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Generalizing the $\mathfrak{bms}_3$ and 2D-conformal algebras by expanding the Virasoro algebra

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Abstract By means of the Lie algebra expansion method, the centrally extended conformal algebra in two dimensions and the $\mathfrak{bms}_3$ algebra are obtained from the Virasoro algebra. We extend this result to construct new families of expanded Virasoro algebras that turn out to be infinite-dimensional lifts of the so-called $\mathfrak{B}_k$, $\mathfrak{C}_k$ and $\mathfrak{D}_k$ algebras recently introduced in the literature in the context of (super)gravity. We also show how some of these new infinite-dimensional symmetries can be obtained from expanded Kač–Moody algebras using modified Sugawara constructions. Applications in the context of three-dimensional gravity are briefly discussed.

1 Introduction

Infinite-dimensional symmetries play a prominent role in different areas of physics. In particular, symmetries of the Virasoro type have had remarkable applications in two-dimensional field theory, fluid mechanics, string theory, soliton theory and gravity among others.

The Virasoro algebra corresponds to the central extension of the algebra of infinitesimal diffeomorphisms of the circle [1]. It was first found in the context of string theory, where it describes the conformal invariance of the two-dimensional worldsheet swept out by strings. This is due to the fact that the conformal algebra in two dimensions is infinite dimensional and its central extension is given by two copies the Virasoro algebra. Therefore, the Virasoro symmetry appears in any physical system with conformal invariance defined on a two-dimensional space. Examples of this are two-dimensional sigma models [2], spin lattice models near criticality [3], integrable systems of the KdV type [4], the asymptotic structure of the $\mathcal{S}$-matrix in General Relativity [5] and the asymptotic symmetries of three-dimensional gravity. In the last case, Brown and Henneaux [6] showed that for suitable boundary conditions the asymptotic symmetry of three-dimensional Einstein gravity with negative cosmological constant is given by two copies of the Virasoro algebra. The presence of the centrally extended 2D-conformal symmetry at infinity was the first hint of an holographic duality, which was later conjectured by Maldacena in the context of strings [7]. This remarkable discovery has lead to a number of important subsequent results that could shed light into the quantum nature of gravity [8,9].

The Virasoro algebra is closely related to the Kač–Moody algebra, which corresponds to the central extensions of the loop algebra. In fact, a representation of the Virasoro algebra can be constructed out of quadratic combinations of the generators of the Kač–Moody algebra by means of the Sugawara construction. This is useful when studying WZW models, as it allows one to find the Virasoro symmetry at the level of the energy momentum tensor starting from its current algebra [10]. Furthermore the Drinfeld–Sokolov Hamiltonian reduction relates the WZW model to Liouville theory, which is conformally invariant [11]. In the context of three-dimensional gravity this has had remarkable uses. In fact, when 3D Einstein gravity with negative cosmological constant is formulated as a Chern–Simons theory, it can be written as an $SL(2,\mathbb{R})$ WZW model once the Hamiltonian constraints are solved within the action. Then, upon imposing the Brown–Henneaux boundary conditions, it can be further reduced to Liouville theory at the boundary [12].

The Virasoro symmetry is not the only infinite-dimensional symmetry that appears when studying the asymptotic structure of gravity theories. In four-dimensional General

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Relativity, the BMS group appears as the asymptotic symmetry of the theory at null infinity [13–15]. This symmetry has attracted great attention in the last years regarding the $\mathcal{S}$-Matrix for quantum gravity and its connection with soft theorems and the gravitational memory effect [16]. Remarkably, this enhancement of the Poincaré symmetry can also be found in three dimensions [17]. Indeed, in the case of vanishing cosmological constant, the $\mathfrak{bms}_3$ algebra is found as the asymptotic symmetry of Einstein gravity at null infinity [18]. This algebra is given by the semi-direct sum of the infinitesimal diffeomorphisms on the circle with an abelian ideal of super translations and can be obtained as an Inönü–Wigner (IW) contraction [19,20] of the centrally extended conformal algebra in two dimensions in the very same way as the Poincaré symmetry follows from the AdS_3 symmetry [21]. The $\mathfrak{bms}_3$ algebra can also be obtained from a Sugawara construction of the $\mathfrak{iso}(2,1)$ current algebra associated with the flat WZW model, which in turn follows from an IW contraction of an $\mathfrak{sl}(2)$ Kač–Moody algebra. Along the same lines, a Hamiltonian reduction of the flat WZW model leads to the flat Liouville theory as the classical two-dimensional dual for asymptotically flat 3D Einstein gravity, which is BMS_3 invariant [22–24]. On the other hand, there is an equivalence between symmetries of ultra-relativistic theory and $\mathfrak{bms}_3$ which has been relevant in the extension of the AdS/CFT correspondence [25–28]. In the last years, generalizations of the conformal and $\mathfrak{bms}_3$ algebras together with their Kač–Moody cousins have appeared in the literature in the context of three-dimensional supergravity and higher spin gravity [29–45]. Analogously to the pure gravity cases, these extensions turn out to be connected by IW contractions.

A particular characteristic of the IW contraction is that the starting and resulting algebras have the same number of generators. A natural way to generalize the IW contraction in order to obtain algebras of greater dimension than the starting one is given by the Lie algebra expansion method [46–50]. In particular, the S-expansion procedure formulated in Ref. [50] combines the structure constants of a given Lie algebra with the inner product of an abelian semigroup and has given rise to a number of interesting new symmetries that can be used to formulate gravity theories coupled to matter [51–60]. Such symmetries can be classified into three families of algebras called $\mathfrak{B}_k$, $\mathfrak{C}_k$ and $\mathfrak{D}_k$. $\mathfrak{B}_k$ algebras have been used to obtain General Relativiy from Chern–Simons and Born–Infeld gravity theories in diverse dimensions [51,53,55,56]. In particular, the $\mathfrak{B}_3$ and $\mathfrak{B}_4$ algebras correspond to the Poincaré and Maxwell algebras [61–66]. It is important to note that $\mathfrak{B}_k$ symmetries can be obtained as IW contraction of the $\mathfrak{C}_k$ algebras [54]. The $\mathfrak{C}_k$ family allows one to relate diverse (pure) Lovelock gravities to Chern–Simons and Born–Infeld gravities [58,60]. Alternatively, $\mathfrak{B}_k$ algebras can be obtained as an IW contraction of another set of algebras called $\mathfrak{D}_k$ symmetries, which correspond to direct sums of the form $\mathrm{AdS} \oplus \mathfrak{B}_{k-2}$ [57,59]. Supersymmetric extensions of some of these expanded algebras have been worked out in Refs. [67–74], which can also be obtained through the S-expansion mechanism. It is therefore interesting to study what kind of infinite-dimensional symmetries can be obtained as S-expansions of known infinite-dimensional algebras. In this paper we put forward such study and present new families of infinite-dimensional algebras that can be obtained by applying the semigroup expansion mechanism to the Virasoro algebra. We first show that the centrally extended 2D-conformal algebra and the $\mathfrak{bms}_3$ algebra can be obtained as a semigroup expansion of the Virasoro algebra. Then, by using more general semigroups, we construct new families expanded Virasoro algebras that we name *generalized 2D-conformal algebras* and *generalized* $\mathfrak{bms}_3$ *algebras*. We also show how these new infinite dimensional symmetries can be related by IW contractions. Interestingly these symmetries correspond to infinite dimensional enhancements of the $\mathfrak{B}_k$ and $\mathfrak{C}_k$ algebras. Additionally, we provide an infinite-dimensional lift of the so-called $\mathfrak{D}_k$ algebras. Finally, we study the Sugawara construction connecting expanded Kač–Moody algebras with our expanded Virasoro algebras and present some explicit examples.

The paper is organized as follows: In Sect. 2 we present the general setup to S-expand the Virasoro algebra and obtain the centrally extended 2D-conformal algebra as well as the $\mathfrak{bms}_3$ algebra particular cases. In Sect. 3 we show how the expansion procedure can be used to construct a deformed $\mathfrak{bms}_3$ algebra which corresponds to an infinite-dimensional lift of the Maxwell algebra. In the same way, an infinite-dimensional enhancement of the AdS–Lorentz algebra is constructed, which is given by three copies of the Virasoro algebra and can be related to the deformed $\mathfrak{bms}_3$ symmetry by an IW contraction. In Sect. 4 we introduce the generalized 2D-conformal algebras and generalized $\mathfrak{bms}_3$ algebras. Section 5 is devoted to the construction of the infinite-dimensional lifts of the $\mathfrak{D}_k$ algebras. In Sect. 6 we present (modified) Sugawara construction that allows one to obtain expanded Virasoro algebras from expanded Kac–Moody algebras in the simplest cases. Finally, future applications of these results in the context of 3D gravity theories and WZW models are discussed in Sect. 7.

2 Centrally extended 2D-conformal algebra and $\mathfrak{bms}_3$ algebra as S-expansions

The S-expansion method [50] consists in combining the structure constants of a Lie algebra $\mathfrak{g}$ with the elements of a semigroup S to obtain a new Lie algebra $\mathfrak{G} = S \times \mathfrak{g}$. In this section we show that the centrally extended 2D-conformal algebra and the $\mathfrak{bms}_3$ algebra can be obtained explicitly as an S-expansion of the Virasoro algebra for suitable semigroups.

For details regarding the notation and the S-expansion procedure we refer the reader to the original references [75–82].

2.1 Expanding the Virasoro algebra

The starting point of this construction is the Virasoro algebra $\mathfrak{vir}$,

$$[\ell_m, \ell_n] = (m-n)\ell_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0}, \tag{2.1}$$

together with a semigroup $S = \{\lambda_\alpha\}$, whose inner product is defined by a 2-selector $K_{\alpha\beta}^{\gamma} = K_{\beta\alpha}^{\gamma}$ such that

$$\lambda_\alpha\lambda_\beta = \lambda_\beta\lambda_\alpha = K_{\alpha\beta}^{\gamma}\lambda_\gamma\,. \tag{2.2}$$

We define an S-expanded Virasoro algebra as the direct product $\mathfrak{vir}_{\mathfrak{h}} = S \times \mathfrak{vir}$, where $\mathfrak{h} = S \times \mathfrak{sl}(2)$ is the expansion of the $\mathfrak{sl}(2,\mathbb{R})$ subalgebra of (2.1) generated by subset $\{\ell_{-1}, \ell_0, \ell_1\}$[1]. The generators of $\mathfrak{vir}_{\mathfrak{h}}$ are given by

$$\ell_{(m,\alpha)} = \lambda_\alpha \ell_m \tag{2.3}$$

and satisfy the commutation relations

$$\begin{aligned}\left[\ell_{(m,\alpha)}, \ell_{(n,\beta)}\right] &= (m-n)K_{\alpha\beta}^{\gamma}\,\ell_{(m+n,\gamma)}\\ &\quad+\frac{c_{\alpha\beta}}{12}m(m^2-1)\delta_{m+n,0},\end{aligned} \tag{2.4}$$

where $c_{\alpha\beta}$ denotes a set of central charges given by

$$c_{\alpha\beta} = c\,K_{\alpha\beta}^{\gamma}\lambda_\gamma\,. \tag{2.5}$$

Note that the finite subalgebra $\mathfrak{h}$ of $\mathfrak{vir}_{\mathfrak{h}}$ is spanned by the subset $\left\{\ell_{(-1,\alpha)}, \ell_{(0,\alpha)}, \ell_{(1,\alpha)}\right\}$.

2.2 Centrally extended 2D-conformal algebra

The centrally extended conformal algebra in two dimensions is given by the direct sum of a pair of Virasoro algebras $\mathfrak{vir}\oplus\mathfrak{vir}$, which we will simply denote $\mathfrak{vir}^2$,

$$\begin{aligned}&[\mathcal{L}_m, \mathcal{L}_n] = (m-n)\,\mathcal{L}_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0},\\ &\left[\bar{\mathcal{L}}_m, \bar{\mathcal{L}}_n\right] = (m-n)\,\bar{\mathcal{L}}_{m+n} + \frac{\bar{c}}{12}m(m^2-1)\delta_{m+n,0},\\ &\left[\mathcal{L}_m, \bar{\mathcal{L}}_n\right] = 0\,.\end{aligned} \tag{2.6}$$

This algebra can be obtained as a particular S-expansion of $\mathfrak{vir}$. In fact, let us consider the (semi)group $\mathbb{Z}_2 = \{\lambda_0, \lambda_1\}$, whose multiplication law is given by

$$\begin{array}{c|cc} \lambda_1 & \lambda_1 & \lambda_0\\ \lambda_0 & \lambda_0 & \lambda_1\\ \hline & \lambda_0 & \lambda_1 \end{array} \tag{2.7}$$

and from which the non-vanishing 2-selectors (2.2) can be read off to be $K_{00}^{0} = K_{11}^{0} = K_{01}^{1} = K_{10}^{1} = 1$. Denoting the generators (2.3) and the central charges (2.5) of the corresponding expanded algebra

$$\begin{aligned}&\mathcal{J}_m \equiv \ell_{(m,0)} = \lambda_0\ell_m, \quad c_1 \equiv c_{00} = c_{11} = \lambda_0 c,\\ &\mathcal{P}_m \equiv \ell_{(m,1)} = \lambda_1\ell_m, \quad c_2 \equiv c_{01} = \lambda_1 c,\end{aligned} \tag{2.8}$$

Eq. (2.4) yields

$$\begin{aligned}&[\mathcal{J}_m, \mathcal{J}_n] = (m-n)\,\mathcal{J}_{m+n} + \frac{c_1}{12}\left(m^3-m\right)\delta_{m+n,0},\\ &[\mathcal{J}_m, \mathcal{P}_n] = (m-n)\,\mathcal{P}_{m+n} + \frac{c_2}{12}\left(m^3-m\right)\delta_{m+n,0},\\ &[\mathcal{P}_m, \mathcal{P}_n] = (m-n)\,\mathcal{J}_{m+n} + \frac{c_1}{12}\left(m^3-m\right)\delta_{m+n,0}.\end{aligned} \tag{2.9}$$

It is easy to see that (2.9) is isomorphic to $\mathfrak{vir}^2$ by making the following change of basis:

$$\mathcal{L}_m = \frac{1}{2}\left(\mathcal{P}_m + \mathcal{J}_m\right), \quad \bar{\mathcal{L}}_{-m} = \frac{1}{2}\left(\mathcal{P}_m - \mathcal{J}_m\right), \tag{2.10}$$

which leads to (2.6) with central charges $c = \frac{1}{2}(c_2 + c_1)$ and $\bar{c} = \frac{1}{2}(c_2 - c_1)$.

2.3 $\mathfrak{bms}_3$ algebra

Consider now the expansion of the Viraroso algebra (2.1) using the semigroup $S_E^{(1)} = \{\lambda_0, \lambda_1, \lambda_2\}$, whose elements satisfy the multiplication law

$$\begin{array}{c|ccc} \lambda_2 & \lambda_2 & \lambda_2 & \lambda_2\\ \lambda_1 & \lambda_1 & \lambda_2 & \lambda_2\\ \lambda_0 & \lambda_0 & \lambda_1 & \lambda_2\\ \hline & \lambda_0 & \lambda_1 & \lambda_2 \end{array} \tag{2.11}$$

and where $\lambda_2 \equiv 0_S$ is the zero element of the semigroup such that $0_S\lambda_\alpha = 0_S$. The 0_S-reduced $S_E^{(1)}$-expanded algebra is obtained imposing $0_S \times \ell_{(m,\alpha)} = 0$. Defining the non-vanishing expanded generators (2.3) in the same way as in (2.8), we get

$$\begin{aligned}&[\mathcal{J}_m, \mathcal{J}_n] = (m-n)\,\mathcal{J}_{m+n} + \frac{c_1}{12}\left(m^3-m\right)\delta_{m+n,0},\\ &[\mathcal{J}_m, \mathcal{P}_n] = (m-n)\,\mathcal{P}_{m+n} + \frac{c_2}{12}\left(m^3-m\right)\delta_{m+n,0},\\ &[\mathcal{P}_m, \mathcal{P}_n] = 0,\end{aligned} \tag{2.12}$$

which corresponds to the $\mathfrak{bms}_3$ algebra [18].

Let us also recall that the $\mathfrak{bms}_3$ algebra can be obtained from two copies of the Virasoro algebra as an IW contraction.

[1] This notation might seem awkward, but throughout our presentation it will prove useful to label expanded Virasoro algebras by their corresponding subalgebras $\mathfrak{h}$.

Writing $\mathfrak{vir}^2$ in the form (2.9) and rescaling its generators as

$$\mathcal{J}_m \to \mathcal{J}_m, \quad \mathcal{P}_m \to \sigma\mathcal{P}_m, \quad c_1 \to c_1, \quad c_2 \to \sigma c_2, \tag{2.13}$$

leads to (2.12) in the limit $\sigma \to \infty$. A similar approach is considered in [83] where they implemented the IW contraction using a Grassman parameter. As we will see in the following, this kind of limit procedure will be useful to establish different links between more general expanded Virasoro algebras.

3 Deformed $\mathfrak{bms}_3$ algebra

The centrally extended conformal algebra and its flat limit, the $\mathfrak{bms}_3$ algebra, are not the only symmetries that can be obtained using the expansion method. In the present section we present new infinite-dimensional symmetries which are directly obtained as an S-expansion of the Virasoro algebra. In particular, a deformed $\mathfrak{bms}_3$ algebra as well as three copies of the Virasoro algebra ($\mathfrak{vir}^3$) can be obtained, where the former corresponds to an IW contraction of the latter.

3.1 Deformed $\mathfrak{bms}_3$ as an S-expansion

Let us consider the semigroup $S_E^{(2)} = \{\lambda_0, \lambda_1, \lambda_2, \lambda_3\}$, whose elements satisfy

$$\begin{array}{c|cccc}
\lambda_3 & \lambda_3 & \lambda_3 & \lambda_3 & \lambda_3 \\
\lambda_2 & \lambda_2 & \lambda_3 & \lambda_3 & \lambda_3 \\
\lambda_1 & \lambda_1 & \lambda_2 & \lambda_3 & \lambda_3 \\
\lambda_0 & \lambda_0 & \lambda_1 & \lambda_2 & \lambda_3 \\
\hline
 & \lambda_0 & \lambda_1 & \lambda_2 & \lambda_3
\end{array} \tag{3.1}$$

and where $\lambda_3 = 0_S$ is the zero element. Denoting the generators (2.3) and the central charges (2.5) of the corresponding expanded algebra as

$$\begin{aligned}
\mathcal{J}_m &\equiv \ell_{(m,0)} = \lambda_0 \ell_m, \quad c_1 \equiv c_{00} = c_{11} = \lambda_0 c, \\
\mathcal{P}_m &\equiv \ell_{(m,1)} = \lambda_1 \ell_m, \quad c_2 \equiv c_{01} = \lambda_1 c, \\
\mathcal{Z}_m &\equiv \ell_{(m,2)} = \lambda_2 \ell_m, \quad c_3 \equiv c_{02} = c_{11} = \lambda_2 c,
\end{aligned} \tag{3.2}$$

the 0_S-reduced $S_E^{(2)}$-expanded algebra satisfies the commutation relations

$$\begin{aligned}
[\mathcal{J}_m, \mathcal{J}_n] &= (m-n)\,\mathcal{J}_{m+n} + \frac{c_1}{12}\left(m^3 - m\right)\delta_{m+n,0}, \\
[\mathcal{J}_m, \mathcal{P}_n] &= (m-n)\,\mathcal{P}_{m+n} + \frac{c_2}{12}\left(m^3 - m\right)\delta_{m+n,0}, \\
[\mathcal{P}_m, \mathcal{P}_n] &= (m-n)\,\mathcal{Z}_{m+n} + \frac{c_3}{12}\left(m^3 - m\right)\delta_{m+n,0}, \\
[\mathcal{J}_m, \mathcal{Z}_n] &= (m-n)\,\mathcal{Z}_{m+n} + \frac{c_3}{12}\left(m^3 - m\right)\delta_{m+n,0}, \\
[\mathcal{P}_m, \mathcal{Z}_n] &= 0, \\
[\mathcal{Z}_m, \mathcal{Z}_n] &= 0.
\end{aligned} \tag{3.3}$$

Interestingly, the Maxwell algebra in $(2+1)$ dimensions is spanned by the generators $\mathcal{J}_0, \mathcal{J}_1, \mathcal{J}_{-1}, \mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_{-1}$ and $\mathcal{Z}_0, \mathcal{Z}_1, \mathcal{Z}_{-1}$. This can be made explicit in terms of generators $\{J_a, P_a, Z_a\}$ obtained through the following change of basis:[2]

$$\begin{aligned}
&\mathcal{J}_{-1} = -2J_0, \mathcal{J}_0 = J_2 \quad, \mathcal{J}_1 = J_1, \\
&\mathcal{P}_{-1} = -2P_0, \mathcal{P}_0 = P_2 \quad, \mathcal{P}_1 = P_1, \\
&\mathcal{Z}_{-1} = -2Z_0, \mathcal{Z}_0 = Z_2 \quad, \mathcal{Z}_1 = Z_1.
\end{aligned} \tag{3.4}$$

This means that the deformed $\mathfrak{bms}_3$ algebra (3.3) corresponds to an infinite-dimensional lift of the $(2+1)$-dimensional Maxwell algebra in the very same way as the algebras $\mathfrak{vir}^2$ and $\mathfrak{bms}_3$ are infinite-dimensional lifts of the AdS and the Poincaré algebras in $2+1$ dimensions, respectively.

3.2 Deformed $\mathfrak{bms}_3$ algebra as a limit of $\mathfrak{vir}^3$

Let us consider now $S_{\mathcal{M}}^{(2)} = \{\lambda_0, \lambda_1, \lambda_2\}$ as the relevant abelian semigroup, whose elements satisfy the following multiplication law:

$$\begin{array}{c|ccc}
\lambda_2 & \lambda_2 & \lambda_1 & \lambda_2 \\
\lambda_1 & \lambda_1 & \lambda_2 & \lambda_1 \\
\lambda_0 & \lambda_0 & \lambda_1 & \lambda_2 \\
\hline
 & \lambda_0 & \lambda_1 & \lambda_2
\end{array} \tag{3.5}$$

Unlike the $S_E^{(2)}$ semigroup, there is no zero element in this case. Adopting the same notation (3.2) for the generators of the $S_{\mathcal{M}}^{(2)}$-expanded algebra, we find the following commutation relations:

$$\begin{aligned}
[\mathcal{J}_m, \mathcal{J}_n] &= (m-n)\,\mathcal{J}_{m+n} + \frac{c_1}{12}\left(m^3 - m\right)\delta_{m+n,0}, \\
[\mathcal{J}_m, \mathcal{P}_n] &= (m-n)\,\mathcal{P}_{m+n} + \frac{c_2}{12}\left(m^3 - m\right)\delta_{m+n,0}, \\
[\mathcal{P}_m, \mathcal{P}_n] &= (m-n)\,\mathcal{Z}_{m+n} + \frac{c_3}{12}\left(m^3 - m\right)\delta_{m+n,0}, \\
[\mathcal{J}_m, \mathcal{Z}_n] &= (m-n)\,\mathcal{Z}_{m+n} + \frac{c_3}{12}\left(m^3 - m\right)\delta_{m+n,0}, \\
[\mathcal{Z}_m, \mathcal{Z}_n] &= (m-n)\,\mathcal{Z}_{m+n} + \frac{c_3}{12}\left(m^3 - m\right)\delta_{m+n,0}, \\
[\mathcal{Z}_m, \mathcal{P}_n] &= (m-n)\,\mathcal{P}_{m+n} + \frac{c_2}{12}\left(m^3 - m\right)\delta_{m+n,0}.
\end{aligned} \tag{3.6}$$

Note that the AdS–Lorentz algebra in $2+1$ dimensions, also known as the semi-simple extension of the Poincaré algebra [84], is the subalgebra of (3.6) spanned by the generators $\mathcal{J}_0, \mathcal{J}_1, \mathcal{J}_{-1}, \mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_{-1}$ and $\mathcal{Z}_0, \mathcal{Z}_1, \mathcal{Z}_{-1}$. This can be explicitly seen using the change of basis (3.4), showing that

[2] In this case the Maxwell algebra is realized with a non-diagonal Minkowski metric $\eta_{ab} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

(3.6) defines and infinite-dimensional lift of the AdS–Lorentz algebra in $2+1$ dimensions.

Remarkably, there is a redefinition of the generators of (3.6) that allows one to see its true algebraic structure. In fact, considering the change of basis

$$\mathcal{L}_m = \frac{1}{2}\left(\mathcal{P}_m + \mathcal{Z}_m\right), \quad \bar{\mathcal{L}}_{-m} = \frac{1}{2}\left(\mathcal{P}_m - \mathcal{Z}_m\right), \\ \tilde{\mathcal{L}}_{-m} = \mathcal{J}_m - \mathcal{Z}_m, \tag{3.7}$$

three copies of the Virasoro algebra, which will be denoted $\mathfrak{vir}^3$, are revealed

$$\begin{aligned} \left[\mathcal{L}_m, \mathcal{L}_n\right] &= (m-n)\,\mathcal{L}_{m+n} + \frac{c}{12}\left(m^3 - m\right)\delta_{m+n,0}, \\ \left[\bar{\mathcal{L}}_m, \bar{\mathcal{L}}_n\right] &= (m-n)\,\bar{\mathcal{L}}_{m+n} + \frac{\bar{c}}{12}\left(m^3 - m\right)\delta_{m+n,0}, \\ \left[\tilde{\mathcal{L}}_m, \tilde{\mathcal{L}}_n\right] &= (m-n)\,\tilde{\mathcal{L}}_{m+n} + \frac{\tilde{c}}{12}\left(m^3 - m\right)\delta_{m+n,0}, \end{aligned} \tag{3.8}$$

where the central extensions are given by $c = \frac{1}{2}\left(c_2 + c_3\right)$, $\bar{c} = \frac{1}{2}\left(c_2 - c_3\right)$ and $\tilde{c} = c_1 - c_3$. Additionally, as in the case of the $\mathfrak{bms}_3$ and the 2D-conformal algebra, there is a limit procedure relating $\mathfrak{vir}^3$ and the deformed $\mathfrak{bms}_3$ algebra through an IW contraction. In fact, the rescaling of the generators of (3.6)

$$\mathcal{J}_m \to \mathcal{J}_m, \quad \mathcal{P}_m \to \sigma\mathcal{P}_m, \quad \mathcal{Z}_m \to \sigma^2\mathcal{Z}_m, \\ c_1 \to c_1, \quad c_2 \to \sigma c_2, \quad c_3 \to \sigma^2 c_3, \tag{3.9}$$

leads to the deformed $\mathfrak{bms}_3$ algebra (3.3) in the limit $\sigma \to \infty$.

4 Generalized expanded Virasoro algebras

In the previous sections we have seen how the S expansion mechanism allows one to obtain the centrally extended 2D-conformal algebra and the $\mathfrak{bms}_3$ algebra from the Virasoro algebra. In the context of three-dimensional gravity, the centrally extended 2D-conformal algebra and the $\mathfrak{bms}_3$ algebra correspond to infinite-dimensional lifts of AdS and the Poincaré symmetries in $2+1$ dimensions. Generalizing this results we have subsequently shown how to construct infinite-dimensional lifts of the Maxwell and the AdS–Lorentz algebras in $2+1$ dimensions, which correspond a deformed $\mathfrak{bms}_3$ symmetry in the former case and to three copies of the Virasoro algebra in the latter. As has recently been pointed out in Refs. [54,55,57,68], the Poincaré and the AdS algebras as well as the Maxwell and the AdS–Lorentz algebras correspond to particular cases of the $\mathfrak{B}_k$ and $\mathfrak{C}_k$ algebras for $k=3$ and $k=4$, respectively. Such families of algebras have been of particular interest in the context of gravity. Indeed, as was shown in Refs. [51,53,56], General Relativity can be obtained as a particular limit of a Chern–Simons and a Born–Infeld gravity theory using the $\mathfrak{B}_k$ symmetries. On the other hand, the $\mathfrak{C}_k$ algebras allow one to recover the pure Lovelock Lagrangian from Chern–Simons and Born–Infeld theories [58,60].

The results obtained up to this point clearly suggest that, in the same way as their respective finite subalgebras, the $\mathfrak{bms}_3$ and $\mathfrak{vir}^2$ algebras as well as the deformed $\mathfrak{bms}_3$ and the $\mathfrak{vir}^3$ algebras should correspond to particular cases of certain families of generalized infinite-dimensional symmetries. In this section we present the general scheme that leads to such families of expanded Virasoro algebras.

4.1 Generalized $\mathfrak{bms}_3$ algebras

Let $S_E^{(k-2)} = \{\lambda_0, \lambda_1, \ldots, \lambda_{k-1}\}$ be the finite abelian semigroup whose elements satisfy the following multiplication law:

$$\lambda_\alpha \lambda_\beta = \begin{cases} \lambda_{\alpha+\beta} & \text{if} \quad \alpha+\beta \leq k-2, \\ \lambda_{k-1} & \text{if} \quad \alpha+\beta > k-2, \end{cases} \tag{4.1}$$

where $\lambda_{k-1} = 0_s$ is the zero element of the semigroup. The $S_E^{(k-2)}$-expanded Virasoro algebra (2.4) in this case is given by

$$\left[\ell_{(m,\alpha)}, \ell_{(n,\beta)}\right] = \begin{cases} (m-n)\ell_{(m+n,\alpha+\beta)} \\ +\dfrac{c_{\alpha+\beta+1}}{12} m(m^2-1)\delta_{m+n,0} & \text{if} \quad \alpha+\beta \leq k-2, \\ 0 & \text{if} \quad \alpha+\beta > k-2, \end{cases} \tag{4.2}$$

where we have defined $c_{\alpha+\beta+1} \equiv c_{\alpha\beta}$. Following the notation introduced in Sect. 2, the algebra (4.2) will be denoted by $\mathfrak{vir}_{\mathfrak{B}_k}$, as the subalgebra $\mathfrak{h}$ generated by $\left\{\ell_{(-1,\alpha)}, \ell_{(0,\alpha)}, \ell_{(1,\alpha)}\right\}$ corresponds to the $\mathfrak{B}_k$ algebra in $2+1$ dimensions [51,85]. It is easy to see that (4.1) always contains an abelian ideal spanned by the subset of generators

$$\mathcal{A} = \left\{\ell_{(m,\tilde{\alpha})}\right\}, \quad \tilde{\alpha} = \left[\frac{k}{2}\right], \ldots, k-2$$

and for which

$$\left[\ell_{(m,\tilde{\alpha})}, \ell_{\left(m,\tilde{\beta}\right)}\right] = 0, \\ \left[\ell_{(m,\tilde{\alpha})}, \ell_{(m,\alpha)}\right] \in \mathcal{A} \; + \text{central terms}. \tag{4.3}$$

For this reason the $\mathfrak{vir}_{\mathfrak{B}_k}$ algebra will be referred to as *generalized* $\mathfrak{bms}_3$ *algebra*. This algebra corresponds to an infinite dimensional lift of the $\mathfrak{B}_k$ algebra in $2+1$ dimensions, which can be made explicit by redefining the generators in the form

$$\mathcal{J}_m^i \equiv \ell_{(m,i)} = \lambda_i \ell_m, \\ \mathcal{P}_m^{\bar{i}} \equiv \ell_{(m,\bar{i})} = \lambda_{\bar{i}} \ell_m, \tag{4.4}$$

where i takes even values and $\bar{i}$ takes odd values. Here we identify the following cases:

- For $k-2=2N$ the abelian ideal $\mathcal{A}$ is generated by

$$\mathcal{A}=\begin{cases}\mathcal{P}_m^{N+1},\mathcal{J}_m^{N+2},\dots,\mathcal{P}_m^{2N-1},\mathcal{J}_m^{2N} & \text{for } N \text{ even,}\\ \mathcal{J}_m^{N+1},\mathcal{P}_m^{N+2},\dots,\mathcal{P}_m^{2N-1},\mathcal{J}_m^{2N} & \text{for } N \text{ odd.}\end{cases}$$

- For $k-2=2N+1$ the abelian ideal $\mathcal{A}$ is generated by

$$\mathcal{A}=\begin{cases}\mathcal{P}_m^{N+1},\mathcal{J}_m^{N+2},\dots,\mathcal{J}_m^{2N},\mathcal{P}_m^{2N+1} & \text{for } N \text{ even,}\\ \mathcal{J}_m^{N+1},\mathcal{P}_m^{N+2},\dots,\mathcal{J}_m^{2N},\mathcal{P}_m^{2N+1} & \text{for } N \text{ odd.}\end{cases}$$

Using the definition (4.4), we write (4.2) in the form

$$\begin{aligned}\left[\mathcal{J}_m^i,\mathcal{J}_n^j\right]&=(m-n)\,\mathcal{J}_{m+n}^{i+j}\\&\quad+\frac{c_{i+j+1}}{12}\left(m^3-m\right)\delta_{m+n,0},\quad \text{for}\quad i+j\le k-2,\\ \left[\mathcal{J}_m^i,\mathcal{P}_n^{\bar{i}}\right]&=(m-n)\,\mathcal{P}_{m+n}^{i+\bar{i}}\\&\quad+\frac{c_{i+\bar{i}+1}}{12}\left(m^3-m\right)\delta_{m+n,0},\quad \text{for}\quad i+\bar{i}\le k-2,\\ \left[\mathcal{P}_m^{\bar{i}},\mathcal{P}_n^{\bar{j}}\right]&=(m-n)\,\mathcal{J}_{m+n}^{\bar{i}+\bar{j}}\\&\quad+\frac{c_{\bar{i}+\bar{j}+1}}{12}\left(m^3-m\right)\delta_{m+n,0},\quad \text{for}\quad \bar{i}+\bar{j}\le k-2,\\ \text{others}&=0.\end{aligned}\tag{4.5}$$

As mentioned before, the $\mathfrak{B}_k$ algebra in $2+1$ dimensions is a subalgebra of (4.5) spanned by the generators $\mathcal{J}_0^i,\mathcal{J}_1^i,\mathcal{J}_{-1}^i$ and $\mathcal{P}_0^{\bar{i}},\mathcal{P}_1^{\bar{i}},\mathcal{P}_{-1}^{\bar{i}}$. Additionally, when written in this form it is straightforward to see that setting $k=3$ leads to the $\mathfrak{bms}_3$ algebra (2.12), while $k=4$ reproduces the deformed $\mathfrak{bms}_3$ algebra (3.3). Thus, $\mathfrak{bms}_3$ and its corresponding deformation can be classified into the infinite family of generalized $\mathfrak{bms}_3$ algebras $\mathfrak{vir}_{\mathfrak{B}_k}$.

4.2 Generalized 2D-conformal algebras

Another family of expanded Virasoro algebras can be obtained by choosing a different semigroup. Let us consider $S_{\mathcal{M}}^{(k-2)}=\{\lambda_0,\lambda_1,\dots,\lambda_{k-2}\}$ as the relevant abelian semigroup whose elements satisfy

$$\lambda_\alpha\lambda_\beta=\begin{cases}\lambda_{\alpha+\beta} & \text{if}\quad \alpha+\beta\le k-2,\\ \lambda_{\alpha+\beta-2\left[\frac{k-1}{2}\right]} & \text{if}\quad \alpha+\beta>k-2.\end{cases}\tag{4.6}$$

Then the $S_{\mathcal{M}}^{(k-2)}$-expanded algebra (2.4) takes the form

$$\left[\ell_{(m,\alpha)},\ell_{(n,\beta)}\right]=\begin{cases}(m-n)\ell_{(m+n,\alpha+\beta)}\\ \quad+\frac{c_{\alpha+\beta+1}}{12}m(m^2-1)\delta_{m+n,0} & \text{if}\quad \alpha+\beta\le k-2,\\ (m-n)\ell_{\left(m+n,\alpha+\beta-2\left[\frac{k-1}{2}\right]\right)}\\ \quad+\frac{c_{\alpha+\beta-2\left[\frac{k-1}{2}\right]+1}}{12}m(m^2-1)\delta_{m+n,0} & \text{if}\quad \alpha+\beta>k-2,\end{cases}\tag{4.7}$$

and corresponds to $\mathfrak{vir}_{\mathfrak{C}_k}$, as its subalgebra $\mathfrak{h}$ is given by the $\mathfrak{C}_k$ algebra in $2+1$ dimensions [58,60]. This algebra corresponds to an infinite-dimensional lift of the $\mathfrak{C}_k$ algebra, which can be explicitly seen by redefining the generators in the form (4.4), yielding

$$\begin{aligned}\left[\mathcal{J}_m^i,\mathcal{J}_n^j\right]&=(m-n)\,\mathcal{J}_{m+n}^{\{i+j\}}+\frac{c_{\{i+j\}+1}}{12}\left(m^3-m\right)\delta_{m+n,0},\\ \left[\mathcal{J}_m^i,\mathcal{P}_n^{\bar{j}}\right]&=(m-n)\,\mathcal{P}_{m+n}^{\{i+\bar{j}\}}+\frac{c_{\{i+\bar{j}\}+1}}{12}\left(m^3-m\right)\delta_{m+n,0},\\ \left[\mathcal{P}_m^{\bar{i}},\mathcal{P}_n^{\bar{j}}\right]&=(m-n)\,\mathcal{J}_{m+n}^{\{\bar{i}+\bar{j}\}}+\frac{c_{\{\bar{i}+\bar{j}\}+1}}{12}\left(m^3-m\right)\delta_{m+n,0},\end{aligned}\tag{4.8}$$

where $\{\cdots\}$ means

$$\{i+j\}=\begin{cases}i+j & \text{if}\quad i+j\le k-2,\\ i+j-2\left[\frac{k-1}{2}\right] & \text{if}\quad i+j>k-2.\end{cases}\tag{4.9}$$

As remarked before, the $\mathfrak{C}_k$ algebra in $2+1$ dimensions is the subalgebra of $\mathfrak{vir}_{\mathfrak{C}_k}$ spanned by the generators $\mathcal{J}_0^i,\mathcal{J}_1^i,\mathcal{J}_{-1}^i$ and $\mathcal{P}_0^{\bar{i}},\mathcal{P}_1^{\bar{i}},\mathcal{P}_{-1}^{\bar{i}}$. When written in the form (4.8) it is clear that setting $k=3$ leads to the centrally extended 2D-conformal algebra (2.9), while the case $k=4$ leads to the $\mathfrak{vir}^3$ algebra (3.6). Therefore $\mathfrak{vir}_{\mathfrak{C}_k}$ will be referred to as the (centrally extended) *generalized 2D-conformal algebra*. As in the cases $k=3$ and $k=4$ studied in the previous sections, the generalized 2D-conformal algebra can be related to the generalized $\mathfrak{bms}_3$ one through an IW contraction. In fact, rescaling the generators of (4.8) in the form

$$\begin{aligned}\mathcal{J}_m^i&\to\sigma^i\mathcal{J}_m^i,\quad \mathcal{P}_m^{\bar{j}}\to\sigma^{\bar{j}}\mathcal{P}_m^{\bar{j}},\\ c_{i+1}&\to\sigma^i c_{i+1},\quad c_{\bar{i}+1}\to\sigma^{\bar{i}}c_{\bar{i}+1},\end{aligned}\tag{4.10}$$

leads to the generalized $\mathfrak{bms}_3$ algebra (4.5) in the limit $\sigma\to\infty$.

The fact that the $\mathfrak{vir}_{\mathfrak{C}_k}$ reduces to two and three copies of the Virasoro algebra in the cases $k=3$ and $k=4$, respectively, might make one think that it could generally be written as $k-1$ copies of the Virasoro algebra. However, this is not true. Let us consider, for instance, the $\mathfrak{vir}_{\mathfrak{C}_5}$ algebra. Renaming its generators as $\mathcal{J}_m^0\equiv\mathcal{J}_m$, $\mathcal{P}_m^1\equiv\mathcal{P}_m$, $\mathcal{J}_m^2\equiv\mathcal{Z}_m$ and $\mathcal{P}_m^3\equiv\mathcal{R}_m$, this algebra can be directly read off from (4.8) to be

$$\begin{aligned}\left[\mathcal{J}_m,\mathcal{J}_n\right]&=(m-n)\,\mathcal{J}_{m+n}+\frac{c_1}{12}\left(m^3-m\right)\delta_{m+n,0},\\ \left[\mathcal{J}_m,\mathcal{P}_n\right]&=(m-n)\,\mathcal{P}_{m+n}+\frac{c_2}{12}\left(m^3-m\right)\delta_{m+n,0},\\ \left[\mathcal{P}_m,\mathcal{P}_n\right]&=(m-n)\,\mathcal{Z}_{m+n}+\frac{c_3}{12}\left(m^3-m\right)\delta_{m+n,0},\\ \left[\mathcal{J}_m,\mathcal{Z}_n\right]&=(m-n)\,\mathcal{Z}_{m+n}+\frac{c_3}{12}\left(m^3-m\right)\delta_{m+n,0},\\ \left[\mathcal{Z}_m,\mathcal{Z}_n\right]&=(m-n)\,\mathcal{J}_{m+n}+\frac{c_1}{12}\left(m^3-m\right)\delta_{m+n,0},\\ \left[\mathcal{Z}_m,\mathcal{P}_n\right]&=(m-n)\,\mathcal{R}_{m+n}+\frac{c_4}{12}\left(m^3-m\right)\delta_{m+n,0},\\ \left[\mathcal{J}_m,\mathcal{R}_n\right]&=(m-n)\,\mathcal{R}_{m+n}+\frac{c_4}{12}\left(m^3-m\right)\delta_{m+n,0},\end{aligned}$$

$$\begin{aligned}
[\mathcal{Z}_m, \mathcal{R}_n] &= (m-n)\,\mathcal{P}_{m+n} + \frac{c_2}{12}\left(m^3-m\right)\delta_{m+n,0}, \\
[\mathcal{R}_m, \mathcal{R}_n] &= (m-n)\,\mathcal{Z}_{m+n} + \frac{c_3}{12}\left(m^3-m\right)\delta_{m+n,0}, \\
[\mathcal{P}_m, \mathcal{R}_n] &= (m-n)\,\mathcal{J}_{m+n} + \frac{c_1}{12}\left(m^3-m\right)\delta_{m+n,0},
\end{aligned} \tag{4.11}$$

which cannot be redefined as four copies of the Virasoro algebra by means of a generalization of (2.10) or (3.7).

5 Infinite-dimensional $\mathfrak{D}_k$-like algebras

In [57] new expanded algebras were presented as a family of Maxwell-like algebras. Inspired by this construction, in this section we consider the expansion of the Virasoro algebra by means of the semigroup $S_D^{(k-2)}$, defined by the product rule

$$\lambda_\alpha \lambda_\beta = \begin{cases} \lambda_{\alpha+\beta} & \text{if } \alpha+\beta \le k-2, \\ \lambda_{(\alpha+\beta-(k-1))\mathrm{mod}2\,+(k-3)} & \text{if } \alpha+\beta > k-2. \end{cases} \tag{5.1}$$

Using the notation (4.4) for the expanded generators, the $S_D^{(k-2)}$-expanded algebra (2.4) can be written in the form

$$\begin{aligned}
\left[\mathcal{J}^i_m, \mathcal{J}^j_n\right] &= (m-n)\,\mathcal{J}^{\{i+j\}}_{m+n} + \frac{c_{\{i+j\}+1}}{12}\left(m^3-m\right)\delta_{m+n,0}, \\
\left[\mathcal{J}^i_m, \mathcal{P}^{\bar{j}}_n\right] &= (m-n)\,\mathcal{P}^{\{i+\bar{j}\}}_{m+n} + \frac{c_{\{i+\bar{j}\}+1}}{12}\left(m^3-m\right)\delta_{m+n,0}, \\
\left[\mathcal{P}^{\bar{i}}_m, \mathcal{P}^{\bar{j}}_n\right] &= (m-n)\,\mathcal{J}^{\{\bar{i}+\bar{j}\}}_{m+n} + \frac{c_{\{\bar{i}+\bar{j}\}+1}}{12}\left(m^3-m\right)\delta_{m+n,0},
\end{aligned} \tag{5.2}$$

where $\{\cdots\}$ means

$$\{i+j\} = \begin{cases} i+j & \text{if } i+j \le k-2, \\ (i+j-(k-1))\mathrm{mod}2 \\ \quad +(k-3) & \text{if } i+j > k-2. \end{cases} \tag{5.3}$$

These algebra corresponds to $\mathfrak{vir}_{\mathfrak{D}_k}$, as their subalgebra $\mathfrak{h}$ is given by the $\mathfrak{D}_k$ algebra in 2+1 dimensions [57] and provides with an infinite-dimensional lift of it. Interestingly, this kind of algebras can be written as the direct sum of two copies of the Virasoro algebra and a generalized $\mathfrak{bms}_3$ algebra, i.e., $\mathfrak{vir}_{\mathfrak{D}_k} = \mathfrak{vir}^2 \oplus \mathfrak{vir}_{\mathfrak{B}_{k-2}}$, when an appropriate change of basis is considered. Furthermore, an IW contraction of $\mathfrak{vir}_{\mathfrak{D}_k}$ using the rescaling (4.10) leads to the generalized $\mathfrak{bms}_3$ algebra $\mathfrak{vir}_{\mathfrak{B}_k}$. In the following, a few simple examples will be worked out.

5.1 $\mathfrak{vir}^2 \oplus \mathfrak{bms}_3$

The simplest case to consider[3] is $k=5$, for which (5.1) yields the $\mathfrak{vir}_{\mathfrak{D}_5}$ algebra:

$$\begin{aligned}
[\mathcal{J}_m, \mathcal{J}_n] &= (m-n)\,\mathcal{J}_{m+n} + \frac{c_1}{12}\left(m^3-m\right)\delta_{m+n,0}, \\
[\mathcal{J}_m, \mathcal{P}_n] &= (m-n)\,\mathcal{P}_{m+n} + \frac{c_2}{12}\left(m^3-m\right)\delta_{m+n,0}, \\
[\mathcal{P}_m, \mathcal{P}_n] &= (m-n)\,\mathcal{Z}_{m+n} + \frac{c_3}{12}\left(m^3-m\right)\delta_{m+n,0}, \\
[\mathcal{J}_m, \mathcal{Z}_n] &= (m-n)\,\mathcal{Z}_{m+n} + \frac{c_3}{12}\left(m^3-m\right)\delta_{m+n,0}, \\
[\mathcal{Z}_m, \mathcal{Z}_n] &= (m-n)\,\mathcal{Z}_{m+n} + \frac{c_3}{12}\left(m^3-m\right)\delta_{m+n,0}, \\
[\mathcal{Z}_m, \mathcal{P}_n] &= (m-n)\,\mathcal{R}_{m+n} + \frac{c_4}{12}\left(m^3-m\right)\delta_{m+n,0}, \\
[\mathcal{J}_m, \mathcal{R}_n] &= (m-n)\,\mathcal{R}_{m+n} + \frac{c_4}{12}\left(m^3-m\right)\delta_{m+n,0}, \\
[\mathcal{R}_m, \mathcal{Z}_n] &= (m-n)\,\mathcal{R}_{m+n} + \frac{c_4}{12}\left(m^3-m\right)\delta_{m+n,0}, \\
[\mathcal{R}_m, \mathcal{P}_n] &= (m-n)\,\mathcal{Z}_{m+n} + \frac{c_3}{12}\left(m^3-m\right)\delta_{m+n,0}, \\
[\mathcal{R}_m, \mathcal{R}_n] &= (m-n)\,\mathcal{Z}_{m+n} + \frac{c_3}{12}\left(m^3-m\right)\delta_{m+n,0},
\end{aligned} \tag{5.4}$$

where we have defined

$$\mathcal{J}_m \equiv \mathcal{J}^0_m, \quad \mathcal{P}_m \equiv \mathcal{P}^1_m \quad \mathcal{Z}_m \equiv \mathcal{J}^2_m, \quad \mathcal{R}_m \equiv \mathcal{P}^3_m. \tag{5.5}$$

The Maxwell-like algebra $\mathfrak{D}_5$ in 2 + 1 dimensions [57] is spanned by the generators $\mathcal{J}_0, \mathcal{J}_1, \mathcal{J}_{-1}, \mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_{-1}, \mathcal{Z}_0, \mathcal{Z}_1, \mathcal{Z}_{-1}$ and $\mathcal{R}_0, \mathcal{R}_1, \mathcal{R}_{-1}$. The algebraic structure of the $\mathfrak{vir}_{\mathfrak{D}_5}$ algebra can be made manifest by performing a suitable change of basis. Indeed, the following redefinition:

$$\mathcal{L}_m = \frac{1}{2}\left(\mathcal{R}_m + \mathcal{Z}_m\right), \quad \bar{\mathcal{L}}_{-m} = \frac{1}{2}\left(\mathcal{R}_m - \mathcal{Z}_m\right), \tag{5.6}$$

reproduces the centrally extended 2D-conformal algebra (2.6) with central charges $c = \frac{1}{2}(c_4+c_3)$ and $\bar{c} = \frac{1}{2}(c_4-c_3)$, while

$$\tilde{\mathcal{J}}_m = \frac{1}{2}\left(\mathcal{J}_m - \mathcal{Z}_m\right), \quad \tilde{\mathcal{P}}_m = \frac{1}{2}\left(\mathcal{P}_m - \mathcal{R}_m\right), \tag{5.7}$$

leads to the $\mathfrak{bms}_3$ algebra (2.12) with central charges $c_1 = \frac{1}{2}(c_1-c_3)$ and $c_2 = \frac{1}{2}(c_2-c_4)$. Since the set of generators $\left\{\mathcal{L}_m, \bar{\mathcal{L}}_m\right\}$ commutes with the set $\left\{\tilde{\mathcal{J}}_m, \tilde{\mathcal{P}}_m\right\}$, this shows that the $\mathfrak{vir}_{\mathfrak{D}_5}$ algebra is given by the direct sum of these two subalgebras, namely, $\mathfrak{vir}^2 \oplus \mathfrak{bms}_3$.

5.2 $\mathfrak{vir}^2 \oplus$ deformed $\mathfrak{bms}_3$

In the case $k=6$, the S-expanded algebra $\mathfrak{vir}_{\mathfrak{D}_6}$ is given by

$$\begin{aligned}
[\mathcal{J}_m, \mathcal{J}_n] &= (m-n)\,\mathcal{J}_{m+n} + \frac{c_1}{12}\left(m^3-m\right)\delta_{m+n,0}, \\
[\mathcal{J}_m, \mathcal{P}_n] &= (m-n)\,\mathcal{P}_{m+n} + \frac{c_2}{12}\left(m^3-m\right)\delta_{m+n,0}, \\
[\mathcal{P}_m, \mathcal{P}_n] &= (m-n)\,\mathcal{Z}_{m+n} + \frac{c_3}{12}\left(m^3-m\right)\delta_{m+n,0},
\end{aligned}$$

[3] The semigroup (5.1) is defined for $k > 3$ and $k = 4$ just gives the semigroup $S^{(2)}_{\mathcal{M}}$, which was already studied in Sect. 3.

$$
\begin{aligned}
[\mathcal{J}_m, \mathcal{Z}_n] &= (m-n)\,\mathcal{Z}_{m+n} + \frac{c_3}{12}\left(m^3-m\right)\delta_{m+n,0},\\
[\mathcal{Z}_m, \mathcal{Z}_n] &= (m-n)\,\mathcal{M}_{m+n} + \frac{c_5}{12}\left(m^3-m\right)\delta_{m+n,0},\\
[\mathcal{Z}_m, \mathcal{P}_n] &= (m-n)\,\mathcal{R}_{m+n} + \frac{c_4}{12}\left(m^3-m\right)\delta_{m+n,0},\\
[\mathcal{J}_m, \mathcal{R}_n] &= (m-n)\,\mathcal{R}_{m+n} + \frac{c_4}{12}\left(m^3-m\right)\delta_{m+n,0},\\
[\mathcal{R}_m, \mathcal{Z}_n] &= (m-n)\,\mathcal{R}_{m+n} + \frac{c_4}{12}\left(m^3-m\right)\delta_{m+n,0},\\
[\mathcal{R}_m, \mathcal{P}_n] &= (m-n)\,\mathcal{M}_{m+n} + \frac{c_5}{12}\left(m^3-m\right)\delta_{m+n,0},\\
[\mathcal{R}_m, \mathcal{R}_n] &= (m-n)\,\mathcal{M}_{m+n} + \frac{c_5}{12}\left(m^3-m\right)\delta_{m+n,0},\\
[\mathcal{J}_m, \mathcal{M}_n] &= (m-n)\,\mathcal{M}_{m+n} + \frac{c_5}{12}\left(m^3-m\right)\delta_{m+n,0},\\
[\mathcal{M}_m, \mathcal{P}_n] &= (m-n)\,\mathcal{R}_{m+n} + \frac{c_4}{12}\left(m^3-m\right)\delta_{m+n,0},\\
[\mathcal{M}_m, \mathcal{Z}_n] &= (m-n)\,\mathcal{M}_{m+n} + \frac{c_5}{12}\left(m^3-m\right)\delta_{m+n,0},\\
[\mathcal{M}_m, \mathcal{R}_n] &= (m-n)\,\mathcal{R}_{m+n} + \frac{c_4}{12}\left(m^3-m\right)\delta_{m+n,0},\\
[\mathcal{M}_m, \mathcal{M}_n] &= (m-n)\,\mathcal{M}_{m+n} + \frac{c_5}{12}\left(m^3-m\right)\delta_{m+n,0},
\end{aligned}
\tag{5.8}
$$

where we have defined

$$
\begin{aligned}
&\mathcal{J}_m \equiv \mathcal{J}_m^0, \quad \mathcal{P}_m \equiv \mathcal{P}_m^1, \quad \mathcal{Z}_m \equiv \mathcal{J}_m^2,\\
&\mathcal{R}_m \equiv \mathcal{P}_m^3, \quad \mathcal{M}_m \equiv \mathcal{J}_m^4.
\end{aligned}
\tag{5.9}
$$

The Maxwell-like algebra $\mathfrak{D}_6$ [57] is spanned by the generators $\mathcal{J}_0, \mathcal{J}_1, \mathcal{J}_{-1}, \mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_{-1}, \mathcal{Z}_0, \mathcal{Z}_1, \mathcal{Z}_{-1}, \mathcal{R}_0, \mathcal{R}_1, \mathcal{R}_{-1}$ and $\mathcal{M}_0, \mathcal{M}_1, \mathcal{M}_{-1}$. The algebraic structure of this algebra can be unveiled by performing a suitable change of basis. In fact, two copies of the Virasoro algebra with central charges $c = \frac{1}{2}(c_4 + c_5)$ and $\bar{c} = \frac{1}{2}(c_4 - c_5)$ can be recovered considering the redefinition

$$
\mathcal{L}_m = \frac{1}{2}(\mathcal{R}_m + \mathcal{M}_m), \quad \bar{\mathcal{L}}_{-m} = \frac{1}{2}(\mathcal{R}_m - \mathcal{M}_m). \tag{5.10}
$$

On the other hand, the change of basis

$$
\begin{aligned}
&\check{\mathcal{J}}_m = \frac{1}{2}(\mathcal{J}_m - \mathcal{Z}_m), \quad \check{\mathcal{P}}_m = \frac{1}{2}(\mathcal{P}_m - \mathcal{R}_m),\\
&\check{\mathcal{Z}}_m = \frac{1}{2}(\mathcal{Z}_m - \mathcal{M}_m)
\end{aligned}
\tag{5.11}
$$

reproduces the deformed $\mathfrak{bms}_3$ algebra (3.3) with central charges $c_1 = \frac{1}{2}(c_1 - c_3)$, $c_2 = \frac{1}{2}(c_2 - c_4)$ and $c_3 = \frac{1}{2}(c_3 - c_5)$. Thus, as the generators (5.10) commute with the generators (5.11), the $S_D^{(6)}$-expanded Virasoro algebra $\mathfrak{vir}_{\mathfrak{D}_6}$ is isomorphic to $\mathfrak{vir}^2 \oplus$ deformed $\mathfrak{bms}_3$. This procedure can be generalized for higher values of k, showing that $\mathfrak{vir}_{\mathfrak{D}_k} = \mathfrak{vir}^2 \oplus \mathfrak{vir}_{\mathfrak{B}_{k-2}}$ holds generically.

6 Sugawara construction and expanded Virasoro algebras

The Kač–Moody algebra $\hat{\mathfrak{g}}_k$ corresponds to the central extension of the loop algebra of a semi-simple Lie algebra $\mathfrak{g}$ and is given by

$$
\left[j_m^a, j_n^b\right] = i f^{ab}{}_c j_{m+n}^c + kmg^{ab}\delta_{m,-n}, \tag{6.1}
$$

where $f_{abc} = -f_{bac}$ correspond to the structure constants of $\mathfrak{g}$ and k denotes its central extension. The Sugawara construction allows one to construct a representation of the Virasoro algebra out of bilinear combinations of the generators of the Kač–Moody algebra by defining

$$
\ell_m = \frac{1}{2(k + C_{\mathfrak{g}})} g_{ab} \sum_n : j_n^a j_{m-n}^b :, \tag{6.2}
$$

where $C_{\mathfrak{g}}$ is the dual Coxeter number of $\mathfrak{g}$, g_{ab} is the corresponding Killing–Cartan metric and normal ordering :: is defined as

$$
\sum_n : A_n B_{m-n} := \sum_{n \leq -1} A_n B_{m-n} + \sum_{n > -1} B_{m-n} A_n\,.
$$

In fact, one can easily check that such definition implies that ℓ_m has conformal weight one, $\left[\ell_m, j_n^a\right] = -n j_{m+n}^a$, and satisfies the Virasoro algebra (2.1) with central charge

$$
c = \frac{k \dim \mathfrak{g}}{k + C_{\mathfrak{g}}}, \tag{6.3}
$$

where $\dim \mathfrak{g} = g_{ab} g^{ab}$ is the dimension of $\mathfrak{g}$.

6.1 Modified Sugawara construction

The modified Sugawara construction consists in defining new Virasoro generators

$$
\tilde{\ell}_m = \ell_m + img_{ab}\alpha^a j_m^b + \frac{1}{2}k\alpha^2\delta_{m,0},
$$

where $\alpha \in \mathfrak{g}$. Provided the generators ℓ_m satisfy $\mathfrak{vir}$ with central charge c, the modified generators $\tilde{\ell}_n$ form a new representation of the Virasoro algebra, i.e.

$$
\left[\tilde{\ell}_m, \tilde{\ell}_n\right] = (m-n)\tilde{\ell}_{m+n} + \frac{\tilde{c}}{12}\left(m^3 - m\right)\delta_{m,-n},
$$

with a shifted central charge given by

$$
\tilde{c} = c + 12k\alpha^2\,.
$$

In the following, we will show how the expanded Virasoro algebras presented in the previous sections can be obtained from Kač–Moody algebras associated with the $\mathfrak{B}_k$ and $\mathfrak{C}_k$ algebras through generalized (modified) Sugawara constructions.

6.2 $\mathfrak{bms}_3$ and the Sugawara construction

Let us consider the following Kač–Moody-like algebra with a semi-direct product structure:

$$\begin{aligned}
\left[j_m^a, j_n^b\right] &= i f^{ab}{}_c j_{m+n}^c + k_1 m g^{ab} \delta_{m,-n}, \\
\left[j_m^a, p_n^b\right] &= i f^{ab}{}_c p_{m+n}^c + k_2 m g^{ab} \delta_{m,-n}, \\
\left[p_m^a, p_n^b\right] &= 0,
\end{aligned} \tag{6.4}$$

which can be obtained from an S-expansion of (6.1) using the semigroup $S_E^{(1)}$ (see the appendix). Now we introduce the following quadratic combinations of the generators j_m^a and p_m^a:

$$\begin{aligned}
\mathcal{P}_m &= \frac{1}{2k_2} g_{ab} \sum_n : p_n^a p_{m-n}^b :, \\
\mathcal{J}_m &= \frac{1}{2k_2} g_{ab} \sum_n : \left(j_n^a p_{m-n}^b + p_n^a j_{m-n}^b\right) : - \frac{k_1 + 2C_{\mathfrak{g}}}{k_2} \mathcal{P}_m .
\end{aligned}$$

Using the affine current algebra (6.4) it is easy to see that

$$\begin{aligned}
\left[\mathcal{J}_m, j_n^a\right] &= -n j_{m+n}^a, \\
\left[\mathcal{J}_m, p_n^a\right] &= -n p_{m+n}^a, \\
\left[\mathcal{P}_m, j_n^a\right] &= -n p_{m+n}^a, \\
\left[\mathcal{P}_m, p_n^a\right] &= 0,
\end{aligned} \tag{6.5}$$

and that $\mathcal{J}_m$ and $\mathcal{P}_m$ satisfy the commutation relations

$$\begin{aligned}
\left[\mathcal{J}_m, \mathcal{J}_n\right] &= (m-n)\mathcal{J}_{m+n} + \frac{\dim\mathfrak{g}}{6}\left(m^3 - m\right)\delta_{m,-n}, \\
\left[\mathcal{J}_m, \mathcal{P}_n\right] &= (m-n)\mathcal{P}_{m+n}, \\
\left[\mathcal{P}_m, \mathcal{P}_n\right] &= 0,
\end{aligned} \tag{6.6}$$

which corresponds to the $\mathfrak{bms}_3$ algebra (2.12) with central charges $c_1 = 2\dim\mathfrak{g}$ and $c_2 = 0$. The central charge c_1 is familiar from the study of abelian Kač–Moody algebras [1] and manifests here due to the abelian ideal generated by p_m^a. Now we can use the modified Sugawara construction to obtain the fully centrally extended $\mathfrak{bm}3_3$ algebra from (6.4). Indeed defining new generators:

$$\begin{aligned}
\widetilde{\mathcal{J}}_m &= \mathcal{J}_m + i m g_{ab} \alpha^a j_m^b + \frac{1}{2} k_1 \alpha^2 \delta_{m,0}, \\
\widetilde{\mathcal{P}}_m &= \mathcal{P}_m + i m g_{ab} \alpha^a p_m^b + \frac{1}{2} k_2 \alpha^2 \delta_{m,0},
\end{aligned}$$

one can show that

$$\begin{aligned}
\left[\widetilde{\mathcal{J}}_m, \widetilde{\mathcal{J}}_n\right] &= (m-n)\widetilde{\mathcal{J}}_{m+n} + \frac{\tilde{c}_1}{12}\left(m^3 - m\right)\delta_{m,-n}, \\
\left[\widetilde{\mathcal{J}}_m, \widetilde{\mathcal{P}}_n\right] &= (m-n)\widetilde{\mathcal{P}}_{m+n} + \frac{\tilde{c}_2}{12}\left(m^3 - m\right)\delta_{m,-n}, \\
\left[\widetilde{\mathcal{P}}_m, \widetilde{\mathcal{P}}_n\right] &= 0,
\end{aligned}$$

where

$$\begin{aligned}
\tilde{c}_1 &= 2\dim\mathfrak{g} + 12 k_1 \alpha^2, \\
\tilde{c}_2 &= 12 k_2 \alpha^2.
\end{aligned}$$

This result can be understood as the quantum version of the Sugawara construction described in [22,37] where $\mathfrak{bms}_3$ is realized as a Poisson algebra for the central charges of asymptotically flat three-dimensional Einstein gravity.

6.3 $\mathfrak{vir}^2$ algebra

The $\mathfrak{bms}_3$ algebra can also be obtained from the Sugawara construction associated with a $\mathbb{Z}_2$-expansion of the Kač–Moody algebra, after an IW contraction. In fact, using the semigroup $\mathbb{Z}_2 = S_{\mathcal{M}}^{(1)}$ to expand (6.1) (see the appendix), we get

$$\begin{aligned}
\left[j_m^a, j_n^b\right] &= i f^{ab}{}_c j_{m+n}^c + k_1 m g^{ab} \delta_{m,-n}, \\
\left[j_m^a, p_n^b\right] &= i f^{ab}{}_c p_{m+n}^c + k_2 m g^{ab} \delta_{m,-n}, \\
\left[p_m^a, p_n^b\right] &= i f^{ab}{}_c j_{m+n}^c + k_1 m g^{ab} \delta_{m,-n}.
\end{aligned} \tag{6.7}$$

Redefining the generators as $j_m^a = l_m^a + \bar{l}_{-m}^a$ and $p_m^a = l_m^a - \bar{l}_{-m}^a$, this algebra can be written as the product of two identical Kač–Moody algebras with levels $k = \frac{1}{2}(k_1 + k_2)$ and $\bar{k} = \frac{1}{2}(k_1 - k_2)$, i.e.,

$$\begin{aligned}
\left[l_m^a, l_n^b\right] &= i f^{ab}{}_c l_{m+n}^c + k m g^{ab} \delta_{m,-n}, \\
\left[\bar{l}_m^a, \bar{l}_n^b\right] &= i f^{ab}{}_c \bar{l}_{m+n}^c + \bar{k} m g^{ab} \delta_{m,-n}, \\
\left[l_m^a, \bar{l}_n^b\right] &= 0 .
\end{aligned}$$

This means that, considering two independent Sugawara constructions

$$\begin{aligned}
\ell_m &= \frac{1}{2(k + C_{\mathfrak{g}})} g_{ab} \sum_n : l_n^a l_{m-n}^b :, \\
\bar{\ell}_m &= \frac{1}{2(\bar{k} + C_{\mathfrak{g}})} g_{ab} \sum_n : \bar{l}_n^a \bar{l}_{m-n}^b :,
\end{aligned}$$

one can trivially obtain the $\mathfrak{vir}^2$ algebra (2.6) with central charges $c = \frac{k\dim\mathfrak{g}}{k + C_{\mathfrak{g}}}$ and $\bar{c} = \frac{\bar{k}\dim\mathfrak{g}}{\bar{k} + C_{\mathfrak{g}}}$. Using (2.10), one can define

$$\begin{aligned}
\mathcal{P}_m &= \frac{\sigma}{2(k + C_{\mathfrak{g}})} g_{ab} \sum_n : \left(l_n^a l_{m-n}^b + \mu \bar{l}_n^a \bar{l}_{-m-n}^b\right) :, \\
\mathcal{J}_m &= \frac{1}{2(k + C_{\mathfrak{g}})} g_{ab} \sum_n : \left(l_n^a l_{m-n}^b - \mu \bar{l}_n^a \bar{l}_{-m-n}^b\right) :,
\end{aligned} \tag{6.8}$$

where $\mu = \frac{k+C_{\mathfrak{g}}}{\bar{k}+C_{\mathfrak{g}}}$. The bilinears (6.8) satisfy the $\mathfrak{bms}_3$ algebra (2.12) in the limit $\sigma \to \infty$ with central charges $c_1 = \frac{(k-\mu\bar{k})}{k+C_{\mathfrak{g}}}\dim\mathfrak{g}$ and $c_2 = \frac{(k+\mu\bar{k})}{k+C_{\mathfrak{g}}}\dim\mathfrak{g}$.

6.4 Deformed $\mathfrak{bms}_3$ algebra from a Sugawara construction

The Sugawara construction presented before can be generalized in order to recover the deformed $\mathfrak{bms}_3$ algebra (3.3) from an expanded Kač–Moody algebra. In this case we introduce the following deformed current algebra:

$$
\begin{aligned}
\left[j_m^a, j_n^b\right] &= if^{ab}{}_c j_{m+n}^c + k_1 m g^{ab}\delta_{m,-n},\\
\left[j_m^a, p_n^b\right] &= if^{ab}{}_c p_{m+n}^c + k_2 m g^{ab}\delta_{m,-n},\\
\left[j_m^a, z_n^b\right] &= if^{ab}{}_c z_{m+n}^c + k_3 m g^{ab}\delta_{m,-n},\\
\left[p_m^a, p_n^b\right] &= if^{ab}{}_c z_{m+n}^c + k_3 m g^{ab}\delta_{m,-n},\\
\left[p_m^a, z_n^b\right] &= 0,\\
\left[z_m^a, z_n^b\right] &= 0,
\end{aligned} \tag{6.9}
$$

which corresponds to an S-expansion of the Kač–Moody algebra (6.4) with the semigroup $S_E^{(2)}$ given in (3.1) (see the appendix). Now we define the following quadratic combinations of its generators:

$$
\begin{aligned}
\mathcal{Z}_m &= \frac{1}{2k_3} g_{ab}\sum_n : z_n^a z_{m-n}^b :,\\
\mathcal{P}_m &= \frac{1}{2k_3} g_{ab}\sum_n : \left(p_n^a z_{m-n}^b + z_n^a p_{m-n}^b\right) : -\frac{k_2}{k_3}\mathcal{Z}_m,\\
\mathcal{J}_m &= \frac{1}{2k_3} g_{ab}\sum_n : \left(p_n^a p_{m-n}^b + j_n^a z_{m-n}^b + z_n^a j_{m-n}^b\right)\\
&\quad : -\frac{k_2}{k_3}\mathcal{P}_m - \frac{k_1+3C_{\mathfrak{g}}}{k_3}\mathcal{Z}_m.
\end{aligned} \tag{6.10}
$$

The commutators of $\mathcal{J}_m$, $\mathcal{P}_m$ and $\mathcal{Z}_m$ with the generators of (6.9) read

$$
\begin{aligned}
&\left[\mathcal{J}_m, j_n^a\right] = -n j_{m+n}^a, \quad \left[\mathcal{P}_m, j_n^a\right] = -n p_{m+n}^a,\\
&\qquad\qquad \left[\mathcal{Z}_m, j_n^a\right] = -n z_{m+n}^a,\\
&\left[\mathcal{J}_m, p_n^a\right] = -n p_{m+n}^a, \quad \left[\mathcal{P}_m, p_n^a\right] = -n z_{m+n}^a,\\
&\qquad\qquad \left[\mathcal{Z}_m, p_n^a\right] = 0,\\
&\left[\mathcal{J}_m, z_n^a\right] = -n z_{m+n}^a, \quad \left[\mathcal{P}_m, z_n^a\right] = 0, \quad \left[\mathcal{Z}_m, z_n^a\right] = 0s.
\end{aligned} \tag{6.11}
$$

Using these relations the algebra of the bilinears (6.10) turns out to be given by

$$
\begin{aligned}
\left[\mathcal{J}_m, \mathcal{J}_n\right] &= (m-n)\mathcal{J}_{m+n} + \frac{3\dim\mathfrak{g}}{12}\left(m^3-m\right)\delta_{m,-n},\\
\left[\mathcal{J}_m, \mathcal{P}_n\right] &= (m-n)\mathcal{P}_{m+n},\\
\left[\mathcal{J}_m, \mathcal{Z}_n\right] &= (m-n)\mathcal{Z}_{m+n},\\
\left[\mathcal{P}_m, \mathcal{P}_n\right] &= (m-n)\mathcal{Z}_{m+n},\\
\left[\mathcal{P}_m, \mathcal{Z}_n\right] &= 0,\\
\left[\mathcal{Z}_m, \mathcal{Z}_n\right] &= 0,
\end{aligned} \tag{6.12}
$$

which corresponds to the deformed $\mathfrak{bms}_3$ algebra (3.3) with central charges $c_1 = 3\dim\mathfrak{g}$, $c_2 = c_3 = 0$. In order to obtain the fully centrally extended deformed $\mathfrak{bms}_3$ algebra from the deformed affine current algebra (6.9), we introduce the following modified Sugawara construction:

$$
\begin{aligned}
\widetilde{\mathcal{J}}_m &= \mathcal{J}_m + i m g_{ab}\alpha^a j_m^b + \frac{1}{2}k_1\alpha^2\delta_{m,0},\\
\widetilde{\mathcal{P}}_m &= \mathcal{P}_m + i m g_{ab}\alpha^a p_m^b + \frac{1}{2}k_2\alpha^2\delta_{m,0},\\
\widetilde{\mathcal{Z}}_m &= \mathcal{Z}_m + i m g_{ab}\alpha^a z_m^b + \frac{1}{2}k_3\alpha^2\delta_{m,0}.
\end{aligned} \tag{6.13}
$$

These generators satisfy the deformed $\mathfrak{bms}_3$ algebra,

$$
\begin{aligned}
\left[\widetilde{\mathcal{J}}_m, \widetilde{\mathcal{J}}_n\right] &= (m-n)\widetilde{\mathcal{J}}_{m+n} + \frac{\tilde{c}_1}{12}\left(m^3-m\right)\delta_{m,-n},\\
\left[\widetilde{\mathcal{J}}_m, \widetilde{\mathcal{P}}_n\right] &= (m-n)\widetilde{\mathcal{P}}_{m+n} + \frac{\tilde{c}_2}{12}\left(m^3-m\right)\delta_{m,-n},\\
\left[\widetilde{\mathcal{J}}_m, \widetilde{\mathcal{Z}}_n\right] &= (m-n)\widetilde{\mathcal{Z}}_{m+n} + \frac{\tilde{c}_3}{12}\left(m^3-m\right)\delta_{m,-n},\\
\left[\widetilde{\mathcal{P}}_m, \widetilde{\mathcal{P}}_n\right] &= (m-n)\widetilde{\mathcal{Z}}_{m+n} + \frac{\tilde{c}_3}{12}\left(m^3-m\right)\delta_{m,-n},\\
\left[\widetilde{\mathcal{P}}_m, \widetilde{\mathcal{Z}}_n\right] &= 0,\\
\left[\widetilde{\mathcal{Z}}_m, \widetilde{\mathcal{Z}}_n\right] &= 0\,,
\end{aligned}
$$

where the central charges are given by

$$
\begin{aligned}
\tilde{c}_1 &= 3\dim\mathfrak{g} + 12k_1\alpha^2,\\
\tilde{c}_2 &= 12k_2\alpha^2,\\
\tilde{c}_3 &= 12k_3\alpha^2\,.
\end{aligned}
$$

6.5 $\mathfrak{vir}^3$ algebra

The deformed $\mathfrak{bms}_3$ algebra can also be obtained as an IW contraction of the Sugawara construction associated with an S-expansion of the Kač–Moody algebra using the semigroup $S_{\mathcal{M}}^{(2)}$. In fact the $S_{\mathcal{M}}^{(2)}$-expanded Kač–Moody algebra is given by (see the appendix)

$$
\begin{aligned}
\left[j_m^a, j_n^b\right] &= if^{ab}{}_c j_{m+n}^c + k_1 m g^{ab}\delta_{m,-n},\\
\left[j_m^a, p_n^b\right] &= if^{ab}{}_c p_{m+n}^c + k_2 m g^{ab}\delta_{m,-n},\\
\left[j_m^a, z_n^b\right] &= if^{ab}{}_c z_{m+n}^c + k_3 m g^{ab}\delta_{m,-n},\\
\left[p_m^a, p_n^b\right] &= if^{ab}{}_c z_{m+n}^c + k_3 m g^{ab}\delta_{m,-n},\\
\left[z_m^a, p_n^b\right] &= if^{ab}{}_c p_{m+n}^c + k_2 m g^{ab}\delta_{m,-n},\\
\left[z_m^a, z_n^b\right] &= if^{ab}{}_c z_{m+n}^c + k_3 m g^{ab}\delta_{m,-n},
\end{aligned} \tag{6.14}
$$

which, through the redefinitions $z_m^a = l_m^a + \bar{l}_{-m}^a$, $p_m^a = l_m^a - \bar{l}_{-m}^a$ and $j_m^a = \tilde{l}_m^a + l_m^a + \bar{l}_{-m}^a$, can be written as the direct product of three identical commuting Kač–Moody algebras with levels $k = \frac{k_3+k_2}{2}$, $\bar{k} = \frac{k_3-k_2}{2}$ and $\tilde{k} = k_1 - k_3$:

$$\begin{aligned}
\left[l_m^a, l_n^b\right] &= i f^{ab}{}_c l_{m+n}^c + k m g^{ab} \delta_{m,-n}, \\
\left[\bar{l}_m^a, \bar{l}_n^b\right] &= i f^{ab}{}_c \bar{l}_{m+n}^c + \bar{k} m g^{ab} \delta_{m,-n}, \\
\left[\tilde{l}_m^a, \tilde{l}_n^b\right] &= i f^{ab}{}_c \tilde{l}_{m+n}^c + \tilde{k} m g^{ab} \delta_{m,-n}\,.
\end{aligned}$$

Therefore, considering three independent Sugawara constructions

$$\begin{aligned}
\ell_m &= \frac{1}{2(k + C_{\mathfrak{g}})} g_{ab} \sum_n : l_n^a l_{m-n}^b :, \\
\bar{\ell}_m &= \frac{1}{2(\bar{k} + C_{\mathfrak{g}})} g_{ab} \sum_n : \bar{l}_n^a \bar{l}_{m-n}^b :, \\
\tilde{\ell}_m &= \frac{1}{2(\tilde{k} + C_{\mathfrak{g}})} g_{ab} \sum_n : \tilde{l}_n^a \tilde{l}_{m-n}^b :,
\end{aligned}$$

one can trivially obtain the $\mathfrak{vir}^3$ algebra (3.8) with central charges $c = \frac{k \dim \mathfrak{g}}{k+C_{\mathfrak{g}}}$, $\bar{c} = \frac{\bar{k} \dim \mathfrak{g}}{\bar{k}+C_{\mathfrak{g}}}$ and $\tilde{c} = \frac{\tilde{k} \dim \mathfrak{g}}{\tilde{k}+C_{\mathfrak{g}}}$. This means that, using the relation (3.7), one can define

$$\begin{aligned}
\mathcal{Z}_m &= \frac{\sigma^2}{2(k + C_{\mathfrak{g}})} g_{ab} \sum_n : \left(l_n^a l_{m-n}^b - \mu \bar{l}_n^a \bar{l}_{-m-n}^b \right) :, \\
\mathcal{P}_m &= \frac{\sigma}{2(k + C_{\mathfrak{g}})} g_{ab} \sum_n : \left(l_n^a l_{m-n}^b + \mu \bar{l}_n^a \bar{l}_{-m-n}^b \right) :, \\
\mathcal{J}_m &= \frac{1}{2(k + C_{\mathfrak{g}})} g_{ab} \sum_n : \left(l_n^a l_{m-n}^b - \mu \bar{l}_n^a \bar{l}_{-m-n}^b + \nu \tilde{l}_n^a \tilde{l}_{-m-n}^b \right) :,
\end{aligned} \tag{6.15}$$

where $\mu = \frac{k+C_{\mathfrak{g}}}{\bar{k}+C_{\mathfrak{g}}}$ and $\nu = \frac{k+C_{\mathfrak{g}}}{\tilde{k}+C_{\mathfrak{g}}}$. It is easy to verify that the bilinear combinations (6.15) satisfy the deformed $\mathfrak{bms}_3$ algebra (3.3) in the limit $\sigma \to \infty$ with central charges $c_1 = \frac{(k-\mu\bar{k})}{k+C_{\mathfrak{g}}} \dim \mathfrak{g}$, $c_2 = \frac{(k+\mu\bar{k})}{k+C_{\mathfrak{g}}} \dim \mathfrak{g}$ and $c_3 = \frac{\left(k+\nu\tilde{k}\right)}{k+C_{\mathfrak{g}}} \dim \mathfrak{g}$.

6.6 Generalization

Following the same steps as described above, one can in principle always find a generalized (modified) Sugawara construction that, given a semigroup S, allows one to pass from the S-expanded Kač–Moody algebra to the corresponding S-expanded Virasoro algebra. As we have seen, the Sugawara construction for the $\mathfrak{bms}_3$ algebra and for the deformed $\mathfrak{bms}_3$ algebra are quite cumbersome and therefore their generalization for $\mathfrak{vir}_{\mathfrak{B}_k}$ with $k > 4$ will not be given here. In the case of the generalized conformal algebras $\mathfrak{vir}_{\mathfrak{C}_k}$, the Sugawara constructions presented here have been somewhat straightforward, as the cases $k = 3$ and $k = 4$ correspond the direct product of two and three copies of the Virasoro algebra, respectively. However, as we have stressed in Sect. 4.2, for $k > 4$ it is not true anymore that the $\mathfrak{vir}_{\mathfrak{C}_k}$ algebras can be written as products of single copies of the Virasoro algebra and therefore the Sugawara construction will be more complicated.

7 Comments and further developments

In this paper we have presented the general setup to obtain new infinite-dimensional algebras by applying the S-expansion method to the Virasoro algebra. Interestingly, the algebras obtained here contain known finite algebras as subalgebras and inherit the way they are related between each other. Indeed, the following diagram summarizes the IW contractions that relate the Poincaré, AdS, Maxwell and AdS–Lorentz algebras in $2+1$ dimensions as well as their relation with the Lorentz algebra through different S-expansions:

$$\begin{tikzcd}
\boxed{AdS_3} \arrow[dd, "\text{IW}"] & & \boxed{AdS_3 \oplus \mathcal{L}orentz} \arrow[dd, "\text{IW}"] \\
 & \boxed{\mathcal{L}orentz} \arrow[ul, "\mathbb{Z}_2"'] \arrow[ur, "S_{\mathcal{M}}^{(2)}"] \arrow[dl, "S_E^{(1)}"] \arrow[dr, "S_E^{(2)}"'] & \\
\boxed{\text{Poincaré}} \arrow[rr, "\text{Deformation}", "+\text{ Enlargement}"'] & & \boxed{\text{Maxwell}}
\end{tikzcd}$$

In the first part of this article we have shown that the centrally extended 2D-conformal algebra $\mathfrak{vir}^2$ as well as the $\mathfrak{bms}_3$ algebra can be obtained as S-expansions of the Virasoro algebra using the semigroups $\mathbb{Z}_2$ and $S_E^{(1)}$, respectively. Subsequently we showed that, using the semigroups $S_{\mathcal{M}}^{(2)}$ and $S_E^{(2)}$, the S-expansion leads to three copies of the Virasoro algebra in the former case and to a deformed $\mathfrak{bms}_3$ algebra in the latter case. These algebras correspond to infinite-dimensional lifts of the AdS–Lorentz and Maxwell algebras and, furthermore, the deformed $\mathfrak{bms}_3$ algebra can be obtained as an IW contraction of $\mathfrak{vir}^3$. This means that the infinite-dimensional symmetries presented here satisfy the same IW contraction and expansion relations as their finite-dimensional subalgebras presented in the previous diagram, i.e.,

$$\begin{tikzcd}
\boxed{\mathfrak{vir}^2} \arrow[dd, "\text{IW}"] & & \boxed{\mathfrak{vir}^3} \arrow[dd, "\text{IW}"] \\
 & \boxed{\mathfrak{vir}} \arrow[ul, "\mathbb{Z}_2"'] \arrow[ur, "S_{\mathcal{M}}^{(2)}"] \arrow[dl, "S_E^{(1)}"] \arrow[dr, "S_E^{(2)}"'] & \\
\boxed{\mathfrak{bms}_3} \arrow[rr, "\text{Deformation}", "+\text{ Enlargement}"'] & & \boxed{\text{Deformed } \mathfrak{bms}_3}
\end{tikzcd}$$

In Sect. 4, we have generalized the previous results by considering expansions of the Virasoro algebra with the semigroups $S_{\mathcal{M}}^{(k-2)}$ and $S_E^{(k-2)}$ to obtain two sets of families of infinite-dimensional algebras that we have called generalized $\mathfrak{bms}_3$ algebras and generalized 2D-conformal algebras. These families are denoted, respectively, by $\mathfrak{vir}_{\mathfrak{C}_k}$ and $\mathfrak{vir}_{\mathfrak{C}_k}$, and reduce to the infinite-dimensional algebras previously discussed for $k = 3$ and $k = 4$. Furthermore, they turn out to be related by an IW contraction for every value of k

$$\begin{tikzcd} & \boxed{\text{Generalized 2D-conformal algebra } (\mathfrak{vir}_{\mathfrak{C}_k})} \arrow[dd, "\text{IW}"] \\ \boxed{\mathfrak{vir}} \arrow[ur, "S_{\mathcal{M}}^{(k-2)}"] \arrow[dr, "S_{E}^{(k-2)}"'] & \\ & \boxed{\text{Generalized } \mathfrak{bms}_3 \text{ algebra } (\mathfrak{vir}_{\mathfrak{B}_k})} \end{tikzcd}$$

In Sect. 5 we have introduced another family of infinite-dimensional algebras, $\mathfrak{vir}_{\mathfrak{D}_k}$, which can be obtained by expanding the Virasoro algebra using the semigroup $S_D^{(k-2)}$, and showed the simplest examples explicitly. These algebras can always be written as the direct product $\mathfrak{vir}^2 \oplus \mathfrak{B}_{k-2}$ after a suitable change of basis.

In Sect. 6 the Sugawara construction has been applied to expanded Kač–Moody algebras to obtain the expanded Virasoro algebras and the cases $k = 3$ and $k = 4$ have been worked out explicitly. This result is remarkable as it means that these new infinite-dimensional symmetries could be related to some kind of generalized WZW theories whose current algebras are given by expanded Kač–Moody algebras. In that case the algebras $\mathfrak{vir}_{\mathfrak{B}_k}$ or $\mathfrak{vir}_{\mathfrak{C}_k}$ should be recovered as the Poisson algebras for the stress-energy momentum tensor components in the very same way as it happens for $\mathfrak{vir}^2$ and $\mathfrak{bms}_3$.

In the context of gravity, upon imposing suitable boundary conditions, the algebras $\mathfrak{vir}^2$ and $\mathfrak{bms}_3$ appear as the asymptotic symmetries of asymptotically AdS and Asymptotically flat three-dimensional Einstein gravity, respectively. We conjecture that the new infinite-dimensional algebras $\mathfrak{vir}_{\mathfrak{B}_k}$, $\mathfrak{vir}_{\mathfrak{C}_k}$ and $\mathfrak{vir}_{\mathfrak{D}_k}$ obtained here correspond the asymptotic symmetries of 3D gravity theories invariant under the algebras $\mathfrak{B}_k$, $\mathfrak{C}_k$ or $\mathfrak{D}_k$ when suitable boundary conditions for the fields content are adopted. These theories of gravity can be straightforwardly constructed by considering Chern–Simons actions invariant under these algebras. This will be the subject of a subsequent article.

On the other hand, it is well known that the KdV system possesses a Virasoro symmetry related to the KdV hierarchy [86]. This result can be used to construct an infinite set of boundary conditions for 3D gravity [87]. Along this line it would be interesting to evaluate the existence of integrable systems associated with expanded Virasoro symmetries and they hierarchies as well as their relations to boundary conditions for gravity theories invariant under the algebras $\mathfrak{B}_k$ or $\mathfrak{C}_k$.

Another natural generalization of our results is to extend the expansion method to $\mathcal{N}$-extended supersymmetric extension of asymptotic symmetries. In particular, it would be interesting to study S-expanded super Virasoro symmetries. However, this would require a more subtle treatment than the one introduced here. Indeed, one cannot naively consider the expansion of a super Virasoro structure. The general setup and the respective supergravity models will be presented in a future paper. As an ending remark: it would be worth exploring the expansion procedure to higher spin extension of gravity theories in $2 + 1$ dimensions.

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Appendix

A Generalized Kač–Moody algebras

Let us consider the Kač–Moody algebra $\hat{\mathfrak{g}}_k$,

$$\left[j_m^a, j_n^b \right] = i f^{ab}{}_c j_{m+n}^c + k m g^{ab} \delta_{m,-n}, \tag{A.1}$$

which corresponds to the central extension of the loop algebra of a semi-simple Lie algebra $\mathfrak{g}$. One can show that two families of Kač–Moody-like algebras can be obtained applying diverse semigroups S to $\hat{\mathfrak{g}}_k$. Let $S_E^{(k-2)} = \{\lambda_0, \lambda_1, \ldots, \lambda_{k-1}\}$ be the finite abelian semigroup whose elements satisfy (4.1) and $\lambda_{k-1} = 0_s$ is the zero element of the semigroup. Then the $S_E^{(k-2)}$-expanded algebra is given by

$$\left[j_{(m,\alpha)}^a, j_{(n,\beta)}^b \right] = \begin{cases} i f^{ab}{}_c j_{(m+n,\alpha+\beta)}^c \\ \quad + k_{\alpha+\beta+1} m g^{ab} \delta_{m,-n} & \text{if} \quad \alpha + \beta \leq k-2 \\ 0 & \text{if} \quad \alpha + \beta > k-2 \end{cases} \tag{A.2}$$

where we have defined $k_{\alpha+\beta+1} \equiv k_{\alpha\beta} = k K_{\alpha\beta}^{\gamma} \lambda_\gamma$. One can see that the $S_E^{(k-2)}$-expanded algebras always contain an abelian ideal generated by the set

$$\mathcal{A} = \left\{ j_{(m,\tilde{\alpha})}^a \right\}, \quad \tilde{\alpha} = \left[\frac{k}{2} \right], \ldots, k-2,$$

and for which

$$\left[j_{(m,\tilde{\alpha})}^a, j_{\left(m,\tilde{\beta}\right)}^b \right] = 0,$$

$$\left[j^a_{(m,\tilde{\alpha})}, j^b_{(m,\alpha)}\right] \in \mathcal{A} + \text{central terms.} \tag{A.3}$$

In particular, the Kač–Moody-like structure appears by redefining the generators in the form

$$\begin{aligned} j_m^{a(i)} &\equiv j^a_{(m,i)} = \lambda_i \ell_m, \\ p_m^{a(\bar{\imath})} &\equiv j^a_{(m,\bar{\imath})} = \lambda_{\bar{\imath}} \ell_m, \end{aligned} \tag{A.4}$$

where i takes even values and $\bar{\imath}$ takes odd values. This allows one to write the $S_E^{(k-2)}$-expanded algebras in the form

$$\begin{aligned} \left[j_m^{a(i)}, j_n^{b(j)}\right] &= i f^{ab}{}_c j^{c(i+j)}_{m+n} \\ &\quad + k_{i+j+1} m g^{ab} \delta_{m,-n} \quad \text{for } i+j \le k-2, \\ \left[j_m^{a(i)}, p_n^{b(\bar{\imath})}\right] &= i f^{ab}{}_c p^{c(i+\bar{\imath})}_{m+n} \\ &\quad + k_{i+\bar{\imath}+1} m g^{ab} \delta_{m,-n} \quad \text{for } i+\bar{\imath} \le k-2, \\ \left[p_m^{a(\bar{\imath})}, p_n^{b(\bar{\jmath})}\right] &= i f^{ab}{}_c j^{c(\bar{\imath}+\bar{\jmath})}_{m+n} \\ &\quad + k_{\bar{\imath}+\bar{\jmath}+1} m g^{ab} \delta_{m,-n} \quad \text{for } \bar{\imath}+\bar{\jmath} \le k-2, \\ \text{others} &= 0. \end{aligned} \tag{A.5}$$

Let us note that, for $k = 3$, the semigroup corresponds to the $S_E^{(1)}$ whose elements satisfy (2.11) and the commutation relations (A.5) reduce to the affine current algebra given by (6.4). The case $k = 4$ reproduce the $S_E^{(2)}$-expanded algebra whose generators satisfy (6.9).

An alternative family of generalized Kač–Moody algebras can be obtained applying the $S_{\mathcal{M}}^{(k-2)} = \{\lambda_0, \lambda_1, \ldots, \lambda_{k-2}\}$ semigroup to $\hat{\mathfrak{g}}_k$. Considering the multiplication law of the semigroup (4.6) one can show that the $S_{\mathcal{M}}^{(k-2)}$-expanded algebra takes the form

$$\begin{aligned} &\left[j^a_{(m,\alpha)}, j^b_{(n,\beta)}\right] \\ &= \begin{cases} i f^{ab}{}_c j^c_{(m+n,\alpha+\beta)} + k_{\alpha+\beta+1} m g^{ab} \delta_{m,-n} & \text{if } \alpha+\beta \le k-2, \\ i f^{ab}{}_c j^c_{\left(m+n,\alpha+\beta-2\left[\frac{k-1}{2}\right]\right)} \\ \quad + k_{\alpha+\beta-2\left[\frac{k-1}{2}\right]+1} m g^{ab} \delta_{m,-n} & \text{if } \alpha+\beta > k-2. \end{cases} \end{aligned} \tag{A.6}$$

One can redefine the generators in the form (A.4) leading to the following generalized affine current algebra:

$$\begin{aligned} \left[j_m^{a(i)}, j_n^{b(j)}\right] &= i f^{ab}{}_c j^{c\{i+j\}}_{m+n} + k_{\{i+j\}+1} m g^{ab} \delta_{m,-n}, \\ \left[j_m^{a(i)}, p_n^{b(\bar{\imath})}\right] &= i f^{ab}{}_c p^{c\{i+\bar{\imath}\}}_{m+n} + k_{\{i+\bar{\imath}\}+1} m g^{ab} \delta_{m,-n}, \\ \left[p_m^{a(\bar{\imath})}, p_n^{b(\bar{\jmath})}\right] &= i f^{ab}{}_c j^{c\{\bar{\imath}+\bar{\jmath}\}}_{m+n} + k_{\{\bar{\imath}+\bar{\jmath}\}+1} m g^{ab} \delta_{m,-n}, \end{aligned} \tag{A.7}$$

where $\{\cdots\}$ means the following:

$$\{i+j\} = \begin{cases} i+j & \text{if } i+j \le k-2, \\ i+j-2\left[\frac{k-1}{2}\right] & \text{if } i+j > k-2. \end{cases} \tag{A.8}$$

Interestingly the $k = 3$ and $k = 4$ cases reproduce two and three copies of Kač–Moody algebras, respectively. However, for $k \ge 5$ the commutation relations of the generalized Kač–Moody algebra obtained here become non-trivial and are given by (A.7).

It is important to mention that the two families of generalized Kač–Moody algebras presented here are related through the IW contraction. Indeed, considering the rescaling of the generators satisfying a $S_{\mathcal{M}}^{(k-2)}$-expanded algebra (A.7)

$$\begin{aligned} &j_m^0 \to j_m^0, \; j_m^i \to \sigma^i j_m^i, \; p_m^{\bar{\imath}} \to \sigma^{\bar{\imath}} p_m^{\bar{\imath}}, \\ &k_1 \to k_1, k_{i+1} \to \sigma^i k_{i+1}, k_{\bar{\imath}+1} \to \sigma^{\bar{\imath}} k_{\bar{\imath}+1}, \end{aligned}$$

the limit $\sigma \to \infty$ leads to the $S_E^{(k-2)}$-expanded algebra (A.5).

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