Generating Chaotic Attractors With Multiple Merged Basins of Attraction: A Switching Piecewise-Linear Control Approach

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***Abstract*—This paper presents several new chaos generators, switching piecewise-linear controllers, which can generate some new chaotic attractors with two or three merged basins of attraction from a given three-dimensional linear autonomous system within a wide range of parameter values. Based on this success, chaotic attractors with n merged basins of attraction are further generated using a formalized controller design methodology. Basic dynamical behaviors of the controlled chaotic system are then investigated via both theoretical analysis and numerical simulation. To that end, the underlying chaos-generation mechanism is further explored by analyzing the parameterization of the controlled system and the dynamics of the system orbits.**

***Index Terms*—Basin of attraction, chaos generation, piecewise-linear controller, switching system.**

I. Introduction

In THE last few years, experience has shown that chaos can actually be useful and can also be well controlled [1]–[3]. As a result, the study of chaotic dynamics has seen an expansion from the traditional concern of understanding and analyzing chaos to the new attempt of controlling and utilizing it. Recently, exploiting chaotic dynamics in high-tech and industrial engineering applications has attracted more and more interest, in which much attention has focused on effectively creating chaos [3]–[5], particularly using simple devices such as nonlinear circuits [6]–[8] and switching piecewise-linear controllers [9]–[15].

It is well known that piecewise functions can easily create various chaotic attractors [8]–[18]. Typical examples include the n-scroll circuits [16], [17]. Motivated by many examples of this type , Lü *et al.* [10] introduced a switching piecewise-linear controller [10], which can create chaos from a given linear autonomous system within a wide range of parameter values; Zheng *et al.* [11] further modified the controller to generate two chaotic attractors simultaneously. These case studies have shown that simple analog chaos generators indeed have strong capability of generating chaos.

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This paper furthers the studies of [10], [11], to investigate the generation of chaotic attractors with multiple merged basins of attraction in a simple control system. The key is to redesign the controller that can generate a chaotic attractor with a single basin of attraction. Two new switching schemes will be introduced into the piecewise-linear controller previously studied in [10] and [11], thereby endorsing the redesigned controller an ability of generating chaotic attractors with two or three merged basins of attraction, from a given simple linear autonomous system. Furthermore, a formalized design procedure is suggested for generating chaotic attractors with n merged basins of attraction. Finally, basic dynamical behaviors of the controlled chaotic system are investigated in some detail, by employing some mathematical tools developed in [19]. In particular, the underlying chaos generation mechanism in switching systems is explored, by analyzing the parameterization of the controlled system and the dynamics of the system orbits.

II. Switching Controlled Chaotic System

Consider the following simple linear controlled system:

$$\dot{X} = AX + U \tag{1}$$

where $X = (x\ y\ z)^T$ and

$$A = \begin{pmatrix} a & b & 0 \\ -b & a & 0 \\ 0 & 0 & c \end{pmatrix}$$

with a switching piecewise-linear controller

$$U = f(X) = \begin{cases} k\begin{pmatrix} -x \\ -y \\ d \end{pmatrix}, & \text{if } z + \sqrt{x^2+y^2} > k \\ 0, & \text{otherwise} \end{cases} \tag{2}$$

where a, b, c, d, k are real parameters. This controlled system (1)–(2) can generate chaos within a wide range of parameter values [9].

Zheng *et al.* [11] has further improved the controller (2) to be the following one:

$$f(X) = \begin{cases} k\begin{pmatrix} -x \\ -y \\ d \end{pmatrix}, & \text{if } z>0 \text{ and } z+\sqrt{x^2+y^2} > k \\ m\begin{pmatrix} -x \\ -y \\ e \end{pmatrix}, & \text{if } z<0 \text{ and } z-\sqrt{x^2+y^2} < -m \\ 0, & \text{otherwise} \end{cases} \tag{3}$$

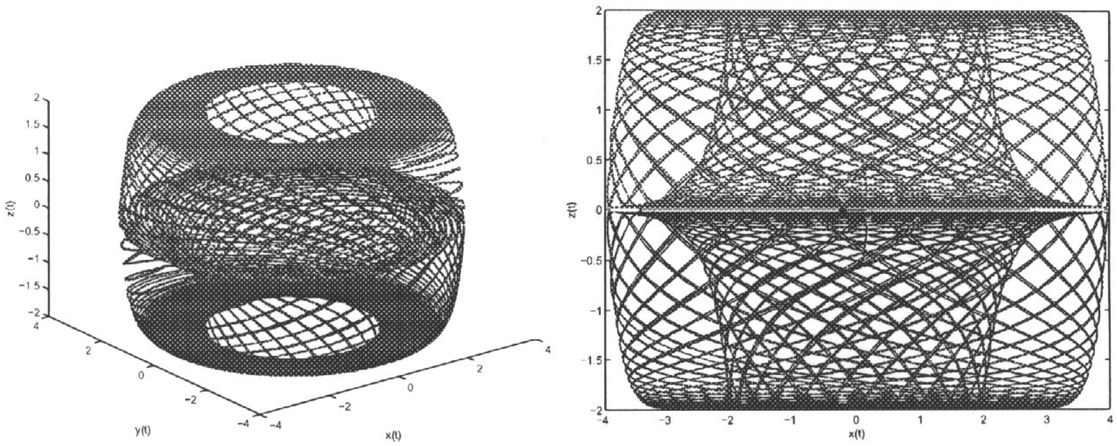


Fig. 1. Upper and lower chaotic attractors generated by the switching controller (3).

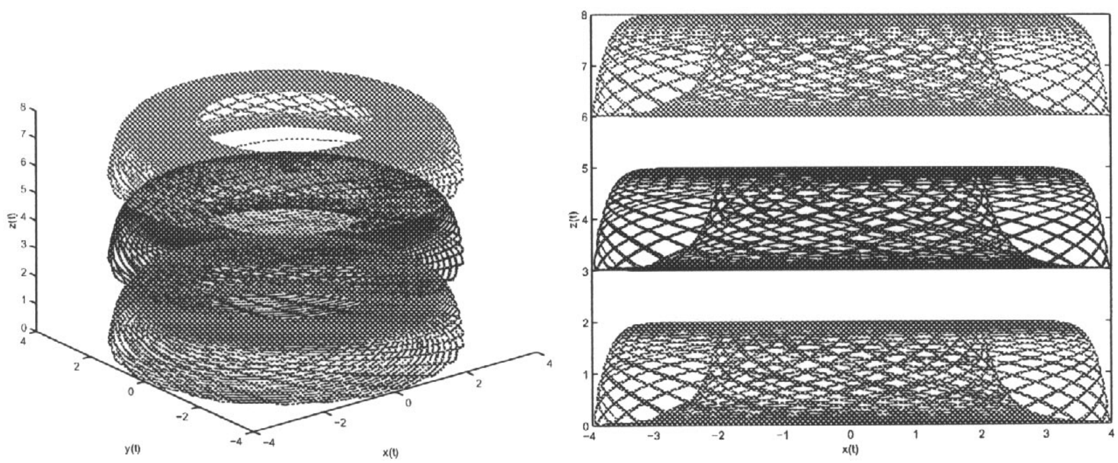


Fig. 2. Three chaotic attractors created by the switching controller (4).

where a, b, c, d, e, k, m are real parameters. Under this controller, the controlled system (1)–(3) can simultaneously generate two chaotic attractors, an upper attractor and a lower attractor, as shown in Fig. 1, when $a = 3$, $b = 20$, $c = -20$, $d = 10$, $e = -10$, $k = 4$, and $m = 4$ [11]. The maximum Lyapunov exponents of the two attractors are both $LE = 1.4374$. It is noticed that $z = 0$ is the invariant manifold of (1)–(3).

A closer look at the controller (3) reveals that it has two switchings: one is on the surface $z+\sqrt{x^2+y^2} = k$ ($z > 0$) denoted by S_1, and the other is on the surface $z-\sqrt{x^2+y^2} = -m$ ($z < 0$) denoted by S_2. Taking parameters $k = m$ and $d = -e$, it is found that the above two switching planes S_1 and S_2 are symmetrical about the invariant manifold $z = 0$. Furthermore, for any point (x, y, z) of the upper attractor, $(x, y, -z)$ belongs to the lower attractor, that is, the upper attractor and the lower attractor are symmetrical with respect to the plane $z = 0$ shown in Fig. 1. In fact, the lower attractor can be attained by turning the upper attractor around the plane $z = 0$ by 180 deg.

Similarly, one can easily create n attractors simultaneously in the switching system (1) by two transforms, parallel displacement and rotation. In fact, since the chaotic attractor is bounded by a finite sphere, one can partition the whole space into n disjoint subspaces, and then duplicate the original attractor, the upper attractor or the lower attractor into every subspace. For example, if one partitions the whole space into two subspaces $\{(x, y, z)|z > 0\}$ and $\{(x, y, z)|z < 0\}$, and then duplicates the upper attractor into subspace $\{(x, y, z)|z < 0\}$, and finally overturns the upper attractor, then, two attractors are obtained simultaneously as shown in Fig. 1.

Now, to further generate three chaotic attractors simultaneously in (1), the controller (2) is restructured as follows:

$$f(X)=\begin{cases} \text{if } z \le h_1 \\ \quad \begin{cases} k\begin{pmatrix} -x \\ -y \\ d \end{pmatrix}, & \text{if } z+\sqrt{x^2+y^2} > k \\ 0, & \text{otherwise} \end{cases} \\ \text{if } h_1 < z \le h_2 \\ \quad \begin{cases} k\begin{pmatrix} -x \\ -y \\ d-ch_1/k \end{pmatrix}, & \text{if, } z-h_1+\sqrt{x^2+y^2} > k \\ 0, & \text{otherwise} \end{cases} \\ \text{if, } z > h_2 \\ \quad \begin{cases} k\begin{pmatrix} -x \\ -y \\ d-ch_2/k \end{pmatrix}, & \text{if } z-h_2+\sqrt{x^2+y^2} > k \\ 0, & \text{otherwise} \end{cases} \end{cases} \tag{4}$$

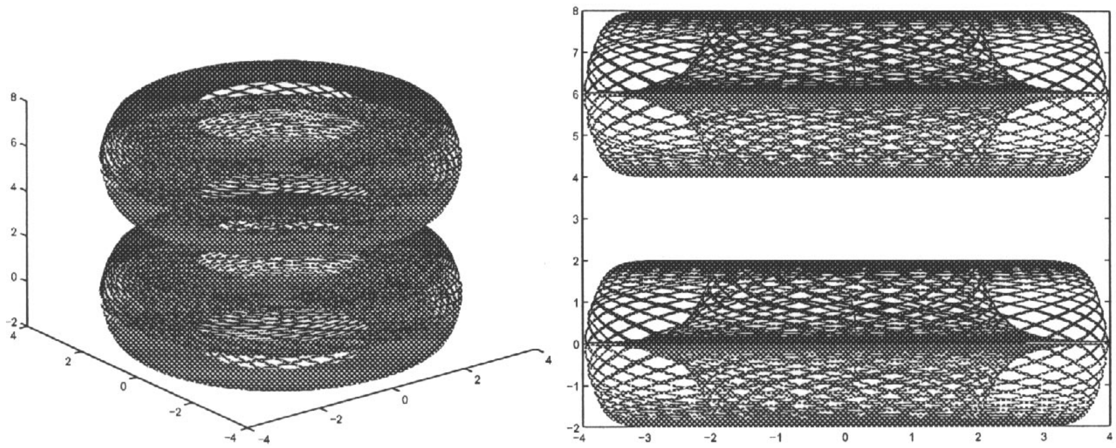


Fig. 3. Four chaotic attractors (upper and lower) generated by the switching controller (5).

where a, b, c, d, k, h_1, h_2 are real parameters. The controlled system (1)–(4) can simultaneously generate three chaotic attractors as shown in Fig. 2, when $a = 3$, $b = 20$, $c = -20$, $d = 10$, $k = 4$, $h_1 = 3$, and $h_2 = 6$. It is noticed that $z = 0$, $z = 3$, and $z = 6$ are all invariant manifolds of system (1)–(4).

To simultaneously create four chaotic attractors (upper and lower) in system (1), the controller (3) is furthermore restructured as follows:

$$f(X) = \begin{cases} \text{if } z \le h_1 & \begin{cases} k\begin{pmatrix} -x \\ -y \\ d \end{pmatrix}, & \text{if } z > 0 \text{ and } z + \sqrt{x^2+y^2} > k \\ m\begin{pmatrix} -x \\ -y \\ e \end{pmatrix}, & \text{if } z < 0 \text{ and } z - \sqrt{x^2+y^2} < -m \\ 0, & \text{otherwise} \end{cases} \\ \text{if } z > h_1 & \begin{cases} k\begin{pmatrix} -x \\ -y \\ d - \dfrac{c(h+h_1)}{k} \end{pmatrix}, & \text{if } z-(h+h_1)>0 \text{ and } \\ & z-h-h_1+\sqrt{x^2+y^2}>k \\ m\begin{pmatrix} -x \\ -y \\ e - \dfrac{c(h+h_1)}{m} \end{pmatrix}, & \text{if } z-(h+h_1)<0 \text{ and } \\ & z-h-h_1-\sqrt{x^2+y^2}<-m, \\ 0, & \text{otherwise} \end{cases} \end{cases} \tag{5}$$

where a, b, c, d, e, k, h, h_1, m are real parameters. The controlled system (1)–(5) can simultaneously generate four chaotic attractors (upper and lower) as shown in Fig. 3, when $a = 3$, $b = 20$, $c = -20$, $d = -e = 10$, $k = m = 4$, $h_1 = 3$, and $h = 3$.

The above procedure can be carried on and on, so as to generate n attractors using only parallel displacement and rotation transformations. Since the controlled system (1) has a natural symmetry under the coordinates transform $(x, y, z) \rightarrow (-x, -y, z)$, one can partition the whole space along the z axis into n subspaces, with heights $h_1, h_2, \ldots, h_n$, respectively, as shown in Fig. 4. Then, one can duplicate the original attractor into every subspaces, thereby generating n attractors simultaneously in (1). It is noticed that one only needs to segment and displace the z axis. In fact, $z - h_i$ ($i = 1, \ldots, n$) were used above to substitute for z in the controller (2) or (3), and it is very easy to modify the controller this way.

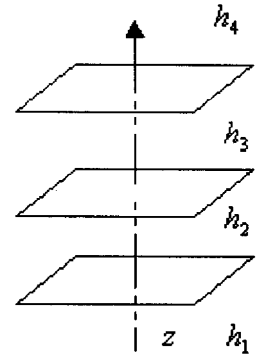


Fig. 4. Subspaces of (1).

However, it must be noted that the above n attractors are independent of one another. That is, there is no system orbit that connects all attractors together.

What is more interesting is actually a single and yet complex chaotic attractor that has multiple merged basins of attraction. This is the topic of study in the next section.

III. Generating Chaotic Attractors With Multiple Merged Basins of Attraction

In this section, the above-designed controllers are further restructured for generating chaotic attractors with multiple merged basins of attraction in the controlled system (1).

A closer look at the controller (3) reveals that it has two switchings on the surfaces S_1 and S_2 separately, which are responsible for the creation of the two chaotic attractors. The switching on the surface S_1 is responsible for creating the upper-chaotic attractor; while the switching on the surface S_2

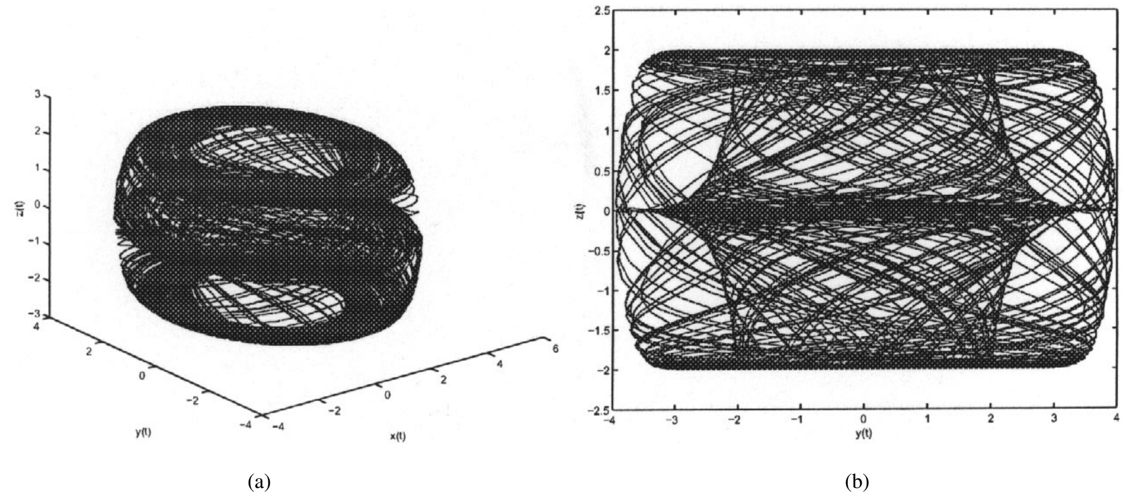


Fig. 5. Chaotic attractor with two merged basins of attraction, generated by the switching controller (6).

creates the lower-chaotic attractor. It is noticed that the plane $z = 0$ is the invariant manifold of (1), which partitions the whole space into two invariant subspaces, $\{(x, y, z)|z > 0\}$ and $\{(x, y, z)|z < 0\}$. It means that the orbit of the upper attractor will remain in the subspace $\{(x, y, z)|z > 0\}$, and not enter into the subspace $\{(x, y, z)|z < 0\}$. Similarly, the orbit of the lower attractor will not enter the subspace $\{(x, y, z)|z > 0\}$.

To connect together the orbits of the upper and lower attractors, one must use control to force the orbit of the upper-attractor to go through the plane $z = 0$ and then enter into the subspace $\{(x, y, z)|z < 0\}$. At the same time, the control should force the orbit of the lower attractor to go through the plane $z = 0$ and then return to the subspace $\{(x, y, z)|z > 0\}$. Based on this observation, $\delta\,(0,\,0,\,-\text{sign}(z))^\top$ is used to substitute for 0 in the controller (3), therefore yielding the following new controller:

$$f(X) = \begin{cases} k\begin{pmatrix} -x \\ -y \\ d \end{pmatrix}, & \text{if } z > 0 \text{ and } z + \sqrt{x^2+y^2} > k \\ m\begin{pmatrix} -x \\ -y \\ e \end{pmatrix}, & \text{if } z < 0 \text{ and} \\ & \quad z - \sqrt{x^2+y^2} < -m \\ \delta\begin{pmatrix} 0 \\ 0 \\ -\text{sign}(z) \end{pmatrix}, & \text{otherwise} \end{cases} \tag{6}$$

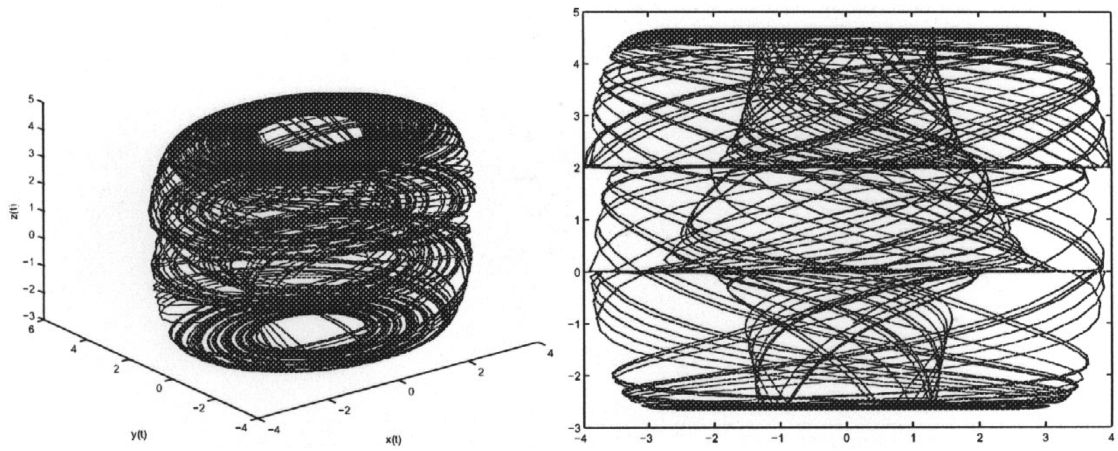
where $a, b, c, d, e, k, m, \delta$ are all real parameters.

In the controller (6), $\delta > 0$ is the control gain, and $\text{sign}(z)$ is the control direction. When $z > 0$, that is, when the orbit is in the subspace $\{(x, y, z)|z > 0\}$, the negative control $-\delta < 0$ is added to force the orbit to enter into the subspace $\{(x, y, z)|z < 0\}$; when $z < 0$, that is, when the orbit is in the subspace $\{(x, y, z)|z < 0\}$, the positive control $\delta > 0$ is added to force the orbit to enter into the subspace $\{(x, y, z)|z > 0\}$. Thus, one can actually connect the orbits of the upper and lower attractors together, thereby forming a single and complex chaotic attractor. The result is shown in Fig. 5(a), where $a = 3, b = 20$, $c = -20, d = -e = 10, k = m = 4$, and $\delta = 1$. Fig. 5(b) is the x–z plane projection of the attractor.

It is clear from Fig. 5 that the chaotic attractor has two merged basins of attraction, the upper basin of attraction and the lower basin of attraction. If the orbit starts from inside of the subspace $\{(x, y, z)|z > 0\}$, then, it will run inside the upper basin of attraction for some time, then move into the subspace $\{(x, y, z)|z < 0\}$ and run inside the lower basin of attraction for some time, and finally return to the subspace $\{(x, y, z)|z > 0\}$ and then restart a new cycle.

A closer look at the controller (6) reveals that it has three switching planes: S_1, S_2, and $z = 0$, in which the two switching planes S_1 and S_2 are responsible for the generation of two chaotic attractors, the upper chaotic attractor and the lower chaotic attractor; while the switching plane $z = 0$ is responsible for the connection of these two chaotic attractors.

Similarly, one can generate a chaotic attractor with three merged basins of attraction. In doing so, the controller (6) is furthermore modified. One first partitions the whole space into two subspaces $\{(x, y, z)|z > h\}$ and $\{(x, y, z)|z < h\}$. In subspace $\{(x, y, z)|z < h\}$, the controlled system (1) has a chaotic attractor with two merged basins of attraction, as seen in Fig. 5, while in subspace $\{(x, y, z)|z > h\}$, (1) has an upper attractor, as seen in Fig. 1. To generate the intended chaotic attractor with three merged basins of attraction, one has to connect the upper attractor to the chaotic attractor with two merged basins of attraction. Control is used to force the orbit of the upper attractor to go through the plane $z = h$ and then enter into the subspace $\{(x, y, z)|z < h\}$. At the same time, the control also forces the orbit of the chaotic attractor, the one that has two merged basins of attraction, to go through the plane $z = h$ and then return to the subspace $\{(x, y, z)|z > h\}$.



(a) (b)

Fig. 6. Chaotic attractor with three merged basins of attraction, generated by the switching controller (7).

To do so, $\delta\ (0, 0, -\text{sign}(z-h) - ch/\delta)^\top$ is employed to substitute for 0 in the controller (2), resulting in the following new controller:

$$f(X) = \begin{cases} \text{if } z < h & \begin{cases} k\begin{pmatrix} -x \\ -y \\ d \end{pmatrix}, & \text{if } z>0 \text{ and } z+\sqrt{x^2+y^2}>k \\ m\begin{pmatrix} -x \\ -y \\ e \end{pmatrix}, & \text{if } z<0 \text{ and } z-\sqrt{x^2+y^2}<-m \\ \delta\begin{pmatrix} 0 \\ 0 \\ -\text{sign}(z) \end{pmatrix}, & \text{otherwise} \end{cases} \\ \text{if } z \geq h, & \begin{cases} k\begin{pmatrix} -x \\ -y \\ d-\dfrac{ch}{k} \end{pmatrix}, & \text{if } z-h+\sqrt{x^2+y^2}>k \\ \delta\begin{pmatrix} 0 \\ 0 \\ -\text{sign}(z-h)-\dfrac{ch}{\delta} \end{pmatrix}, & \text{otherwise} \end{cases} \end{cases} \tag{7}$$

where $a, b, c, d, e, k, m, h, \delta$ are all real parameters.

In controller (7), $\delta > 0$ is the control gain, and $\text{sign}(z)$ and $\text{sign}(z-h)$ are the control directions. When $z > h$, that is, when the orbit is in the subspace $\{(x, y, z)|z > h\}$, the negative control $-\delta < 0$ is added to force the orbit to enter into the subspace $\{(x, y, z)|z < h\}$; when $z < h$, that is, when the orbit is in the subspace $\{(x, y, z)|z < h\}$, the positive control $\delta > 0$ is added to force the orbit to enter into the subspace $\{(x, y, z)|z > h\}$. This way, the orbits of the upper attractor and the chaotic attractor with two merged basins of attraction are connected together, forming a single but more complex chaotic attractor. The result is shown in Fig. 6(a), where $a = 3$, $b = 20$, $c = -15$, $d = -e = 10$, $k = m = 4$, $h = 2$, and $\delta = 5$. Fig. 6(b) is the x–z plane projection of the attractor.

It is clear from Fig. 6 that the chaotic attractor has three merged basins of attraction: two upper basins of attraction and one lower basin of attraction. If the orbit starts from inside of the subspace $\{(x, y, z)|z > 2\}$, then it will run inside the first upper basin of attraction for some time, then go into the subspace $\{(x, y, z)|0 < z < 2\}$ and then run inside the second upper basin of attraction for some time, and then go into the subspace $\{(x, y, z)|z < 0\}$ and run inside the lower basin of attraction for some time, and then return to the subspace $\{(x, y, z)|0 < z < 2\}$ and run inside the second upper basin of attraction for some time, and finally return to the subspace $\{(x, y, z)|z > 2\}$ and then restart a new cycle.

It is clear that the controller (7) has five switching planes, $z = 0$, $z = 2$, $z + \sqrt{x^2+y^2} = k$, $z - \sqrt{x^2+y^2} = -m$, and $z - h + \sqrt{x^2+y^2} = k$. Among them, three switching planes, $z + \sqrt{x^2+y^2} = k$, $z - \sqrt{x^2+y^2} = -m$, and $z - h + \sqrt{x^2+y^2} = k$, are responsible for generating three chaotic attractors separately, and the other two switching planes, $z = 0$ and $z = 2$, are responsible for connecting the above three chaotic attractors together so as to form a single and complex chaotic attractor.

Similarly, chaotic attractors with n merged basins of attraction can also be generated. The formalized design method is summarized as follows.

1) Partition the whole space into n subspaces along the z-axis, as shown in Fig. 4.
2) Duplicate the original attractors, the upper-attractor and the lower-attractor, to every subspaces.
3) Use the switching control strategy to connect all the n independent attractors, so as to form a single complex chaotic attractor, as depicted by Fig. 7.

Here, the switching control strategy can be chosen as $\delta\,\text{sign}(z - h_i)$, where the height h_i (between two neighboring subspaces) should be smaller than the height of a single chaotic attractor.

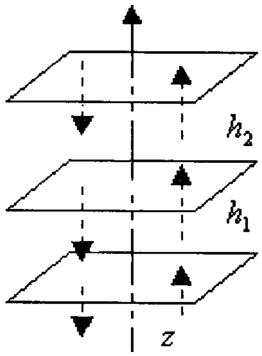


Fig. 7. Illustrative sketch for the connection of orbit.

IV. Dynamical Behaviors of Switching Controlled System

Dynamical behaviors, such as symmetry, dissipativity, fixed points, and the structure of system orbit of the chaotic system (1) under the control of the switching piecewise-linear controller (6) and (7), respectively, are further investigated in this section.

A. Symmetry

Obviously, (1), controlled by the switching piecewise-linear controller (6) or (7), has a natural symmetry under the coordinates transform $(x, y, z) \to (-x, -y, z)$, which persists for all values of the system parameters.

B. Dissipativity and Existence of Attractor

First, consider the controlled system (1) with controller (6), where it is assumed that $a > 0$, $c < -2a$, $k > 0$, and $m > 0$.

The variation of the volume $V(t)$ of a small element, $\delta\Omega(t) = \delta x\, \delta y\, \delta z$, in the state space is determined by the divergence of the flow

$$\nabla V = \frac{\partial \dot{x}}{\partial x} + \frac{\partial \dot{y}}{\partial y} + \frac{\partial \dot{z}}{\partial z}$$

which is

$$\nabla V = \begin{cases} 2a+c-2k < 0, & \text{for } z > 0,\ z+\sqrt{x^2+y^2} > k \\ 2a+c-2m < 0, & \text{for } z < 0,\ z-\sqrt{x^2+y^2} < -m \\ 2a+c < 0, & \text{otherwise.} \end{cases}$$

Therefore, (1) is dissipative at an exponential contraction rate

$$\delta\Omega(t) = \begin{cases} e^{(2a+c-2k)t}, & \text{for } z > 0,\ z+\sqrt{x^2+y^2} > k \\ e^{(2a+c-2m)t}, & \text{for } z < 0,\ z-\sqrt{x^2+y^2} < -m \\ e^{(2a+c)t}, & \text{otherwise.} \end{cases}$$

Hence, a volume element V_0 is contracted by the flow into a volume element $V_0 e^{\nabla V t}$ in time t. That is, each volume containing the system orbit shrinks to zero as $t \to \infty$ at an exponential rate, ∇V, which is independent of x, y, z. Consequently, all system orbits will ultimately be confined to a specific subset of zero volume and the asymptotic motion settles onto an attractor.

Similarly, consider the controlled system (1) with the controller (7), where it is assumed that $a > 0$, $c < -2a$, $k > 0$, $h > 0$, and $m > 0$. In this case, one has

$$\nabla V = \begin{cases} 2a+c-2m<0, & \text{for } z<0,\ z-\sqrt{x^2+y^2}<-m \\ 2a+c-2k<0, & \text{for } \begin{array}{l} 0<z<h,\ z+\sqrt{x^2+y^2}>k \\ z>h,\ z-h+\sqrt{x^2+y^2}>k \end{array} \\ 2a+c<0, & \text{otherwise.} \end{cases}$$

Therefore, the controlled system (1)–(7) is also dissipative and all the system orbits will ultimately be confined to a specific subset of zero volume, with the asymptotic motion settling onto an attractor.

C. Equilibria and Stability

Again, first consider the controlled system (1)–(6), where it is assumed that $b \neq 0$, $c \neq 0$, $m > 0$, and $k > 0$. Then the following conditions hold.

i) If $-(d/c) > 1$, $e/c > 1$ and $\delta/c > 0$, then the controlled system has four equilibria: $S_1(0, 0, -(kd/c))$, $S_2(0, 0, -(me/c))$, $S_3(0, 0, \delta/c)$, and $S_4(0, 0, -(\delta/c))$.

ii) If $-(d/c) > 1$, $e/c > 1$ and $\delta/c < 0$, then the controlled system has two equilibria: $S_1(0, 0, -(kd/c))$ and $S_2(0, 0, -(me/c))$.

iii) If $-(d/c) < 1$, $e/c < 1$ and $\delta/c > 0$, then the controlled system has two equilibria: $S_3(0, 0, \delta/c)$ and $S_4(0, 0, -(\delta/c))$.

iv) If $-(d/c) < 1$, $e/c < 1$ and $\delta/c < 0$, then the controlled system has no equilibrium point.

v) If $-(d/c) > 1$, $e/c < 1$ and $\delta/c > 0$, then the controlled system has three equilibria: $S_1(0, 0, -(kd/c))$, $S_3(0, 0, \delta/c)$, and $S_4(0, 0, -(\delta/c))$.

vi) If $-(d/c) > 1$, $e/c < 1$ and $\delta/c < 0$, then the controlled system has one equilibrium point: $S_1(0, 0, -(kd/c))$.

vii) If $-(d/c) < 1$, $e/c > 1$ and $\delta/c > 0$, then the controlled system has three equilibria: $S_2(0, 0, -(me/c))$, $S_3(0, 0, \delta/c)$, and $S_4(0, 0, -(\delta/c))$.

viii) If $-(d/c) < 1$, $e/c > 1$ and $\delta/c < 0$, then the controlled system has one equilibrium point: $S_2(0, 0, -(me/c))$.

Now, consider the equilibria $S_3(0, 0, \delta/c)$ and $S_4(0, 0, -(\delta/c))$. The system Jacobian J at these two points are both equal to

$$J = \begin{pmatrix} a & b & 0 \\ -b & a & 0 \\ 0 & 0 & c \end{pmatrix} \tag{8}$$

which has eigenvalues $\lambda_{1,2} = a \pm b\mathrm{i}$ and $\lambda_3 = c$. Hence, the stability of the two equilibria, $S_3(0, 0, \delta/c)$ and $S_4(0, 0, -(\delta/c))$, can be summarized as follows.

i) If $a > 0$ or $c > 0$, then these two equilibria are both unstable.

ii) If $a < 0$ and $c < 0$, then these two equilibria are both stable.

Furthermore, it is noticed that with $c<0$, $a=0$ is a Hopf bifurcation point.

Next, when $-(d/c)>1$, the system Jacobian at the equilibrium $S_1(0,\,0,\,-(kd/c))$ is

$$J=\begin{pmatrix} a-k & b & 0\\ -b & a-k & 0\\ 0 & 0 & c\end{pmatrix} \tag{9}$$

and its eigenvalues are $\lambda_{1,\,2}=a-k\pm bi$ and $\lambda_3=c$. Obviously, with $c<0$, $a=k$ is a Hopf bifurcation point. And the stability of S_1 is classified as follows.

i) If $a>k$ and $c\neq 0$, then S_1 is unstable.
ii) If $a<k$ and $c>0$, then S_1 is unstable.
iii) If $a<k$ and $c<0$, then S_1 is stable.

Similarly, when $e/c>1$, the system Jacobian for the equilibrium $S_2(0,\,0,\,-(me/c))$ is

$$J=\begin{pmatrix} a-m & b & 0\\ -b & a-m & 0\\ 0 & 0 & c\end{pmatrix} \tag{10}$$

which has eigenvalues $\lambda_{1,\,2}=a-m\pm bi$ and $\lambda_3=c$. Clearly, with $c<0$, $a=m$ is a Hopf bifurcation point, and the stability of this equilibrium, $S_2(0,\,0,\,-(me/c))$, is summarized as follows.

i) If $a>m$ and $c\neq 0$, then S_2 is unstable.
ii) If $a<m$ and $c>0$, then S_2 is unstable.
iii) If $a<m$ and $c<0$, then S_2 is stable.

In a similar manner, one can discuss the controlled system (1)–(7). Assume that $b\neq 0$, $c\neq 0$, $m>0$, $h>0$ and $k>0$. Then the following hold.

i) If $-(d/c)>1$, $e/c>1$ and $\delta/c>0$, then the controlled system has six equilibria: $S_1(0,\,0,\,-(kd/c))$, $S_2(0,\,0,\,-(me/c))$, $S_3(0,\,0,\,\delta/c)$, $S_4(0,\,0,\,-(\delta/c))$, $S_5(0,\,0,\,h-(kd/c))$, and $S_6(0,\,0,\,h+(\delta/c))$.
ii) If $-(d/c)>1$, $e/c>1$ and $\delta/c<0$, then the controlled system has three equilibria: $S_1(0,\,0,\,-(kd/c))$, $S_2(0,\,0,\,-(me/c))$, and $S_5(0,\,0,\,h-(kd/c))$.
iii) If $-(d/c)<1$, $e/c<1$ and $\delta/c>0$, then the controlled system has three equilibria: $S_3(0,\,0,\,\delta/c)$, $S_4(0,\,0,\,-(\delta/c))$, and $S_6(0,\,0,\,h+(\delta/c))$.
iv) If $-(d/c)<1$, $e/c<1$ and $\delta/c<0$, then the controlled system has no equilibrium point.
v) If $-(d/c)>1$, $e/c<1$ and $\delta/c>0$, then the controlled system has five equilibria: $S_1(0,\,0,\,-(kd/c))$, $S_3(0,\,0,\,\delta/c)$, $S_4(0,\,0,\,-(\delta/c))$, $S_5(0,\,0,\,h-(kd/c))$, and $S_6(0,\,0,\,h+(\delta/c))$.
vi) If $-(d/c)>1$, $e/c<1$ and $\delta/c<0$, then the controlled system has two equilibria: $S_1(0,\,0,\,-(kd/c))$ and $S_5(0,\,0,\,h-(kd/c))$.
vii) If $-(d/c)<1$, $e/c>1$ and $\delta/c>0$, then the controlled system has four equilibria: $S_2(0,\,0,\,-(me/c))$, $S_3(0,\,0,\,\delta/c)$, $S_4(0,\,0,\,-(\delta/c))$, and $S_6(0,\,0,\,h+(\delta/c))$.
viii) If $-(d/c)<1$, $e/c>1$ and $\delta/c<0$, then the controlled system has one equilibrium point: $S_2(0,\,0,\,-(me/c))$.

Obviously, the equilibria S_3, S_4, S_6 have the same stability, which can be summarized as follows.

i) If $a>0$ or $c>0$, then these three equilibria are all unstable.
ii) If $a<0$ and $c<0$, then these three equilibria are all stable.

Next, for the equilibria S_1 and S_5, they also have the same stability as follows.

i) If $a>k$ and $c\neq 0$, then S_1 and S_5 are both unstable.
ii) If $a<k$ and $c>0$, then S_1 and S_5 are both unstable.
iii) If $a<k$ and $c<0$, then S_1 and S_5 are both stable.

Finally, the equilibrium S_2 has the following stability.

i) If $a>m$ and $c\neq 0$, then S_2 is unstable.
ii) If $a<m$ and $c>0$, then S_2 is unstable.
iii) If $a<m$ and $c<0$, then S_2 is stable.

V. Qualitative Analysis of Switching Controlled System

A. Qualitative Analysis of Controlled System (1)–(6)

Consider the controlled system (1)–(6). Define four regions, Σ_1: $\{(x,\,y,\,z)|z>0,\, z+\sqrt{x^2+y^2}>k\}$, Σ_2: $\{(x,\,y,\,z)|z>0,\, z+\sqrt{x^2+y^2}\leq k\}$, Σ_3: $\{(x,\,y,\,z)|z<0,\, z-\sqrt{x^2+y^2}\geq -m\}$, and Σ_4: $\{(x,\,y,\,z)|z<0,\, z-\sqrt{x^2+y^2}<-m\}$. The controlled system (1)–(6) is then parameterized. When $z>0$ and $z+\sqrt{x^2+y^2}>k$, (1) is

$$\begin{cases}\dot{x}=(a-k)x+by\\ \dot{y}=-bx+(a-k)y\\ \dot{z}=cz+kd.\end{cases} \tag{11}$$

Let $x=r\cos\theta$, $y=r\sin\theta$. Then, (11) becomes

$$\begin{cases}\dot{r}=(a-k)r\\ \dot{\theta}=-b\\ \dot{z}=c\left(z+\dfrac{kd}{c}\right).\end{cases} \tag{12}$$

Hence, solving (12) gives the solution

$$\begin{cases}r=r_0e^{(a-k)t}\\ \theta=\theta_0-bt\\ z=\left(z_0+\dfrac{kd}{c}\right)e^{ct}-\dfrac{kd}{c}.\end{cases} \tag{13}$$

Similarly, when $z<0$ and $z-\sqrt{x^2+y^2}<-m$, the solution of system (1) is

$$\begin{cases}r=r_0e^{(a-m)t}\\ \theta=\theta_0-bt\\ z=\left(z_0+\dfrac{me}{c}\right)e^{ct}-\dfrac{me}{c}.\end{cases} \tag{14}$$

For $z\geq 0$ and $z+\sqrt{x^2+y^2}\leq k$, on the other hand, the solution is

$$\begin{cases}r=r_0e^{at}\\ \theta=\theta_0-bt\\ z=\left(z_0-\dfrac{\delta}{c}\right)e^{ct}+\dfrac{\delta}{c}.\end{cases} \tag{15}$$

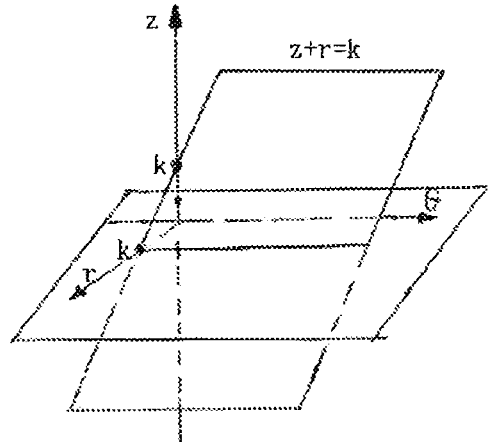


Fig. 8. Upper switching plane of the controlled system (1).

Finally, the solution for $z < 0$ and $z - \sqrt{x^2+y^2} \geq -m$ is

$$\begin{cases} r = r_0 e^{at} \\ \theta = \theta_0 - bt \\ z = \left(z_0 + \dfrac{\delta}{c}\right) e^{ct} - \dfrac{\delta}{c}. \end{cases} \tag{16}$$

Thus, the controlled system (1)–(6) can be classified into the four systems, from (13)–(16).

The system parameters must satisfy $a > 0, c < 0, a < k, a < m, \delta > 0$ for generating chaos in system (1). And, since the upper switching plane is $z + r = k (z > 0)$, as shown in Fig. 8, when the initial point (r_0, θ_0, z_0) is above the plane $z + r = k (z > 0)$, the dynamical behavior of (1) satisfies (13). That is, when $t \to +\infty$, one has $r = r_0 e^{(a-k)t} \to 0$, $z \to -(kd/c)$, and $z + r \to -(kd/c)$.

To create chaos in (1), the system orbit must go through the plane $z + r = k$ at a certain instant t_1, for which $0 < d < -c$. After this instant t_1, the orbit of (1) goes into region Σ_2, and the dynamical equation is described by (15). When $t \to \infty$, one has $r = r_0 e^{at} \to \infty$ and $z \to \delta/c < 0$. Hence, the orbit will go through the plane $z = 0$ and then go into region Σ_3 at some instant t_2. In region Σ_3, the orbit satisfies (16). So, when $t \to \infty$, one has $r = r_0 e^{at} \to \infty$, $z \to -(\delta/c) > 0$, and $z - r \to -\infty < -m$. Therefore, the orbit will go through the switching plane $z - r = -m$ at a certain instant t_3 and then enter into region Σ_4. Now, the orbit satisfies (14). When $t \to +\infty$, one has $r = r_0 e^{(a-m)t} \to 0$, $z \to -(me/c)$, and $z - r \to -(me/c)$.

In order to make (1) create chaos, the system orbit must go through the switching plane $z - r = -m$ at some instant t_4, for which $c < e < 0$. After this instant t_4, the orbit goes into region Σ_3. Since $z \to -(\delta/c) > 0$ as $t \to \infty$, the orbit will go through the plane $z = 0$ and then enter into region Σ_2 again at a certain instant t_5. Similarly, since $z + r \to +\infty$ as $t \to \infty$, the orbit will go through the plane $z + r = k$ and then return to the original region Σ_1 again at some instant t_6.

The system orbit will repeat the above motions again and again, eventually forming a single but complex chaotic attractor.

According to the above theoretical analysis, a *necessary condition* for chaos generation for the controlled system (1)–(6) is: $a > 0, c < 0, 0 < d < -c, c < e < 0, a < k \leq a - c$, $a < m \leq a - c$, and $\delta > 0$.

Therefore, for any initial value $(r_0, \theta_0, z_0) \in \Sigma_1$, as $t \to +\infty$, the orbit of the controlled system (1)–(6) will go through three switching planes $z + \sqrt{x^2+y^2} = k$, $z = 0$, and $z - \sqrt{x^2+y^2} = -m$, repeatedly for infinitely many times. The system has different dynamical behaviors inside the four different regions, $\Sigma_i, i = 1, \ldots, 4$, whose dynamical equations are given by (13), (15), (16), and (14), respectively. When $t \to +\infty$, the system changes its dynamical behaviors (folding and stretching dynamics) repeatedly as the orbit goes through the four regions alternately and repeatedly, leading to very complex dynamics such as the appearance of bifurcations and chaos.

Finally, some numerical results are presented. Let $a = 3$, $b = 20$, $c = -20$, $d = -e = 10$, $k = m = 4$, and $\delta = 1$. The controlled system (1)–(6) has a chaotic attractor with two merged basins of attraction, as seen in Fig. 5. Fig. 9(a) shows the directions of the orbit of the attractor, indicated by the arrows therein. From Fig. 9(a), one can see that the orbit runs through in the following sequence:

$$\Sigma_1 \to \Sigma_2 \to \Sigma_3 \to \Sigma_4 \to \Sigma_3 \to \Sigma_2 \to \Sigma_1.$$

B. Qualitative Analysis of Controlled System (1)–(7)

Now, consider the controlled system (1)–(7). Define six regions, Σ_1: $\{(x, y, z) | z > h, z - h + \sqrt{x^2+y^2} > k\}$, Σ_2: $\{(x, y, z) | z > h, z - h + \sqrt{x^2+y^2} \leq k\}$, Σ_3: $\{(x, y, z) | 0 < z \leq h, z + \sqrt{x^2+y^2} > k\}$, Σ_4: $\{(x, y, z) | 0 < z \leq h, z + \sqrt{x^2+y^2} \leq k\}$, Σ_5: $\{(x, y, z) | z < 0, z - \sqrt{x^2+y^2} \geq -m\}$, and Σ_6: $\{(x, y, z) | z < 0, z - \sqrt{x^2+y^2} < -m\}$.

Similarly, parameterize the controlled system (1) and solve for its solutions. If $z \geq h$ and $z - h + r > k$, one has

$$\begin{cases} r = r_0 e^{(a-k)t} \\ \theta = \theta_0 - bt \\ z = \left(z_0 + \dfrac{kd}{c} - h\right) e^{ct} + h - \dfrac{kd}{c}. \end{cases} \tag{17}$$

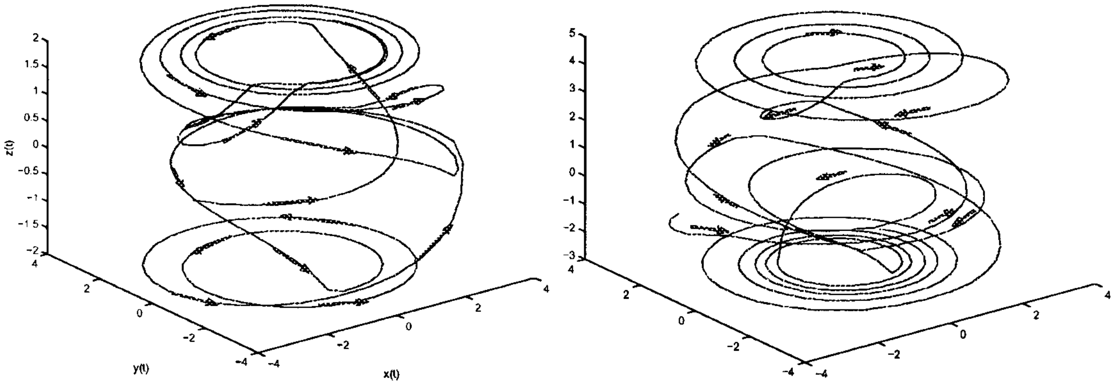
If $z \geq h$ and $z - h + r \leq k$, the solution is

$$\begin{cases} r = r_0 e^{at} \\ \theta = \theta_0 - bt \\ z = \left(z_0 - h - \dfrac{\delta}{c}\right) e^{ct} + h + \dfrac{\delta}{c}. \end{cases} \tag{18}$$

For $0 < z < h$ and $z + r > k$, the solution is the same as (13). For $0 < z < h$ with $z + r \leq k$, for $z \leq 0$ with $z - r > -m$, and for $z \leq 0$ with $z - r \leq -m$, the solutions are the same as (15), (16) and (14), respectively.

To generate chaos in (1)–(7), the parameters must satisfy $a > 0, c < 0, a < k, a < m, h > 0, \delta > 0$. The first upper switching plane is $z - h + r = k (z > h)$. When the initial point (r_0, θ_0, z_0) is above the plane $z - h + r = k (z > h)$, the dynamical behavior of (1) satisfies (17). That is, when $t \to +\infty$, one has $r = r_0 e^{(a-k)t} \to 0$, $z \to h - (kd/c)$, and $z + r \to h - (kd/c)$.

In order for (1) to generate chaos, the orbit of (1) must go through the plane $z - h + r = k$ at a certain instant t_1, for which $0 < d < -c$. After this instant t_1, the system orbit goes into



(a) (b)

Fig. 9. Structure of system orbit of the switching controlled system. (a) Two merged basins of attraction. (b) Three merged basins of attraction.

region Σ_2, and the governing dynamical equation is (18). When $t \to \infty$, one has $r = r_0 e^{at} \to \infty$, $z \to h + (\delta/c) < h$. Hence, the orbit will go through the plane $z = h$ and then go into region Σ_3 at some instant t_2. In region Σ_3, the orbit satisfies (13). And, when $t \to +\infty$, one has $r = r_0 e^{(a-k)t} \to 0$, $z \to -(kd/c)$, and $z + r \to -(kd/c)$.

The orbit of (1) must go through the plane $z + r = k$ at a certain instant t_3 for creating chaos in system (1). After this instant t_3, the system orbit goes into region Σ_4, and the dynamical equation is governed by (15). When $t \to \infty$, one has $r = r_0 e^{at} \to \infty$ and $z \to (\delta/c) < 0$. Hence, the orbit will go through the plane $z = 0$ and then go into region Σ_5 at some instant t_4. In region Σ_5, the orbit satisfies (16). So, when $t \to \infty$, one has $r = r_0 e^{at} \to \infty$, $z \to -(\delta/c) > 0$, and $z - r \to -\infty < -m$. Hence, the orbit will go through the switching plane $z - r = -m$ at a certain instant t_5 and then enter into region Σ_6, in which the orbit satisfies (14). When $t \to +\infty$, one has $r = r_0 e^{(a-m)t} \to 0$, $z \to -(me/c)$, and $z - r \to -(me/c)$.

For generating chaos in (1), the system orbit must go through the switching plane $z - r = -m$ at some instant t_6, for which $c < e < 0$. After this instant t_6, the system orbit goes into region Σ_5. Since $z \to -(\delta/c) > 0$ as $t \to \infty$, the orbit will go through the plane $z = 0$ and then enter into region Σ_4 again at a certain instant t_7. Since $z + r \to +\infty$ as $t \to \infty$, the orbit will go through the plane $z + r = k$ and then return to region Σ_3 at some instant t_8. In Σ_3, $z \to -(kd/c)$ as $t \to \infty$, so the system orbit must go through the plane $z = h$ at a certain instant t_9. After this instant t_9, the system orbit goes into region Σ_2, where the dynamics are governed by (18). Finally, notice that $z - h + r \to +\infty$ as $t \to \infty$. Therefore, the orbit will go through the plane $z - h + r = k$ and then return to the original region Σ_1 again at a certain instant t_{10}. The system orbit will then repeat the above process again and again, eventually forming a single but complex chaotic attractor.

According to the above theoretical analysis, a *necessary condition* for generating chaos in the controlled system (1)–(7) is: $a > 0$, $c < 0$, $0 < d < -c$, $c < e < 0$, $a < k \le a - c$, $a < m \le a - c$, $0 < h \le -(kd/c)$, and $\delta > 0$.

For any initial value $(r_0, \theta_0, z_0) \in \Sigma_1$, as $t \to +\infty$, the orbit of the controlled system (1)–(7) will go through five switching planes $z - h + r = k$, $z = h$, $z + r = k$, $z = 0$, and $z - r = -m$, repeatedly for infinitely many times. The controlled system has different dynamical behaviors in these six different regions, Σ_i, $i = 1, \ldots, 6$, whose dynamical equations are given by (17), (18), (13), (15), (16), and (14), respectively. When $t \to +\infty$, the system changes its dynamical behaviors (folding and stretching dynamics) repeatedly, as the orbit goes through the six regions repeatedly, leading to complex dynamics such bifurcations and chaos.

Finally, some numerical results are presented. Let $a = 3$, $b = 20$, $c = -15$, $d = -e = 10$, $k = m = 4$, $h = 2$, and $\delta = 5$. The controlled system (1)–(7) has a chaotic attractor with three merged basins of attraction, as seen in Fig. 6. Fig. 9(a) shows the directions of the attractor orbit, indicated by the arrows therein. From Fig. 9(b), one can see that the orbit runs along the following route:

$$\Sigma_1 \to \Sigma_2 \to \Sigma_3 \to \Sigma_4 \to \Sigma_5 \to \Sigma_6 \to \Sigma_5 \to \Sigma_4 \to \Sigma_3 \to \Sigma_2 \to \Sigma_1.$$

VI. Conclusions

This paper has presented several new chaos generators, i.e., some new switching piecewise-linear controllers. These chaos generators are simple in structure but are capable of generating complex chaotic attractors with multiple merged basins of attraction from a given three-dimensional linear autonomous system within a wide range of parameter values. Basic dynamical behaviors of the controlled chaotic system have also been investigated via both theoretical analysis and numerical simulation. Moreover, the underlying chaos generation mechanism has been explored by analyzing the parameterization of the controlled system and the dynamics of the system orbits.

It has been known, and verified once again in this paper, that abundant complex dynamical behaviors can be generated by piecewise-linear functions if designed appropriately. This paper provides a simple and viable design method for generating some seemingly complicated chaotic attractors with multiple merged basins of attraction. Although this chaos synthesis method is mainly focused on a special class of piecewise-linear systems, it seems to have great potential to be further generalized to some (piecewise but not necessarily piecewise-linear) switching controlled systems. Therefore, this initiative should motivate more research efforts in the studies of switching systems for chaos generation.

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