GENERALIZED WILCZYNSKI INVARIANTS FOR NON-LINEAR ORDINARY DIFFERENTIAL EQUATIONS

BORIS DOUBROV

Abstract. We show that classical Wilczynski–Se-ashi invariants of linear systems of ordinary differential equations are generalized in a natural way to contact invariants of non-linear ODEs. We explore geometric structures associated with equations that have vanishing generalized Wilczynski invariants and establish relationship of such equations with deformation theory of rational curves on complex algebraic surfaces.

1. Introduction

This paper is devoted to a very important class of contact invariants of (non-linear) systems of ordinary differential equations. They can be considered as a direct generalization of classical Wilczynski invariants of linear differential equations, which are, in its turn, closely related to projective invariants of non-parametrized curves.

The construction of these invariants is based on the fundamental idea of approximating the non-linear objects by linear. In case of differential equations we can consider linearization of a given non-linear equation along each solution. Roughly speaking, this linearization describes the tangent space to the solution set of the given equation. Note that in general this set is not Hausdorff, but we can still speak about linearization of the equation without any loss of generality.

Unlike the class of all non-linear equations of a fixed order, which is stable with respect contact transformations, the class of linear equations forms a category with a much smaller set of morphisms. This makes it possible to describe the set of all invariants of the linear equations explicitly. The main goal of this paper is to show the general geometric procedure that extends these invariants to the class of non-linear equations via the notion of the linearization.

1991 *Mathematics Subject Classification.* 34A26, 53B15.

Key words and phrases. Differential invariants, projective curves, non-linear equations, twistor spaces, symmetries of differential equations.

In short, the major result of this paper is that *invariants of the linearization of a given non-linear equation are contact invariants of this equation.*

In fact, these invariants have been known for ordinary differential equations of low order: third order equations [3, 14], fourth order ODEs [1, 8, 6], systems of second order [7, 9], but it was not understood up to now that these invariants come from the linearization of non-linear equations.

In the theory of linear differential equations Wilczynski invariants generate all invariants and, in particular, are responsible for the trivialization of the equation. In non-linear case the generalized Wilczynski invariants form only part of the generators in the algebra of all contact invariants. However, the equations with vanishing generalized Wilczynski invariants have a remarkable property, that its solution space carries a natural geometric structure (see [1] and [6]). We discuss this structure in more detail in Section 5 for a single ODE. Finally, in Section 6 we show that all results of this paper can be extended to the case of systems of ordinary differential equations.

Acknowledgments. This paper is based in large on the lectures given at the IMA workshop "Symmetries and overdetermined systems of partial differential equations". I would like to thank organizers of this workshop for inviting me to give these lectures. I also would like to thank Eugene Ferapontov, Rod Gover and Igor Zelenko for corrections and valuable discussions on the topic of this paper.

2. Naive approach

As an example, let us outline this idea for the case of a single non-linear ODE of order ≥ 3. Note that all equations of smaller order are contact equivalent to each other and, thus, do not have any non-trivial invariants.

The complete set of invariants for a single linear ODE was described in the classical work of Wilczynski [18]. Namely, consider the class of linear homogeneous differential equations

$$y^{(n+1)} + p_n(x)y^{(n)} + \dots + p_0(x)y = 0, \tag{1}$$

viewed up to all invertible transformations of the form:

$$(x, y) \mapsto (\lambda(x), \mu(x)y). \tag{2}$$

In fact, these are the most general transformations preserving the class of linear equations.

A function I of the coefficients $p_i(x)$ and their derivatives is called *a relative invariant of weight l* if it is transformed by the rule $I \mapsto$

$(\lambda')^l I$ under the change of variables (2). In particular, if such relative invariant vanishes identically for the initial equation, it will also vanish for the transformed one. Relative invariants of weight 0 are called *(absolute) invariants of the linear equation*.

E. Wilczynski [18] gave the complete description of all relative (and, thus, absolute) invariants of linear ODEs of any order. It is well-known that each equation (1) can be brought by the transformation (2) to the so-called Laguerre–Forsyth canonical form:

$$y^{(n+1)} + q_{n-2}(x)y^{(n-2)} + \cdots + q_0(x)y = 0.$$

The set of transformations preserving this canonical form is already a finite-dimensional Lie group:

$$(x, y) \mapsto \left(\frac{ax+b}{cx+d}, \frac{ey}{(cx+d)^{n+1}}\right).$$

This group acts on coefficients $q_0, \dots, q_{n-2}$ of the canonical form, and the relative invariants of this action are identified with the relative invariants of the general linear equation. The simplest $n-1$ relative invariants $\theta_3, \dots, \theta_{n+1}$ linear in $q_i^{(j)}$ have the form:

(3)
$$\theta_k = \sum_{j=1}^{k-2} (-1)^{j+1} \frac{(2k-j-1)!(n-k+j)!}{(k-j)!(j-1)!} q_{n-k+j}^{(j-1)} \qquad k = 3, \dots, n+1.$$

Each invariant θ_k has weight k. Wilczynski proved that all other invariants can be expressed in terms of the invariants θ_k and their derivatives.

Although these formulas express invariants in terms of coefficients of the canonical form, they can be written explicitly in terms of the initial coefficients of the general equation. Moreover, it can be shown that each of these invariants θ_i is polynomial in terms of functions $p_i(x)$ and their derivatives.

Equivalence theory of linear ODEs is intimately related with the projective theory of non-parametrized curves. Namely, let $\{y_0(x), \dots, y_n(x)\}$ be a fundamental set of solutions of a linear equation $\mathcal{E}$ given by (1). Consider the curve

$$L_{\mathcal{E}} = \{[y_0(x) : y_1(x) : \cdots : y_n(x)] \mid x \in \mathbb{R}\}$$

in the n-dimensional projective space $\mathbb{R}\mathrm{P}^n$. Since the solutions do not vanish simultaneously for any $x \in \mathbb{R}$, this curve is well-defined. Moreover, since the fundamental set of solutions is linearly independent, this curve is *non-degenerate*, i.e., it is not contained in any hyperplane.

Finally, since the set of fundamental solutions is defined up to any non-degenerate linear transformation, we see that the curve $L_{\mathcal{E}}$ is defined up to projective transformations.

Consider now what happens with this curve, if we apply the transformations (2) to the equation $\mathcal{E}$. The transformations $(x, y) \mapsto (x, \mu(x)y)$ do not change the curve, since they just multiply each solution by $\mu(x)$. The transformations $(x, y) \mapsto (\lambda(x), y)$ are equivalent to reparametrizations of $L_{\mathcal{E}}$.

Thus, we see that to each linear equation $\mathcal{E}$ we can assign the set of projectively-equivalent non-degenerate curves in $\mathbb{R}\mathrm{P}^n$. It is easy to see that this correspondence is one-to-one. Indeed, having a non-degenerate curve in $\mathbb{R}\mathrm{P}^n$ we can fix a parameter x on it and write it explicitly as $[y_0(x) : y_1(x) : \dots : y_n(x)]$, where the coordinates $y_i(x)$ are defined modulo a non-zero multiplier $\mu(x)$. Since the curve is non-degenerate, the set of functions $\{y_i(x)\}_{i=0,\dots,n}$ is linearly independent and defines a unique linear equation $\mathcal{E}$ having these functions as the set of fundamental solutions.

Each relative invariant I of weight k can be naturally interpreted as a section of the line bundle $S^k(TL)$ invariant with respect to projective transformations. In particular, relative invariants of weight 0 are just projective differential invariant of non-parametrized curves in the projective space. First examples of such invariants were constructed by Sophus Lie [12] for curves on the projective plane and then generalized by Halphen [10] to the case of projective spaces of higher dimensions. Note that they can be constructed via the standard Cartan moving frame method.

E. Wilczynski also proved the following result characterizing the equations with vanishing invariants:

Theorem 1. *Let $\mathcal{E}$ be a linear ODE given by* (1)*. The following conditions are equivalent:*

(1) invariants $\theta_3, \dots, \theta_{n+1}$ vanish identically.
(2) the equation $\mathcal{E}$ is equivalent to the trivial equation $y^{(n+1)} = 0$.
(3) the curve $L_{\mathcal{E}}$ is an open part of the normal rational curve in $\mathbb{R}\mathrm{P}^n$.
(4) the symmetry algebra of $L_{\mathcal{E}}$ is isomorphic to the subalgebra $\mathfrak{sl}(2, \mathbb{R}) \subset \mathfrak{sl}(n+1, \mathbb{R})$ acting irreducibly on $\mathbb{R}^{n+1}$.

Let us show how to extend Wilczynski invariants to arbitrary non-linear ordinary differential equations via the notion of linearization. Indeed, consider now an arbitrary non-linear ODE solved with respect to the highest derivative:

$$y^{(n+1)} = f(x, y, y', \dots, y^{(n)}). \tag{4}$$

Let $y_0(x)$ be any solution of this equation. Then we can consider the linearization of (4) along this solution. It is a linear equation

$$h^{(n+1)} = \frac{\partial f}{\partial y^{(n)}} h^{(n)} + \dots + \frac{\partial f}{\partial y} h, \tag{5}$$

where all coefficients are evaluated at the solution $y_0(x)$. It describes all deformations $y_\epsilon(x) = y_0(x) + \epsilon h(x)$ of the solution $y_0(x)$, which satisfy the equation (4) modulo $o(\epsilon)$.

Consider now Wilczynski invariants $\theta_3, \dots, \theta_{n+1}$ of the linearization (5). They are polynomial in terms of the coefficients $\frac{\partial f}{\partial y^{(i)}}$ and their derivatives (evaluated at the solution $y_0(x)$). In general, let $F(x, y, y', \dots, y^{(n)})$ be any function of x, $y(x)$ and its derivatives evaluated at any solution of the equation (4). Then, differentiating it by x is equivalent to applying the operator of total derivative:

$$D = \frac{\partial}{\partial x} + y' \frac{\partial}{\partial y} + \dots + y^{(n)} \frac{\partial}{\partial y^{(n-1)}} + f \frac{\partial}{\partial y^{(n)}}.$$

Thus, analytically, Wilczynski invariants of (5) can be expressed as polynomials in terms of $D^j \left(\frac{\partial f}{\partial y^{(i)}} \right)$ $(0 \leq i \leq n,\ j \geq 0)$ and are independent of the solution $y_0(x)$ we started with. More precisely, there are $(n-1)$ well-defined expressions $W_3, \dots, W_{n+1}$ polynomial in terms of $D^j \left(\frac{\partial f}{\partial y^{(i)}} \right)$, which, being evaluated at any solution $y_0(x)$ of the equation (4), give Wilczynski invariants of its linearization along $y_0(x)$. We call these expressions $W_3, \dots, W_{n+1}$ the *generalized Wilczynski invariants of non-linear ordinary differential equations.*

To understand, what geometric objects correspond to generalized Wilczynski invariants we need to consider the jet interpretation of differential equations. Let $J^i = J^i(\mathbb{R}^2)$ be the space of all jets of order i of (non-parametrized) curves on the plane. Then for each curve L on the plane we can define its lift $L^{(i)}$ to the jet space J^i of order i. Then the equation (4) can be considered as a submanifold $\mathcal{E} \subset J^{n+1}$ of codimension 1, and, by the existence and uniqueness theorem for ODEs, all lifts of its solutions form a one-dimensional foliation on $\mathcal{E}$. Let us denote its tangent one-dimensional distribution by E. Let (x, y) be any local coordinate system on the plane. Then it naturally defines a local coordinate system $(x, y_0, y_1, \dots, y_i)$ on J^i such that the lift of the graph $(x, y(x))$ is equal to $(x, y(x), y'(x), \dots, y^{(i)})$. In these coordinates the submanifold $\mathcal{E}$ is given by equation $y_{n+1} = f(x, y_0, y_1, \dots, y_n)$, functions $(x, y_0, y_1, \dots, y_n)$ form a local coordinate system on $\mathcal{E}$ and the distribution E is generated by the vector field

$$D = \frac{\partial}{\partial x} + y_1 \frac{\partial}{\partial y_0} + \dots + y_n \frac{\partial}{\partial y_{n-1}} + f \frac{\partial}{\partial y}_n .$$

The expressions $W_3, \dots, W_{n+1}$ can be naturally interpreted as functions on the equation manifold $\mathcal{E}$. In fact, each invariant W_i, being a relative invariant of weight i of all linearizations, defines a section $\mathcal{W}_i$ of the line bundle $S^i E^*$ by the formula:

$$\mathcal{W}_i(D, D, \dots, D) = W_i.$$

The largest set of invertible transformations preserving the class of ODEs of fixed order i consists of so-called contact transformations, which are the most general transformations of J^i preserving the class of lifts of plane curves. The main result of this paper can be formulated as follows:

Theorem 2. *The sections $\mathcal{W}_i$, $i = 3, \dots, n+1$ of the line bundles $S^i E^*$ are invariant with respect to contact transformations of J^{n+1}.*

3. Algebraic model of Wilczynski invariants

In this section we give an alternative algebraic description of Wilczynski invariants, which is due to Se-ashi [15, 16]. There are two main reasons for providing an algebraic picture behind Wilczynski invariants. First, it can be easily generalized to invariants of systems of ODEs or even to more general classes of linear finite type equations (see [15]). Wilczynski himself described only invariants for a single ODE of arbitrary order and for linear systems of second order ODEs. It seems that analytic methods become too elaborate to proceed with systems of higher order.

Second, Se-ashi's construction gives alternative analytic formulas for computing Wilczynski invariants, which are independent of Laguerre-Forsyth canonical form. In particular, this explains why Wilczynski invariants are polynomial in the initial coefficients and their derivatives, while the coefficients of the canonical form are not.

Denote by V_n the set $S^n(\mathbb{R}^2)$ of all homogeneous polynomials of degree n in two variables v_1, v_2. The standard $GL(2, \mathbb{R})$-action on $\mathbb{R}^2$ is naturally extended to V_n and turns it into an irreducible $GL(2, \mathbb{R})$-module. Denote by $\rho_n \colon GL(2, \mathbb{R}) \to GL(V_n)$ the corresponding representation mapping.

Consider the corresponding action of $\mathfrak{gl}(2,\mathbb{R})$ on $V_n = S^n(\mathbb{R}^2)$. Denote by X, Y, H, Z the following basis in $\mathfrak{gl}(2,\mathbb{R})$:

$$X = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right), H = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right), Y = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right), Z = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right).$$

Then the action of these basis elements on V_n is equivalent to the action of the following vector fields on $\mathbb{R}_n[v_1, v_2]$:

$$X = v_1\frac{\partial}{\partial v_2},\ H = v_1\frac{\partial}{\partial v_1} - v_2\frac{\partial}{\partial v_2},\ Y = v_2\frac{\partial}{\partial v_1},\ Z = v_1\frac{\partial}{\partial v_1} + v_2\frac{\partial}{\partial v_2}.$$

In the sequel we shall identify $\mathfrak{gl}(2,\mathbb{R})$ with its image in $\mathfrak{gl}(V_n)$ defined by this action.

We define a gradation on V_n such that the polynomials $E_0 = v_1^n$, $E_1 = v_1^{n-1}v_2$, $\dots$, $E_n = v_2^n$ have degrees $-n-1$, $-n$, $\dots$, -1 respectively. Then the elements X, H, Y, Z define operators of degrees -1, 0, 1 and 0 respectively. Denote also by $V_n^{(i)}$ the set of all elements in V_n of degree $\le i$. These subspaces define a filtration on V_n:

$$\{0\} = V_n^{(-n-2)} \subset V_n^{(-n-1)} \subset \dots \subset V_n^{(-1)} = V_n.$$

Let E be a one-dimensional vector bundle over a one-dimensional manifold M with a local coordinate x. Denote by $J^n(E)$ the n-th order jet bundle of E, which is a $(n+1)$-dimensional vector bundle over M. Then any $(n+1)$-th order linear homogeneous ODE can be considered as a connection on $J^n(E)$ such that all its solutions, being lifted to $J^n(E)$, are horizontal. Let $\mathcal{F}(J^n(E))$ be the frame bundle of $J^n(E)$. Since $\dim V_n = n+1$, we can identify $\mathcal{F}(J^n(E))$ as a set of all isomorphisms $\phi_x\colon V_n \to J^n_x(E)$. This turns $\mathcal{F}(J^n(E))$ into a principle $GL(V_n)$-bundle over M.

Denote by ω the corresponding connection form on $\mathcal{F}(J^n(E))$. In brief, the main idea of the Se-ashi works is that Wilczynski invariants can be interpreted in terms of a natural reduction of the $GL(V_n)$-bundle $\mathcal{F}(J^n(E))$ to some G-subbundle P characterized by the following conditions:

1. G is the image of the lower-triangular matrices in $GL(2,\mathbb{R})$ under the representation $\rho_n\colon GL(2,\mathbb{R}) \to GL(V_n)$;
2. $\omega|_P$ takes values in the subspace $\langle X, H, Z, Y, Y^2, \dots, Y^n\rangle$.

The form $\omega|_P$ can be decomposed into the sum of $\omega_{\mathfrak{gl}}$ with values in $\mathfrak{gl}(2,\mathbb{R}) \subset \mathfrak{gl}(V_n)$ and $\sum_{i=2}^n \omega_i Y^i$. Then $\omega_{\mathfrak{gl}}$ defines a flat projective structure on the manifold M, while the forms ω_i (or, more precisely, their values on the vector field $\frac{\partial}{\partial x}$) coincide up to the constant with Wilczynski invariants θ_{i+1}.

Let us describe this reduction in more detail. Denote by $G^{(0)}$ the subgroup of $GL(V_n)$ consisting of all elements preserving the filtration

$V_n^{(i)}$ on V_n introduced above. These are exactly all elements of $GL(V_n)$ represented by lower-triangular matrices in the basis $\{E_0, \dots, E_n\}$. For $k \geq 1$ denote by $\mathfrak{gl}^{(k)}(V_n)$ the following subalgebra in $\mathfrak{gl}(V_n)$:

$$\mathfrak{gl}^{(k)}(V_n) = \{\phi \in \mathfrak{gl}(V_n) \mid \phi(V_n^{(i)}) \subset V_n^{(i+k)}\}.$$

Let $GL^{(k)}(V_n)$ be the corresponding unipotent subgroup in $GL(V_n)$. Define the subgroups $G^{(k)} \subset G^{(0)}$ as the products $GL^{(0)}(2, \mathbb{R})GL^{(k)}(V_n)$ for each $k \geq 0$, where $GL^{(0)}(2, \mathbb{R})$ is the intersection of $G^{(0)}$ with $\rho_n(GL(2, \mathbb{R}))$. Denote also by W the subspace in $\mathfrak{gl}(V_n)$ spanned by the endomorphisms $Y^2, \dots, Y^n$ (here $Y \subset \mathfrak{gl}(2, \mathbb{R})$ is identified with the corresponding element in $\mathfrak{gl}(V_n)$).

The reduction $P \subset \mathcal{F}(J^n(E))$ is constructed via series of reductions $P_{k+1} \subset P_k$, where P_k is a principal $G^{(k)}$-bundle characterized by the following conditions:

a) P_0 consists of all frames $\phi_x \colon V_n \to J^n_x(E)$, which map the filtration of V_n into the filtration on each fiber $J^n_x(E)$;
b) for $k \geq 1$ the form $\omega_k = \omega|_{P_k}$ takes values in the subspace $W_k = W + \mathfrak{gl}^{(k-1)}(V_n) + \mathfrak{gl}(2, \mathbb{R})$.

At the end of this procedure we arrive at the principal bundle $P = P_{n+1}$ with the structure group $G = G_{n+1}$ and the 1-from $\omega = \omega_{n+1}$ with values in $W + \mathfrak{gl}(2, \mathbb{R})$.

In fact, the second condition can be considered as a definition of P_k for $k \geq 1$. Indeed, let $(x, y_0, y_1, \dots, y_n)$ be a local coordinate system on $J^n(E)$, such that the n-jet of the section $y(x)$ of $J(E)$ is given by $y_i(x) = y^{(i)}(x)$. Then the connection on $J^n(E)$ corresponding to the linear homogeneous equation (1) is defined as an annihilator of the forms:

$$\theta_i = dy_i - y_{i+1}dx,$$
$$\theta_n = dy_n + \left(\sum_{i=0}^{n} p_i(x)y_i\right) dx.$$

If $s \colon x \mapsto (y_0(x), \dots, y_n(x))$ is any section of $J^n(E)$, then the covariant derivative of s along the vector field ∂_x has the form:

$$\nabla_{\partial_x} s = (y_0'(x) - y_1(x), y_1'(x) - y_2(x), \dots, y_n'(x) + \sum_{i=0}^{n} p_i(x)y_i(x)).$$

Let s_i, $i = 0, \dots, n$, be the standard sections defined by $y_j(x) = \delta_{ij}$. Then $s = \{s_0, \dots, s_n\}$ is a local section of the frame bundle P_0. Let ω be the connection form. The pull-back $s^*\omega$ can be written in this coordinate system as:

$$s^*\omega = \begin{pmatrix} 0 & -1 & 0 & \dots & 0 \\ 0 & 0 & -1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & -1 \\ p_0(x) & p_1(x) & p_2(x) & \dots & p_n(x) \end{pmatrix} dx$$

Then Se-ashi reduction theorem says that there exists such gauge transformation $C\colon M \to G^{(0)}$ that

$$C^{-1}\big(s^*\omega\big)C + C^{-1}dC = \big(-X + \alpha H + \beta Z + \gamma Y + \sum_{i=2}^{n} \bar\theta_{i+1} Y^i\big)\, dx,$$

and the set of such transformations forms a principal G-bundle. As shown in [15], the coefficients $\bar\theta_{i+1}$ coincide with classical Wilczynski invariants θ_{i+1} defined by (3) up to the constant and some polynomial expression of invariants of lower weight:

$$\bar\theta_{i+1} = c\theta_{i+1} + P_{i+1}(\theta_3, \dots, \theta_i),$$

where P_{i+1} is some fixed polynomial without free term. In particular, Theorem 1 remains true if we substitute invariants θ_i with their modified versions $\bar\theta_i$.

We shall not repeat the computations from Se-ashi work [15], just mentioning that it is based on the following simple technical fact. Namely, for any $k \geq 1$ consider the subspace $\mathfrak{gl}_k(V_n)$ of all operators of degree k and the mapping $\operatorname{ad}_k(X)\colon \mathfrak{gl}_k(V_n) \to \mathfrak{gl}_{k-1}(V_n)$, $A \mapsto [X, A]$. Then we have the decomposition $\mathfrak{gl}_k(V_n) = \langle Y^k \rangle \oplus \operatorname{Im} \operatorname{ad}_{k+1} X$ for all $k \geq 1$, which allows to carry effectively the reduction from P_k to P_{k+1}. On each step such reduction involves only the operation of solving linear equations with constant coefficients and differentiation. In particular, this proves, that Wilczynski invariants are polynomial in terms of the coefficients $p_0(x), \dots, p_n(x)$ of the initial equation (1).

In general, we can not make further reductions without assumption that some of these invariants do not vanish. And if all these invariants vanish, then our equation is equivalent to the trivial equation $y^{(n+1)} = 0$.

4. Generalization of Wilczynski invariants to non-linear ODEs

Let $\mathcal{E}$ be now an arbitrary non-linear ODE of order $(n+1)$. As above, we identify it with a submanifold of codimension 1 in the jet space J^{n+1}. Then in local coordinates $(x, y_0, \dots, y_{n+1})$ the equation

$\mathcal{E}$ is given by $y_{n+1} = f(x, y_0, \dots, y_n)$, and the functions $(x, y_0, \dots, y_n)$ form a local coordinate system on $\mathcal{E}$.

Let us recall that there is a canonical contact C^k distribution defined on each jet space $J^k(\mathbb{R}^2)$, which is generated by tangent lines to all lifts of curves from the plane. In local coordinates it is generated by two vector fields:

$$C^k = \left\langle \frac{\partial}{\partial y_k}, \ \frac{\partial}{\partial x} + y_1 \frac{\partial}{\partial y_0} + \dots y_k \frac{\partial}{\partial y_{k-1}} \right\rangle.$$

We also have natural projections $\pi_{k,l}\colon J^k(\mathbb{R}^2) \to J^l(\mathbb{R}^2)$ for all $l < k$.

The contact distribution C^{n+1} defines a line bundle on the equation $\mathcal{E}$ as follows $E_p = T_p\mathcal{E} \cap C_p^{n+1}$ for all $p \in \mathcal{E}$. Integral curves of this bundle are precisely lifts of the solutions of the given ODE. In local coordinates this line bundle is generated by the vector field:

$$D = \frac{\partial}{\partial x} + y_1 \frac{\partial}{\partial y_0} + \dots + y_n \frac{\partial}{\partial y_{n-1}} + f \frac{\partial}{\partial y_n},$$

which defines also the total derivative operator.

We are interested in the frames on the normal bundle to E, that is the bundle $N(\mathcal{E}) = T\mathcal{E}/E$. We have a natural filtration of this bundle defined by means of the projections $\pi_{n+1,i}\colon \mathcal{E} \to J^i(\mathbb{R}^2)$ for $i < n$. Namely, we define N_i as the intersection of $\pi_{n+1,i}^* C^i$ with $T\mathcal{E}$ modulo the line bundle E for all $i = 0, \dots, n$. Then it is easy to see that the sequence

$$N = N_0 \supset N_1 \supset \dots N_n \supset 0,$$

is strictly decreasing and $\dim N_i = n + 1 - i$. In local coordinates we have

$$N_i = \left\langle \frac{\partial}{\partial y_i}, \dots, \frac{\partial}{\partial y_n} \right\rangle + E.$$

The vector field D defines a first order operator on N, which is compatible with this filtration, i.e. $D(N_i) \subset N_{i-1}$ for all $i = 1, \dots, n$.

As in the case of linear equations, we can define the frame bundle, consisting of all maps $\phi_p\colon V_n \to N_p$ from the $GL(2, R)$-module V_n into N_p, which preserve the filtrations. This defines a $G^{(0)}$-bundle P_0 over $\mathcal{E}$.

To proceed further with similar reductions, we need to have an analog of the connection form. In general, we don't have any natural connection on the normal bundle $N(\mathcal{E})$. However, we can define the covariant derivative along all vectors lying in the line bundle E generated by D:

$$\nabla_D(X + E) = [D, X] + E, \quad \text{for any } X + E \in N(\mathcal{E}).$$

This gives us a so-called partial connection with the connection form $\omega\colon E \to \mathfrak{gl}(V_n)$. This connection form is sufficient for our purposes. As in the case of linear equations, we can form the similar set of reductions P_i of the principle bundle P_0. Each such bundle P_i, being restricted to any solution L, coincides with the principle bundle constructed from the linearization of $\mathcal{E}$ along this solution. As a result, we get the set of well-defined generalized Wilczynski invariants $\mathcal{W}_i$, where each $\mathcal{W}_i$ is a global section of the bundle S^iE^*.

Since the bundle $N(\mathcal{E})$ and the connection form ω are defined totally in terms of the contact geometry of the jet space J^{n+1} and the reduction procedure does not depend on any external data, we see that generalized Wilczynski invariants are contact invariants of the original equation. It can be formulated rigorously in the following way:

Theorem 3. *Assume that $n \geq 2$. Let $\phi\colon J^{n+1} \to J^{n+1}$ be a local contact transformation establishing the local equivalence of two equations $\mathcal{E}$ and $\overline{\mathcal{E}}$. Let E ($\overline{E}$) be the line bundle on $\mathcal{E}$ (resp. $\overline{\mathcal{E}}$) defining the solution foliation. Let $\mathcal{W}_i \in \Gamma(S^iE^*)$ (resp. $\overline{\mathcal{W}}_i \in \Gamma(S^i\overline{E}^*)$) be generalized Wilczynski invariants of $\mathcal{E}$ (resp. $\overline{\mathcal{E}}$). Then we have $\phi_*(E) = \overline{E}$ and $\phi^*\overline{\mathcal{W}}_i = \mathcal{W}_i$ for all $i = 3, \dots, n+1$.*

This theorem was first proven in the work [5] via direct computation of the canonical Cartan connection associated with any ordinary differential equation of order ≥ 4 (see [17, 4] for more details).

5. Equations with vanishing Wilczynski invariants

5.1. **Structure of the solution space.** Suppose now that all generalized Wilczynski invariants vanish identically for a given equation $\mathcal{E}$. For linear equations this would mean that the given equation is trivializable, and the associated curve $L_{\mathcal{E}}$ is an open part of the rational normal curve in $\mathbb{R}\mathrm{P}^n$.

Even though the curve $L_{\mathcal{E}}$ is defined up to contact transformations, we can define geometric structures on the solution space itself, which will be independent of the choice of fundamental solutions. Indeed, let $V(\mathcal{E})$ be the solution space of the linear equation (1). Then at each point x_0 on the line we can define a subspace $V_{x_0}(\mathcal{E})$ in $V(\mathcal{E})$ consisting of all solutions vanishing at x_0. Thus, assigning the line $V_{x_0}^{\perp}$ in the dual space $V(\mathcal{E})^*$ to each point x_0 we get a well-defined curve in the projective space $\mathrm{P}V_{\mathcal{E}}^*$.

Using the duality principle, we can also construct the curve in the projective space $\mathrm{P}V_{\mathcal{E}}$. To do this, we can consider the line $l_{x_0}(\mathcal{E})$ in $V(\mathcal{E})$ consisting of all solutions vanishing at x_0 with their derivatives up to order $n-1$. The mapping $x \mapsto l_{x_0}(\mathcal{E})$ defines a non-degenerate curve in the projective space $\mathrm{P}V(\mathcal{E})$.

Suppose now, that all Wilczynski invariants of the linear equation $\mathcal{E}$ vanish identically and consider the symmetry algebra $\mathfrak{g} \subset \mathfrak{sl}(V(\mathcal{E}))$ of the constructed curve. Fixing any fundamental set of solutions and applying Theorem 1, we can show that this Lie algebra is isomorphic to $\mathfrak{sl}(2,\mathbb{R})$ and acts irreducibly on $V(\mathcal{E})$. Let G be the corresponding subgroup in $SL(V(\mathcal{E})$ and let $\bar{G}$ be its product with the central subgroup in $GL(V(\mathcal{E}))$. Then $\bar{G}$ is naturally isomorphic to $GL(2,\mathbb{R})$, and the $\bar{G}$-module $V(\mathcal{E})$ is equivalent to the $GL(2,\mathbb{R})$-module $S^n(\mathbb{R}^2)$. Thus, any equation with vanishing Wilczynski invariants defines a $GL(2,\mathbb{R})$-structure on its solution space.

For a non-linear equation the vanishing of generalized Wilczynski invariants is not sufficient for being trivializable. The paper [5] describes the extra set of invariants that should vanish to guarantee the trivializability of the equation:

Theorem 4 ([5])**.** *The equation* (4) *is contact equivalent to the trivial equation* $y^{(n+1)} = 0$ *if and only if its generalized Wilczynski invariants vanish identically and in addition:*

for $n = 2$*:* $f_{2222} = 0$*;*
for $n = 3$*:* $f_{333} = 6f_{233} + f_{33}^2 = 0$*;*
for $n = 4$*:* $f_{44} = 6f_{234} - 4f_{333} - 3f_{34}^2 = 0$*;*
for $n = 5$*:* $f_{55} = f_{45} = 0$*;*
for $n \geq 6$*:* $f_{n,n} = f_{n,n-1} = f_{n-1,n-1} = 0.$

Yet, even if the equation is not trivializable, its linearization at each solution is trivializable, and we can construct a family of associated rational normal curves. It will define a $GL(2,\mathbb{R})$-structure on the solution space $\mathcal{S}$, if it is a Hausdorff manifold.

Let $\mathcal{E}$ be an arbitrary non-linear ODE. Suppose that its solution set $\mathcal{S}$ is Hausdorff (i.e., there exist a quotient of $\mathcal{E}$ by the foliation formed by all solutions). Let $y_0(x)$ be any solution of the non-linear ODE. This is just a point in the manifold $\mathcal{S}$. The tangent space $T_{y_0}\mathcal{S}$ to $\mathcal{S}$ at $y_0(x)$ can be naturally identified with a solution space of the linearization of $\mathcal{E}$ along $y_0(x)$. By the above, we can naturally construct a curve l in the projectivization of $T_{y_0}\mathcal{S}$, which means that we have a well-defined two-dimensional cone C_{y_0} in each tangent space $T_{y_0}\mathcal{S}$.

Suppose now that all generalized Wilczynski invariants of $\mathcal{E}$ vanish identically. Then linearizations of $\mathcal{E}$ along all solutions are trivializable, and all cones $C_{y_0} \subset T_{y_0}\mathcal{S}$ are locally equivalent to the cone in $\mathbb{R}^{n+1}$ corresponding to the normal curve in $\mathbb{R}\mathrm{P}^n$.

Thus, we arrive at the following result.

Theorem 5. *Let $\mathcal{E}$ be an arbitrary (non-linear) ODE with vanishing generalized Wilczynski invariants. Suppose that its solution space $\mathcal{S}$ is Hausdorff and, hence, is a smooth manifold. Then there exists a natural irreducible $GL(2,\mathbb{R})$-structure on $\mathcal{S}$.*

This structure was constructed in [6] in a slightly different way and is called a paraconformal structure.

5.2. **Examples from twistor theory.** In general, explicit examples of equations with vanishing Wilczynski are very difficult to construct. However, as suggested by N. Hitchin [11], there is a large class of examples coming from twistor theory.

The whole theory above is also valid in complex analytic category. Consider an arbitrary complex surface S with a rational curve L on it. Suppose that the normal bundle of L has Grothendieck type $\mathcal{O}(n)$. Then Kodaira theory states that this rational curve L is included into a complete $(n+1)$-parameter family $\mathcal{L} = \{L_a\}$ of all deformations of L.

The family $\mathcal{L}$ uniquely defines an ordinary differential equation of order $(n+1)$ such that all these curves are its solutions. In more detail, we have to consider the lifts of the curves L_a to the jet space $J^{n+1}(S)$, where they will form a submanifold of codimension 1. This defines an ordinary differential equation $\mathcal{E}$.

Theorem 6. *Let $\mathcal{E}$ be an ordinary differential equation defining the complete set of deformations of a rational curve L on a complex surface S. Then all generalized Wilczynski invariant of $\mathcal{E}$ vanish identically.*

Proof. Each generalized Wilczynski invariant $\mathcal{W}_i$, being restricted to the lift of any solution L_a, defines a global section of the line bundle $S^iT^*L_a$, which has Grothendieck type $\mathcal{O}(-2i)$. Since the only global section of the line bundle of this type is zero, we see that all generalized Wilczynski invariants will vanish identically. □

Consider a number of explicit examples.

Example 1. Let $S = \mathbb{C}\mathrm{P}^2$ and let L be a quadric in S. Then the normal bundle of L has type $\mathcal{O}(4)$, and the complete family of deformations is 5-dimensional. Clearly, all quadrics in $\mathbb{C}\mathrm{P}^2$ belong to this family and depend on exactly 5 parameters.

So, we see that the family $\mathcal{L}$ in this case is a family of all quadrics on the complex projective plane. The differential equation of all quadrics is well-known and has the form:

$$9(y'')^2y^{(5)} - 45y''y'''y^{(4)} + 40(y''')^3 = 0.$$

By above, we have $\mathcal{W}_3 = \mathcal{W}_4 = \mathcal{W}_5 = 0$ for this equation. However, this equation is not trivializable, since, for example, its symmetry algebra is only 8-dimensional, while the trivial equation $y^{(5)} = 0$ has a 9-dimensional symmetry algebra.

Example 2. Let S be a rational surface $S_k = \mathrm{P}(\mathcal{O}(0) + \mathcal{O}(k))$ viewed as a projective bundle over $\mathbb{C}\mathrm{P}^1$. Then all sections of this bundle with a fixed intersection number l with fibers form a complete family of deformations depending on $k + l + 1$ parameter. In the appropriate coordinate system on S_k they can be explicitly written as:

$$y(x) = \frac{a_0 + a_1 x + \cdots + a_k x^k}{b_0 + b_1 x + \cdots + b_l x^l}.$$

Assume that $k \geq l$. (Otherwise we can substitute y with $1/y$.) This set of rational curves defines the following differential equation:

$$F_{k,l} = \begin{vmatrix} z_{k-l+1} & z_{k-l+2} & \cdots & z_{k+1} \\ z_{k-l+2} & z_{k-l+3} & \cdots & z_{k+2} \\ \vdots & \vdots & \ddots & \vdots \\ z_{k+1} & z_{k+2} & \cdots & z_{k+l+1} \end{vmatrix} = 0, \qquad \text{where } z_i = \frac{(-1)^i}{i!} y^{(i)}.$$

For example, in the simplest case $l = 0$ we get just the trivial equation $y^{(k+1)} = 0$, while for $l = 1$ we get the following equation:

$$(k+1) y^{(k)} y^{(k+2)} - (k+2)(y^{(k+1)})^2 = 0.$$

For $k \geq 2$ the symmetry algebra of this equation is $k+2$-dimensional, and, hence, it is not trivializable.

For $l \geq 2$ the equation $F_{k,l} = 0$ is never trivializable, since, for example, being solved with respect to the highest derivative $y^{(k+l+1}$, it is not linear in term $y^{(k+l)}$.

Since all solutions of the equation $F_{k,l} = 0$ are by construction rational curves, all its generalized Wilczynski invariants vanish identically.

Example 3. Consider the standard symplectic form σ on the 4-dimensional vector space $\mathbb{C}^4$. Then it defines the contact structure on $\mathbb{C}\mathrm{P}^3$ invariant with respect to the induced action of $SP(4, \mathbb{C})$. It is easy to see that the set of all rational normal curves in $\mathbb{C}\mathrm{P}^3$, which are at the same time integral curves of the contact distribution, depends on 7 parameters and forms the complete family of deformations of any of such curves.

The equation describing this set is given by:

$$\begin{aligned} 10(y''')^3 y^{(7)} - 70(y''')^2 y^{(4)} y^{(6)} - 49(y''')^2 (y^{(5)})^2 \\ + 280 y''' (y^{(4)})^2 y^{(5)} - 175 (y^{(4)})^4 = 0. \end{aligned}$$

Again, since all solutions of this equation are rational curves, all its generalized Wilczynski invariants vanish identically. Yet, this equation is not trivializable, since its symmetry algebra is 10-dimensional (in fact, it coincides with $\mathfrak{sp}(4,\mathbb{C})$), while the trivial equation $y^{(7)}=0$ has 11-dimensional symmetry algebra.

Note that equations from the examples above have the common property that they have the symmetry algebra of submaximal dimension [13].

6. Wilczynski invariants for systems of ODEs

Since all results of this paper are based on two ideas, namely the notion of linearization and the invariants and structures associated with linear equations, they are directly generalized to systems of ordinary differential equations and even to equations of finite type.

The generalization of Wilczynski invariants to the systems of ODEs was obtained by Se-ashi [16]. Consider an arbitrary system of linear ordinary differential equations:

$$y^{(n+1)} + P_n(x)y^{(n)} + \dots + P_0(x)y(x) = 0,$$

where $y(x)$ is an $\mathbb{R}^m$-valued vector function. The canonical Laguerre-Forsyth form of these equations is defined by conditions $P_n = 0$ and $\operatorname{tr} P_{n-1} = 0$. Then, as in the case of a singe ODE, the following expressions:

(6)

$$\Theta_k = \sum_{j=1}^{k-1} (-1)^{j+1} \frac{(2k-j-1)!(n-k+j)!}{(k-j)!(j-1)!} P_{n-k+j}^{(j-1)}, \quad k = 2, \dots, n+1.$$

are the $\operatorname{End}(\mathbb{R}^m)$-valued relative invariants, where each invariant θ_i has weight i. Note that unlike the case of a single ODE, the first non-trivial Wilczynski invariant has weight 2.

We can also consider a characteristic curve associated with any linear system, which will take values in the Grassmann manifold $\operatorname{Gr}_m(\mathbb{R}^{(n+1)m})$. Se-ashi also proved that the system is trivializable if and only if all these invariants vanish identically, or, equivalently, if the characteristic curve is an open part of the curve defined by a trivial equation. The symmetry group of this curve is isomorphic to the direct product of $SL(2,\mathbb{R})$ and $GL(m,\mathbb{R})$ with an action on $\mathbb{R}^{(n+1)m}$ equivalent to the natural action on $S^{(n+1)}(\mathbb{R}^2) \otimes \mathbb{R}^m$.

All that leads us immediately to the following generalization of Theorems 2 and 5.

Theorem 7. *Let $\mathcal{E}$ be an arbitrary (non-linear) system of m ordinary differential equations of order $(n+1)$. Then there exist generalized Wilczynski invariants $\mathcal{W}_2, \dots, \mathcal{W}_{n+1}$ that, being restricted to each solution, coincide with Wilczynski invariants of the linearization along this solution.*

Suppose that all these invariants vanish identically and the solution space $\mathcal{S}$ is a smooth manifold. Then there exists a natural irreducible $SL(2,\mathbb{R}) \times GL(m,\mathbb{R})$ structure on $\mathcal{S}$.

Remark 1. Let us note that in the case of systems of ODEs the invariant $\mathcal{W}_i$ is a section the vector bundle $S^i E^* \otimes \operatorname{End}(V_n)$, where V_n is a subbundle of the normal bundle $N(\mathcal{E}) = T\mathcal{E}/E$ defined as $\pi^*_{n+1,n} C^n \cap T\mathcal{E}$ modulo E. In case of a single ODE the bundle V_n is one-dimensional, and $\operatorname{End}(V_n)$ can be identified with the trivial bundle over $\mathcal{E}$.

Example 4. Consider the simplest non-trivial case of second order systems of ODEs. The linear homogeneous system can be written as:

$$y'' = A(x)y' + B(x)y, \tag{7}$$

where $y(x)$ is the unknown $\mathbb{R}^m$-valued function and $A(x), B(x) \in \operatorname{End}(\mathbb{R}^m)$. The set of all transformations preserving the class of such systems has the form $(x, y) \mapsto (\lambda(x), \mu(x)y)$, where $\lambda(x)$ is a local line reparametrization and $\mu(x) \in GL(m,\mathbb{R})$.

We can bring the equation (7) into the semi-canonical form with vanishing coefficient $A(x)$ by means of a certain gauge transformation $(x, y) \mapsto (x, \mu(x)y)$. Then, applying a reparametrization $(x, y) \mapsto (\lambda(x), y)$ and an appropriate gauge transformation to bring the equation back to the semi-canonical form, we can make trace of $B(x)$ vanish.

The traceless part of the coefficient $B(x)$ in the semicanonical form gives us the Wilczynski invariant Θ_2 for linear systems of second order. Explicitly, it can be written as:

$$\Theta_2 = \Phi(x) - 1/m \operatorname{tr} \Phi(x), \quad \text{where } \Phi(x) = B(x) - \frac{1}{2}A'(x) + \frac{1}{4}A(x)^2. \tag{8}$$

According to Se-ashi [16], this is the only Wilczynski invariant available for systems of second order, and the equation (7) is trivializable if and only if the invariant Θ_2 vanishes identically.

Consider now an arbitrary non-linear system of second order:

$$y_i'' = f_i(x, y_j, y_k'), \qquad i = 1, \dots, m.$$

Then its generalized Wilczynski invariant can be obtained from (8) by substituting $A(x)$ with the matrix $\left(\frac{\partial f_i}{\partial y_k'}\right)$, $B(x)$ with the matrix $\left(\frac{\partial f_i}{\partial y_j}\right)$, and the usual derivative d/dx with operator of total derivative:

$$D = \frac{\partial}{\partial x} + \sum_{i=1}^{m} y_i' \frac{\partial}{\partial y_i} + \sum_{j=1}^{m} f_j \frac{\partial}{\partial y_j'}.$$

Denote by W_2 the invariant we get in this way. It coincides (up to the constant multiplier) with the invariant constructed by M. Fels [7].

If the invariant W_2 vanishes identically, then we get the $GL(2,\mathbb{R}) \times GL(m,R)$ structure on the solution space. This structure is also called Segre structure and was first constructed for solution space of second order ODEs with vanishing generalized Wilczynski invariant by D. Grossman [9].

References

[1] R. Bryant, *Two exotic holonomies in dimension four, path geometries, and twistor theory*, Proc. Symp. Pure Math. **53** (1991), pp. 33-88.

[2] E. Cartan, *Sur les variétés à connexion projective*, Bull. Soc. Math. France, **52** (1924), pp. 205–241.

[3] S.-S. Chern, *The geometry of the differential equation* $y''' = F(x, y, y', y'')$, Sci. Rep. Nat. Tsing Hua Univ., **4** (1950), pp. 97–111.

[4] B. Doubrov, B. Komrakov, T. Morimoto, *Equivalence of holonomic differential equations*, Lobachevskij Journal of Mathematics, **3** (1999), pp. 39–71.

[5] B. Doubrov, *Contact trivialization of ordinary differential equations*, Differential Geometry and Its applications, 2001, pp. 73–84.

[6] M. Dunajski, P. Tod, *Paraconformal geometry of n-th order ODEs, and exotic holonomy in dimension four*, J. Geom. Phys., **56** (2006), pp. 1790-1809.

[7] M. Fels, *The equivalence problem for systems of second order ordinary differential equations*, Proc. London Math. Soc., **71** (1995), no. 1, pp. 221–240.

[8] M. Fels, *The inverse problem of the calculus of variations for scalar fourth-order ordinary differential equations*, Trans. Amer. Math. Soc., **348** (1996), pp. 5007–5029.

[9] D. Grossman, *Torsion-free path geometries and integrable second order ODE systems*, Sel. Math., New Ser. **6** (2000), pp. 399-442.

[10] G.H. Halphen, *Sur les invariants differentiels des courbes ganches*, Journ. de l'Ecole Polytechnique. **28** (1880), pp. 1–25.

[11] N. Hitchin, *Complex manifolds and Einstein's equations,* Twistor Geometry and Non-linear Systems, LNM 970, Springer, 1982.

[12] S. Lie, *Klassifikation und Integration von gewönlichen Differentialgleichungen zwischen x, y, die eine Gruppe von Transformationen gestatten, I–IV*, Gesamelte Abhandlungen, v. 5, Leipzig–Teubner, 1924, S. 240–310, 362–427, 432–448.

[13] P.J. Olver, *Symmetry, invariants, and equivalence*, New York, Springer–Verlag, 1995.

[14] H. Sato, A. Y. Yoshikawa, *Third order ordinary differential equations and Legendre connections*, J. Math. Soc. Japan, **50** (1998), pp. 993–1013.

[15] Yu. Se-ashi, *On differential invariants of integrable finite type linear differential equations*, Hokkaido Math. J., **17** (1988), pp. 151–195.
[16] Yu. Se-ashi, *A geometric construction of Laguerre-Forsyth's canonical forms of linear ordinary differential equations*, Adv. Studies in Pure Math., **22** (1993), pp. 265–297.
[17] N. Tanaka, *Geometric theory of ordinary differential equations*, Report of Grant-in-Aid for Scientific Research MESC Japan, 1989.
[18] E.J. Wilczynski, *Projective differential geometry of curves and ruled surfaces*, Leipzig, Teubner, 1905.

Belarussian State University, Skoriny 4, 220050, Minsk, Belarus
E-mail address: `doubrov@islc.org`