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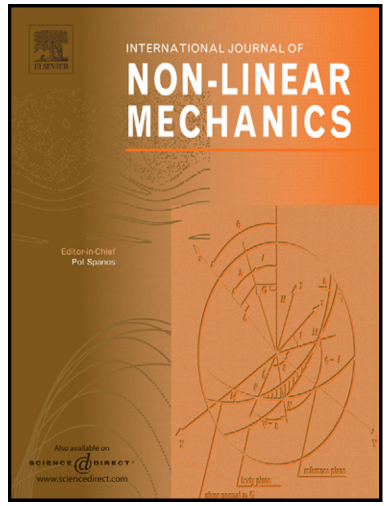


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# Generating functions for volume-preserving transformations

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**Abstract**

A general implicit solution for determining volume-preserving transformations in the $n$-dimensional Euclidean space is obtained in terms of a set of $2n$ generating functions in mixed coordinates. For $n = 2$, the proposed representation corresponds to the classical definition of a potential stream function in a canonical transformation. For $n = 3$, the given solution defines a more general class of isochoric transformations, *when* compared to existing methods based on multiple potentials. Illustrative examples are discussed both in rectangular and in cylindrical coordinates for applications in mechanical problems of incompressible continua. Solving exactly the incompressibility constraint, the proposed representation method is suitable for determining three-dimensional isochoric perturbations to be used in bifurcation theory. Applications in nonlinear elasticity are envisaged for determining the occurrence of complex instability patterns for soft elastic materials.

*Keywords:* Generating function, Canonical transformation, Incompressible material, Bifurcation theory, Nonlinear elasticity.

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# 1 Introduction

Considering a bounded region $\Omega_0$ in the n-dimensional Euclidean space, this work is aimed at defining generating functions for volume-preserving transformations of a set of continuously differentiable functions $u_j = u_j(U_1, U_2, ..., U_n) : \Omega_0 \to \Re$, with $j = 1, 2.., n$. Such an isochoric constraint can be expressed by a non-linear first-order partial differential equation as follows:

$$\mathrm{J}(\mathrm{U}_1, \mathrm{U}_2, ..., \mathrm{U}_\mathrm{n}) = \det \frac{\partial(\mathrm{u}_1, \mathrm{u}_2, ..., \mathrm{u}_\mathrm{n})}{\partial(\mathrm{U}_1, \mathrm{U}_2, ..., \mathrm{U}_\mathrm{n})} = 1 \qquad (1)$$

where $J$ is defined as the Jacobian of the transformation. The cases $n = 2, 3$ are of particular interests in continuum mechanics, because the functions $U_j$,$u_j$ can be treated as the material/spatial components of the position vectors $\mathbf{U}, \mathbf{u} = \mathbf{u}(\mathbf{U})$ in the reference/actual configuration, respectively. In such a case, the Jacobian defined in Eq.(1) corresponds to the determinant of the deformation tensor $\mathbf{F} = \mathrm{Grad}\ \mathbf{u} = \partial\mathbf{u}/\partial\mathbf{U}$, so that the functions $u_j$ determine the deformation fields for an incompressible material. For $n = 2$, the solution of Eq.(1) corresponds to an area-preserving transformation, as reported by Bateman (1918), who ascribed its first formulation to Gauss. Rooney and Carroll (1984) realized that such a solution could be expressed by an implicit representation through the definition of a stream function. Using this change of notation, the governing equations have the structure of Hamilton's canonical equations with one degree of freedom, therefore such a stream function can be regarded as a generating function for a canonical transform of planar coordinates. The extension of this solution to $n \geq 3$ was considered by Carroll (2004), who proposed an implicit representation by the means of $(n-1)$ potential functions, restricted by a set of $(n-1)$ admissibility conditions. Another implicit solution was later proposed by Knops (2005), transforming the problem to a linear first-order non-homogeneous differential equation by using prescribed cofactors in the expanded expression for the Jacobian, recovering the Carroll's expression for $n = 3$. Although representing complete solutions of the differential problem given by Eq.(1), both methods are given in implicit form and their application might be difficult for seeking explicit solutions with given boundary conditions imposed by the mechanical problem under consideration.

This work is organized as follows. In Section 2, the existing description of volume preserving transformation using coupled potential functions is analyzed, underlying its limitations for continuum mechanics applications. In Section 3, the definition of generating functions for volume preserving transformation is given for a general $n$-dimensional problem. The three-dimensional case is particularly examined, highlighting possible applications for stability problems in non-linear elasticity. The results are finally summarized in Section 4.

## 2 Limitations of existing solutions

In this paragraph, the solution for a generic isochoric deformation presented by Carroll (2004) is analyzed. Choosing $n = 3$ for the sake of simplicity, the volume preserving transformation is given in terms of two potential functions $\Phi(X, y, z)$ and $\psi(X, Y, z)$, referring to different mixed coordinate systems. The general solution takes the following implicit form:

$$x = \frac{\partial \Phi(X, y, z)}{\partial y} \tag{2}$$

$$Z = \frac{\partial \psi(X, Y, z)}{\partial Y} \tag{3}$$

$$\frac{\partial \Phi(X, y, z)}{\partial X} = \frac{\partial \psi(X, Y, z)}{\partial z} \tag{4}$$

In order to understand if the solution given by Eqs.(2-4) is able to represent a generic isochoric deformation, the multiplicative decomposition $\mathbf{F} = \mathbf{F}_1\mathbf{F}_2\mathbf{F}_3$ is introduced, representing the local changes of coordinates sketched in Figure 1.

It is straightforward to show that the local deformation gradients between the mixed coor-

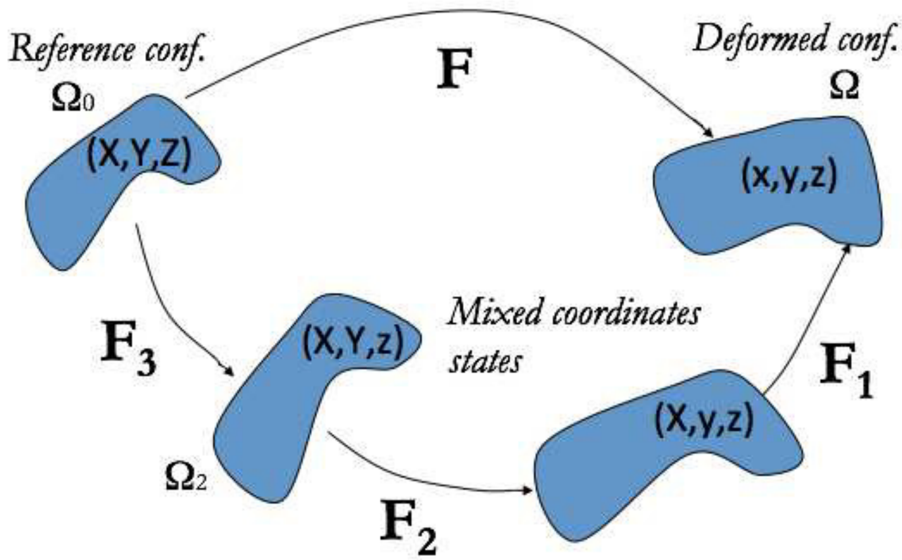


Figure 1: Multiplicative decomposition of the deformation gradient between the reference $(X, Y, Z)$ and the actual $(x, y, z)$ configurations, considering two intermediate states defined in mixed coordinates as $(X, Y, z)$ and $(X, y, z)$.

dinate states can be expressed as:

$$\mathbf{F}_1 = \begin{bmatrix} \Phi_{,yX} & \Phi_{,yy} & \Phi_{,yz} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \quad \mathbf{F}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{\psi_{,YX}}{\psi_{,Yz}} & -\frac{\psi_{,YY}}{\psi_{,Yz}} & \frac{1}{\psi_{,Yz}} \end{bmatrix}; \tag{5}$$

where comma denotes partial differentiation, and the admissibility condition $\psi_{,Yz} \neq 0$ is set to avoid local singularities. Similarly, the tensor $\mathbf{F}_2$ can be given with respect to $y = y(X, Y, z)$ and $Y = Y(X, y, z)$, as follows:

$$\mathbf{F}_2 = \begin{bmatrix} 1 & 0 & 0 \\ y_{,X} & y_{,Y} & y_{,z} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{Y_{,X}}{Y_{,y}} & \frac{1}{Y_{,y}} & -\frac{Y_{,z}}{Y_{,y}} \\ 0 & 0 & 1 \end{bmatrix}; \tag{6}$$

The incompressibility condition for the overall deformation can be derived using Eqs.(5,6) in the following form:

$$\det \mathbf{F} = \frac{\Phi_{,\mathrm{yX}}}{\psi_{,\mathrm{zY}} \cdot \mathrm{Y}_{,\mathrm{y}}} = \frac{\Phi_{,\mathrm{yX}} \cdot \mathrm{y}_{,\mathrm{Y}}}{\psi_{,\mathrm{zY}}} = 1 \tag{7}$$

which is identically satisfied imposing the condition in Eq.(4), together *with* the implicit representation given by Eqs.(2,3).

Using simple differentiation on both sides of Eq.(4) with respect to $Z$, the following identity also holds:

$$\Phi_{,\mathrm{yX}} \frac{\partial \mathrm{y}}{\partial \mathrm{Z}} = (\psi_{,\mathrm{zz}} - \Phi_{,\mathrm{Xz}}) \frac{\partial \mathrm{z}}{\partial \mathrm{Z}} = 0 \tag{8}$$

which reveals that the volume preserving transformation in the solution given by Carroll (2004) imposes $\partial y/\partial Z = 0$, being limited to a particular deformation field. Moreover, such an implicit representation is unable to derive explicitly the expression of the transformation of the $y$ coordinate, limiting its practical utility for finding explicit solutions in continuum mechanics problems. In the following, the use of generating functions is investigated to define a generic n-dimensional isochoric *transformation*.

## 3 Definition of generating functions for volume-preserving transformations

In classical mechanics, canonical transformations are used in order to preserve area changes in the displacements fields, based on the definition of generating functions of mixed (one material, one spatial) coordinates which allow to define implicit relations between coordinates belonging to the same framework (Sewell and Roulstone, 1993). In the following, the definition of generating functions is given for generic volume preserving transformations, first for the three-dimensional case and, secondly, for a general $n$-dimensional problem.

### 3.1 Isochoric displacement fields in rectangular coordinates

The definition of volume-preserving transformations using a three-dimensional generating function is investigated in the following. Dealing with a generic three-dimensional deformation in rectangular coordinates, one can try to extend the classical methodology using a mixed coordinate state $(X, Y, z)$, so that a multiplicative decomposition $\mathbf{F} = \mathbf{F}_a\mathbf{F}_b$ can be imposed, as depicted in Figure 2.

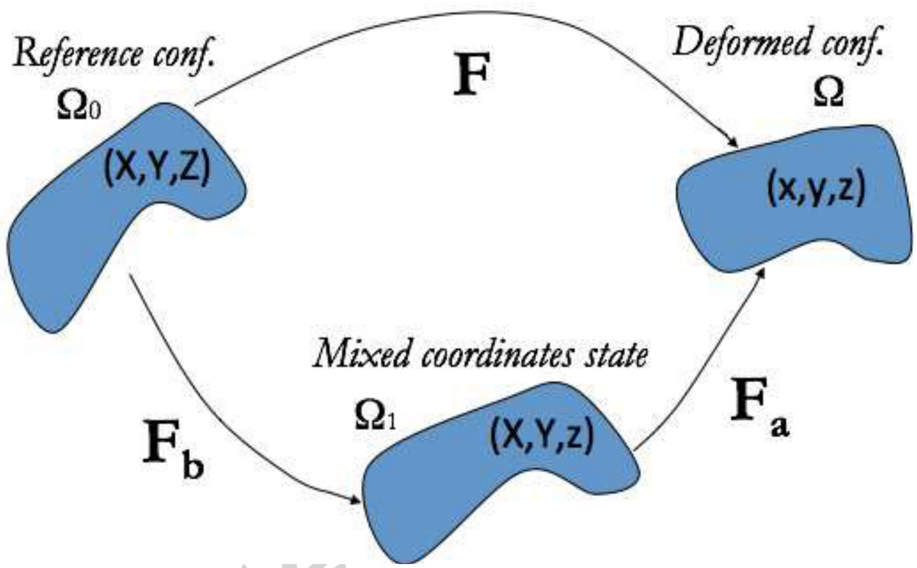


Figure 2: Multiplicative decomposition of the deformation gradient between the reference $(X, Y, Z)$ and the actual $(x, y, z)$ configurations, considering an intermediate state $(X, Y, z)$ defined in mixed coordinates.



Assuming the existence of a generating function $f(X, Y, z)$, the following implicit relations between coordinates are defined as:

$$x = \frac{\partial^2 f(X, Y, z)}{\partial Y \partial z} \tag{9}$$

$$y = \frac{\partial^2 f(X, Y, z)}{\partial X \partial z} \tag{10}$$

where the expression of $Z = Z(X, Y, z)$ has to be determined from the incompressibility constraint. According to the implicit representation given by Eqs.(9, 10), the local deformation tensors can be expressed as follows:

$$\mathbf{F}_a = \begin{bmatrix} f_{,XYz} & f_{,YYz} & f_{,Yzz} \\ f_{,XXz} & f_{,XYz} & f_{,Xzz} \\ 0 & 0 & 1 \end{bmatrix}; \qquad \mathbf{F}_b = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{Z_{,X}}{Z_{,z}} & -\frac{Z_{,Y}}{Z_{,z}} & \frac{1}{Z_{,z}} \end{bmatrix} \tag{11}$$

Looking for isochoric solutions of the differential problem, the incompressibility condition $\det\mathbf{F} = 1$ is fulfilled by choosing the following implicit representation for the $Z$ coordinate:

$$Z = \int^{z} \left(f^2_{,XY\eta}(X,Y,\eta) - f_{,XX\eta}(X,Y,\eta) \cdot f_{,YY\eta}(X,Y,\eta)\right) d\eta + g(X,Y) \tag{12}$$

where g is an arbitrary function, and we must set $\partial Z/\partial z \neq 0$ in order to avoid local singularities. Looking for applications in continuum mechanics, an illustrative example is given by using the following expression for the generating function:

$$\mathrm{f(X,Y,z) = XYz + \epsilon \cdot h(z)\ \sin(k_x\ X)\ \sin(k_y\ Y)} \tag{13}$$

where $h(z)$ is a generic function of $z$. Using the implicit coordinate transformations in Eqs.(9, 10, 12) and considering $\epsilon$ as a small parameter, the displacements fields are defined at first order in $\epsilon$ as follows:

$$\begin{cases} \mathrm{x = X + \epsilon \cdot k_y\ h'(z)\ \sin(k_x X)\ \cos(k_y Y)} \\ \mathrm{y = Y + \epsilon \cdot k_x\ h'(z)\ \cos(k_x X)\ \sin(k_y Y)} \\ \mathrm{Z = z + \epsilon \cdot 2k_y k_x\ h(z)\ \cos(k_x X)\ \cos(k_y Y)} \end{cases} \tag{14}$$

The solution given by Eq.(14) represents a z-dependent sinusoidal perturbation of the $(X,Y)$-planes, having modes $k_x, k_y$ along the axes $X$ and $Y$, as depicted in Figure 3.

Such a perturbation corresponds to the displacements fields in the elastic solution given by Ben Amar and Ciarletta (2010) (see Eqs. 41-46 therein). *In the Appendix A, this transformation is applied to derive the equilibrium equation for the surface instability pattern arising in the*

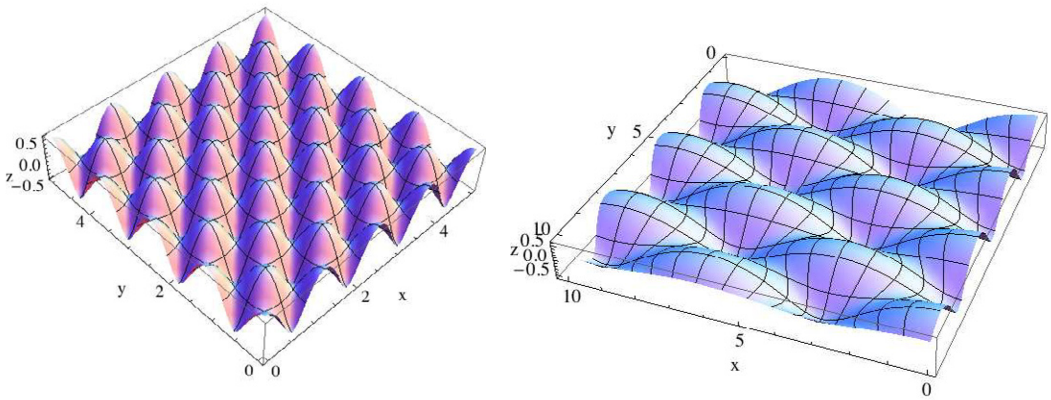


Figure 3: Isochoric deformations of the plane $Z = 0$ obtained using the generating function expressed by Eq.(13). The displacements fields are depicted for $k_x = k_y = 5$ (left, $\epsilon = 0.01$) and for $k_x = 2, k_y = 1$ (right, $\epsilon = 0.15$), setting $h(0) = h^{'}(0) = 1$.

*biaxial growth of a surface-attached soft layer, using a variational method in nonlinear elasticity.* The profiles given by Eq.(14) in the $(x, z)$ and $(y, z)$ planes represent generalized curtate cycloids, whose asymmetry indicates the possibility of cusp formation in the nonlinear regime.

### 3.2 Isochoric displacement fields in cylindrical coordinates

The three-dimensional description of an isochoric transformation in cylindrical coordinates is considered for its importance in growth instabilities of tubular tissues in continuum biomechanics. In particular, the aim of this paragraph is to determine a generating function $f(R, Z, \theta)$ for the volume-preserving transformation. By the means of the multiplicative decomposition through the intermediate mixed coordinate state, the local tensorial components in terms of $r = r(R, Z, \theta)$, $z = z(R, Z, \theta)$ and $\Theta = \Theta(R, Z, \theta)$ read:

$$\mathbf{F}_a = \begin{bmatrix} r_{,R} & r_{,Z} & \frac{r_{,\theta}}{R} \\ z_{,R} & z_{,Z} & \frac{z_{,\theta}}{R} \\ 0 & 0 & \frac{r}{R} \end{bmatrix}; \qquad \mathbf{F}_b = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -R\frac{\Theta_{,r}}{\Theta_{,\theta}} & -R\frac{\Theta_{,z}}{\Theta_{,\theta}} & \frac{1}{\Theta_{,\theta}} \end{bmatrix} \tag{15}$$

so that the incompressibility constraint can be expressed as follows:

$$\det \mathbf{F} = \frac{\mathrm{r\ r_{,R}\ z_{,Z}}}{\mathrm{R}\ \Theta_{,\theta}} = \frac{(\mathrm{r}^2/2)_{,\mathrm{R}}\ \mathrm{z}_{,\mathrm{Z}}}{\mathrm{R}\ \Theta_{,\theta}} = 1 \tag{16}$$

Imposing $\Theta_{,\theta} \neq 0$ for avoiding local singularities, an implicit isochoric transformation can be derived from Eq.(16), having the following properties:

$$r^2 = 2\frac{\partial^2 f(R,Z,\theta)}{\partial Z \partial \theta} \tag{17}$$

$$z = \frac{1}{R}\ \frac{\partial^2 f(R,Z,\theta)}{\partial R \partial \theta} \tag{18}$$

$$\Theta = \frac{1}{\mathrm{R}^2}\int^{\theta}\left(\mathrm{f(R,Z,\eta)}^2_{,\mathrm{RZ}\eta} - \mathrm{f(R,Z,\eta)}_{,\mathrm{RR}\eta}\mathrm{f(R,Z,\eta)}_{,\mathrm{ZZ}\eta} + \frac{\mathrm{f(R,Z,\eta)}_{,\mathrm{R}\eta}\mathrm{f(R,Z,\eta)}_{,\mathrm{ZZ}\eta}}{\mathrm{R}}\right)\mathrm{d}\eta + \mathrm{g(R,Z)} \tag{19}$$

where $g(R,Z)$ is a generic function of the material coordinates. Two illustrative examples of isochoric transformation obtained using Eqs.(17- 19) are presented for possible application in elastic stability problems. First, a generating function is given with the following expression:

$$\mathrm{f(R,Z,\theta)} = \frac{(\mathrm{R}^2+\mathrm{a})\mathrm{Z}\theta}{2} + \epsilon\cdot\sqrt{\mathrm{R}^2+\mathrm{a}}\ \mathrm{h}(\sqrt{\mathrm{R}^2+\mathrm{a}})\ \sin(\mathrm{k_z\ Z})\ \sin(\mathrm{k}_\theta\ \theta) \tag{20}$$

where $h(\sqrt{R^2+a})$ is a generic function of $\sqrt{R^2+a}$ and $a$ is a constant. If $\epsilon$ is a small parameter, such a generating function represents a perturbation on a generic inhomogeneous deformation state, whose displacements fields are defined at first order in $\epsilon$ as:

$$\begin{cases} \mathrm{r} = \sqrt{\mathrm{R}^2+\mathrm{a}} + \epsilon\cdot\mathrm{k_z k_\theta}\ \mathrm{h}'(\sqrt{\mathrm{R}^2+\mathrm{a}})\ \cos(\mathrm{k_z Z})\ \cos(\mathrm{k}_\theta\theta) \\ \mathrm{z} = \mathrm{Z} + \epsilon\cdot\mathrm{k}_\theta\left(\frac{\mathrm{h}(\sqrt{\mathrm{R}^2+\mathrm{a}})}{\sqrt{\mathrm{R}^2+\mathrm{a}}} + \mathrm{h}'(\sqrt{\mathrm{R}^2+\mathrm{a}})\right)\ \cos(\mathrm{k}_\theta\theta)\ \sin(\mathrm{k_z Z}) \\ \Theta = \theta + \epsilon\cdot 2\mathrm{k_Z}\ \left(\frac{\mathrm{h}(\sqrt{\mathrm{R}^2+\mathrm{a}})}{\sqrt{\mathrm{R}^2+\mathrm{a}}} + \mathrm{h}'(\sqrt{\mathrm{R}^2+\mathrm{a}})\right)\ \sin(\mathrm{k}_\theta\theta)\ \cos(\mathrm{k_z Z}) \end{cases} \tag{21}$$

The isochoric transformation described by Eq.(21) describes a sinusoidal perturbation of an axisymmetric elastic solution both in the longitudinal and the circumferential directions, having modes $k_z$ and $k_\theta$, respectively. The shape of a perturbed cylindrical surface is shown in Figure 4 (left); the application of such an isochoric transformation is suitable for a morphoelastic analysis of villi formation in the intestinal mucosa.

A second example is given considering the following expression of the generating function:

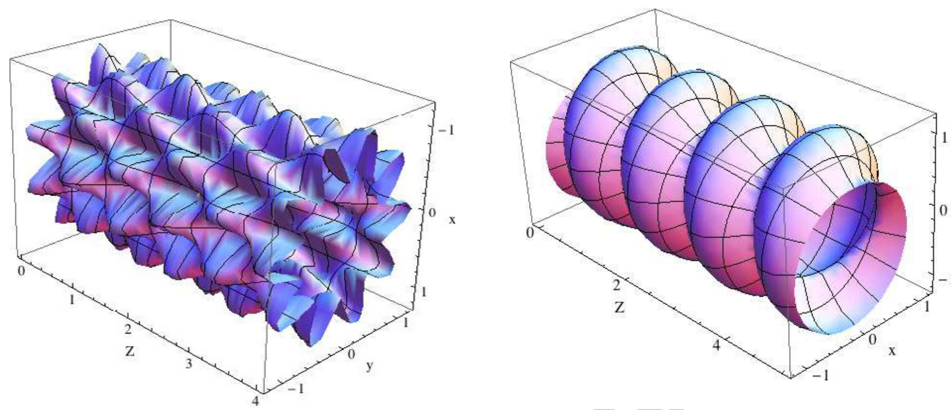

Figure 4: Isochoric deformations of the surface $R = 1$ obtained using the generating functions expressed by Eq.(20) (left, with $k_z = 4$, $k_\theta = 10$, $\epsilon = 0.005$) and Eq.(20)(right, with $k_z = 5$, $k_\theta = 1$, $\epsilon = 0.05$). The solutions are calculated setting $a = 0$ and $h(1) = h^{'}(1) = 1$.

$$\mathrm{f(R,Z,\theta)} = \frac{\mathrm{(R^2+a)Z\theta}}{2} + \epsilon\cdot\sqrt{\mathrm{R^2+a}}\ \mathrm{h}(\sqrt{\mathrm{R^2+a}})\ \sin(\mathrm{k_z\ Z - k_\theta\ \theta}) \tag{22}$$

Using Eqs.(17, 18, 19), the implicit transformations of coordinates at first order in $\epsilon$ read:

$$\begin{cases} \mathrm{r} = \sqrt{\mathrm{R^2+a}} + \epsilon\cdot \mathrm{k_z k_\theta h}(\sqrt{\mathrm{R^2+a}})\ \sin(\mathrm{k_z\ Z - k_\theta\ \theta}) \\ \mathrm{z} = \mathrm{Z} - \epsilon\cdot \mathrm{k_\theta}\left(\frac{\mathrm{h}(\sqrt{\mathrm{R^2+a}})}{\sqrt{\mathrm{R^2+a}}} + \mathrm{h}^{'}(\sqrt{\mathrm{R^2+a}})\right)\ \cos(\mathrm{k_z\ Z - k_\theta\ \theta}) \\ \Theta = \theta + \epsilon\cdot 4\mathrm{k_z}\ \left(\frac{\mathrm{h}(\sqrt{\mathrm{R^2+a}})}{\sqrt{\mathrm{R^2+a}}} + \mathrm{h}^{'}(\sqrt{\mathrm{R^2+a}})\right)\ \sin(\mathrm{k_\theta}\theta/2)\ \sin(\mathrm{k_z\ Z - k_\theta\ \theta/2}) \end{cases} \tag{23}$$

It is straightforward to show that Eq.(23) represents an helicoidal perturbation of an axisymmetric elastic solution characterized by an inhomogeneous deformation state, where $k_\theta$ defines the number of perturbed helices having longitudinal wavenumber $k_z$. The helicoidal deformation described by Eqs.(22, 23) for a cylindrical surface with circular section is shown in Figure

4 (right). *Finally, it is useful to highlight that, while the displacement fields in Eq.(14) can be rewritten explicitly, the two isochoric transformations in Eqs.(21,23) are intrinsically implicit.*

### 3.3 General solution in the n-dimensional case

The general case of a transformation of $n$ functions $u_i = u_i(U_1, ..., U_j, ..., U_n)$ of $n$ variables $U_j$, with $i, j = 1, 2, ..., n$ is considered in the following. Fixing the Jacobian of the transformation equal to one, the differential problem is given as follows:

$$\mathrm{J} = \det \begin{vmatrix} \frac{\partial u_1}{\partial U_1} & \frac{\partial u_1}{\partial U_2} & \cdots & \frac{\partial u_1}{\partial U_n} \\ \frac{\partial u_2}{\partial U_1} & \cdots & \cdots & \frac{\partial u_2}{\partial U_n} \\ \vdots & & & \vdots \\ \frac{\partial u_n}{\partial U_1} & \frac{\partial u_n}{\partial U_2} & \cdots & \frac{\partial u_n}{\partial U_n} \end{vmatrix} = 1 \tag{24}$$

Fixing a mixed coordinate state in the n-dimensional case, we can define a generating function $\Gamma = \Gamma(U_1, ..., U_j, ..., U_{n-1}, u_n)$ giving the following implicit representation of the spatial coordinates:

$$u_k = \frac{\partial^{(n-1)}\Gamma}{\partial U_1 ... \partial U_j ... \partial u_n} \qquad \mathrm{k} = 1, ..., \mathrm{n}-1; \quad \mathrm{j} = 1, ..., \mathrm{n}-1; \quad \mathrm{j} \neq \mathrm{k} \tag{25}$$

Using the same methodology of the three-dimensional case, a multiplicative decomposition $\mathbf{F} = \mathbf{F}_a\mathbf{F}_b$ can be imposed, which reads:

$$\mathbf{F}_a = \begin{bmatrix} \frac{\partial^{(n)}\Gamma}{\partial U_1 ... \partial U_j ... \partial u_n} & \frac{\partial^{(n)}\Gamma}{\partial^2 U_2 ... \partial U_j ... \partial u_n} & \cdots & \frac{\partial^{(n)}\Gamma}{\partial U_2 ... \partial U_j ... \partial^2 u_n} \\ \vdots & & & \vdots \\ \frac{\partial^{(n)}\Gamma}{\partial^2 U_1 \partial U_2 ... \partial U_{n-2} \partial u_n} & \cdots & \frac{\partial^{(n)}\Gamma}{\partial U_1 \partial U_2 ... \partial U_{n-1} \partial u_n} & \frac{\partial^{(n)}\Gamma}{\partial U_1 \partial U_2 ... \partial U_{n-2} \partial^2 u_n} \\ 0 & 0 & \cdots & 1 \end{bmatrix} \tag{26}$$

$$\mathbf{F}_b = \begin{bmatrix} 1 & 0 & ... & 0 \\ 0 & 1 & ... & 0 \\ \vdots & & 1 & \vdots \\ -\frac{\partial U_n/\partial U_1}{\partial U_n/\partial u_n} & -\frac{\partial U_n/\partial U_2}{\partial U_n/\partial u_n} & ... & \frac{1}{\partial U_n/\partial u_n} \end{bmatrix} \tag{27}$$

Recalling Eq.(24), a general solution for the n-dimensional case can be expressed as follows:

$$U_n = \int^{u_n} \det\mathbf{F}_{\rm a}(\Gamma(\rm U_1, ..., U_j, ..., U_{n-1}, \eta))d\eta + G(U_1, ..., U_j, ..., U_{n-1}) \tag{28}$$

where $G$ is a generic function of the material coordinates. Given the arbitrary choice of the mixed coordinate state, there exist $2n$ of such generating functions, expressed as $\Gamma(U_1, .., U_j, ..., U_n, u_k)$, and $\Gamma(u_1, .., u_j, ..., u_n, U_k)$ for $k = 1, .., n$ and $j \neq k$), for defining a general isochoric transformations in the n-dimensional case. Taking n=2 in Eqs.(25,28), the solution is given by $u_1 = \partial\Gamma(u_2, U_1)/\partial u_2$ and $U_2 = \partial\Gamma(u_2, U_1)/\partial U_1$, which is the well-known canonical transform proposed by Rooney and Carroll (1984). *It is worth noticing that, in the particular case of pseudo-plane deformations, such an implicit solution allows an explicit representation, first given by Hill and Shield (1986), which extends to nonlinear elasticity a well-known result for viscous incompressible fluids.*

# 4 Discussion and concluding remarks

In this work, the definition of generating functions for volume preserving transformations is given for a general n-dimensional problem in the Euclidean space. Compared to existing implicit solutions, it is shown that the proposed representation defines a more general set of isochoric transformations. In the case n=3, illustrative examples are discussed both in rectangular and in cylindrical coordinates for applications in mechanical problems of incompressible continua. *Because the proposed representation solves exactly the isochoric constraint, its application in hyperelasticity does not require the introduction of a Lagrange multiplier ensuring incompressibility. As shown in the Appendix A,* a complex boundary value problem in nonlinear elasticity can be transformed into a fully variational formulation, having several advantages in dealing with stability problems, *when* compared to the classical incremental deformation method (Ciarletta and Ben Amar, 2011). A main advantage of using generating functions for isochoric perturbations is the possibility to describe an asymmetric pattern in the linear stability analysis. In the nonlinear regime, the implicit representation therefore allows to take into account the formation of local singularities in the elastic solution.

Although the applicability of an arbitrary generating function is constrained by explicitly solving Eq.(28) in closed form, a particular choice of its mathematical expression can be made *a priori* in order to fulfil some boundary conditions prescribed by the mechanical problem. Finally, this feature *might be* particularly important for stability problems in nonlinear elasticity, allowing to build three-dimensional isochoric perturbations, as shown in Figure 3 and 4. *Future applications will be focused on the construction of variational formulations in nonlinear elasticity, with potential applications for the analysis of pattern formation during the growth of soft tissues.*



## A Analysis of wrinkling formation in biaxial constrained growth

*In this work, a method for defining isochoric transformations by the definition of generating functions is proposed. An application is derived in the following to determine the occurrence of a surface wrinkling on a growing material. A soft layer with thickness $H$ and widths $L_x, L_y >> H$ is attached on a fixed substrate at $z = 0$ and confined laterally by rigid walls, undergoing a volume increase with an isotropic growth rate $g$. The incompatibility of such a growth with the geometrical constraint induces biaxial residual strains in the plane $(x, y)$, possibly leading to wrinkling formation. Taking a reference configuration $(gX, gY, gZ)$ in Figure 2, we can introduce a general form of Eq.(13) to define the following generating function for an isochoric transformation:*

$$\mathrm{f(X,Y,z) = XYz} + \epsilon \cdot \phi(\mathrm{X,Y,z}) \tag{A.1}$$

*where $\phi$ represents a general perturbation of the homogeneous elastic solution ($x = X, y = Y, z = g^3 Z$). Substituting the expression in Eqs.(9,10,12) into the tensorial objects defined in Eq.(11), the elastic deformation tensor* $\mathbf{F}$ *at the first order in* $\epsilon$ *reads:*

$$\mathbf{F} = \frac{1}{g}\begin{bmatrix} (1+\epsilon\phi_{,XYz}) & (\epsilon\phi_{,YYz}) & g^3\,(\epsilon\phi_{,Yzz}) \\ (\epsilon\phi_{,XXz}) & (1+\epsilon\phi_{,XYz}) & g^3\,(\epsilon\phi_{,Xzz}) \\ -2\epsilon\phi_{,XXY} & -2\epsilon\phi_{,XYY} & g^3\,(1-2\epsilon\phi_{,XYz}) \end{bmatrix} \tag{A.2}$$

*while the incompressibility constraint* $det\mathbf{F} = 1$ *is identically satisfied at any order in* $\epsilon$*. Assuming a neo-Hookean constitutive behavior for the soft layer, the total strain energy of the body can be written as follows:*

$$\int_{\Omega_\mathrm{i}} \Psi(\mathrm{X,Y,z})\mathrm{d}\Omega_\mathrm{i} = \mu \int_{\mathrm{X}=-\mathrm{L_x}/2}^{\mathrm{L_x}/2}\int_{\mathrm{Y}=-\mathrm{L_y}/2}^{\mathrm{L_y}/2}\int_{\mathrm{z}=0}^{\mathrm{H}} \mathrm{g}^3\mathrm{det}\mathbf{F_a}\cdot(\mathrm{tr}(\mathbf{F}^\mathrm{T}\mathbf{F})-3)\mathrm{dXdYdz} \tag{A.3}$$

*where* $\Omega_i$ *indicates the body volume in the intermediate configuration, and* $\mu$ *is the elastic shear modulus. Using Eqs.(11,12,A.1), the expression of the strain energy density* $\Psi$ *in Eq.(A.3) at the second order in* $\epsilon$ *is given by:*

$$\begin{aligned}\Psi = {} & \mathrm{g}\mu(2-3\mathrm{g}^2+\mathrm{g}^6) - 2\epsilon\mathrm{g}\mu(-4+3\mathrm{g}^2+\mathrm{g}^6)\phi_{,\mathrm{XYz}} + \mathrm{g}\mu\epsilon^2\left[\mathrm{g}^6(\phi_{,\mathrm{Yzz}}^2+\phi_{,\mathrm{Xzz}}^2) + (12-3\mathrm{g}^2+3\mathrm{g}^6)\phi_{,\mathrm{XYz}}^2 + \right.\\ & \left. +(g^6+3g^2-2)\phi_{,XXz}\phi_{,YYz} + 4(\phi_{,XXY}^2+\phi_{,XYY}^2-\phi_{,XYY}\phi_{,Xzz}-\phi_{,XXY}\phi_{,Yzz}) + \phi_{,YYz}^2+\phi_{,XXz}^2\right]\end{aligned} \tag{A.4}$$

*Performing an arbitrary variation* $\delta\phi$ *in Eq.(A.4), the volumetric Euler-Lagrange equation at the second order in* $\epsilon$ *is obtained in the following form:*

$$g^6(\phi_{,XXzzzz}+\phi_{,YYzzzz}) + \phi_{,XXXXzz} + \phi_{,YYzzzz} + 4(\phi_{,XXYYYY}+\phi_{,XXXXYY}) + (2+4g^6)\phi_{,XXYYzz} = 0 \tag{A.5}$$

*Setting* $\phi(X,Y,z) = h(z)\ sin(k_x X)\ sin(k_y Y)$ *as in Eq.(13), the Euler-Lagrange equation can be transformed in a forth-order ordinary differential equation on* $h(z)$*, which reads:*

$$g^6(k_x^2+k_y^2)h''''(z) - (k_x^4 + 2(1+2g^6)k_x^2k_y^2 + k_y^4)h''(z) + 4k_x^2k_y^2(k_x^2+k_y^2)h(z) = 0 \tag{A.6}$$

*where* $k_x = 2\pi m/L_x$, $k_y = 2\pi m/L_y$, *and* $n, m$ *are integer numbers for satisfying the no-sliding conditions at the side surfaces. Considering* $k_x = k_y = k_3$ *in Eq.(A.6), one obtains the same equilibrium equation found in Ben Amar and Ciarletta (2010) using the method of incremental elastic deformations. The differential problem defined in Eq.(A.6) requires four boundary conditions: two are given by the vanishing of the perturbation at the fixed substrate (*$h(0) = h'(0) = 0$*), while the remaining two can be obtained variationally for arbitrary variations* $\delta\phi$ *at the surface* $Z = H$*. The solution of the elastic problem is out of the scopes of this work, and further derivations are neglected here for the sake of simplicity.*

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>A general implicit solution for determining volume-preserving transformations in the n-dimensional Euclidean space is obtained.> For n = 2, it corresponds to the classical definition of a potential stream function in a canonical transformation. > For n = 3, it defines a more general class of isochoric transformations, if compared to existing methods based on multiple potentials. > this representation method is suitable for determining three-dimensional isochoric perturbations in bifurcation theory. > Applications in nonlinear elasticity are envisaged for determining complex instability patterns for soft elastic materials.