

Generalizing trigonometric functions from different points of view

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In this survey we shall explore (one definition of) generalized trigonometric functions from different standpoints and illustrate the roles they play in various branches of mathematics. We start from the analytic point of view and for each $p \in (1, \infty)$ introduce a function \sin_p^{-1} by an integral formula, which is just an extension of the well known integral representation of arcsin, and then use it to define generalized sine, cosine and tangent functions (labelled \sin_p , \cos_p and \tan_p respectively). Numerous properties of these functions, such as an identity of Pythagorean type, are exhibited. Then we consider the unit circle in \mathbf{R}^2 with the l_p norm and define generalized trigonometric functions as is done in the standard case when the l_2 norm is used. We show that these functions coincide with those introduced earlier. In the third section we consider the integral operator $T : L^p(I) \rightarrow L^p(I)$ given by $Tf(x) = \int_0^x f(t)dt$, where $I = (0, 1)$, and look at the problem of finding an extremal function (an element of the unit sphere of $L^p(I)$ at which the norm of T is attained). It turns out that the extremal functions are given by \cos_p . The following section deals with the Dirichlet eigenvalue problem for the p -Laplacian on a bounded interval: all eigenfunctions are expressible by means of \sin_p functions, which corresponds exactly to the classical situation when $p = 2$. After establishing a connection with approximation theory, we conclude with a review of other ways in which the classical trigonometric functions have been generalized.

In the literature a variety of different definitions of generalized trigonometric functions can be found (see [7], [8]): all extend the classical functions and preserve some of their properties. It becomes clear that no single definition preserves all the classical properties and that which definition is adopted depends on the applications envisaged. Our focus on a particular choice reflects our research interests.

1 Analytic point of view

It is well known from basic calculus that

$$\int_0^1 \frac{1}{\sqrt{1-t^2}} dt = \pi/2 \tag{1}$$

and

$$\int_0^x \frac{1}{\sqrt{1-t^2}} dt, \quad 0 \leq x \leq 1 \quad (2)$$

define a (differentiable) function on $[0, 1]$. Since $\frac{1}{\sqrt{1-t^2}}$ is positive on $(0, 1)$, the function is increasing and one-to-one from $[0, 1]$ to $[0, \pi/2]$. This function is $\arcsin(x)$ and can be used to define the function \sin on $[0, \pi/2]$. By standard extension procedures we can define the \sin function on $(-\infty, \infty)$.

Obviously this can be extended. Let $1 < p < \infty$ and define a (differentiable) function $F_p : [0, 1] \rightarrow \mathbf{R}$ by

$$F_p(x) = \int_0^x \frac{1}{\sqrt[p]{1-t^p}} dt, \quad 0 \leq x \leq 1. \quad (3)$$

Since F_p is strictly increasing it is a one-to-one function on $[0, 1]$ with range $[0, \pi_p/2]$, where

$$\pi_p = 2 \int_0^1 \frac{1}{\sqrt[p]{1-t^p}} dt, \quad 0 \leq x \leq 1. \quad (4)$$

The inverse of F_p on $[0, \pi_p/2]$ we denote by \sin_p and extend as in the case of \sin ($p=2$) to $[0, \pi_p]$ by defining

$$\sin_p(x) = \sin_p(\pi_p - x) \quad \text{for } x \in [\pi_p/2, \pi_p];$$

further extension is achieved by oddness and $2\pi_p$ -periodicity on the whole of \mathbf{R} . By this means we obtain a differentiable function on \mathbf{R} which coincides with \sin when $p = 2$.

Corresponding to this we define a function \cos_p by the prescription

$$\cos_p(x) = \frac{d}{dx} \sin_p(x), \quad x \in \mathbf{R}. \quad (5)$$

Clearly \cos_p is even, $2\pi_p$ -periodic and odd about π_p ; and $\cos_2 = \cos$. If $x \in [0, \pi_p/2]$, then from the definition it follows that

$$\cos_p(x) = (1 - (\sin_p(x))^p)^{1/p}. \quad (6)$$

Moreover, the asymmetry and periodicity show that

$$|\sin_p(x)|^p + |\cos_p(x)|^p = 1, \quad x \in \mathbf{R}. \quad (7)$$

Fig.1 below gives the graphs of \sin_p and \cos_p for $p = 1.2$ and 6 .

From (4) it follows that

$$\frac{\pi_p}{2} = p^{-1} \int_0^1 (1-s^p)^{-1/p} s^{1/p-1} ds = p^{-1} B(1-1/p, 1/p) = p^{-1} \Gamma(1-1/p) \Gamma(1/p),$$

where B is the Beta function, Γ is the Gamma function and

$$\pi_p = \frac{2\pi}{p \sin(\pi/p)}. \quad (8)$$

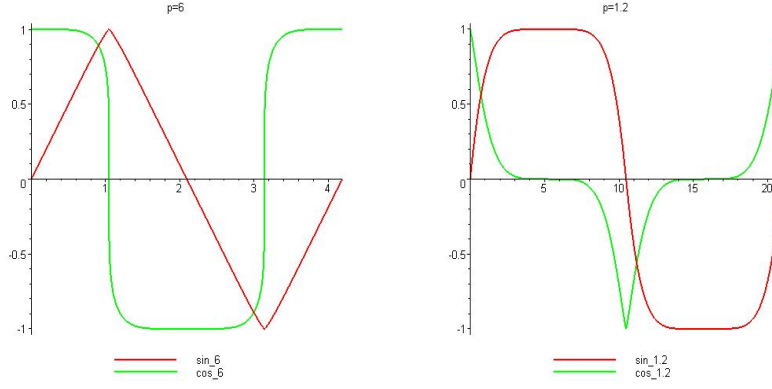


Figure 1: \sin_6, \cos_6 and $\sin_{1.2}, \cos_{1.2}$

Clearly $\pi_2 = \pi$ and, with $p' = p/(p-1)$,

$$p\pi_p = 2\Gamma(1/p')\Gamma(1/p) = p'\pi_{p'}. \quad (9)$$

Using (8) and (9) we see that π_p decreases as p increases, with

$$\lim_{p \rightarrow 1} \pi_p = \infty, \quad \lim_{p \rightarrow \infty} \pi_p = 2, \quad \lim_{p \rightarrow 1} (p-1)\pi_p = \lim_{p \rightarrow 1} \pi_{p'} = 2. \quad (10)$$

The dependence of π_p on p is illustrated in Fig. 2.

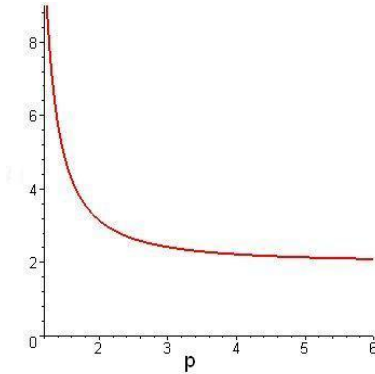


Figure 2: $y = \pi_p$

The generalized tangent function is defined as in the classical case:

$$\tan_p x = \frac{\sin_p x}{\cos_p x}. \quad (11)$$

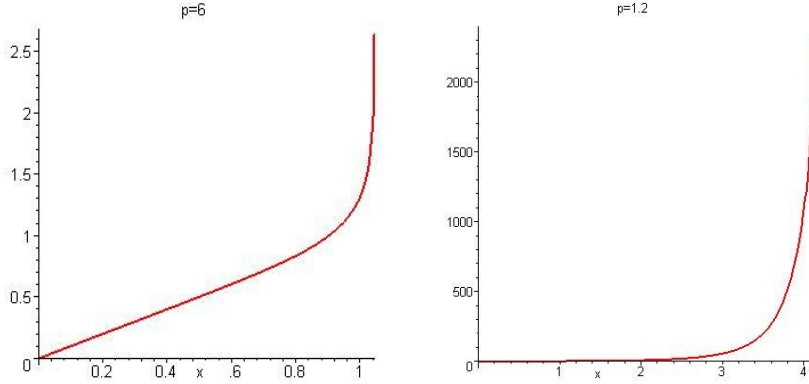


Figure 3: $y = \tan_6(x)$, $[0, \pi_6/2)$ $y = \tan_{1.2}(x)$, $[0, \pi_{1.2}/2)$

Fig.3 indicates the behaviour of \tan_p when $p = 1.2$ and 6 .

Obviously $\tan_p x$ is defined for all $x \in \mathbf{R}$ except for the points $(k + 1/2)\pi_p$ ($k \in \mathbf{Z}$); it is odd, π_p -periodic and $\tan_p 0 = 0$. Use of (7) shows that on $(-\pi_p/2, \pi_p/2)$, \tan_p has derivative $1 + |\tan_p x|^p$.

It follows that

$$\frac{d}{dx}(\tan_p^{-1} x) = \frac{1}{1 + |x|^p}$$

and on $(-\pi_p/2, \pi_p/2)$,

$$\tan_p^{-1}(x) = \int_0^x \frac{1}{1 + |t|^p} dt$$

Evidently $\tan_2^{-1} x = \arctan x$.

2 Geometric point of view

Here we start by recalling the definition of the sin and cos functions via the unit circle in the plane \mathbf{R}^2 with the l_2 metric, and then generalize this for \mathbf{R}^2 with the l_p metric.

Given $r > 0$, then $S_r = \{(x, y) \in \mathbf{R}^2; x^2 + y^2 = r^2\}$ is a circle in the plane \mathbf{R}^2 with the l_2 metric.

Each point in \mathbf{R}^2 can be described by rectangular coordinates (x, y) or polar coordinates (r, ϕ) . The relation between these different coordinates is described by the functions sin, cos and tan.

The connection between polar and rectangular coordinates is given by:

$$x = r \cos(\phi) \tag{12}$$

$$y = r \sin(\phi). \tag{13}$$

Due to the l_2 metric we have

$$r^2 = x^2 + y^2;$$

and ϕ is related to x and y by means of the *tan* function:

$$\phi = \tan^{-1}(y/x), \text{ when } x, y > 0.$$

Let us consider the case $p \neq 2$.

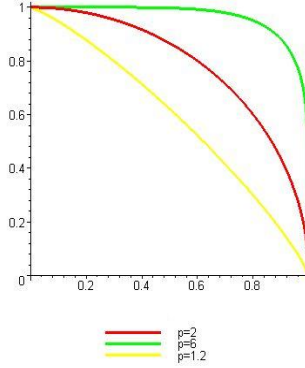


Figure 4: The first quadrant of S_1 for $p = 2, 6, 1.2$

Then the analogue of the circle is the p -circle $S_r = \{(x, y) \in \mathbf{R}^2; |x|^p + |y|^p = r^p\}$ and we expect the following identities:

$$x = r \cos_p(\phi) \tag{14}$$

$$y = r \sin_p(\phi). \tag{15}$$

From this follows $|\sin_p(\phi)|^p + |\cos_p(\phi)|^p = 1$, so that $\cos_p(\phi) = (1 - (\sin_p(\phi))^p)^{1/p}$ when $x, y > 0$. Fig. 4 above shows how the shape of S_1 changes with p . When $p \neq 2$, the p -circle S_r is not symmetric with respect to rotation and then the obvious one-to-one relation between the length of a curve segment on S_r and its angle, as we have in the l_2 metric, does not exist. Instead of this we will consider the quite natural condition

$$\frac{d}{d\phi} \sin_p(\phi) = \cos_p(\phi),$$

setting

$$\phi = 0 \text{ when } (x, y) = (1, 0),$$

and suppose that ϕ increases when (x, y) moves on S_r in the anticlockwise direction.

Then

$$\frac{d}{dt} \sin_p^{-1}(t) = \frac{1}{\cos_p(\sin_p^{-1}(t))} = \frac{1}{1 - t^p} \text{ when } 0 \leq t < 1,$$

from which it follows that the \sin_p and \cos_p functions defined in this section are the same as those defined in Section 1.

We define the \tan_p function as in Section 1:

$$\tan_p(\phi) := \frac{\sin_p(\phi)}{\cos_p(\phi)}$$

and we have, as in the l_2 case,

$$\phi = \tan_p^{-1}(y/x), \text{ when } x, y > 0.$$

3 An integral operator and generalized trigonometric functions

In this section we concentrate on the most simple integral operator. On the interval $I = [0, 1]$ let

$$Tf(x) := \int_0^x f(t)dt. \quad (16)$$

At first we consider T as a map from $L_2(0, 1)$ into $L_2(0, 1)$. It is obvious that T is compact and that there exists a function in $L_2(0, 1)$ at which the norm of T is attained. In this case it is quite simple to show that $\|T|_{L_2(0, 1)} \rightarrow L_2(0, 1)\| = 2/\pi$ and that the norm is attained when:

$$f(t) = \cos\left(\frac{\pi x}{2}\right) \frac{\pi}{2}$$

so that

$$Tf(t) = \sin\left(\frac{\pi x}{2}\right).$$

When $p \neq 2$ then again T is a compact map from $L_p(0, 1)$ into $L_p(0, 1)$ and there exists a function at which the norm is attained. In [6] it was proved that

$$\|T|_{L_p(0, 1)} \rightarrow L_p(0, 1)\| = \frac{(p' + p)^{1 - \frac{1}{p'} + \frac{1}{p}} (p')^{1/p} p^{1/p'}}{B\left(\frac{1}{p'}, \frac{1}{p}\right)}$$

and that the norm is attained when:

$$f(t) = \cos_p\left(\frac{\pi_p x}{2}\right) \frac{\pi_p}{2} \quad \text{and} \quad Tf(t) = \sin_p\left(\frac{\pi_p x}{2}\right).$$

This leads us, again, to the generalized trigonometric functions.

4 Eigenfunctions for the p-Laplacian

Consider the following classical Dirichlet problem on $(0, 1)$:

$$\left. \begin{aligned} \Delta u + \lambda u &= 0 && \text{on } (0, 1), \\ u(0) = 0, u(1) &= 0. \end{aligned} \right\} \quad (17)$$

It is well known that all eigenvalues are of the form:

$$\lambda_n = (n\pi)^2, \quad n \in \mathbf{N}$$

with corresponding eigenfunctions

$$u_n(t) = \sin(n\pi t), \quad n \in \mathbf{N}.$$

We recall the definition of the p -Laplacian which is a natural extension of the Laplacian:

$$\Delta_p u = (|u'|^{p-2}u')'.$$

Evidently $\Delta_2 u = \Delta u$. Then the analogue of (17) is the eigenvalue problem

$$\left. \begin{aligned} \Delta_p u + \lambda |u|^{p-2}u &= 0 \quad \text{on } (0, 1), \\ u(0) = 0, u(1) &= 0. \end{aligned} \right\} \quad (18)$$

In [4] it is shown that all eigenvalues of this problem are of the form

$$\lambda_n = (n\pi_p)^p \frac{p}{p'}$$

with corresponding eigenfunctions

$$u_n(t) = \sin_p(n\pi_p t).$$

Once more we see the appearance of our generalized trigonometric functions.

Let us note that the literature on the p -Laplacian and operators that resemble it in some sense is enormous. Here we mention only a few works, beyond those already cited, that seem of particular relevance to our approach. Of particular interest is the excellent survey paper by Lindqvist [11]; see also the book [5]. In [1] a Sturm-Liouville theory is developed for the one-dimensional p -Laplacian, following on from the work of [3]; see also [12].

5 The approximation theory point of view and the generalization of trigonometric functions

Let $1 < p < \infty$ and $-\infty < a < b < \infty$. Consider the Sobolev embedding on $I = [a, b]$,

$$E : W_0^{1,p}(I) \rightarrow L^p(I), \quad (19)$$

where $W_0^{1,p}(I)$ is the Sobolev space of functions on the interval I with zero trace equipped with the following norm:

$$\|u\|_{W_0^{1,p}(I)} := \left(\int_0^1 |u'(t)|^p dt \right)^{1/p}.$$

The Sobolev embedding is one of the most useful maps in Analysis. It is well known that (19) is a compact map and that more detailed information about its

compactness plays an important role in different branches of mathematics. The properties of compact maps can be well described by using the Kolmogorov, Bernstein and Gel'fand n -widths together with the approximation numbers.

We recall the definitions of these quantities:

Definition 5.1 *Let $T : X \rightarrow Y$ be a bounded operator, where X and Y are Banach spaces, and let $n \in \mathbf{N}$.*

(i) *The Kolmogorov n -width $d_n(T)$ of T is*

$$d_n(T) = d_n(T(X), Y) = \inf_{X_n} \sup_{\|x\|_X \leq 1} \inf_{y \in X_n} \|Tx - y\|_Y$$

where the infimum is taken over all n -dimensional subspaces X_n of X .

(ii) *The Gel'fand n -width $d^n(T)$ of T is*

$$d^n(T) = d^n(T(X), Y) = \inf_{L^n} \sup_{\|x\|_X \leq 1, x \in L^n} \|Tx\|_Y$$

where the infimum is taken over all subspaces L_n of codimension n of X .

(iii) *The Bernstein n -width $b_n(T)$ of T is*

$$b_n(T) = b_N(T(X), Y) = \sup_{X_{n+1}} \inf_{Tx \in X_{n+1}, Tx \neq 0} \|Tx\|_Y / \|x\|_X$$

where X_{n+1} is any subspace of $\text{span}\{Tx : x \in X\}$ of dimension $\geq n + 1$.

(iv) *The approximation number $a_n(T)$ of T is*

$$a_n(T) = \inf \|T - F|X \rightarrow Y\|,$$

where the infimum is taken over all bounded linear maps $F : X \rightarrow Y$ with rank less than n .

In our case we have $X = W_0^{1,p}(I)$ and $Y = L^p(I)$. Then since $L^p(I)$ has the approximation property for $1 \leq p \leq \infty$, E is compact if and only if $a_m(E) \rightarrow 0$ as $m \rightarrow \infty$.

Define $I_0 = \left[a, a + \frac{|I|}{2n} \right]$, $I_n = \left[b - \frac{1}{2} \frac{|I|}{n}, b \right]$ and $I_i = \left[a + (i - \frac{1}{2}) \frac{|I|}{n}, a + (i + \frac{1}{2}) \frac{|I|}{n} \right]$ for $1 < i < n$. Then $\{I_i\}_{i=0}^n$ is a covering of I and we use it to define the following map:

$$R_n f = \sum_{i=1}^{n-1} P_i f$$

where

$$P_i f(x) = \chi_{I_i}(x) f \left(a + i \frac{|I|}{n} \right).$$

It is obvious that R_n is a map from $W_0^{1,p}(I)$ into $L^p(I)$ with rank $= n - 1$.

The following theorem was proved in [2].

Theorem 5.2 *Let $1 < p < \infty$. Then*

$$s_n(E) = \frac{|I|}{n\pi_p} \cdot \left(\frac{p'}{p}\right)^{1/p}$$

and

$$s_n(E) = \|(E - R_n)g\|_{L^p(I)}, \quad \text{where } g(x) = \sin_p\left(\frac{x-a}{\pi_p} \frac{|I|}{n}\right)$$

and $s_n(E)$ stands for any of the following: $a_n(E)$, $d_n(E)$, $d^n(E)$ or $b_n(E)$.

This theorem provides us with information about the image of the unit ball of $W_0^{1,p}(I)$ in the space $L^p(I)$.

We can see from this theorem that the largest element in $BW_0^{1,p}(I) := \{f; \|f\|_{W_0^{1,p}} \leq 1\}$ in the $L^p(I)$ norm is

$$f_1(x) := \frac{\sin_p\left(\frac{x-a}{\pi_p}|I|\right)}{\|\sin_p\left(\frac{x-a}{\pi_p}|I|\right)\|_{W_0^{1,p}(I)}}$$

Let us approximate $BW_0^{1,p}(I)$ by a one-dimensional subspace in $L^p(I)$. The most distant element from the optimal one-dimensional approximation is

$$f_2(x) := \frac{\sin_p\left(\frac{x-a}{\pi_p} \cdot \frac{|I|}{2}\right)}{\|\sin_p\left(\frac{x-a}{\pi_p} \frac{|I|}{2}\right)\|_{W_0^{1,p}(I)}}$$

More generally, if we approximate $BW_0^{1,p}(I)$ by an n -dimensional subspace in $L^p(I)$, then the most distant element from the optimal n -dimensional approximation is

$$f_n(x) := \frac{\sin_p\left(\frac{x-a}{\pi_p} \cdot \frac{|I|}{n}\right)}{\|\sin_p\left(\frac{x-a}{\pi_p} \frac{|I|}{n}\right)\|_{W_0^{1,p}(I)}}$$

Also from the previous theorem we have that $\|f_i\|_{L^p(I)} = s_n(E)$

We can see that the functions f_i are playing, in some sense, roles similar to those of the semi-axes of an ellipsoid.

We present below figures which show an image of $BW_0^{1,p}(I)$ restricted to a linear subspace $\text{span}\{f_1, f_2, f_3\}$ in $L^p(I)$.

In the case $p = 2$ we obtain an ellipsoid (here the x, y, z axes correspond to f_1, f_2, f_3).

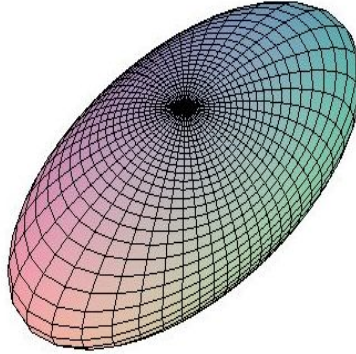


Figure 5: $p = 2$

When $p = 10$ and $p = 1.1$ we have the images below:

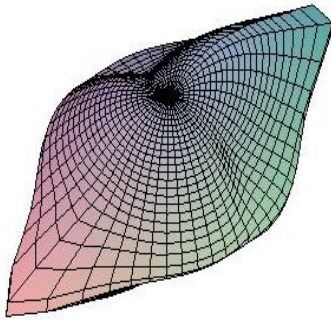
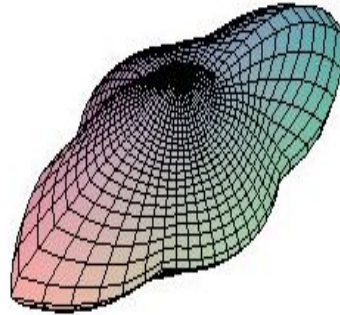


Figure 6: $p = 10$



$p = 1.1$

We can see that the main difference between Fig. 5 and Fig 6 is that the pictures in Fig. 6 are not convex. This suggests that possibly the functions f_1, f_2, f_3 are not orthogonal in the James sense.

We recall the definition of this orthogonality. Let a, b be elements of a Banach space X . We say that a is orthogonal to b in the James sense, written $a \perp^j b$, if $\|a\|_X \leq \|a + \lambda b\|_X$ for every $\lambda \in \mathbf{R}$. In some literature this orthogonality is called Birkhoff orthogonality.

Fig. 7 indicates that for $p = 6$ (similar graphs can be obtained for other $p \neq 2$) the function f_1 is not orthogonal to f_3 and also f_3 is not orthogonal to f_1 .

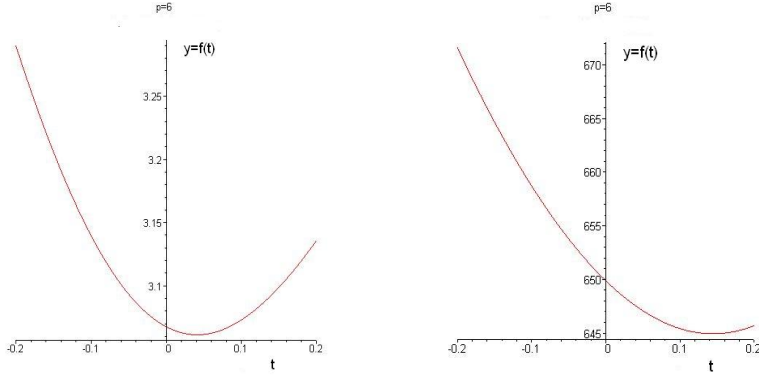


Figure 7: $f(t) = \|f_1 + tf_3\|_p$

$f(t) = \|f_3 + tf_1\|_p$

6 Some other definitions of generalized trigonometric functions

The definition of the generalised trigonometric functions that we have chosen is only one of several that can be found in the literature, which is now quite extensive and goes back at least as far as the 1879 work of Lundberg (see [10]): details of the various approaches can be found in the papers of Lindqvist [9] and of Lindqvist and Peetre [7]; see also [8]. In [10] a beautiful account is given of the history of such work, with especial reference to that of Lundberg. To illustrate these alternative methods we consider first the approach taken in [7], [8]. Let $p \in (1, \infty)$ and set

$$\widetilde{\pi}_p = p \int_0^1 \frac{dt}{(1-t^p)^{(p-1)/p}}.$$

On $(0, \widetilde{\pi}_p/p)$ define functions $S_p, C_p,$ and T_p by:

$$x = \int_0^{S_p(x)} \frac{dt}{(1-t^p)^{(p-1)/p}}, \quad x = \int_{C_p(x)}^1 \frac{dt}{(1-t^p)^{(p-1)/p}}, \quad x = \int_0^{T_p(x)} \frac{dt}{(1+t^p)^{2/p}}$$

and extend them to \mathbf{R} as was done in Section 1. Note that $\widetilde{\pi}_p = pS_p^{-1}(1) = pC_p^{-1}(0)$.

Then we have, on $(0, \widetilde{\pi}_p/p)$,

$$\begin{aligned} S_p(x)^p + C_p(x)^p &= 1, & T_p(x) &= \frac{S_p(x)}{C_p(x)} \\ S_p'(x) &= C_p(x)^{p-1}, & C_p'(x) &= -S_p(x)^{p-1} \\ S_p\left(\frac{\widetilde{\pi}_p}{p} - x\right) &= C_p(x), \end{aligned} \tag{20}$$

$$S_p\left(\frac{\widetilde{\pi}_p}{2p}\right) = \frac{1}{\sqrt[p]{2}} = C_p\left(\frac{\widetilde{\pi}_p}{2p}\right). \quad (21)$$

When $p = 2$, we see that $S_2(x) = \sin(x)$, $C_2(x) = \cos(x)$, $T_2(x) = \tan(x)$.

Another way of proceeding is given in [9] (see also the earlier paper [13]). In this we set

$$\hat{\pi}_p = \frac{2\sqrt[p]{p-1}}{p \sin \frac{\pi}{p}} \pi.$$

On $(0, \hat{\pi}_p/2)$ we define functions $\hat{S}_p(x)$, $\hat{C}_p(x)$, and $\hat{T}_p(x)$ by:

$$x = \int_0^{\hat{S}_p(x)} \frac{dt}{(1 - \frac{t^p}{p-1})^{1/p}}, \quad x = \int_{\hat{C}_p(x)}^{\sqrt[p]{p-1}} \frac{dt}{(1 - \frac{t^p}{p-1})^{1/p}},$$

$$x = \int_0^{\hat{T}_p(x)} \frac{dt}{1 + \frac{t^p}{p-1}}$$

and extend them to \mathbf{R} as in Section 1. Note that

$$\sqrt[p]{p-1} = \hat{S}_p\left(\frac{\hat{\pi}_p}{2}\right) = \hat{C}_p(0).$$

When $p = 2$ clearly $\hat{S}_2(x) = \sin(x)$, $\hat{C}_2(x) = \cos(x)$, $\hat{T}_2(x) = \tan(x)$. We have on $(0, \hat{\pi}_p/2)$:

$$\frac{(\hat{C}_{p'}(x))^{p'}}{p'-1} + \frac{(\hat{S}_p(x))^p}{p-1} = 1$$

and

$$\frac{d\hat{S}_p(x)}{dx} = (p-1)^{1/p} (\hat{C}_{p'}(x))^{p'-1}$$

$$\frac{d\hat{C}_p(x)}{dx} = -(p-1)^{1/p} (\hat{S}_{p'}(x))^{p'-1}$$

$$\hat{S}_p(x) = \hat{C}_p\left(\frac{\hat{\pi}}{2} - x\right), \quad \hat{C}_p(x) = \hat{S}_p\left(\frac{\hat{\pi}}{2} - x\right),$$

while for \hat{T}_p we have

$$\hat{T}_p(x) = \frac{\hat{S}_p(x)}{\frac{d}{dx}(\hat{S}_p(x))} = \frac{\hat{S}_p(x)}{(p-1)^{1/p} (\hat{C}_{p'})^{p'-1}},$$

and also

$$\frac{d}{dx}(\hat{T}_p(x)) = 1 + \frac{(\hat{T}_p(x))^p}{p-1}.$$

However, it is important to recognise that whatever definition is adopted, good features will be accompanied by less pleasant ones. For example, the

definition used in the earlier sections leads to the fine Pythagorean identity (7) and has a close connection with the Dirichlet problem for the p -Laplacian, but the derivative of \cos_p is given by a somewhat complicated formula and $\pi_p \rightarrow \infty$ as $p \rightarrow 1$. The definition given first in the present section leads to the Pythagorean relation and to the property (20), but the derivative of S_p involves a power of C_p . As for the last definition, while $\widehat{\pi}_p$ remains bounded as p varies, to set against that is the fact that the Pythagorean relation is not aesthetically pleasing and the derivatives of \widehat{S}_p and \widehat{C}_p are given by expressions without much appeal. The choice of definition to be made depends on how best the features of the corresponding generalised function fit in with the particular application envisaged.

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