# Generating chaos with a switching piecewise-linear controller

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This paper introduces a new chaos generator, a switching piecewise-linear controller, which can create chaos from a three-dimensional linear system within a wide range of parameter values. Basic dynamical behaviors of the chaotic controlled system are investigated in some detail. © *2002 American Institute of Physics.* [DOI: 10.1063/1.1478079]

**Over the last decade, knowing that chaos can actually be useful and can be well controlled, the intensive study of chaotic dynamics has evolved from the traditional trend of understanding and analyzing chaos to the new attempt of controlling and utilizing it. Recently, there has been increasing interest in exploiting chaotic dynamics in various engineering and technological applications, whereas much attention has focused on effectively generating chaos via simple physical devices such as simple nonlinear circuits. For electronic engineers, it has been known that piecewise-linear functions can be used to generate various chaotic attractors such as *n*-scroll attractors in the simple Chua's circuit. This motivates the present study of the problem of generating new chaotic attractors by using a switching type of piecewise-linear controller. The designed controller can create chaos from a linear system within a wide range of parameter values, demonstrating that simple analog chaos generators indeed have strong capability of chaos generation. Basic dynamical behaviors of the new chaotic system are also investigated in some details in the present paper.**

## I. INTRODUCTION

Over the last decade, the study of chaotic dynamics has evolved from the traditional trend of understanding and analyzing chaos to the new attempt of controlling and utilizing it.[1–3] Recently, there has been increasing interest in exploiting chaotic dynamics in engineering applications, such as electrical engineering, telecommunications, computing and information processing, material engineering, etc., whereas much attention has focused on effectively generating chaos via simple devices such as circuitry design.[4–8]

It has been well known that just like the $n$-scroll Chua's circuit,[9] piecewise-linear function can easily generate various chaotic attractors.[10–12] In many cases, we need to generate chaos purposely. Notice that there are some attempts of using the changes of phase space location of the system orbits.[13–19] Motivated by many examples of this type and the need of applications, the present paper studies the problem of generating a new chaotic attractor by designing a switching piecewise-linear controller. This controller can create chaos from a linear system within a wide range of parameter values, and covers the chaotic attractors found in some other reports.[12] Moreover, basic dynamical behaviors of the chaotic controlled system are investigated in detail, by employing some effective mathematical tools.[20]

## II. THE NEW CHAOS GENERATOR AND ITS CONTROLLED SYSTEM

Consider the following linear controlled system:

$$\dot{X} = AX + f(X), \tag{1}$$

where

$$A = \begin{pmatrix} a & b & 0 \\ -b & a & 0 \\ 0 & 0 & c \end{pmatrix},$$

with a switching piecewise-linear controller

$$f(X) = k\begin{pmatrix} -x \\ -y \\ d \end{pmatrix} \quad \text{if} \quad z + \sqrt{x^2+y^2} > k, \qquad = 0, \quad \text{otherwise}, \tag{2}$$

for a real parameter $k > 0$.

The controller embedded in this system is switched on, when the state variables travel through the surface $z + \sqrt{x^2+y^2} = k$ in the state space. This controlled system has

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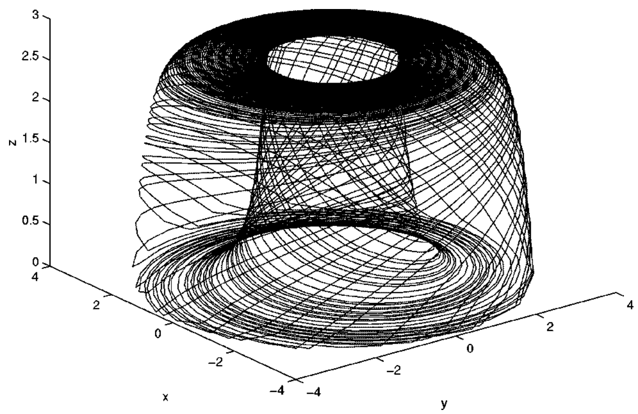


FIG. 1. The chaotic attractor generated by the switching controller.

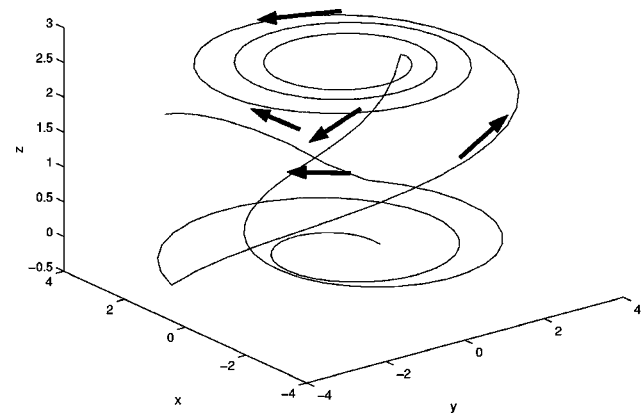


FIG. 2. The orbit of system (1).

a chaotic attractor, as shown in Fig. 1, when $a=3$, $b=20$, $c=-20$, $k=4$, $d=10$. The maximum Lyapunov exponent of this attractor is $LE=1.5963$.

It is remarked that the chaotic attractor reported by Yang and Li[12] is a special case of this new chaotic family.

## III. DYNAMICAL BEHAVIORS OF THE CHAOTIC CONTROLLED SYSTEM

Some basic dynamical behaviors of the chaotic controlled system (1) are investigated here by both theoretical analysis and numerical simulation.

### A. Symmetry and dissipativity

System (1) has a natural symmetry under the coordinates transform $(x,y,z)\rightarrow(-x,-y,z)$, which persists for all values of the system parameters.

In the following, assume that $a>0$, $c<-2a$, $k>0$.

The variation of the volume $V(t)$ of a small element, $\delta\Omega(t)=\delta x\,\delta y\,\delta z$ in the state space, is determined by the divergence of the flow:

$$\nabla V=\frac{\partial\dot{x}}{\partial x}+\frac{\partial\dot{y}}{\partial y}+\frac{\partial\dot{z}}{\partial z},$$

which is

$$\nabla V=\begin{cases}2a+c-2k<0 & \text{for } z+\sqrt{x^2+y^2}>k,\\ 2a+c<0 & \text{otherwise.}\end{cases}$$

Hence, system (1) is dissipative, with an exponential contraction rate

$$\delta\Omega(t)=\begin{cases}e^{2a+c-2k} & \text{for } z+\sqrt{x^2+y^2}>k,\\ e^{2a+c} & \text{otherwise.}\end{cases}$$

As a result, a volume element $V_0$ is contracted by the flow into a volume element $V_0e^{\nabla Vt}$ in time $t$. Namely, each volume containing the system trajectory shrinks to zero as $t\rightarrow\infty$ at an exponential rate, $\nabla V$, which is independent of $x$, $y$, $z$. Thus, all system orbits will ultimately be confined to a specific subset having zero volume and the asymptotic motion settles onto an attractor.

### B. System equilibria

In this section, assume that $b\neq0$, $c\neq0$, $k>0$.

The equilibria of system (1) are found by solving the three equations $\dot{x}=\dot{y}=\dot{z}=0$, which gives

(1) if $-d/c>1$, then the system has two equilibria, (0,0,0) and $(0,0,-kd/c)$;
(2) if $-d/c<1$, then the system has a unique equilibrium, (0,0,0).

Consider the equilibrium (0,0,0). The system Jacobian matrix $J$ at this point is

$$J=\begin{pmatrix}a & b & 0\\ -b & a & 0\\ 0 & 0 & c\end{pmatrix}, \tag{3}$$

which has eigenvalues $\lambda_{1,2}=a\pm b\mathrm{i}$ and $\lambda_3=c$.

Therefore, the stability of the equilibrium (0,0,0) can be clarified

(1) if $a>0$ or $c>0$, then this equilibrium is unstable;
(2) if $a<0$ and $c<0$, then this equilibrium is stable.

At the same time, it is noticed that with $c<0$, $a=0$ is a Hopf bifurcation point.

Similarly, for the equilibrium $(0,0,-kd/c)$, the system Jacobian is

$$J=\begin{pmatrix}a-k & b & 0\\ -b & a-k & 0\\ 0 & 0 & c\end{pmatrix}, \tag{4}$$

and its eigenvalues are $\lambda_{1,2}=a-k\pm b\mathrm{i}$ and $\lambda_3=c$. Hence, the stability of this equilibrium, $(0,0,-kd/c)$ is classified as follows:

(1) if $a>k$ or $c>0$, then this equilibrium is unstable;
(2) if $a<k$ and $c<0$, then this equilibrium is stable.

Also, with $c<0$, $a=k$ is a Hopf bifurcation point.

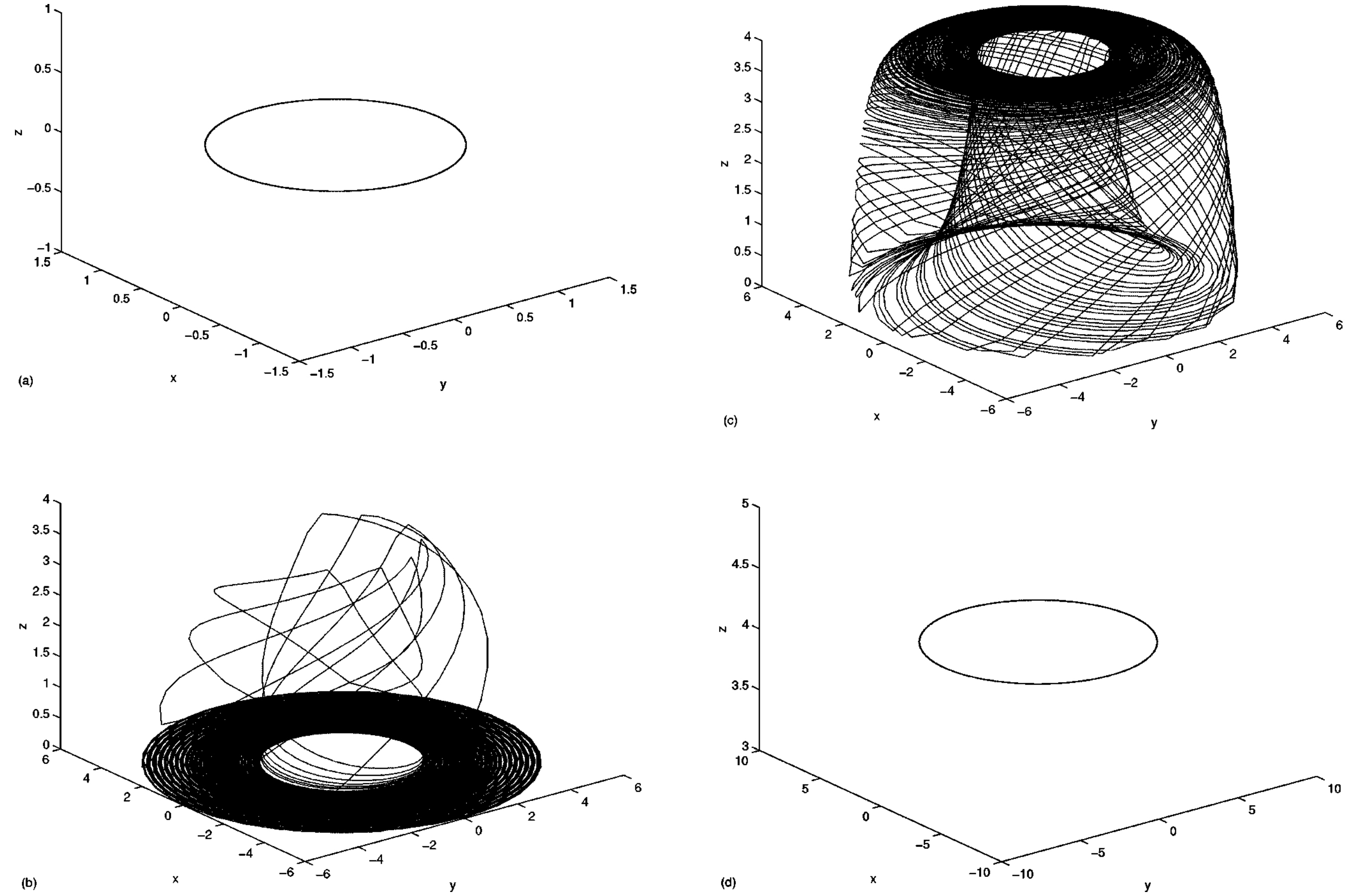


FIG. 3. Phase portraits of system (1). (a) $a=0$; (b) $a=0.1$; (c) $a=5$; (d) $a=6$.

### C. Dynamical analysis of the switching controlled system

In this section, assume that $a>0$, $k>0$, $c<0$, $d>0$, such that $a<k\leqslant a-c$ and $d<-c$. Define two regions $\Sigma$, $\{(x,y,z)|z+\sqrt{x^2+y^2}\leqslant k\}$; $\overline{\Sigma}$, $\{(x,y,z)|z+\sqrt{x^2+y^2}>k\}$.

When $z+\sqrt{x^2+y^2}\leqslant k$, system (1) becomes

$$\dot{x}=ax+by, \quad \dot{y}=-bx+ay, \quad \dot{z}=cz. \tag{5}$$

According to the third equation of system (5), $z=z(0)e^{ct}$. Thus, when $t\to+\infty$, one has $z\to 0$. Let $V=x^2+y^2$. Then,

$$\dot{V}=2x\dot{x}+2y\dot{y}=2a(x^2+y^2)=2aV,$$

so $V=V(0)e^{2at}$. That is, when $t\to+\infty$, $V(t)\to+\infty$, so that $f(t)=z+\sqrt{x^2+y^2}\to+\infty$. However, notice that system (5) must satisfy $f(t)=z+\sqrt{x^2+y^2}\leqslant k$. Hence, when $t$ gets to some particular instant, $t_1$, $f(t_1)>k$, so that system (5) fails to hold. The orbit system will then go through the plane $z+\sqrt{x^2+y^2}=k$ and then switch into region $\overline{\Sigma}$. After this instant, the system becomes

$$\begin{aligned}
\dot{x}&=(a-k)x+by,\\
\dot{y}&=-bx+(a-k)y,\\
\dot{z}&=cz+kd.
\end{aligned} \tag{6}$$

For this system, let $V=x^2+y^2+[z+(kd/c)]^2$. Then,

$$\begin{aligned}
\dot{V}&=2\left[x\dot{x}+y\dot{y}+\left(z+\frac{kd}{c}\right)\dot{z}\right]\\
&=2\left[(a-k)(x^2+y^2)+c\left(z+\frac{kd}{c}\right)^2\right]\\
&=2\left[(a-k)V+(c+k-a)\left(z+\frac{kd}{c}\right)^2\right]\\
&\leqslant 2(a-k)V.
\end{aligned}$$

Hence, $V(t)\leqslant V(0)e^{2(a-k)t}\to 0$, as $t\to+\infty$. Therefore, when $t\to+\infty$, $f(t)=z+\sqrt{x^2+y^2}\to-(kd/c)<k$. That is, when $t$ reaches a particular instant, $t_2$, $f(t_2)<k$. But system (6) holds if and only if $f(t)=z+\sqrt{x^2+y^2}>k$. Therefore, the system orbits will go through the plane $z+\sqrt{x^2+y^2}=k$, and then go back to region $\Sigma$.

Under condition $0<d<-c$, system (1) has a unique equilibrium, (0,0,0), and it is unstable. Furthermore, from the above analysis, it can be seen that when $t\to+\infty$, system (1)

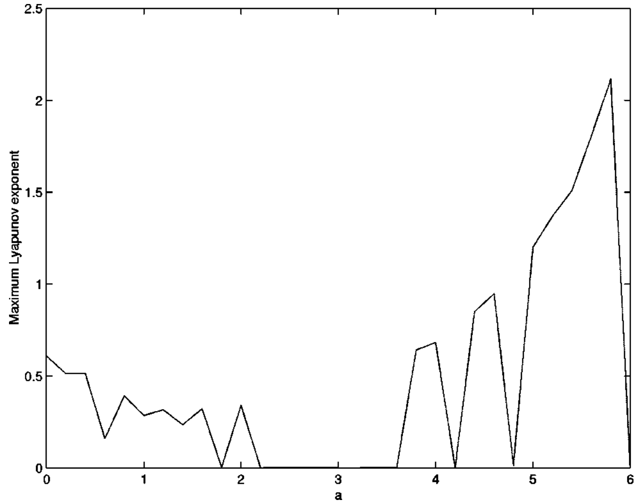


FIG. 4. The maximum Lyapunov exponents of system (1).

changes dynamical behaviors as the orbits go through the plane $z+\sqrt{x^2+y^2}=k$ repeatedly. That is, system (1) has folding and stretching dynamics repeatedly, leading to complex dynamics such as the appearance of bifurcations and chaos.

Figure 2 shows the directions of the system orbit, denoted by the arrows, where the corresponding parameters are $a=3$, $b=20$, $c=-20$, $k=4$, $d=10$. For $(x_0,y_0,z_0)=(0.1,1,-0.1)$, when $t<0.5s$, the orbit runs inside region $\Sigma$; when $0.5s<t<1.6s$, it will be in region $\overline{\Sigma}$; when $1.6s<t<2s$, it runs back into region $\Sigma$, and so on.

### D. Dynamical structures with parameters variation

Now, the dynamical behaviors of the system (1) is investigated numerically.

#### *1. Variation of parameter a*

Fix parameters $b=20$, $c=-15$, $k=6$, $d=10$, and let $a$ vary. The system dynamical behaviors are summarized in the following.

(i) When $a<0$, the system orbit converges to a point.
(ii) When $a=0$, there is a limit cycle, as shown in Fig. 3(a).
(iii) When $0.1<a<6$, there is a chaotic region in which the maximum Lyapunov exponents are as shown in Fig. 4.
(iv) When $a=6$, there is a limit cycle, as shown in Fig. 3(d).
(v) When $a>6$, the system orbit does not converge.

#### *2. Variation of parameter b*

Fix parameters $a=3$, $c=-15$, $k=8$, $d=10$, and let $b$ vary. Numerical simulations show that system (1) is chaotic or chaoslike for almost all values of $b$ (Fig. 5).

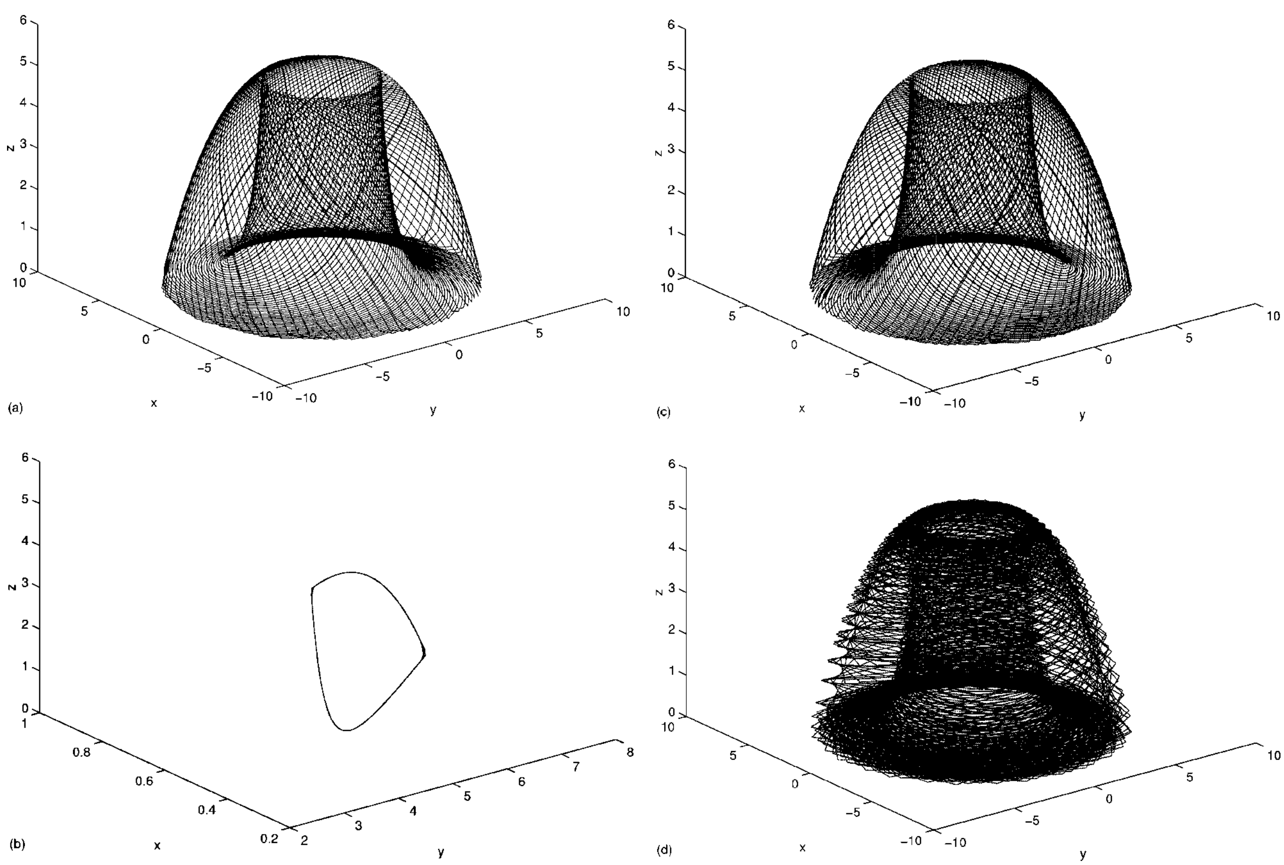


FIG. 5. Phase portraits of system (1). (a) $b=-10$; (b) $b=0$; (c) $b=10$; (d) $b=100$.

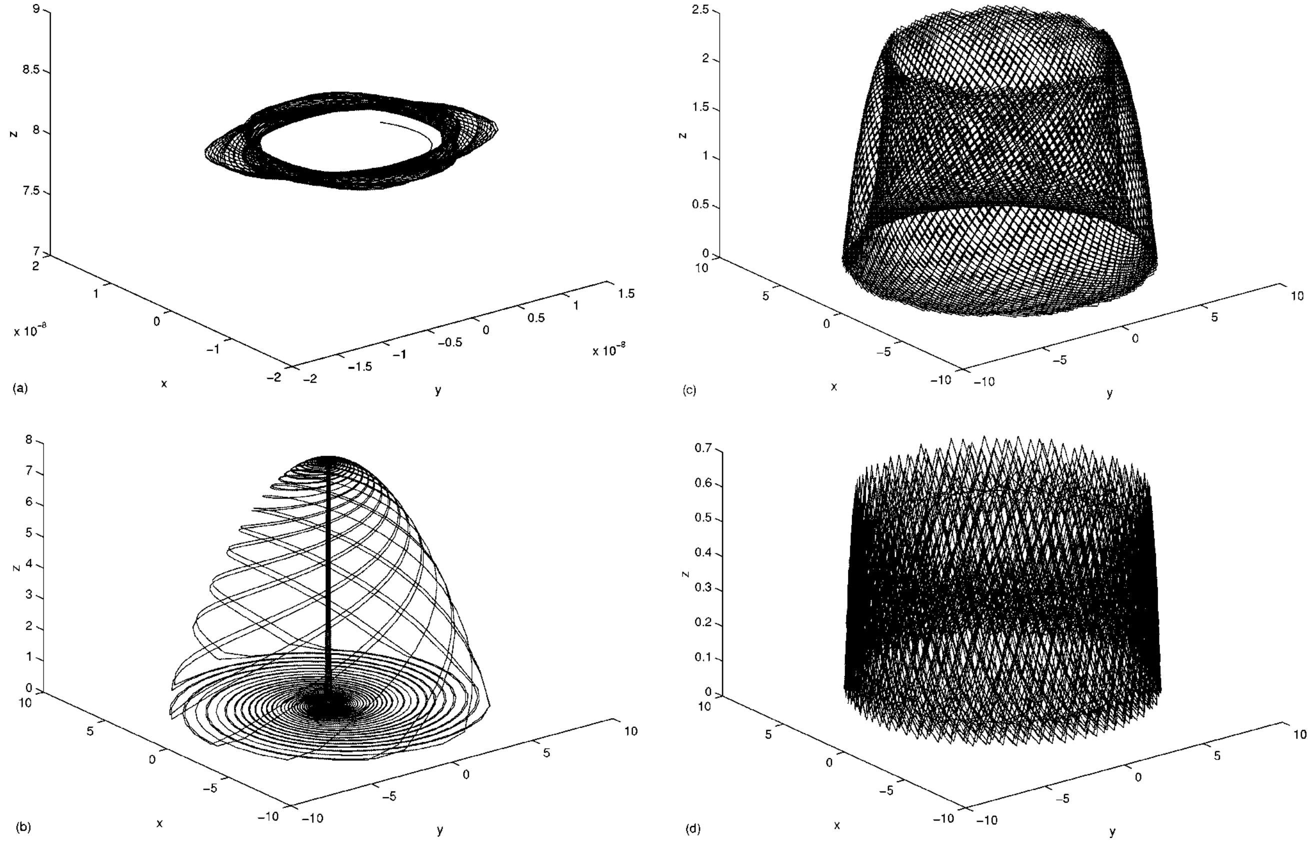


FIG. 6. Phase portraits of system (1). (a) $c=-10$; (b) $c=-10.1$; (c) $c=-30$; (d) $c=-100$.

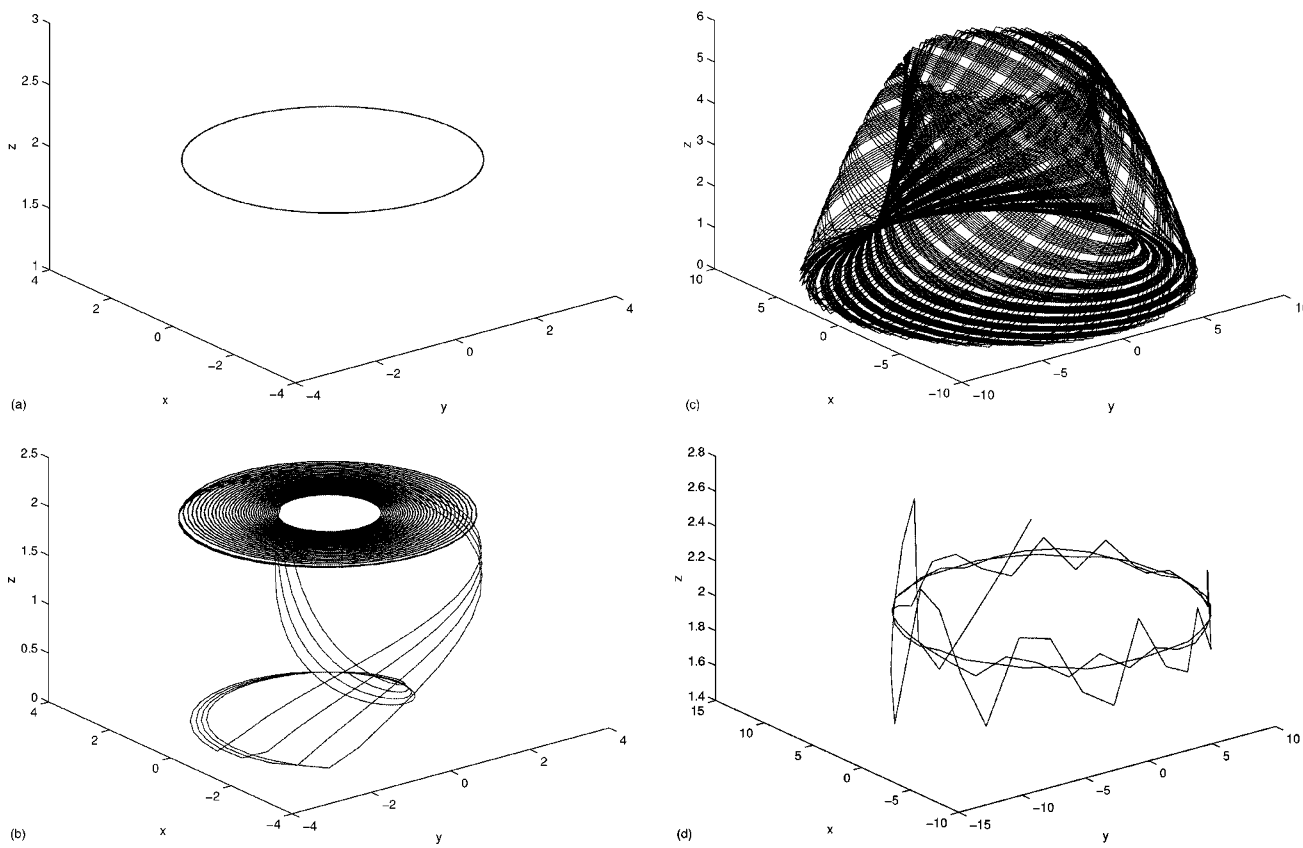


FIG. 7. Phase portraits of system (1). (a) $k=3$; (b) $k=3.1$; (c) $k=10$; (d) $k=12$.

### 3. Variation of parameter c

Fix parameters $a=3$, $b=20$, $k=8$, $d=10$, and let $c$ vary. When $c \geqslant 0$, the system orbit does not converge; when $-10 < c < 0$, it converges to a point; when $c < -10$, the system is chaotic or chaoslike (Fig. 6).

### 4. Variation of parameter k

Fix parameters $a=3$, $b=20$, $c=-15$, $d=10$, and let $k$ vary. Numerical simulations show that when $k<3$, the orbit of system (1) does not converge; when $k=3$ or $k=12$, there is a limit cycle, as shown in Fig. 7; when $3<k<12$, it is chaotic or chaoslike.

## IV. CONCLUSIONS

A new chaos generating controller has been introduced and investigated, which is a simple piecewise-linear function. Dynamical behaviors of the chaotic controlled system have also been analyzed, both theoretically and numerically. It has been known that abundant complex dynamical behaviors can be generated by piecewise-linear functions when designed appropriately; however, there does not seem to be a general methodology that can provide a generic design for a controller, therefore further research into the subject is still important and insightful. Although this paper provides one more class of systems that fall into this category of piecewiselinear switching control, the new finding is unique and quite interesting both theoretically and practically, in terms of new chaos generation techniques and possible engineering applications of chaos.


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