $D_{\varphi,u}^n : \mathcal{N}_\alpha^p \to \mathcal{H}_{\mu,0}^\infty$ *is bounded*;
(ii) $D_{\varphi,u}^n : \mathcal{N}_\alpha^p \to \mathcal{H}_{\mu,0}^\infty$ *is compact*;
(iii) $u(z) \in \mathcal{H}_{\mu,0}^\infty$ *and for all $c > 0$,*

$$(11) \qquad \lim_{|z|\to 1} \frac{\mu(|z|)|u(z)|}{(1-|\varphi(z)|^2)^n} \exp\Big[\frac{c}{(1-|\varphi(z)|^2)^{\frac{2+\alpha}{p}}}\Big] = 0.$$

Proof. (ii)⇒(i). It is obvious.

(i)⇒(iii). Suppose that $D_{\varphi,u}^n : \mathcal{N}_\alpha^p \to \mathcal{H}_{\mu,0}^\infty$ is bounded. For $f(z) = z^n \in \mathcal{N}_\alpha^p$, it follows that $u(z) \in \mathcal{H}_{\mu,0}^\infty$. Since $D_{\varphi,u}^n : \mathcal{N}_\alpha^p \to \mathcal{H}_\mu^\infty$ is bounded, by Corollary 3, we conclude that the condition (9) hold for all $c > 0$. Thus, for each $c, \varepsilon > 0$, there exists a number $t \in (0,1)$ such that

$$(12) \qquad \frac{\mu(|z|)|u(z)|}{(1-|\varphi(z)|^2)^n} \exp\Big[\frac{c}{(1-|\varphi(z)|^2)^{\frac{2+\alpha}{p}}}\Big] < \varepsilon,$$

whenever $t < |\varphi(z)| < 1$.

Moreover, from $u(z) \in \mathcal{H}^\infty_{\mu,0}$, we infer that there exists a number $r \in (0,1)$ such that for $r < |z| < 1$,

$$\mu(|z|)|u(z)| < \frac{\varepsilon}{(1-t^2)^n} \exp\Big[\frac{-c}{(1-t^2)^{\frac{2+\alpha}{p}}}\Big]. \tag{13}$$

Therefore, if $r < |z| < 1$ and $|\varphi(z)| > t$, then by (12) we have

$$\frac{\mu(|z|)|u(z)|}{(1-|\varphi(z)|^2)^n} \exp\Big[\frac{c}{(1-|\varphi(z)|^2)^{\frac{2+\alpha}{p}}}\Big] < \varepsilon.$$

If $r < |z| < 1$ and $|\varphi(z)| \le t$, then by (13) we have

$$\frac{\mu(|z|)|u(z)|}{(1-|\varphi(z)|^2)^n} \exp\Big[\frac{c}{(1-|\varphi(z)|^2)^{\frac{2+\alpha}{p}}}\Big] \le \frac{\mu(|z|)|u(z)|}{(1-t^2)^n} \exp\Big[\frac{c}{(1-t^2)^{\frac{2+\alpha}{p}}}\Big] < \varepsilon.$$

In other words, for each $\varepsilon > 0$, there exists an $r \in (0,1)$ such that for $r < |z| < 1$,

$$\frac{\mu(|z|)|u(z)|}{(1-|\varphi(z)|^2)^n} \exp\Big[\frac{c}{(1-|\varphi(z)|^2)^{\frac{2+\alpha}{p}}}\Big] < \varepsilon,$$

which implies that (11) holds for all $c > 0$.

(iii)$\Rightarrow$(ii). Suppose that $u(z) \in \mathcal{H}^\infty_{\mu,0}$ and (11) holds for all $c > 0$. From Lemma 2, $D^n_{\varphi,u} : \mathcal{N}^p_\alpha \to \mathcal{H}^\infty_{\mu,0}$ is compact if and only if

$$\lim_{|z|\to 1} \sup_{f \in B_{\mathcal{N}^p_\alpha}} \mu(|z|)|(D^n_{\varphi,u} f)(z)| = \lim_{|z|\to 1} \sup_{f \in B_{\mathcal{N}^p_\alpha}} \mu(|z|)|u(z)||f^{(n)}(\varphi(z))| = 0, \tag{14}$$

where $B_{\mathcal{N}^p_\alpha} = \{g \in \mathcal{N}^p_\alpha : \|g\|_{\mathcal{N}^p_\alpha} \le 1\}$ is the unit ball in the space $\mathcal{N}^p_\alpha$.

On the other hand, by Lemma 1, we have

$$\mu(|z|)|u(z)||f^{(n)}(\varphi(z))| \le \frac{\mu(|z|)|u(z)|}{(1-|\varphi(z)|^2)^n} \exp\Big[\frac{C\|f\|_{\mathcal{N}^p_\alpha}}{(1-|\varphi(z)|^2)^{\frac{2+\alpha}{p}}}\Big]. \tag{15}$$

Taking the supremum in (15) over the unit ball $B_{\mathcal{N}^p_\alpha}$, and letting $|z| \to 1$, from (11) we see that (14) hold and hence $D^n_{\varphi,u} : \mathcal{N}^p_\alpha \to \mathcal{H}^\infty_{\mu,0}$ is compact. The proof is completed. □

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