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Generalized Wintgen inequality for statistical submanifolds in statistical manifolds of constant curvature

M. Evren Aydin[1] · Adela Mihai[2] · Ion Mihai[3]

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Abstract The Wintgen inequality (1979) is a sharp geometric inequality for surfaces in the 4-dimensional Euclidean space involving the Gauss curvature (intrinsic invariant) and the normal curvature and squared mean curvature (extrinsic invariants), respectively. De Smet et al. (Arch. Math. (Brno) 35:115–128, 1999) conjectured a generalized Wintgen inequality for submanifolds of arbitrary dimension and codimension in Riemannian space forms. This conjecture was proved by Lu (J. Funct. Anal. 261:1284–1308, 2011) and by Ge and Tang (Pac. J. Math. 237:87–95, 2008), independently. In the present paper we establish a generalized Wintgen inequality for statistical submanifolds in statistical manifolds of constant curvature.

Keywords Wintgen inequality · Statistical manifold · Statistical submanifold · Dual connections

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✉ Ion Mihai
imihai@fmi.unibuc.ro

M. Evren Aydin
meaydin@firat.edu.tr

Adela Mihai
adela_mihai@fmi.unibuc.ro

[1] Department of Mathematics, Faculty of Science, Firat University, Elazig 23200, Turkey

[2] Department of Mathematics and Computer Science, Technical University of Civil Engineering Bucharest, 020396 Bucharest, Romania

[3] Department of Mathematics, Faculty of Mathematics and Computer Science, University of Bucharest, 010014 Bucharest, Romania

1 Introduction

For surfaces M^2 of the Euclidean space $\mathbb{E}^3$, the Euler inequality $G \leq \|H\|^2$ is fulfilled, where G is the (intrinsic) Gauss curvature of M^2 and $\|H\|^2$ is the (extrinsic) squared mean curvature of M^2.

Furthermore, $G = \|H\|^2$ everywhere on M^2 if and only if M^2 is totally umbilical, or still, by a theorem of Meusnier, if and only if M^2 is (a part of) a plane $\mathbb{E}^2$ or, it is (a part of) a round sphere S^2 in $\mathbb{E}^3$.

In 1979, Wintgen [25] proved that the Gauss curvature G, the squared mean curvature $\|H\|^2$ and the normal curvature $G^{\perp}$ of any surface M^2 in $\mathbb{E}^4$ always satisfy the inequality

$$G \leq \|H\|^2 - |G^{\perp}|;$$

the equality holds if and only if the ellipse of curvature of M^2 in $\mathbb{E}^4$ is a circle.

The Whitney 2-sphere satisfies the equality case of the Wintgen inequality identically.

A survey containing recent results on surfaces satisfying identically the equality case of Wintgen inequality can be read in [5].

Later, the Wintgen inequality was extended by Rouxel [20] and by Guadalupe and Rodriguez [10] independently, for surfaces M^2 of arbitrary codimension m in real space forms $\widetilde{M}^{2+m}(c)$; namely

$$G \leq \|H\|^2 - |G^{\perp}| + c.$$

The equality case was also investigated.

A corresponding inequality for totally real surfaces in n-dimensional complex space forms was obtained in [13]. The equality case was studied and a non-trivial example of a totally real surface satisfying the equality case identically was given.

In 1999, De Smet et al. [7] formulated the conjecture on Wintgen inequality for submanifolds of real space forms, which is also known as the *DDVV conjecture*.

This conjecture was proven by the authors for submanifolds M^n of arbitrary dimension $n \geq 2$ and codimension 2 in real space forms $\tilde{M}^{n+2}(c)$ of constant sectional curvature c.

Recently, the DDVV conjecture was finally settled for the general case by Lu [12] and independently by Ge and Tang [9].

One of the present authors obtained generalized Wintgen inequalities for Lagrangian submanifolds in complex space forms [14] and Legendrian submanifolds in Sasakian space forms [15], respectively. Moreover, two of the present authors established in [3] a version of the Euler inequality and the Wintgen inequality for statistical surfaces in statistical manifolds of constant curvature.

In this paper, using the sectional curvature defined in [19], we derive a generalized Wintgen inequality for statistical submanifolds in statistical manifolds of constant curvature.

2 Statistical manifolds and their submanifolds

A *statistical manifold* is a Riemannian manifold $(\tilde{M}^{n+k}, \tilde{g})$ of dimension $(n + k)$, endowed with a pair of torsion-free affine connections $\tilde{\nabla}$ and $\tilde{\nabla}^*$ satisfying

$$Z\tilde{g}(X, Y) = \tilde{g}(\tilde{\nabla}_Z X, Y) + \tilde{g}(X, \tilde{\nabla}^*_Z Y), \tag{2.1}$$

for any $X, Y, Z \in \Gamma(T\tilde{M})$. The connections $\tilde{\nabla}$ and $\tilde{\nabla}^*$ are called *dual connections* (see [1,17,22]), and it is easily shown that $(\tilde{\nabla}^*)^* = \tilde{\nabla}$. The pair $(\tilde{\nabla}, \tilde{g})$ is said to be a *statistical structure*. If $(\tilde{\nabla}, \tilde{g})$ is a statistical structure on $\tilde{M}^{n+k}$, so is $(\tilde{\nabla}^*, \tilde{g})$ [1,24].

On the other hand, any torsion-free affine connection $\tilde{\nabla}$ always has a dual connection given by

$$\tilde{\nabla} + \tilde{\nabla}^* = 2\tilde{\nabla}^0, \tag{2.2}$$

where $\tilde{\nabla}^0$ is Levi-Civita connection on $\tilde{M}^{n+k}$.

Denote by $\tilde{R}$ and $\tilde{R}^*$ the curvature tensor fields of $\tilde{\nabla}$ and $\tilde{\nabla}^*$, respectively.

A statistical structure $(\tilde{\nabla}, \tilde{g})$ is said to be of constant curvature $c \in \mathbb{R}$ if

$$\tilde{R}(X, Y)Z = c\{\tilde{g}(Y, Z)X - \tilde{g}(X, Z)Y\}. \tag{2.3}$$

A statistical structure $(\tilde{\nabla}, \tilde{g})$ of constant curvature 0 is called a *Hessian structure.*

The curvature tensor fields $\tilde{R}$ and $\tilde{R}^*$ of dual connections satisfy

$$\tilde{g}(\tilde{R}^*(X, Y)Z, W) = -\tilde{g}(Z, \tilde{R}(X, Y)W). \tag{2.4}$$

From (2.4) it follows immediately that if $(\tilde{\nabla}, \tilde{g})$ is a statistical structure of constant curvature c, then $(\tilde{\nabla}^*, \tilde{g})$ is also a statistical structure of constant curvature c. In particular, if $(\tilde{\nabla}, \tilde{g})$ is Hessian, so is $(\tilde{\nabla}^*, \tilde{g})$ [8].

On a Hessian manifold $(\tilde{M}^{n+k}, \tilde{\nabla})$, let $\gamma = \tilde{\nabla}^0 - \tilde{\nabla}$. The tensor field Q of type (1,3) defined by the covariant differential $Q = \tilde{\nabla}\gamma$ of γ is said to be the *Hessian curvature tensor* for $\tilde{\nabla}$ (see [21]).

By using the Hessian curvature tensor Q, a Hessian sectional curvature can be defined on a Hessian manifold.

A Hessian manifold has constant Hessian sectional curvature $\tilde{c}$ if and only if (see [21])

$$Q(X, Y, Z, W) = \frac{\tilde{c}}{2}[g(X, Y)g(Z, W) + g(X, W)g(Y, Z)],$$

for all vector fields on $\tilde{M}^{n+k}$.

If $(\tilde{M}^{n+k}, \tilde{g})$ is a statistical manifold and M^n a submanifold of dimension n of $\tilde{M}^{n+k}$, then (M^n, g) is also a statistical manifold with the induced connection by $\tilde{\nabla}$ and induced metric g. In the case that $(\tilde{M}^{n+k}, \tilde{g})$ is a semi-Riemannian manifold, the induced metric g has to be non-degenerate. For details, see [23,24].

In the geometry of Riemannian submanifolds (see [4]), the fundamental equations are the Gauss and Weingarten formulas and the equations of Gauss, Codazzi and Ricci.

Let denote the set of the sections of the normal bundle to M^n by $\Gamma(TM^{n\perp})$.

In our case, for any $X, Y \in \Gamma(TM^n)$, according to [24], the corresponding Gauss formulas are

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{2.5}$$

$$\tilde{\nabla}^*_X Y = \nabla^*_X Y + h^*(X, Y), \tag{2.6}$$

where $h, h^* : \Gamma(TM^n) \times \Gamma(TM^n) \to \Gamma(TM^{n\perp})$ are symmetric and bilinear, called the *imbedding curvature tensor* of M^n in $\tilde{M}^{n+k}$ for $\tilde{\nabla}$ and the *imbedding curvature tensor* of M^n in $\tilde{M}^{n+k}$ for $\tilde{\nabla}^*$, respectively.

In [24], it is also proved that (∇, g) and (∇^*, g) are dual statistical structures on M^n.

Since h and h^* are bilinear, we have the linear transformations A_ξ and A^*_ξ on TM^n defined by

$$g(A_\xi X, Y) = \tilde{g}(h(X, Y), \xi), \tag{2.7}$$

$$g(A^*_\xi X, Y) = \tilde{g}(h^*(X, Y), \xi), \tag{2.8}$$

for any $\xi \in \Gamma(TM^{n\perp})$ and $X, Y \in \Gamma(TM^n)$. Further, see [24], the corresponding Weingarten formulas are

$$\tilde{\nabla}_X \xi = -A^*_\xi X + \nabla^\perp_X \xi, \tag{2.9}$$

$$\tilde{\nabla}^*_X \xi = -A_\xi X + \nabla^{*\perp}_X \xi, \tag{2.10}$$

for any $\xi \in \Gamma(TM^{n\perp})$ and $X \in \Gamma(TM^n)$. The connections $\nabla^\perp_X$ and $\nabla^{*\perp}_X$ given by (2.9) and (2.10) are Riemannian dual connections with respect to induced metric on $\Gamma(TM^{n\perp})$.

Let $\{e_1, \ldots, e_n\}$ and $\{\xi_1, \ldots, \xi_k\}$ be orthonormal tangent and normal frames, respectively, on M^n. Then the mean curvature vector fields are defined by

$$H = \frac{1}{n}\sum_{i=1}^{n} h\left(e_i, e_i\right) = \frac{1}{n}\sum_{\alpha=1}^{k}\left(\sum_{i=1}^{n} h^\alpha_{ii}\right)\xi_\alpha, \quad h^\alpha_{ij} = \tilde{g}\left(h\left(e_i, e_j\right), \xi_\alpha\right), \tag{2.11}$$

and

$$H^* = \frac{1}{n}\sum_{i=1}^{n} h^*\left(e_i, e_i\right) = \frac{1}{n}\sum_{\alpha=1}^{k}\left(\sum_{i=1}^{n} h^{*\alpha}_{ii}\right)\xi_\alpha, \quad h^{*\alpha}_{ij} = \tilde{g}\left(h^*\left(e_i, e_j\right), \xi_\alpha\right), \tag{2.12}$$

for $1 \leq i, j \leq n$ and $1 \leq \alpha \leq k$ (see also [6]).

The corresponding Gauss, Codazzi and Ricci equations are given by the following result.

Proposition 2.1 [24] *Let $\tilde{\nabla}$ and $\tilde{\nabla}^*$ be dual connections on $\tilde{M}^{n+k}$ and ∇ the induced connection by $\tilde{\nabla}$ on M^n. Let $\tilde{R}$ and R be the Riemannian curvature tensors for $\tilde{\nabla}$ and ∇, respectively. Then,*

$$\begin{aligned}\tilde{g}(\tilde{R}(X,Y)Z,W) &= g(R(X,Y)Z,W) + \tilde{g}(h(X,Z),h^*(Y,W)) \\ &\quad - \tilde{g}(h^*(X,W),h(Y,Z)), \qquad (2.13)\\ (\tilde{R}(X,Y)Z)^{\perp} &= \nabla^{\perp}_X h(Y,Z) - h(\nabla_X Y,Z) - h(Y,\nabla_X Z) \\ &\quad - \{\nabla^{\perp}_Y h(X,Z) - h(\nabla_Y X,Z) - h(X,\nabla_Y Z)\}, \\ \tilde{g}(R^{\perp}(X,Y)\xi,\eta) &= \tilde{g}(\tilde{R}(X,Y)\xi,\eta) + g([A^*_\xi, A_\eta]X,Y), \qquad (2.14)\end{aligned}$$

where $R^{\perp}$ is the Riemannian curvature tensor of $\nabla^{\perp}$ on $TM^{n\perp}$, $\xi,\eta \in \Gamma(TM^{n\perp})$ and $[A^*_\xi, A_\eta] = A^*_\xi A_\eta - A_\eta A^*_\xi$.

For the equations of Gauss, Codazzi and Ricci with respect to the connection $\tilde{\nabla}^*$ on M^n, we have

Proposition 2.2 [24] *Let $\tilde{\nabla}$ and $\tilde{\nabla}^*$ be dual connections on $\tilde{M}^{n+k}$ and ∇^* the induced connection by $\tilde{\nabla}^*$ on M^n. Let $\tilde{R}^*$ and R^* be the Riemannian curvature tensors for $\tilde{\nabla}^*$ and ∇^*, respectively. Then,*

$$\begin{aligned}\tilde{g}(\tilde{R}^*(X,Y)Z,W) &= g(R^*(X,Y)Z,W) + \tilde{g}(h^*(X,Z),h(Y,W)) \\ &\quad - \tilde{g}(h(X,W),h^*(Y,Z)), \qquad (2.15)\\ (\tilde{R}^*(X,Y)Z)^{\perp} &= \nabla^{*\perp}_X h^*(Y,Z) - h^*(\nabla^*_X Y,Z) - h^*(Y,\nabla^*_X Z) \\ &\quad - \{\nabla^{*\perp}_Y h^*(X,Z) - h^*(\nabla^*_Y X,Z) - h^*(X,\nabla^*_Y Z)\}, \\ \tilde{g}(R^{*\perp}(X,Y)\xi,\eta) &= \tilde{g}(\tilde{R}^*(X,Y)\xi,\eta) + g([A_\xi, A^*_\eta]X,Y), \qquad (2.16)\end{aligned}$$

where $R^{*\perp}$ is the Riemannian curvature tensor of $\nabla^{\perp *}$ on $TM^{n\perp}$, $\xi,\eta \in \Gamma(TM^{n\perp})$ and $[A_\xi, A^*_\eta] = A_\xi A^*_\eta - A^*_\eta A_\xi$.

Geometric inequalities for statistical submanifolds in statistical manifolds with constant curvature were obtained in [2].

3 Statistical surfaces in statistical manifolds of constant curvature

Let $(\tilde{M}^3, \tilde{g})$ be a 3-dimensional statistical manifold of constant curvature c and M^2 a surface of $\tilde{M}$. Denote the Gauss curvature, the mean curvature and the dual mean curvature of M, by G, H and H^*, respectively. In [3], a version of the Euler inequality for statistical surfaces was given.

Proposition 3.1 [3] *Let M^2 be a surface in a 3-dimensional statistical manifold of constant curvature c. Then its Gauss curvature satisfies:*

$$G \leq 2\|H\| \cdot \|H^*\| - c. \qquad (3.1)$$

Some examples of statistical surfaces satisfying the equality case of the above Euler inequality can be provided by the following.

Example 1 (*A trivial example*) Recall Lemma 5.3 of Furuhata [8].

Let $(\mathbb{H}, \tilde{\nabla}, \tilde{g})$ be a Hessian manifold of constant Hessian sectional curvature $\tilde{c} \neq 0$, (M, ∇, g) a trivial Hessian manifold and $f : M \longrightarrow \mathbb{H}$ a statistical immersion of codimension one. Then one has:

$$A^* = 0, \quad h^* = 0, \quad \|H^*\| = 0.$$

Thus, if $\dim M = 2$, the immersion f of codimension one satisfies the equality case of the statistical version of the Euler inequality given by Proposition 3.1.

Example 2 Let $(\mathbb{H}^3, \tilde{g})$ be the upper half space of constant sectional curvature -1, i.e.,

$$\mathbb{H}^3 = \{y = (y^1, y^2, y^3) \in \mathbb{R}^3 : y^3 > 0\}, \quad \tilde{g} = (y^3)^{-2} \sum_{k=1}^{3} dy^k dy^k.$$

An affine connection $\tilde{\nabla}$ on $\mathbb{H}^3$ is given by

$$\tilde{\nabla}_{\frac{\partial}{\partial y^3}} \frac{\partial}{\partial y^3} = (y^3)^{-1} \frac{\partial}{\partial y^3}, \quad \tilde{\nabla}_{\frac{\partial}{\partial y^i}} \frac{\partial}{\partial y^j} = 2\delta_{ij} (y^3)^{-1} \frac{\partial}{\partial y^3}, \quad \tilde{\nabla}_{\frac{\partial}{\partial y^i}} \frac{\partial}{\partial y^3} = \tilde{\nabla}_{\frac{\partial}{\partial y^3}} \frac{\partial}{\partial y^j} = 0,$$

where $i, j = 1, 2$. The curvature tensor field $\tilde{R}$ of $\tilde{\nabla}$ is identically zero, i.e., $c = 0$. Thus $(\mathbb{H}^3, \tilde{\nabla}, \tilde{g})$ is a Hessian manifold of constant Hessian sectional curvature 4 (see [21]).

Now let consider a horosphere M^2 in $\mathbb{H}^3$ having null Gauss curvature, i.e., $G \equiv 0$ (for details, see [11]). If $f : M^2 \longrightarrow \mathbb{H}^3$ is a statistical immersion of codimension one, then, by using Lemma 4.1 of [16], we deduce $A^* = 0$, and then $H^* = 0$. This implies that the horosphere M^2 satisfies the equality case of the statistical version of the Euler inequality given by Proposition 3.1.

More generally, let consider a 4-dimensional statistical manifold of constant curvature c, i.e. $(\tilde{M}^4, c)$, and a surface M^2 of $\tilde{M}^4$. We respectively denote the Gauss curvature, the normal curvature and the Gauss curvature with respect to the Levi-Civita connection by G, $G^{\perp}$ and G^0. Similarly, we respectively denote the mean vector field, the dual mean curvature and the sectional curvature with respect to the Levi-Civita connection by H, H^* and $\tilde{K}^0$. We have the following Wintgen inequalities.

Theorem 3.2 [3] *Let M^2 be a statistical surface in a 4-dimensional statistical manifold $(\tilde{M}^4, c)$ of constant curvature c. Then*

$$G + |G^{\perp}| + 2G^0 \leq \frac{1}{2}(\|H\|^2 + \|H^*\|^2) - c + 2\tilde{K}^0(e_1 \wedge e_2).$$

In particular, for $c = 0$ we derive the following.

Corollary 3.3 [3] *Let M^2 be a statistical surface of a Hessian 4-dimensional statistical manifold $\tilde{M}^4$ of Hessian curvature 0. Then*:

$$G + |G^{\perp}| + 2G^0 \leq \frac{1}{2}(\|H\|^2 + \|H^*\|^2).$$

4 Wintgen inequality for statistical submanifolds

Let M^n be an n-dimensional statistical submanifold of a $(n+m)$-dimensional statistical manifold $(\tilde{M}^{n+m}, c)$ of constant curvature c.

The sectional curvature K on M^n is defined by [3] (see also [18,19])

$$K(X \wedge Y) = \frac{1}{2}[g(R(X,Y)X,Y) + g(R^*(X,Y)X,Y)],$$

for any orthonormal vectors $X, Y \in T_pM^n,\ p \in M^n$.

In the case of the Levi-Civita connection, the above definition coincides (up to the sign) to the standard definition of the sectional curvature.

Let $p \in M^n$ and $\{e_1, e_2, \ldots, e_n\}$ an orthonormal basis of T_pM^n. Then the normalized scalar curvature ρ is defined by (see [7]):

$$\begin{aligned}\rho &= \frac{2}{n(n-1)} \sum_{1 \leq i < j \leq n} K\left(e_i \wedge e_j\right) \\ &= \frac{1}{n(n-1)} \sum_{1 \leq i < j \leq n} \left[g\left(R\left(e_i, e_j\right) e_i, e_j\right) + g\left(R^*\left(e_i, e_j\right) e_i, e_j\right)\right].\end{aligned}$$

By using the Gauss equations for the dual connections $\tilde{\nabla}$ and $\tilde{\nabla}^*$, respectively, we obtain

$$\begin{aligned}\rho = \frac{1}{n(n-1)} \sum_{1 \leq i < j \leq n} &\left[-c - g\left(h\left(e_i, e_i\right), h^*\left(e_j, e_j\right)\right) + g\left(h^*\left(e_i, e_j\right), h\left(e_i, e_j\right)\right)\right. \\ &\left. - c - g\left(h^*\left(e_i, e_i\right), h\left(e_j, e_j\right)\right) + g\left(h\left(e_i, e_j\right), h^*\left(e_i, e_j\right)\right)\right].\end{aligned}$$

Denoting as usual by

$$\begin{gathered} h_{ij}^r = g\left(h\left(e_i, e_j\right), \xi_r\right), \quad h_{ij}^{*r} = g\left(h^*\left(e_i, e_j\right), \xi_r\right), \\ \forall i, j = 1, \ldots, n \text{ and } r = 1, \ldots, m, \end{gathered}$$

the above equation becomes

$$\rho = -c + \frac{1}{n(n-1)} \sum_{r=1}^{m} \sum_{1 \leq i < j \leq n} \left(2h_{ij}^r h_{ij}^{*r} - h_{ii}^{*r} h_{jj}^r - h_{ii}^r h_{jj}^{*r}\right). \tag{4.1}$$

On the other hand, the normalized normal scalar curvature $\rho^{\perp}$ is defined by (see also [3]):

$$\rho^{\perp}=\frac{1}{n\left(n-1\right)}\left\{\sum_{1\leq r<s\leq m}\sum_{1\leq i<j\leq n}\left[g\left(R^{\perp}\left(e_i,e_j\right)\xi_r,\xi_s\right)+g\left(R^{*\perp}\left(e_i,e_j\right)\xi_r,\xi_s\right)\right]^2\right\}^{\frac{1}{2}}.$$

The Ricci equations for the dual connections $\tilde{\nabla}$, and $\tilde{\nabla}^*$, respectively, imply

$$\rho^{\perp}=\frac{1}{n\left(n-1\right)}\left\{\sum_{1\leq r<s\leq m}\sum_{1\leq i<j\leq n}\left[g\left(\left[A^*_{\xi_r},A_{\xi_s}\right]e_i,e_j\right)+g\left(\left[A_{\xi_r},A^*_{\xi_s}\right]e_i,e_j\right)\right]^2\right\}^{\frac{1}{2}}$$

or equivalently,

$$\rho^{\perp}=\frac{1}{n\left(n-1\right)}\left\{\sum_{1\leq r<s\leq m}\sum_{1\leq i<j\leq n}\left[\sum_{k=1}^{n}\left(h^s_{ik}h^{*r}_{jk}-h^{*r}_{ik}h^s_{jk}+h^{*s}_{ik}h^r_{jk}-h^r_{ik}h^{*s}_{jk}\right)\right]^2\right\}^{\frac{1}{2}}.$$

It follows that

$$\rho^{\perp}=\frac{1}{n\left(n-1\right)}\left\{\sum_{1\leq r<s\leq m}\sum_{1\leq i<j\leq n}\left[\sum_{k=1}^{n}\left(\left(h^s_{ik}+h^{*s}_{ik}\right)\left(h^r_{jk}+h^{*r}_{jk}\right)-h^s_{ik}h^r_{jk}-h^{*s}_{ik}h^{*r}_{jk}\right.\right.\right.$$
$$\left.\left.\left.-\left(h^r_{ik}+h^{*r}_{ik}\right)\left(h^s_{jk}+h^{*s}_{jk}\right)+h^r_{ik}h^s_{jk}+h^{*r}_{ik}h^{*s}_{jk}\right)\right]^2\right\}^{\frac{1}{2}}.$$

It is known that the components of the second fundamental form h^0 of M^n with respect to the Levi-Civita connection $\tilde{\nabla}^0$ are given by $2h^{0r}_{ik}=h^r_{ik}+h^{*r}_{ik}$, $\forall i,k=1,\ldots,n$, $r=1,\ldots,m$. Then we can write

$$\rho^{\perp}=\frac{1}{n\left(n-1\right)}\left\{\sum_{1\leq r<s\leq m}\sum_{1\leq i<j\leq n}\left[\sum_{k=1}^{n}\left(4\left(h^{0s}_{ik}h^{0r}_{jk}-h^{0r}_{ik}h^{0s}_{jk}\right)+\left(h^r_{ik}h^s_{jk}-h^s_{ik}h^r_{jk}\right)\right.\right.\right.$$
$$\left.\left.\left.+\left(h^{*r}_{ik}h^{*s}_{jk}-h^{*s}_{ik}h^{*r}_{jk}\right)\right)\right]^2\right\}^{\frac{1}{2}}. \tag{4.2}$$

We shall use the algebraic inequality

$$(a+b+c)^2\leq 3(a^2+b^2+c^2),\quad \forall a,b,c\in\mathbb{R}.$$

Therefore

$$\rho^{\perp} \leq \frac{3}{n\,(n-1)} \left\{ \sum_{1 \leq r < s \leq m} \sum_{1 \leq i < j \leq n} \left(16 \left[\sum_{k=1}^{n} \left(h_{ik}^{0s} h_{jk}^{0r} - h_{ik}^{0r} h_{jk}^{0s} \right) \right]^2 \right.\right.$$
$$\left.\left. + \left[\sum_{k=1}^{n} \left(h_{ik}^{r} h_{jk}^{s} - h_{ik}^{s} h_{jk}^{r} \right) \right]^2 + \left[\sum_{k=1}^{n} \left(h_{ik}^{*r} h_{jk}^{*s} - h_{ik}^{*s} h_{jk}^{*r} \right) \right]^2 \right) \right\}^{\frac{1}{2}}. \quad (4.3)$$

Recall an inequality from [12] (see also [14])

$$\sum_{r=1}^{m} \sum_{1 \leq i < j \leq n} \left(h_{ii}^{r} - h_{jj}^{r} \right)^2 + 2n \sum_{r=1}^{m} \sum_{1 \leq i < j \leq n} \left(h_{ij}^{r} \right)^2$$
$$\geq 2n \left[\sum_{1 \leq i < j \leq n} \sum_{1 \leq r < s \leq m} \left(\sum_{k=1}^{n} (h_{jk}^{r} h_{ik}^{s} - h_{ik}^{r} h_{jk}^{s}) \right)^2 \right]^{\frac{1}{2}}.$$

Similarly, we have

$$\sum_{r=1}^{m} \sum_{1 \leq i < j \leq n} \left(h_{ii}^{*r} - h_{jj}^{*r} \right)^2 + 2n \sum_{r=1}^{m} \sum_{1 \leq i < j \leq n} \left(h_{ij}^{*r} \right)^2$$
$$\geq 2n \left[\sum_{1 \leq i < j \leq n} \sum_{1 \leq r < s \leq m} \left(\sum_{k=1}^{n} (h_{jk}^{*r} h_{ik}^{*s} - h_{ik}^{*r} h_{jk}^{*s}) \right)^2 \right]^{\frac{1}{2}}$$

and

$$\sum_{r=1}^{m} \sum_{1 \leq i < j \leq n} \left(h_{ii}^{0r} - h_{jj}^{0r} \right)^2 + 2n \sum_{r=1}^{m} \sum_{1 \leq i < j \leq n} \left(h_{ij}^{0r} \right)^2$$
$$\geq 2n \left[\sum_{1 \leq i < j \leq n} \sum_{1 \leq r < s \leq m} \left(\sum_{k=1}^{n} (h_{jk}^{0r} h_{ik}^{0s} - h_{ik}^{0r} h_{jk}^{0s}) \right)^2 \right]^{\frac{1}{2}}.$$

Summing up the above three inequalities, from (4.3) we obtain

$$\rho^{\perp} \leq \frac{3}{2n^2\,(n-1)} \sum_{r=1}^{m} \sum_{1 \leq i < j \leq n} \left[\left(h_{ii}^{r} - h_{jj}^{r} \right)^2 + \left(h_{ii}^{*r} - h_{jj}^{*r} \right)^2 + 16 \left(h_{ii}^{0r} - h_{jj}^{0r} \right)^2 \right]$$
$$+ \frac{3}{n\,(n-1)} \sum_{r=1}^{m} \sum_{1 \leq i < j \leq n} \left[\left(h_{ij}^{r} \right)^2 + \left(h_{ij}^{*r} \right)^2 + 16 \left(h_{ij}^{0r} \right)^2 \right]. \quad (4.4)$$

Also, we can write

$$n^2 \|H\|^2 = \sum_{r=1}^{m} \left(\sum_{i=1}^{n} h_{ii}^r \right)^2 = \frac{1}{n-1} \sum_{r=1}^{m} \sum_{1 \le i < j \le n} \left(h_{ii}^r - h_{jj}^r \right)^2 + \frac{2n}{n-1} \sum_{r=1}^{m} \sum_{1 \le i < j \le n} h_{ii}^r h_{jj}^r$$

and similarly,

$$n^2 \left\| H^* \right\|^2 = \frac{1}{n-1} \sum_{r=1}^{m} \sum_{1 \le i < j \le n} \left(h_{ii}^{*r} - h_{jj}^{*r} \right)^2 + \frac{2n}{n-1} \sum_{r=1}^{m} \sum_{1 \le i < j \le n} h_{ii}^{*r} h_{jj}^{*r}$$

and

$$n^2 \left\| H^0 \right\|^2 = \frac{1}{n-1} \sum_{r=1}^{m} \sum_{1 \le i < j \le n} \left(h_{ii}^{0r} - h_{jj}^{0r} \right)^2 + \frac{2n}{n-1} \sum_{r=1}^{m} \sum_{1 \le i < j \le n} h_{ii}^{0r} h_{jj}^{0r}.$$

Substituting in (4.4), we get

$$\begin{aligned}
\rho^{\perp} \le\ & \frac{3}{2} \|H\|^2 + \frac{3}{2} \left\| H^* \right\|^2 + 24 \left\| H^0 \right\|^2 \\
& - \frac{3}{n(n-1)} \sum_{r=1}^{m} \sum_{1 \le i < j \le n} \left(h_{ii}^r h_{jj}^r + h_{ii}^{*r} h_{jj}^{*r} + 16 h_{ii}^{0r} h_{jj}^{0r} \right) \\
& + \frac{3}{n(n-1)} \sum_{r=1}^{m} \sum_{1 \le i < j \le n} \left[\left(h_{ij}^r \right)^2 + \left(h_{ij}^{*r} \right)^2 + 16 \left(h_{ij}^{0r} \right)^2 \right] \\
=\ & \frac{3}{2} \|H\|^2 + \frac{3}{2} \left\| H^* \right\|^2 + 24 \left\| H^0 \right\|^2 \\
& - \frac{3}{n(n-1)} \sum_{r=1}^{m} \sum_{1 \le i < j \le n} \left[\left(h_{ii}^r + h_{ii}^{*r} \right) \left(h_{jj}^r + h_{jj}^{*r} \right) - h_{ii}^{*r} h_{jj}^r - h_{ii}^r h_{jj}^{*r} + 16 h_{ii}^{0r} h_{jj}^{0r} \right] \\
& + \frac{3}{n(n-1)} \sum_{r=1}^{m} \sum_{1 \le i < j \le n} \left[\left(h_{ij}^r + h_{ij}^{*r} \right)^2 - 2 h_{ij}^r h_{ij}^{*r} + 16 \left(h_{ij}^{0r} \right)^2 \right].
\end{aligned}$$

Using again $2h_{ij}^{0r} = h_{ij}^r + h_{ij}^{*r}$, $\forall i, j = 1, \dots, n$, $r = 1, \dots, m$, we obtain

$$\begin{aligned}
\rho^{\perp} \le\ & \frac{3}{2} \|H\|^2 + \frac{3}{2} \left\| H^* \right\|^2 + 24 \left\| H^0 \right\|^2 \\
& - \frac{3}{n(n-1)} \sum_{r=1}^{m} \sum_{1 \le i < j \le n} \left[20 h_{ii}^{0r} h_{jj}^{0r} - h_{ii}^{*r} h_{jj}^r - h_{ii}^r h_{jj}^{*r} - 20 (h_{ij}^{0r})^2 + 2 h_{ij}^r h_{ij}^{*r} \right].
\end{aligned} \tag{4.5}$$

Substituting (4.1) in (4.5), one leads to

$$\rho^{\perp} \leq \frac{3}{2}\|H\|^2 + \frac{3}{2}\left\|H^*\right\|^2 + 24\left\|H^0\right\|^2 - 3\rho - 3c - \frac{60}{n(n-1)}\sum_{r=1}^{m}\sum_{1\leq i<j\leq n}\left[h_{ii}^{0r}h_{jj}^{0r} - \left(h_{ij}^{0r}\right)^2\right]. \quad (4.6)$$

If we denote by

$$\tilde{\rho}^0 = \frac{2}{n(n-1)}\sum_{1\leq i<j\leq n}\tilde{R}^0\left(e_i, e_j, e_i, e_j\right),$$

the Gauss equation for the Levi-Civita connection $\tilde{\nabla}^0$ gives

$$\tilde{\rho}^0 = \rho^0 - \frac{2}{n(n-1)}\sum_{r=1}^{m}\sum_{1\leq i<j\leq n}\left[h_{ii}^{0r}h_{jj}^{0r} - \left(h_{ij}^{0r}\right)^2\right]. \quad (4.7)$$

From (4.6) and (4.7) we obtain

$$\rho^{\perp} \leq \frac{3}{2}\|H\|^2 + \frac{3}{2}\left\|H^*\right\|^2 + 24\left\|H^0\right\|^2 - 3\rho - 3c + 30\left(\tilde{\rho}^0 - \rho^0\right).$$

Summarizing, we proved the following generalized Wintgen inequality.

Theorem 4.1 *Let M^n be a submanifold in a statistical manifold $(\tilde{M}^{n+m}, c)$ of constant curvature c. Then*

$$\rho^{\perp} + 3\rho \leq \frac{15}{2}\|H\|^2 + \frac{15}{2}\|H^*\|^2 + 12g(H, H^*) - 3c + 30(\tilde{\rho}^0 - \rho^0).$$

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