Generating mathematical knowledge in the classroom through proof, refutation, and abductive reasoning

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Abstract

Proving and refuting are fundamental aspects of mathematical practice that are intertwined in mathematical activity in which conjectures and proofs are often produced and improved through the back-and-forth transition between attempts to prove and disprove. One aspect underexplored in the education literature is the connection between this activity and the construction by students of knowledge, such as mathematical concepts and theorems, that is new to them. This issue is significant to seeking a better integration of mathematical practice and content, emphasised in curricula in several countries. In this paper, we address this issue by exploring how students generate mathematical knowledge through discovering and handling refutations. We first explicate a model depicting the generation of mathematical knowledge through *heuristic refutation* (revising conjectures/proofs through discovering and addressing counterexamples) and draw on a model representing different types of abductive reasoning. We employed both models, together with the literature on the teachers' role in orchestrating whole-class discussion, to analyse a series of classroom lessons involving secondary school students (aged 14–15 years, Grade 9). Our analysis uncovers the process by which the students discovered a counterexample invalidating their proof and then worked via creative abduction where a certain theorem was produced to cope with the counterexample. The paper highlights the roles played by the teacher in supporting the students' work and the importance of careful task design. One implication is better insight into the form of activity in which students learn mathematical content while engaging in mathematical practice.

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1 Introduction

Proving and refuting are fundamental aspects of mathematical practice (Lakatos, 1976). Both processes are closely inter-related, and conjectures and proofs are often produced and improved through the back-and-forth transition between attempts to prove and disprove. Such mathematical activity also leads to the development of mathematical knowledge and theory.

Mathematics education research has examined different aspects of student activity involving refutations. Studies have investigated the role of counterexamples in learning proof (Stylianides & Stylianides, 2009), students' performance in finding counterexamples (Küchemann & Hoyles, 2006; Lee, 2016), and the ways in which students and teachers produce and accept counterexamples (Weber, 2009; Zazkis & Chernoff, 2008). Other studies have explicitly addressed the inter-related activities of conjecturing, proving, and refuting such as how students modify their conjectures, sometimes by capitalising on their proofs, when faced with counterexamples (Balacheff, 1991; Larsen & Zandieh, 2008).

In this paper, we focus on this kind of interplay between proving and refuting—specifically, the notion of *heuristic refutation* that Komatsu and Jones (2019) employed from Lakatos (1976) and de Villiers (2010) to refer to mathematical activity involving the discovery of counterexamples to conjectures/proofs and the revision of the conjectures/proofs by addressing such counterexamples. The meaning of the term *heuristic* in this context is thus not associated with the more typical use derived from Polya's (1957) work on problem solving but is rather based on Lakatos's (1976) work in the philosophy of mathematics.[1] Lakatos conducted case studies in the history of mathematics and argued that mathematics develops through a zig-zag path among conjectures, proofs, and refutations. He introduced the term heuristic refutations to refer to refutations that stimulate the growth of knowledge, such as the improvement of conjectures and the development of mathematical concepts. The adjective heuristic is used to emphasise the type of mathematical activity engaged in after refutations to revise the refuted conjectures and proofs, and not the refutations themselves.

A key question to resolve in this context is how students deal with refutations and *simultaneously* generate mathematical knowledge that is new to them, including mathematical concepts and theorems. This question is underexplored in the literature noted above, in which students certainly improved conjectures to address counterexamples, but the focus was not on creating mathematical knowledge that was new to the students.[2] Although research has shown the role of counterexamples in fostering students' concept formation (Roh & Lee, 2017; Zaslavsky & Shir, 2005), the relationship between refutations and student knowledge construction has not been considered in the area of argumentation and proof. The lack of research in this area is partly because of the context of existing studies where the students already had knowledge necessary to address the counterexamples. However, students are not necessarily familiar with the relevant knowledge. Both Buchbinder and Zaslavsky (2011), and Ko and Knuth (2009), reported on cases

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[1] For Polya (1957), heuristic reasoning is reasoning useful to discover the solution to the present problem. When discussing heuristics in the case of proving, Polya generally considered a situation where a statement to be proved (obtained by conjecturing or being given) was fixed and not revised during the proving process. Heuristic in this context means serving to discover the proof of the fixed statement. In contrast, our study follows Lakatos's work to focus on mathematical activity involving the revision of statements and proofs (rather than the discovery of the proofs of statements) and adopts the term heuristic, as did Lakatos (1976), as being useful for this activity. Lakatos (1976), we note, contains a dedication to Polya for the "revival" of mathematical heuristic.

[2] Our aim here is to clarify the difference between our study and existing research; we do not intend any criticism of existing studies.

where students failed to judge the truth/falsity of statements as a result of their inadequate understanding of concepts and inappropriate use of applicable knowledge they had.

In this paper, we address this lack of research by paying attention to mathematical activity involving heuristic refutation together with the construction of mathematical knowledge that is new to students. Our focus is meaningful for two reasons. First, our focus leads to a substantial theoretical contribution. Mathematics education research (Balacheff, 1991; Larsen & Zandieh, 2008; Komatsu, 2016; Reid, 2002) has shown that Lakatos's (1976) work, which originally focused on mathematical research, is also helpful in describing and making sense of how students respond to counterexamples. However, there is another important aspect in Lakatos's research that highlighted not only the discovery and handling of counterexamples but also the development of mathematical knowledge (e.g., mathematical concepts such as polyhedra and uniform convergence). Such interplay between heuristic refutation and knowledge construction has not been examined in the mathematics education literature. In this paper, we extend existing studies by considering the aspect of students' construction of mathematical knowledge, including mathematical theorems.[3]

Second, the consideration of knowledge construction in the context of heuristic refutation provides an implication for educational practice. Mathematics curricula in several countries, including, for example, Japan and England, comprise mathematical content and practice. Moreover, the integration of these two strands is targeted in that students are expected to learn mathematical knowledge and procedures *through* mathematical activity and should be taught to work mathematically through mathematical content (e.g., Department for Education, 2013; Ministry of Education, Culture, Sports, Science and Technology, 2018). Even so, how to combine mathematical content and practice in a concrete way is not clearly described. The series of classroom lessons analysed in this paper presents one approach to attaining this integration of mathematical content and practice, in which a mathematical theorem emerges while students engage in heuristic refutation.

Thus, the research question explored in this paper is: How do students, with the support of the teacher, generate mathematical knowledge new to them through discovering and addressing counterexamples?

To address this research question, we employ research on abduction. Abduction is one type of logical reasoning; it begins with an observation and then devises a hypothesis to explain the observation. We consider abduction because, according to Peirce (1932, 1935), who coined the term, this type of reasoning involves explanation and discovery, which are relevant to our focal mathematical activity. Mathematics education research has recently explored the role of abductive reasoning in the context of conjecturing and proving (e.g., Antonini, 2019; Arzarello & Sabena, 2011; Baccaglini-Frank, 2019; Meyer, 2010; Pedemonte & Reid, 2011). Our study also aims to

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[3] We are aware that the mathematical process described by Lakatos does not represent all aspects of mathematical disciplinary practice (Hanna, 1995), and we also do not think that every aspect of mathematical practice (including Lakatos's description) should inform the teaching of mathematics in classrooms (Weber et al., 2020). We are specifically aware that some of Lakatos's (1976) claims about the generation of mathematical knowledge, such as the development of mathematical concepts, do not truly represent the history of mathematics (Weber et al., 2020); indeed, Lakatos himself later acknowledged the inaccuracy of his description in the case of uniform convergence (Lakatos, 1978, p. 46). Nevertheless, we draw on Lakatos's work to introduce the aspect of knowledge construction; our rationale for this is to gain a better insight into students' activity. As discussed later in the paper, we also think that the integration of heuristic refutation and knowledge construction is practically significant. However, our intention is not to go further in this paper into whether introducing this integration into classrooms leads to the achievement of "authentic" mathematical learning that entirely represents mathematical disciplinary practice.

contribute to the literature on abduction by exploring whether the model of abduction can be employed to analyse a different process, namely discovering and addressing refutations.

We also consider the role of teachers, given its importance in learning mathematics in general (Jackson et al., 2013; Stein et al., 2008), and for proof and proving in particular (Komatsu, 2017; Stylianides & Stylianides, 2009). Teacher support would be particularly vital in challenging situations in which students do not have immediate access to the mathematical knowledge they need in order to tackle the tasks presented to them.

In this paper, we first explicate a model depicting the generation of mathematical knowledge through heuristic refutation. To analyse the knowledge construction process, we draw on research about different kinds of abductive reasoning (as laid out below). After clarifying our view of classroom learning and considering teachers' role in orchestrating whole-class discussion, a series of classroom lessons is analysed where a Japanese secondary class addressed a counterexample to their proof by producing a mathematical theorem. The analysis of the lessons uses the two models (one of the generation of mathematical knowledge through heuristic refutation, and one of the kinds of abduction) and highlights the roles played by the teacher in supporting the students' work.

2 Theoretical framework

2.1 Generation of mathematical knowledge through heuristic refutation

Komatsu and Jones (2019) employed Lakatos's (1976) work to depict the mathematical activity of heuristic refutation as a triplet of conjecturing, proving, and refuting. Whereas it can be said that mathematical knowledge is generated even during heuristic refutation, in that improved conjectures are produced (and such student activity involving improved conjectures has been reported in the literature), in this paper we extend the model of heuristic refutation and propose the framework illustrated by Fig. 1, which adds *generation of mathematical knowledge* as an element.

This framework comprises heuristic refutation represented as a two-dimensional plane (involving conjecturing, proving, and refuting) in reciprocal relationship to the generation of mathematical knowledge (envisaged in the figure as if located above the plane in three dimensions). The arrow to/from the generation of mathematical knowledge relates, in some
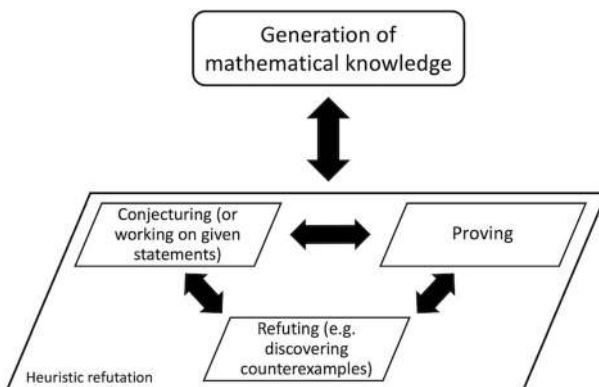

Fig. 1 Generation of mathematical knowledge through heuristic refutation

instances, to the whole notion of heuristic refutation, while in other instances the arrow connects with the individual elements of heuristic refutation such as refuting, the inter-relationship of refuting and conjecturing, the inter-relationship of refuting and proving, and so on.

Mathematical knowledge in this model refers to knowledge pertaining to the conjectures to be proved, and not the conjectures themselves. This is actually the case with Lakatos's case studies; for example, in one of his studies involving polyhedra, the conjecture explored was about the relationship between the numbers of vertices, edges, and faces of polyhedra, and the knowledge produced was the definitions of polyhedron.

It is difficult to fully explain this framework in this limited space, so here we make three specific notes relevant to the empirical part of this paper. First, regarding the element of *generation of mathematical knowledge*, Lakatos (1976) paid particular attention to the definitions of mathematical concepts (e.g., a triangle is a geometrical figure enclosed by three straight lines). We extend this by adding other types of mathematical knowledge, including the properties of mathematical objects, some of which are expressed as theorems (e.g., the sum of the interior angles of a triangle is 180°). In this context, the proof of the generated theorem is also included in the element of generation of mathematical knowledge in our framework, not in the element of proving; the element of proving refers to the proving of original conjectures (or their modified versions).

Second, regarding the element of *refuting*, Lakatos (1976) distinguished two types of counterexamples: *global counterexamples* to conjectures (counterexamples in a conventional sense) and *local counterexamples* to proofs (Komatsu & Jones, 2019; Komatsu et al., 2017). More specifically, a local counterexample is a refutation invalidating a specific step in a proof. Taking, as an example, the statement from number theory that the sum of two odd numbers is always even, some students may represent two odd numbers as $2n + 1$ and $2n + 3$ and claim the conclusion by calculating $(2n + 1) + (2n + 3) = 2(2n + 2)$. While the statement is true, and thus there exists no global counterexample, $3 + 7$ is a local counterexample to this argument because this case cannot be represented as $2n + 1$ and $2n + 3$; the argument can cover only two consecutive odd numbers.

Local counterexamples can occasionally be found in geometric proof. In school geometry, a mathematical statement to be proved is typically described with a given diagram, provided as a generic example that represents a certain general class (Otten et al., 2014). However, this class, although being general to some extent, does not necessarily correspond to the entire domain of the statement, because the drawn diagram, although generic, is still a single and concrete case and thus may inevitably include a hidden assumption that is not applicable to all cases of the statement. In this context, a proof valid for the generic diagram may become invalid in some particular cases. Local counterexamples to the proof can be discovered by transforming the given diagram into different configurations that continue to satisfy the premise of the statement (Komatsu, 2017; Komatsu & Jones, 2019; Komatsu et al., 2017).

The third note about our framework is that there are several patterns of transition from heuristic refutation (the lower part of Fig. 1) to the generation of mathematical knowledge (the upper part). One pattern is that solvers first cannot handle counterexamples with their available knowledge and thus the need arises for new knowledge; once new knowledge is established, the problematic situation is resolved. In this case, the arrow towards the *generation of mathematical knowledge* in Fig. 1 commences in the arrows from *refuting* to *conjecturing/proving* during which solvers work on the revision of conjectures and proofs. As with Lakatos's research, this paper focuses on this process, in which mathematical knowledge is constructed while addressing counterexamples.

To analyse student activity related to this process, we employ research on abduction, employing Eco's (1983, 1986) classification of different kinds of abduction as explained in the next section.

2.2 Types of abduction

Research on abduction has its origins in Peirce (1932, 1935). Abduction can be characterised by its purposes and forms (Reid, 2018), with one purpose being to explain observations. According to Peirce, abduction is as follows:

The surprising fact, C, is observed;

But if A were true, C would be a matter of course,

Hence, there is reason to suspect that A is true. (Peirce, 1935, CP 5.189)

One form of abduction is to start with an observation Q and then infer P from Q and P→Q. This form, generally the typical representation of abduction, can be contrasted with those of deduction and induction; one kind of deduction is to derive Q from P and P→Q, while induction is to conjecture P→Q from P and Q (Peirce, 1932, CP 2.623).

Reid (2018) addresses a further form of abduction, based on the location of punctuation in Peirce's statement above, where this additional form of abduction includes not only the inference of P but also the invention of P→Q from Q. In relation to this form of abduction, Eco (1983, 1986) paid particular attention to the provision of rules in the argument (P→Q in the above) and, by building on Thagard (1978) and Bonfantini and Proni (1983), classified abduction into three kinds: overcoded, undercoded, and creative. *Overcoded abduction* occurs when the rule "is given automatically or quasi-automatically", *undercoded abduction* arises when "the rule must be selected among a series of equiprobable alternatives", while abduction involving the invention of P→Q, i.e., *creative abduction*, is when "the rule acting as an explanation has to be invented *ex novo*" (i.e. literally "anew") (Eco, 1986, pp. 41–42).

Eco (1986) illustrated *creative abduction* with the development of the Copernican theory of the universe. Copernicus was concerned about several weaknesses in the Ptolemaic system that was dominant at that time, such as the inability to provide the accurate duration of 1 year and its complicated way of accounting for the detailed movements of the planets. Copernicus proposed the heliocentric hypothesis (that the sun is located at the centre of the solar system), deriving that these empirical phenomena could be explained if the hypothesis were true.

Eco's classification of abduction has been examined and used in mathematics education studies (Baccaglini-Frank, 2019; Meyer, 2010; Pedemonte & Reid, 2011; Reid, 2018; Sáenz-Ludlow, 2016). In line with those studies, and based on our research question of how students, with the support of the teacher, generate mathematical knowledge new to them, we employ Eco's classification for the analysis of the data.

To operationalise his classification, we follow Pedemonte and Reid (2011) who illustrated the three kinds of abduction with Toulmin's (2003) model of arguments. Figure 2 is our adaptation using the three main components of Toulmin's model. In the context of abduction explaining an observation, a claim (C) is the observation to be explained, datum (D) is a hypothesis proposed to explain the observation, and a warrant (W) describes why the datum
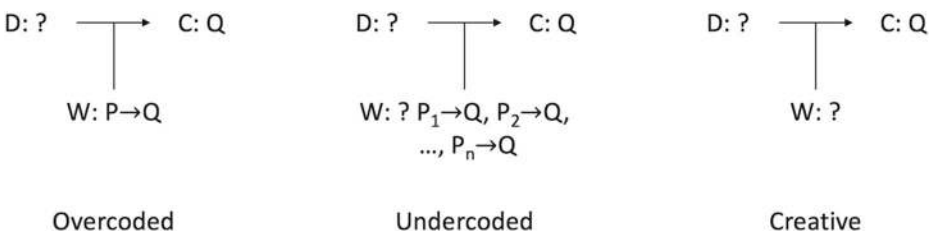

Fig. 2 Three kinds of abduction (adapted from Pedemonte and Reid, 2011)

Table 1 Summary of the definitions of key terms used in this paper

- *Heuristic refutation* is mathematical activity involving the discovery of counterexamples to conjectures/proofs and the revision of the conjectures/proofs by addressing such counterexamples.
- A *global counterexample* is a case refuting a conjecture, and a *local counterexample* is a case invalidating a specific step in a proof.
- *Creative abduction* is to explain an observation by proposing a hypothesis and devising a reason why this hypothesis can explain the observation.

can explain the claim. The question marks in Fig. 2 indicate elements to be sought in each form of abduction. In the case of creative abduction, a warrant is also a hypothesis because it is invented, and thus its truth is not established yet.

In contrast to overcoded abduction (where warrants are already given), the selection/invention of warrants does not necessarily precede the inference of data in undercoded abduction and creative abduction. Eco stated that "it is important to stress that the real problem is not whether to find first the Case [datum] or the Rule [warrant], but rather how to figure out both the Rule and the Case *at the same time*, since they are inversely related, tied together by a sort of chiasmus" (Eco, 1983, p. 203, emphasis in the original, words in square brackets added by us). In the above example, Copernicus may have considered the hypothesis *and* its possibility of explaining empirical phenomena *simultaneously*, or he may have first considered the former and then the latter.

When, in creative abduction, the inference of data precedes the invention of warrants, the form of the abduction becomes similar to that of induction. However, their purposes are different. While the purpose of induction is to make a generalisation P→Q, creative abduction aims to explain an observation Q through a general warrant P→Q that is invented as a means to achieve this aim.

A feature of any form of abduction is that data inferred in the abduction need to be tested because abduction is not deterministic reasoning. In creative abduction, warrants also need to be verified as they are invented (Eco, 1986).

Table 1 is the summary of the definitions of key terms used in this paper. Before proceeding to the empirical part of this paper, we clarify our view of classroom learning and refer to the teachers' role in orchestrating whole-class discussion.

2.3 Teachers' role in orchestrating whole-class discussion

Underpinning our analysis of a series of lessons in a secondary school classroom, we regard whole-class discussion orchestrated by the teacher as one form of *mathematical discussion* proposed by Bartolini Bussi (1996, 1998). One characteristic of the notion of mathematical discussion is the difference between students' voices and the teacher's voice in a whole-class discussion. Although the students and teacher discuss a common mathematical object, the teacher orchestrates the classroom discussion with a particular perspective, different from those of the students. Suppose, for example, that a teacher initiates a whole-class discussion after students work on a mathematical task. The students' focus in the discussion may be on the present task, such as sharing how they solved the task and what challenges they faced. In contrast, the teacher may have a different perspective and consider the students' discussion on the task not as the goal of the lesson itself, but rather as a pedagogical opportunity to build on student thinking to teach more general mathematical knowledge.

In this context, it is crucial to consider the teacher's role in classroom discussion because students' individual work is not necessarily successful, and their discussion may not necessarily move towards the learning goal planned by the teacher. Different roles of teachers in holding and orchestrating whole-class discussion have been identified in the literature. For example, when launching a task at the beginning of a lesson, a teacher introduces a whole-class discussion to support students in developing a taken-as-shared understanding of the task, rather than immediately engaging them in the task (Jackson et al., 2013). After students' individual work, a teacher holds another type of classroom discussion where the teacher purposefully selects, sequences, and connects certain students' ideas that are worth examining, to relate the entire class's activity to significant mathematical ideas (Leatham et al., 2015; Stein et al., 2008). As a representative of the mathematical community, a teacher also "revoices" (e.g., rephrases and elaborates) students' thoughts by using mathematical ideas to connect students' work with the broader mathematical culture (Bartolini Bussi, 1998; Forman & Ansell, 2002).

In all this, the teacher's role in whole-class discussion is delicate: if the teacher's intervention in launching a task is excessive, it can decrease the cognitive demand of the task and limit the scope of the students' activity to a single direction (Jackson et al., 2013). If a teacher does not sufficiently respect students' contributions in a whole-class discussion held after their individual work, the students may consider the teacher's move abrupt and not derived from their work; furthermore, they would not see the connection between their work and the mathematical idea the teacher aims at in the lesson. From a research perspective, it is important to explore how a teacher orchestrates whole-class discussion in terms of balancing between students' contributions and the learning goal of the lesson. Here, the focus is how students, through heuristic refutation with the support of the teacher, generate mathematical knowledge new to them through discovering and addressing counterexamples.

To sum up the theoretical part of this paper, our study explores how the framework combining Figs. 1 and 2 can provide an insight into students' mathematical activity; in particular, the generation of mathematical knowledge through dealing with counterexamples. We analyse a classroom episode for this purpose, by considering the activity of the class as a whole—it is not our focus to investigate what the students might be able to do independent of the class and the teacher. We use the literature on teachers' role in whole-class discussion to examine how the teacher supported the students' activity in the episode.

3 Methods

3.1 Background

The classroom episode that we analyse in this paper was implemented as part of our 3-year design research project titled "Proofs and Refutations in Dynamic Geometry Environments" (PRinDGE). In this project, we addressed the lack of intervention studies focusing on students' experiences with counterexamples (Stylianides et al., 2017) by developing a set of task design principles and aspects of teachers' role that engage students in mathematical activities relevant to the process described by Lakatos (1976). In the first year of the project, we focused on task design and conducted task-based interviews (Komatsu & Jones, 2019). We then included teachers' role and conducted a pilot intervention study in the second year (Komatsu & Jones,

2020). In the third year, we implemented the final intervention study by collaborating with teachers of four classes in Japanese lower secondary schools.

The data analysed in this paper were collected in the third year of the research project. This paper focuses on one class because the students' mathematical activity was almost identical across the four classes. Each intervention comprised a series of three lessons (50 min per lesson).

The intervention was conducted by the mathematics teacher of the class, an expert teacher who had over 20 years' teaching experience in several schools. We first designed the task to be implemented and drafted rough lesson plans. We shared these with the teacher along with Fig. 1 and explained the objective of our design research. After that, the teacher devised detailed lesson plans based on our draft plans and discussed these with us. He then implemented the lessons, and the first author of this paper observed the lessons as a non-participant.

The participating students were 39 Japanese Grade 9 students aged 14–15 years. They had studied Euclidean-type geometry in Grades 8 and 9 and were competent in proof construction using conditions for congruent triangles and similar triangles. They knew several basic geometry theorems with proofs, such as the inscribed angle theorem and the triangle exterior angle theorem. The students, however, were not familiar with either the concept of cyclic quadrilateral or the cyclic quadrilateral theorem (that the sum of opposite angles of a cyclic quadrilateral is 180°, and that an interior angle of a cyclic quadrilateral is equivalent to the exterior angle of the opposite angle). In fact, the goal of the implemented lessons was that, through heuristic refutation, the students discovered and proved the cyclic quadrilateral theorem as mathematical knowledge shown in Fig. 1.

The dynamic geometry environment (DGE) employed in this study was *GeoGebra*, which was available on the school's tablet computers. The students often used tablet computers in their regular lessons, so they were proficient in the basic functions of the DGE, including the construction of geometrical diagrams and the transformation of diagrams by dragging. In the lessons implemented, the students used the DGE only in the latter half of lesson 1, and the teacher used it in lessons 2 and 3 as a demonstration tool during whole-class discussion.

3.2 Data collection and analysis

The lessons were recorded with three video cameras, two of which were fixed to record the entire classroom, and another which was mobile to record individual student work and discussions with neighbours. The data for analysis included video recordings, the transcripts made from these recordings, the students' worksheets, and field notes taken by the first author.

These data were analysed following a procedure similar to that in Knipping and Reid (2015). The data analysis consisted of three phases. The first author and a research assistant conducted each phase of the analysis independently; we then met to compare each analysis and discuss any discrepancies until we reached a consensus.

In phase 1 of the analysis, the implemented lessons were divided into coherent episodes. We then used the model presented in Fig. 1 to examine what element(s) of the model corresponded to each episode. We also considered the types of counterexamples that were produced and the mathematical knowledge that was generated for handling the counterexamples. In phase 2, relevant episodes were analysed with the main elements of Toulmin's model: claims, data, and warrants. In line with Toulmin's model, this included, when necessary, examining instances of *backing* for a warrant if a warrant was in doubt. Although Toulmin's model has *rebuttal* to indicate possible exceptions, we followed

Knipping and Reid (2015) who instead added the element of *refutation* to mean the negation of some part of the argument (e.g., data and claims); in representing students' reasoning, refutations are shown as jagged lines (as in Fig. 6). Elements implicit in students' arguments were interpolated by interpreting students' worksheets and utterances; we explain our interpretations as far as possible in the analysis section. In phase 3 of the analysis, we paid particular attention to the class's activity involving heuristic refutation. The framework of the kinds of abduction (Fig. 2) was employed to analyse the process whereby the class generated mathematical knowledge that was new to the students. In doing so, we also examined how the teacher supported the students' work by employing the literature reviewed in the previous section on different roles of teachers in orchestrating whole-class discussion.

Regarding the relationship between the theoretical framework of this study and our data interpretation, we conceived the framework depicted in Fig. 1 before implementing the lessons. For the study, this figure was shared with the teacher, and the goal of our design research was to find a way to engage students in mathematical activity illustrated by this figure. We did not consider abduction in planning the intervention. It was during the analysis of the data that it became clear that the work of Eco, and related mathematics education research, was pertinent. Through working on our data over a period of time, we found that the framework on the kinds of abduction (Fig. 2), combined with our model in Fig. 1, provided additional insights into the class's mathematical activity.

The students' names in this paper are pseudonyms. The term "students" is used for moments in the transcripts in which multiple students spoke. The students' and teacher's utterances, the students' proofs, and their comments from their worksheets have all been translated into English from Japanese by the first author. Ellipses in the transcripts indicate the omission of irrelevant parts and repetitions for the sake of readability, and words in square brackets are added for clarification.

4 Analysis of the lessons

We first present an overview of the classroom lessons. The students tackled the task shown in Fig. 3. After proving the statement, they engaged in heuristic refutation where they discovered local counterexamples invalidating their proof and attempted to address those counterexamples by revising their proof. During coping with one of the local counterexamples, the class produced the cyclic quadrilateral theorem, together with its proof, as mathematical knowledge.

Task. Put four points A, B, C, and D on a circumference, connect A and C, and B and D with lines, and let E be the intersection point. Can we say that ΔABE and ΔDCE are similar figures?
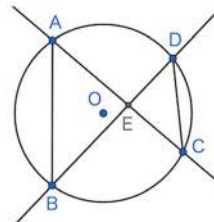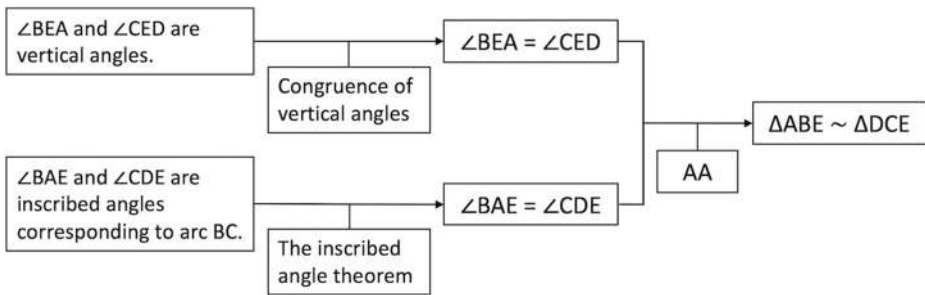
Fig. 3 The task

Fig. 4 Student Sakura's proof (represented using Toulmin's model)

4.1 Statement and initial proof

The teacher began lesson 1 by providing the task shown in Fig. 3. Many students responded saying that the statement, ΔABE ~ ΔDCE, would be true, and then the students individually worked on proving it. One proof that a student, Sakura, shared with the entire class by presenting on the blackboard was as follows:

In triangles ABE and DCE, from vertical angles, ∠BEA = ∠CED … (1)

Since inscribed angles corresponding to arc BC are equal, ∠BAE = ∠CDE … (2)

From (1) and (2), since two pairs of angles are respectively equal, ΔABE ~ ΔDCE

This proof can be represented using Toulmin's model, as in Fig. 4. Some elements in this figure, such as "∠BAE and ∠CDE are inscribed angles corresponding to arc BC," were not explicitly seen in the student's proof, but can be inferred from it. The warrants are represented with abbreviations, including the inscribed angle theorem and AA as a condition for similar triangles (two triangles are similar if two pairs of angles are respectively congruent). The warrant for (1) in the proof is not sufficient, but other students clarified it in their proofs; for example, Nanami wrote on her worksheet, "∠AEB and ∠DEC are vertical angles and thus equal."

4.2 Refutation

Although the task shown in Fig. 3 included the particular configuration of the diagram, the task sentences per se do not specify the positions of points A–D on the circumference. Thus, the statement in the task refers not only to the given diagram, but also to a general class involving different places of points A–D on the circumference. The students explored this general case.

The teacher first used the DGE to change slightly the position of point A in the diagram shown in Fig. 3 and asked the class whether ΔABE and ΔDCE were still similar. Student Ren said that it was true because the above proof was still valid.

The teacher then asked whether the statement, ΔABE ~ ΔDCE, was always true. Haruto said that there was a case in which points A and B coincided. For the investigation, the teacher distributed a tablet computer to each student. The students then constructed the diagram in Fig. 3 by themselves and confirmed that ΔABE did not exist when points A and B coincided. They then examined the teacher's question (i.e., whether the statement, ΔABE ~ ΔDCE, is always true) either individually or through a discussion with their neighbours for approximately 10 min, using the DGE with which they explored various cases by dragging.

In the whole-class discussion that followed, the students shared their thoughts on three cases. First, Miwa showed another degenerate case in which lines AC and BD were parallel, and stated that the intersection point E, and thus ΔABE and ΔDCE, did not appear. The second case is represented in Fig. 5a. Misaki mentioned that the difference between the original diagram in Fig. 3 and this case was the position of the intersection point E; that is, whether point E was inside or outside circle O. The third case, which was presented by Moe, is shown in Fig. 5b where point E is seen outside circle O as well, but its configuration is different from that of Fig. 5a.

At the end of lesson 1, the teacher prompted the students to reflect on the lesson and summarise their thoughts on their worksheets. Many conjectured that the statement, ΔABE ~ ΔDCE, would be true as long as ΔABE and ΔDCE appeared. In Lakatos's terminology, the students *did not* consider the cases shown in Fig. 5 to be *global counterexamples* to the statement.

While several students intuitively arrived at this conjecture without deeply examining how to prove it, some students went further but struggled to find a way to prove it, particularly in the case shown in Fig. 5a. Below are representative comments taken from the students' worksheets:

> Daiki: I am honestly not sure whether it can be proven [that ΔABE and ΔDCE are] similar in the version where A goes beyond B [Fig. 5a], so I want to investigate it. I think I get the one going beyond C [Fig. 5b]. The equality of two [pairs of] angles would be the most useful, so I will use it.
>
> Mao: If I am to prove the second [Fig. 5a], I am not sure how to prove it. I guess I can do it since I guess [the two triangles are] similar from their appearance, but I am not sure about the way of the proof.

These students recognised that their initial proof as depicted in Fig. 4 could not be employed for the case in Fig. 5a and thus acknowledged this case as a *local counterexample* to their proof. This local counterexample was treated more explicitly in lesson 2.

In lesson 2, the students first dealt with the case in Fig. 5b, which was another local counterexample to their initial proof, by showing that ∠BEA = ∠CED (common angles) and ∠EBA = ∠ECD (the inscribed angle theorem). The teacher then initiated a whole-class discussion where the students revisited the case of Fig. 5a:
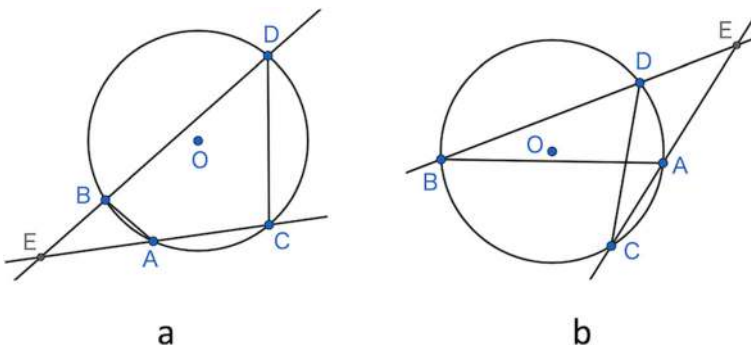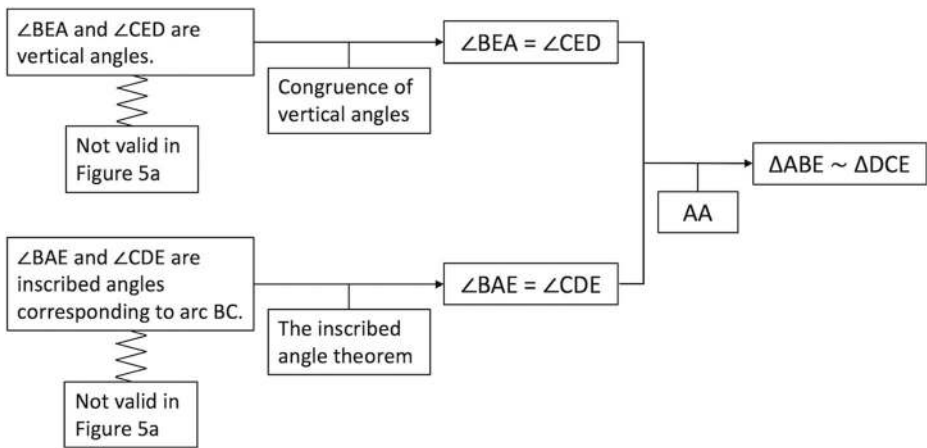

Fig. 5 Configurations showing local counterexamples

Fig. 6 Students' recognition of the local counterexample (represented using Toulmin's model)

285 Teacher: Rin, you came to me to talk about this yesterday.
286 Rin: Yes.
287 Teacher: What did you say yesterday?
288 Rin: Well, the inscribed angle theorem can be used for this [the proof for Fig. 5b].
289 Teacher: This is [proved with] the inscribed angle theorem, but.
290 Rin: I am not sure about this [Fig. 5a].
291 Teacher: You are not sure. [Speaking to the whole class.] Please think about it. Can you also do [prove] this, this [Fig. 5a]?
292 Haruto: Difficult, um.

In this discussion, the teacher purposefully selected Rin, who had come to the teacher after lesson 1 to share her struggle with the case of Fig. 5a. Rin indicated that the inscribed angle theorem could not be employed for the case in Fig. 5a (line 290). She probably noticed that ∠BAE and ∠CDE were no longer inscribed angles corresponding to arc BC. In lesson 1, another student, Takumi, also pointed out that ∠BEA and ∠CED were not vertical angles. These students' responses are illustrated in Fig. 6, where the jagged lines represent the invalidated parts of the proof shown in Fig. 4 (represented using Toulmin's model).

4.3 Generation of hypothesis through creative abduction

The students tackled the case in Fig. 5a, which was really challenging for them. They discussed this case with their neighbours, and the teacher moved around in the classroom to monitor their discussions. The teacher listened to Natsuki and then intentionally selected her to share her thoughts with the class:

315 Teacher: Well, let's listen to Natsuki. Natsuki, are you stuck?
316 Natsuki: [Nodding.]
317 Teacher: Why are you stuck?
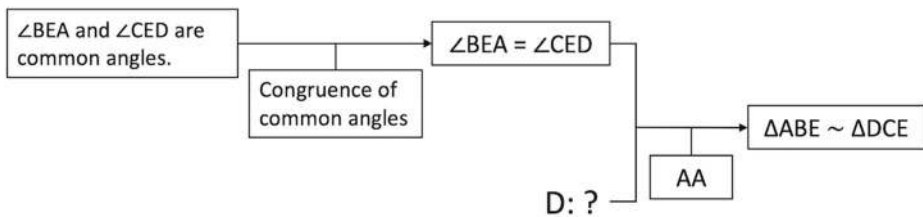318 Natsuki: The second angle is unknown.

Fig. 7 Overcoded abduction being suggested by student Natsuki

319 Teacher: The second angle is unknown. Did you get the first angle? What is the first angle?
320 Natsuki: Common angle.
321 Teacher: Well, everyone, is it OK that this [pointing to ∠E] is the common angle? I see. Angle E, angle E is common.

This exchange is represented (following Toulmin's model) by Fig. 7. Natsuki's reasoning is abductive because she wanted to prove that ΔABE ~ ΔDCE and focused on the second pair of angles for that purpose. Her abductive reasoning is overcoded in terms of Eco's (1986) classification of abduction. She did not clarify the possible condition for the similarity of the triangles, so there may be a possibility that her abductive reasoning was undercoded (in Eco's classification) where she selected AA from AA and SAS. However, this would not be realistic, and she would consider only AA because the students in the class had entirely used AA throughout the lessons.

The class further discussed the second pair of angles:

321 Teacher: Another angle is unknown. Rin, Rin, …, is the common fine? Everyone, fine? The common is OK. The common is OK. Considering the correspondence with, for example, ABE [the teacher misspoke, meaning ∠BAE], which would be equal to this angle [pointing to ∠BAE]?
322 Rin: D.
323 Teacher: Angle D. Rin said that this is unknown. Do you observe that this [∠BAE] and this [∠CDE] are equal, seemingly equal? Make a guess.
324 Students: Yes, they are [equal].

Figure 8 depicts this exchange. The students, probably including the previous student, Natsuki, guessed that ∠BAE = ∠CDE, based on the abductive reasoning mentioned above as well as the visual appearance of the two angles (lines 322–324). However, they had no idea how to prove
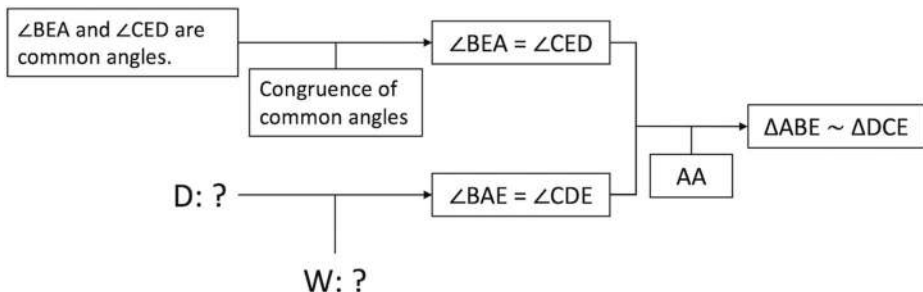

Fig. 8 Creative abduction to be performed

that the two angles were equal; creative abduction was to be performed because of the lack of a warrant (and datum) for showing that ∠BAE = ∠CDE.

The teacher then introduced a term that was new to the students, *cyclic quadrilateral*, in order to rephrase and elaborate on the struggle confronting the students:

326 Teacher: Well, this is a new term. … There are one, two, three, four points on the circumference. What polygon can be made by connecting these four points?
327 Students: Quadrilateral.
328 Teacher: Quadrilateral. This quadrilateral connecting one, two, three, four points on the circumference, we call this a *cyclic quadrilateral*.
329 Students: Ah.
330 Teacher: So, it is OK if it can be proven that this blue circle [∠BAE] and blue circle [∠CDE] are equal, isn't it? … I want to describe it using this term cyclic quadrilateral. … The problem is whether one interior angle of a cyclic quadrilateral is equal to the exterior angle of the opposite interior angle, the next exterior angle. Right?

This class discussion is illustrated in Fig. 9, where the *cyclic quadrilateral hypothesis* is within line 330 above, viz "The problem is whether one interior angle of a cyclic quadrilateral is equal to the exterior angle of the opposite interior angle", that was elaborated by the teacher as a rephrasing of what the students had become aware of through their struggle with the case in Fig. 5a.

The class discussion detailed above can be considered as a case of creative abduction for the students because the students were not familiar with the cyclic quadrilateral theorem and it was their efforts that led to the cyclic quadrilateral hypothesis. While it was the teacher who introduced the concept of cyclic quadrilateral, and elaborated the cyclic quadrilateral hypothesis by rephrasing what the students had become aware of, what the teacher offered in line 330 was not an abrupt intervention irrespective of the students' work. Rather, the cyclic quadrilateral hypothesis was suggested based on the students' thoughts, through rephrasing and elaborating the students' question (of whether ∠BAE = ∠CDE in Fig. 5a) that emerged from their activity (Bartolini Bussi, 1998; Forman & Ansell, 2002). Furthermore, the students did not accept this hypothesis on the authority of the teacher; they recognised that the truth of what the teacher had rephrased from their work was not yet known and thus needed *backing* (as
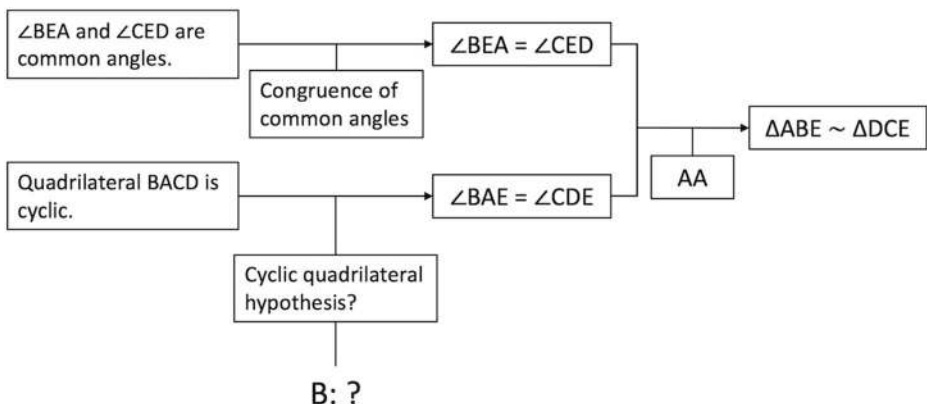

Fig. 9 Creative abduction performed and need recognised for backing (B) to support the hypothesis

such, *cyclic quadrilateral hypothesis* and *B* in Fig. 9 have question marks). This is clearly illustrated in the students' comments below, taken from their reflection worksheets after completing lesson 2:

> Shota: Another one [Fig. 5a] is still not clear and cannot be solved unless I prove another one [the cyclic quadrilateral hypothesis].
>
> Riko: I kind of get the meaning of this [the cyclic quadrilateral hypothesis], so I want to investigate whether it is really true in the next lesson.

Hence, this class activity can be regarded as creative abduction performed by the students with the support of the teacher.

Here, the difference between abductive and inductive reasoning can be discerned. In the above class discussion, the cyclic quadrilateral hypothesis (warrant) was devised almost simultaneously with a new view of the diagram in which cyclic quadrilateral BACD was considered (datum). If the transcripts are examined line-by-line, the reference to the datum precedes that of the warrant. Yet, the goal of the class was to explain that $\angle BAE = \angle CDE$, and the cyclic quadrilateral hypothesis was conjectured for that purpose. In other words, making the conjecture was not the goal per se. Thus, the class's reasoning is abductive and not inductive.

In lesson 3, the students worked on the proof of the cyclic quadrilateral hypothesis, which we analyse in the next section.

4.4 Proof of the generated hypothesis

The teacher began lesson 3 by showing a simplified diagram omitting point E in Fig. 5a. He then set up a specific opportunity where, before getting started on proving, the students thought for a few minutes about how to approach the proof. After that, the teacher held a whole-class discussion in which the students shared their ideas (lines 357–365 are omitted to avoid repetitions):

> 352 Teacher: Well, let's listen. Shota, what would you use?
> 353 Shota: The inscribed angle theorem.
> 354 Teacher: Ah, the inscribed angle theorem, the inscribed angle theorem. … How about Aoi?
> 355 Aoi: Auxiliary lines.
> 356 Teacher: Well, auxiliary lines. Drawing auxiliary lines.
> 366 Teacher: These two. Ayaka?
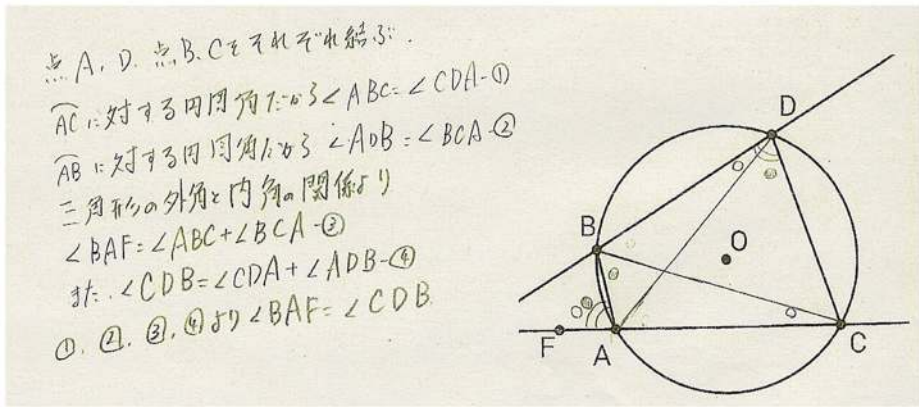> 367 Ayaka: Exterior angles and interior angles of a triangle.
> 368 Teacher: Ah, the relationship between the interior angles and exterior angles of a triangle, triangle. I want you to note down what you didn't write. … Takumi, is there anything else?
> 369 Takumi: Well, central angles.
> 370 Teacher: Oh, central angles, oh. The central angles of a circle, right? … How about Haruna? … Did you write anything else?
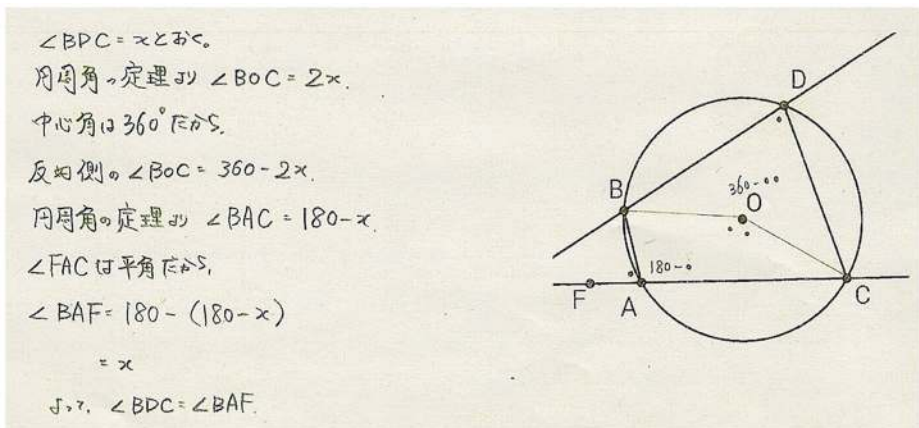> 371 Haruna: No.
> 372 Teacher: Well, does anyone have anything else? OK.

と A, D. 点 B, C を それぞれ結ぶ.

AC に対する円周角になるく∠ABC=∠CDA-①

AB に対する円周角になるく∠ADB=∠BCA-②

三角形の外角と内角の関係より

∠BAF=∠ABC+∠BCA-③

また ∠CDB=∠CDA+∠ADB-④

①, ②, ③, ④ より ∠BAF=∠CDB.

We connect points A and D, and points B and C.
From the inscribed angles corresponding to arc AC, ∠ABC = ∠CDA — ①
From the inscribed angles corresponding to arc AB, ∠ADB = ∠BCA — ②
From the relationship between the exterior angles and interior angles of a triangle,
$\qquad$ ∠BAF = ∠ABC + ∠BCA — ③
In addition, ∠CDB = ∠CDA + ∠ADB — ④
From ①, ②, ③, and ④, ∠BAF = ∠CDB.

Fig. 10 Mizuki's proof (English translation added by us)

Capitalising on these ideas, the students worked individually on proving the cyclic quadrilateral hypothesis for approximately 15 min. When doing so, the teacher suggested focusing on ∠BAF and ∠CDB (see Fig. 10 for the place of point F). Many students succeeded in the construction of the proof, and four students' proofs were shared in the class discussion. Two of

∠BPC=x とおく。

円周角の定理より ∠BOC=2x.

中心角は360°だから,

反対側の∠BOC=360-2x.

円周角の定理より ∠BAC=180-x.

∠FAC は平角だから,

∠BAF=180-(180-x)

$\qquad$ = x

よって, ∠BDC=∠BAF.

We consider ∠BDC = x.
From the inscribed angle theorem, ∠BOC = 2x.
Since the central angle is 360°, the other side ∠BOC = 360 − 2x.
From the inscribed angle theorem, ∠BAC = 180 − x.
Since ∠FAC is a straight angle, ∠BAF = 180 − (180 − x) = x
Hence, ∠BDC = ∠BAF.

Fig. 11 Miwa's proof (English translation added by us)

the proofs are illustrated in Figs. 10 and 11. In Fig. 10, Mizuki employed the inscribed angle theorem and the triangle exterior angle theorem in ∆ABC, showing that ∠ABC = ∠CDA, ∠ADB = ∠BCA, and then ∠BAF = ∠ABC + ∠BCA = ∠CDA + ∠ADB = ∠CDB. In Fig. 11, Miwa let ∠BDC be x and used the inscribed angle theorem, showing that ∠BOC = $2x$, the reflex angle of ∠BOC is $360 - 2x$, ∠BAC = $180 - x$, and then ∠BAF = $180 - (180 - x) = x = $ ∠BDC. Based on these proofs, the students accepted the cyclic quadrilateral hypothesis as a proved theorem.

At the end of the lesson, the class came back to the case shown in Fig. 5a (which was a local counterexample to their initial proof) and confirmed that the original statement, ∆ABE ~ ∆DCE, could be proven with the cyclic quadrilateral theorem.

5 Discussion

The analysis presented in this paper shows that mathematical activity in the classroom can be encapsulated by combining our model representing the generation of mathematical knowledge through heuristic refutation (Fig. 1) and the model of the kinds of abduction (Fig. 2). After proving the statement (prompted by the task in Fig. 3), the students faced a local counterexample invalidating their proof and struggled to deal with it (the arrow from *refuting* to *proving* in Fig. 1). Creative abduction depicted in Fig. 2 was then observed where the cyclic quadrilateral hypothesis was proposed (the move to *generation of mathematical knowledge*) based on the students' efforts. By recognising that this hypothesis needed to be proved, and by proving it, the students succeeded in proving that the original statement was true in the case of the local counterexample (the move back to the arrow from *refuting* to *proving*).

The mathematical knowledge constructed in the class that was new to the students was the cyclic quadrilateral theorem—an interior angle of a cyclic quadrilateral is equivalent to the exterior angle of the opposite angle—together with its proof. Mariotti (2000) proposed the notion of mathematical theorem that consists of a statement, its proof, and the reference theory where the proof works. The class analysed for this paper generated the cyclic quadrilateral theorem in the sense that the statement, the cyclic quadrilateral hypothesis, was developed from the students' activity, and many students succeeded in proving this hypothesis in Euclidean theory.

5.1 Abduction and counterexamples

In this paper, we have considered abduction for analysing the classroom data. Abduction is not deterministic reasoning, and therefore, hypotheses discovered through abduction need to be verified. The lessons analysed in this paper can be understood further by looking into the differences with Peirce's theory in terms of the objects and modes of verification (i.e., what is verified and how it is verified).

The object of verification in Peirce's theory is mainly relevant to data in Toulmin's model. In creative abduction, both data and warrants are invented and thus need verification (Eco, 1986; Pedemonte & Reid, 2011). In our lessons, while the warrant (the cyclic quadrilateral hypothesis) was proven, the datum was not scrutinised. This is because the datum was related to how to view the diagram, and the new term describing this view (cyclic quadrilateral) was straightforward for the students. When discussing abduction, Thagard (1978) distinguished

between inference to cases (data) and inference to rules (warrants), and the abduction in our case was the latter.

Peirce argued that the modes of verification are deduction and induction (Peirce, 1935, CP 6.468–473). Hypotheses discovered through abduction constitute the starting point of deduction. The consequences of the hypotheses are collected by deduction. Induction plays a decisive role in the subsequent stage in which those consequences are compared with empirical phenomena, and the hypotheses are thus corroborated, modified, or rejected. In our lessons, however, the cyclic quadrilateral hypothesis produced in the abductive reasoning was the goal of deduction, not the starting point, where this hypothesis was derived from other known results (Meyer, 2010). Empirical tests of the hypothesis by induction (which were not performed but were possible, given that a DGE was available in the class) are not sufficient to establish the truth of the hypothesis. This represents one characteristic of mathematics where, in contrast to natural science that was Peirce's main focus, the truth of conjectures must be shown by deductive reasoning (although empirical verification plays some role).

Our work in this paper makes a theoretical contribution to the literature both on counter-examples and on abduction. Existing studies on counterexamples have investigated how students and teachers respond to counterexamples with their available knowledge (e.g., Larsen & Zandieh, 2008; Zazkis & Chernoff, 2008). Our study complements those studies by adding another dimension in which students cope with counterexamples through constructing mathematical knowledge that is new to them.

In addition, our study has examined the role of abductive reasoning in a different context. Although existing research has investigated several aspects of the conjecturing and proving processes of students (e.g., Arzarello & Sabena, 2011; Pedemonte & Reid, 2011), this paper illustrates that the model of abduction is also useful in analysing the process involving refutations. Furthermore, we have shown that one type of abductive reasoning, namely creative abduction, can lead to the construction of mathematical knowledge. This finding addresses the gap in the literature pointed out by Rivera (2017), who reviewed research on abduction and stated that "Empirical research in mathematics education along these features [central features in abductive cognition] is needed to fully assess the extent and impact of their power in shaping mathematical knowledge construction" (p. 564). In sum, this paper advances mathematics education research by presenting a case in which mathematical knowledge is constructed through heuristic refutation and abductive reasoning.

This contribution is also significant for educational practice. While mathematics curricula in several countries highlight mathematical content and practice as the goals of students' mathematical learning, integrating these two strands better remains an issue. This issue is particularly important in the area of argumentation and proof, something that Dawkins and Karunakaran (2016) highlight as the need to facilitate "the integration of proving as a *means* of mathematical learning across the curriculum" (p. 74, italics in the original). Heuristic refutation is relevant to mathematical practice, and mathematical theorems are one kind of mathematical knowledge. The lessons analysed in this paper offer an example case of introducing mathematical theorems through engaging students in heuristic refutation, and this illustration shows one way to integrate mathematical content and practice.

Despite the limitations of this study (such as the number of cases examined), abductive reasoning may be observed in other similar contexts. Lakatos (1976) presented several ways to handle global counterexamples, and one method, *lemma incorporation*,

can be described with abduction (Arzarello et al., 1998). Lemma incorporation leads to the invention of new concepts, and this method involves overcoded abduction because the existing proofs are used to identify premises that make the conjectures true. Larsen and Zandieh (2008) showed that a related process was observed in the mathematical activity of a group of undergraduate students. These studies imply that abductive reasoning can play a role in several processes in which mathematical knowledge is produced to deal with refutations.[4]

5.2 Task design and teachers' role

The lessons examined in this paper were not ordinary classroom lessons, but an intervention implemented as part of our design research. In this design research, we conducted a specific task design based on a set of design principles that were theoretically developed and empirically tested (Komatsu & Jones, 2019). The design principles include the use of tasks whose conditions are purposefully unspecified so that refutations can emerge and the provision of tools (DGE in this case) that facilitate the production of counterexamples. Research has shown the importance of task design, and our study reinforces this importance.

While the task shown in Fig. 3 itself is included in many Japanese textbooks, a main characteristic of our task design is how to organise students' activity after solving this task. There are three differences between our approach and the ordinary use of this task. First, we provided an opportunity for the students to draw and explore various diagrams after proving the original statement. Second, for this exploration, we allowed the students to use DGE; the use of DGE is not currently common in Japanese classrooms. Third, we dealt with the case presented in Fig. 5a in a deliberately atypical manner. This case is usually treated in an upper grade where students have *already* learned the cyclic quadrilateral theorem and thus can employ it (Komatsu & Jones, 2019). In contrast, our study introduced this case *with students who were not familiar with this theorem* because we intended to stimulate the students to recognise its necessity as a means for handling a counterexample. As shown in this paper, our task design enabled the students to learn the concept of cyclic quadrilateral and the related theorem through working on heuristic refutation.

In addition to task design, our analysis identified how the teacher played a vital role in supporting students while allowing sufficient space for their exploration and contributions. In lines 285–291 and 315–321, for example, the teacher focused the students' attention on the most challenging point (namely, whether $\angle BAE = \angle CDE$ in Fig. 5a) by orchestrating the whole-class discussions, wherein specific students' thoughts were purposefully selected and pursued (Leatham et al., 2015; Stein et al., 2008). One of the most significant supports from the teacher was observed during the subsequent creative abduction (lines 326–330). The teacher suggested the cyclic quadrilateral hypothesis by building on and revoicing the students' thinking using mathematical terms (Bartolini Bussi, 1998; Forman & Ansell, 2002). The teacher thereby supported the students' activity and also linked their work with standard

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[4] We do not intend to insist that abduction can be observed in all related processes. It appears that abduction is not related to another important means of dealing with global counterexamples shown by Lakatos (1976), namely *increasing content by deductive guessing* (Komatsu, 2016). Increasing content by deductive guessing—in contrast to exception barring and lemma incorporation, where the domain of an original conjecture is restricted to exclude counterexamples—refers to the use of deductive reasoning to produce a more general conjecture that is true even in the case of counterexamples to the original conjecture.

mathematical knowledge (i.e., cyclic quadrilateral and the related theorem) as a representative of the mathematical community.

Another critical move on the part of the teacher was to set up an opportunity in which, before getting started on problem solving, the students examined how to approach the problem and shared their ideas in the whole-class discussion (Jackson et al., 2013). This teaching strategy was observed while proving the cyclic quadrilateral hypothesis formed through creative abduction (lines 352–372). Creative abduction may not lead to the success of the later proofs because the devised warrants must be proven (Pedemonte & Reid, 2011). To support the students in this endeavour, the teacher facilitated the class discussion in which the students exchanged their ideas for proving the hypothesis (i.e., considering auxiliary lines and invoking known results that may be useful). Sharing problem-solving approaches is delicate because doing so may lower the cognitive demand of the task (e.g., the conversion of proof tasks into fill-in-the-blank formats) and may imply the use of a particular approach that funnels student activity into a specific direction (Jackson et al., 2013). Here, the teacher deliberately maintained the balance between helping the students' activity and ensuring space for their exploration, evidenced by the fact that the students produced various types of full proof later, as illustrated in Figs. 10 and 11. Sharing problem-solving approaches in this way is especially helpful in the context of proving, given that one of the major difficulties in proof construction is encountered in the initial phase of proving where students often cannot find how to begin proving and cannot invoke the potentially useful knowledge they have (Moore, 1994; Weber, 2001).

These teachers' roles are frequently observed in Japanese problem-solving-style lessons. It is common for Japanese teachers, certainly expert teachers, to give opportunity for students to think about and share ideas on how to approach problems before actually working on the problems and to orchestrate whole-class discussion by selecting and sharing particular students' ideas (Funahashi & Hino, 2014; Ohtani, 2014). To enact these roles effectively, teachers' pedagogical content knowledge (Shulman, 1986) is crucial. Pedagogical content knowledge involves *knowledge of content and students*, which refers to anticipating what students are likely to think and what they will find challenging, and *knowledge of content and teaching*, which relates to making pedagogical decisions in whole-class discussion, including which of students' ideas should be pursued and what new questions should be posed to foster students' thinking (Ball et al., 2008). The teacher in our study had these sets of knowledge to sufficient degrees. Indeed, when planning the lessons, the teacher expanded the draft lesson plans made by the researchers (us) into detailed lesson plans relatively straightforwardly, and these plans demonstrated his rich pedagogical content knowledge, such as his anticipation of students' thinking and his strategy for orchestrating classroom discussion. The implemented lessons also mostly worked out as we expected.

We have discussed the integration of mathematical content and practice highlighted in curricula in several countries. Our findings show that careful task design and relevant knowledge/roles of teachers are essential to actualise this integration in classrooms.

6 Conclusion

In this paper, we have analysed a series of classroom lessons and described the process by which the students discovered a counterexample invalidating their proof and then worked via creative abduction in which the cyclic quadrilateral theorem was produced to cope with the

counterexample. This finding contributes to the literature by highlighting a connection between mathematical activity involving refutations and the generation of mathematical knowledge. Our work also expands the literature by examining the role of abductive reasoning in heuristic refutation and construction of mathematical knowledge. This study contributes to educational practice by providing an illustration of mathematical activity in which students learn mathematical content while engaging in mathematical practice. In addition to the need for careful task design, our study also highlights the importance of the teacher's role in supporting students working on this mathematical activity. Nevertheless, this study inevitably has some limitations, including the number of cases analysed and our limited focus on certain aspects of the framework shown in Fig. 1.

One direction for future research is to explore other aspects of our framework that were not considered in this paper. For example, whereas we paid attention to local counterexamples invalidating proofs, future studies can examine the possibility of, and process involved in, the generation of mathematical knowledge in the case of global counterexamples refuting conjectures. We also focused on mathematical theorems, as one kind of mathematical knowledge. The model in Fig. 1 encompasses not only knowledge of mathematical content (e.g., definitions and properties) but also knowledge of mathematical methods, such as specific proof methods (e.g., mathematical induction). Indeed, Lakatos (1976), whose work is the basis of our framework, discussed that the mathematician Seidel invented proof analysis as a method for addressing counterexamples. Thus, future studies can consider the production of different types of mathematical knowledge such as mathematical concepts and methods. Furthermore, we dealt with one pattern of transition from heuristic refutation to the generation of mathematical knowledge, where mathematical knowledge is constructed while addressing counterexamples. There would also be another case in which mathematical knowledge is constructed after managing counterexamples and then reflecting on the activity. In this case, the arrow towards the generation of mathematical knowledge in Fig. 1 commences with the whole of the lower part of Fig. 1, where solvers reflect on the entire activity comprising conjecturing, proving, and refuting. Exploring these aspects can serve as a basis for furthering the integration of mathematical content and practice in the area of argumentation and proof.

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