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GENERATING FUNCTION FOR GENERALIZED FUNCTION OF TWO VARIABLES

B. L. SHARMA AND R. F. A. ABIODUN

ABSTRACT. In this paper we obtain a generating function for a generalized function of two variables. The result is very general in character and includes as particular cases some of the results recently given by Meijer [4], Carlitz [7] and Srivastava [5].

In a recent paper, Sharma [2] has defined the generalized function of two variables as follows:

$$S\left[x, y\left|\left[\begin{matrix} m_1, 0 \\ p_1, q_1 \end{matrix}\right]\begin{matrix} a_{p1} \\ b_{q1}\end{matrix}\right|\left(\begin{matrix} n_2, & m_2 \\ p_2, & q_2 \end{matrix}\right)\begin{matrix} c_{p2} \\ d_{q2}\end{matrix}\left|\left(\begin{matrix} n_3, & m_3 \\ p_3, & q_3 \end{matrix}\right)\begin{matrix} e_{p3} \\ f_{q3}\end{matrix}\right|\right]$$

$$(1) \qquad = \frac{1}{(2\pi i)^2}\int_{c_1}\int_{c_2} \frac{\prod_{j=1}^{m1}\Gamma(a_j+s+t)\prod_{j=1}^{m2}\Gamma(1-c_j+s)\prod_{j=1}^{n2}\Gamma(d_j-s)}{\prod_{j=m_1+1}^{p1}\Gamma(1-a_j-s-t)\prod_{j=1}^{q1}\Gamma(b_j+s+t)\prod_{j=m_2+1}^{p2}\Gamma(c_j-s)}$$

$$\cdot \frac{\prod_{j=1}^{m3}\Gamma(1-e_j+t)\prod_{j=1}^{n3}\Gamma(f_j-t)x^s y^t\,ds\,dt}{\prod_{j=n_2+1}^{q2}\Gamma(1-d_j+s)\prod_{j=m_3+1}^{p3}\Gamma(e_j-t)\prod_{j=n_3+1}^{q3}\Gamma(1-f_j+t)}$$

where c_1 and c_2 are two suitable contours and positive integers $p_1, p_2, p_3, q_1, q_2, q_3, m_1, m_2, m_3, n_2$ and n_3 satisfy the following inequalities. $q_2 \geq 1$, $q_3 \geq 1$, $p_1 \geq 0$, $0 \leq m_1 \leq p_1$, $0 \leq m_2 \leq p_2$, $0 \leq n_2 \leq q_2$, $0 \leq m_3 \leq p_3$, $0 \leq n_3 \leq q_3$, $p_1 + p_2 \leq q_1 + q_2$, $p_1 + p_3 \leq q_1 + q_3$. The values $x = 0$ and $y = 0$ are excluded.

We shall make use of the formula [6, Vol. I, Example 212, p. 126]

$$(2) \qquad \sum_{n=0}^{\infty}\binom{a+(b+1)n}{n}t^n = \frac{(1+v)^{a+1}}{1-bv},$$

where v is a function of t given by $v(0) = 0$, $v = t(1+v)^{b+1}$. We establish the following formula:

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(3)
$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}\, S\left[x, y \,\middle|\, \begin{bmatrix} m_1+1, & 0 \\ p_1+1, & q_1+1 \end{bmatrix} \begin{matrix} \alpha - n\beta, a_{p_1} \\ \alpha - n - n\beta, b_{q_1} \end{matrix} \,\middle|\, \begin{pmatrix} n_2, & m_2 \\ p_2, & q_2 \end{pmatrix} \begin{matrix} c_{p_2} \\ d_{q_2} \end{matrix} \,\middle|\, \begin{pmatrix} n_3, & m_3 \\ p_3, & q_3 \end{pmatrix} \begin{matrix} e_{p_3} \\ f_{q_3} \end{matrix} \right] t^n$$
$$= \frac{(1+v)^{1-\alpha}}{1-\beta v}\, S\left[\frac{x}{1+v}, \frac{y}{1+v} \,\middle|\, \begin{bmatrix} m_1, & 0 \\ p_1, & q_1 \end{bmatrix} \begin{matrix} a_{p_1} \\ b_{q_1} \end{matrix} \,\middle|\, \begin{pmatrix} n_2, & m_2 \\ p_2, & q_2 \end{pmatrix} \begin{matrix} c_{p_2} \\ d_{q_2} \end{matrix} \,\middle|\, \begin{pmatrix} n_3, & m_3 \\ p_3, & q_3 \end{pmatrix} \begin{matrix} e_{p_3} \\ f_{q_3} \end{matrix} \right]$$

where v is a function of t defined by $v(0) = 0$, $v = t(1+v)^{\beta+1}$, $|t| < 1$, $|\arg x| < (m_1 + m_2 + n_2 - \tfrac{1}{2}p_1 - \tfrac{1}{2}q_1 - \tfrac{1}{2}p_2 - \tfrac{1}{2}q_2)\pi$, $|\arg y| < (m_1 + m_3 + n_3 - \tfrac{1}{2}p_1 - \tfrac{1}{2}q_1 - \tfrac{1}{2}p_2 - \tfrac{1}{2}q_2)\pi$, $2(m_1 + m_2 + n_2) > p_1 + q_1 + p_2 + q_2$, $2(m_1 + m_3 + n_3) > p_1 + q_1 + p_3 + q_3$.

To prove (3), substitute the contour integral (1) for the generalized function of two variables in the left side of (3), change the order of integration and summation, using (2) and (1); we get the right side of (3).

We mention some of the interesting particular cases of (3). Using the relation due to Sharma [3]

(4)
$$S\left[x, y \,\middle|\, \begin{bmatrix} 0, & 0 \\ 0, & 0 \end{bmatrix}^{-}_{-} \,\middle|\, \begin{pmatrix} n_2, & m_2 \\ p_2, & q_2 \end{pmatrix} \begin{matrix} c_{p_2} \\ d_{q_2} \end{matrix} \,\middle|\, \begin{pmatrix} n_3, & m_3 \\ p_3, & q_3 \end{pmatrix} \begin{matrix} e_{p_3} \\ f_{q_3} \end{matrix} \right]$$
$$= G^{n_2, m_2}_{p_2, q_2}\left[x \,\middle|\, \begin{matrix} c_{p_2} \\ d_{q_2} \end{matrix}\right] G^{n_3, m_3}_{p_3, q_3}\left[y \,\middle|\, \begin{matrix} e_{p_3} \\ f_{q_3} \end{matrix}\right]$$

in (3), we have

(5)
$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}\, S\left[x, y \,\middle|\, \begin{bmatrix} 1, & 0 \\ 1, & 1 \end{bmatrix} \begin{matrix} \alpha - n\beta \\ \alpha - n - n\beta \end{matrix} \,\middle|\, \begin{pmatrix} n_2, & m_2 \\ p_2, & q_2 \end{pmatrix} \begin{matrix} c_{p_2} \\ d_{q_2} \end{matrix} \,\middle|\, \begin{pmatrix} n_3, & m_3 \\ p_3, & q_3 \end{pmatrix} \begin{matrix} e_{p_3} \\ f_{q_3} \end{matrix} \right] t^n$$
$$= \frac{(1+v)^{1-\alpha}}{1-\beta v}\, G^{n_2, m_2}_{p_2, q_2}\left[\frac{x}{1+v} \,\middle|\, \begin{matrix} c_{p_2} \\ d_{q_2} \end{matrix}\right] G^{n_3, m_3}_{p_3, q_3}\left[\frac{y}{1+v} \,\middle|\, \begin{matrix} e_{p_3} \\ f_{q_3} \end{matrix}\right],$$

provided that v is a function of t defined above and $|t| < 1$, $2(m_2 + n_2) > p_2 + q_2$, $2(m_3 + n_3) > p_3 + q_3$, $|\arg x| < (m_2 + n_2 - \tfrac{1}{2}p_2 - \tfrac{1}{2}q_2)\pi$, $|\arg y| < (m_3 + n_3 - \tfrac{1}{2}p_3 - \tfrac{1}{2}q_3)\pi$. Further we use the relation due to Sharma [3]

(6)
$$S\left[x, y \,\middle|\, \begin{bmatrix} 1, & 0 \\ 1, & 1 \end{bmatrix}^{\lambda}_{\mu} \,\middle|\, \begin{pmatrix} 1, & 1 \\ 1, & 1 \end{pmatrix} \begin{matrix} 1-\rho_1 \\ 0 \end{matrix} \,\middle|\, \begin{pmatrix} 1, & 1 \\ 1, & 1 \end{pmatrix} \begin{matrix} 1-\rho_2 \\ 0 \end{matrix} \right]$$
$$= \frac{\Gamma(\lambda)\Gamma(\rho_1)\Gamma(\rho_2)}{\Gamma(\mu)}\, F_1[\lambda; \rho_1, \rho_2; \mu; -x, -y],$$

(for Appell function F_1, see [1, p. 224 (6)]) in (5), we have

$$\sum_{n=0}^{\infty} \frac{(-1)^n\Gamma(\alpha - n\beta)}{n!\Gamma(\alpha - n\beta - n)} F_1[\alpha - n\beta; \rho_1, \rho_2; \alpha - n - n\beta; x, y]t^n = \frac{(1+v)^{1-\alpha}}{1-\beta v}\left(1 - \frac{x}{1+v}\right)^{-\rho_1}\left(1 - \frac{y}{1+v}\right)^{-\rho_2}. \tag{7}$$

In case we use the relation due to Erdélyi [1, Equation (3), p. 1216] and using the notation for hypergeometric functions of higher order and of two variables due to Chaundy [8] in (5), we get

$$\sum_{n=0}^{\infty} \frac{(-1)^n\Gamma(\alpha - n\beta)}{n!\Gamma(\alpha - n - n\beta)} F\begin{bmatrix}\alpha - n\beta; -; -; -x, -y \\ \alpha - n - n\beta; 1+\rho_1; 1+\rho_2\end{bmatrix}t^n = \frac{(1+v)^{1-\alpha}\Gamma(1+\rho_1)\Gamma(1+\rho_2)}{1-\beta v}\left(\frac{x}{1+v}\right)^{-\rho_1/2}\left(\frac{y}{1+v}\right)^{-\rho_2/2} \cdot J_{\rho_1}\left(2\left(\frac{x}{1+v}\right)^{1/2}\right)J_{\rho_2}\left(2\left(\frac{y}{1+v}\right)^{1/2}\right). \tag{8}$$

In case $y = 0$ in (8), it gives

$$\sum_{n=0}^{\infty} \frac{(-1)^n\Gamma(\alpha - n\beta)}{n!\Gamma(\alpha - n - n\beta)} {}_1F_2(\alpha - n\beta; \alpha - n - n\beta, 1+\rho_1; -x)t^n = \frac{\Gamma(1+\rho_1)(1+v)^{1-\alpha}}{1-\beta v}\left(\frac{x}{1+v}\right)^{-\rho_1/2} J_{\rho_1}\left(2\left(\frac{x}{1+v}\right)^{1/2}\right). \tag{9}$$

We use Erdélyi's formula [1, (57), p. 220] in (5) to obtain

$$\sum_{n=0}^{\infty} \frac{(-1)^n\Gamma(\alpha - n\beta)}{n!\Gamma(\alpha - n - n\beta)} F\begin{bmatrix}\alpha - n\beta; 1/2; 1/2; -x, -y \\ \alpha - n - n\beta; 1 \pm \rho_1; 1 \pm \rho_2\end{bmatrix}t^n = \frac{(1+v)^{1-\alpha}\Gamma(1\pm\rho_1)\Gamma(1\pm\rho_2)}{(1-\beta v)} J_{\rho_1}\left(\left(\frac{x}{1+v}\right)^{1/2}\right) \cdot J_{-\rho_1}\left(\left(\frac{x}{1+v}\right)^{1/2}\right)J_{\rho_2}\left(\left(\frac{y}{1+v}\right)^{1/2}\right)J_{-\rho_2}\left(\left(\frac{y}{1+v}\right)^{1/2}\right). \tag{10}$$

Next we use the formula due to Sharma [3]

$$\operatorname*{Lt}_{x\to 0} S\left[x, y\left|\begin{bmatrix}m_1, 0 \\ p_1, q_1\end{bmatrix}_{1-b_{q_1}}^{1-a_{p_1}}\right|\begin{pmatrix}n_2, m_2 \\ p_2, q_2\end{pmatrix}_{d_{q_2}}^{c_{p_2}}\left|\begin{pmatrix}1, 0 \\ 0, 1\end{pmatrix}_0^-\right|\right] = G^{n_2, m_2+m_1}_{p_2+p_1, q_2+q_1}\left[y\left|\begin{matrix}a_1, \cdots, a_{m_1}, c_{p_2}, a_{m_1+1}, \cdots, a_{p_1} \\ d_{q_2}, b_{q_1}\end{matrix}\right.\right] \tag{11}$$

in (3); we have

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} G^{n_1, m_1+1}_{p_1+1, q_1+1}\left(x \,\middle|\, \begin{matrix} 1-\alpha+n\beta;\ c_{p_1} \\ d_{q_1},\ 1-\alpha+n+n\beta \end{matrix}\right) t^n = \frac{(1+v)^{1-\alpha}}{1-\beta v} G^{n_1, m_1}_{p_1, q_1}\left(\frac{x}{1+v} \,\middle|\, \begin{matrix} c_{p_1} \\ d_{q_1} \end{matrix}\right). \tag{12}$$

Also

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} G^{n_1+1, m_1}_{p_1+1, q_1+1}\left(x \,\middle|\, \begin{matrix} c_{p_1},\ \alpha-n-n\beta \\ \alpha-n\beta,\ d_{q_1} \end{matrix}\right) t^n = \frac{(1+v)^{1-\alpha}}{1-\beta v} G^{n_1, m_1}_{p_1, q_1}\left(x(1+v) \,\middle|\, \begin{matrix} c_{p_1} \\ d_{q_1} \end{matrix}\right). \tag{13}$$

$\beta = -1$ and $\alpha = 0$ in (12) reduces to a result due to Meijer [4, (33), p. 483].
$\beta = 0$, $\alpha = 1$ in (12) gives a result due to Meijer [4, (36), p. 485].
$\beta = 0$, $\alpha = 1$ in (13) gives a result due to Meijer [4, (37), p. 485].
$\beta = -1$, $\alpha = 0$ in (13) reduces to a result due to Meijer [4, (46), p. 487].

If we use the formula [1, (1), p. 215] in (12) we get a result due to Srivastava [5, (9), p. 591]. If we specialize the parameters of (12) we get the result due to Carlitz [7, (8), p. 826].

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF IFE, ILE-IFE, NIGERIA