Generation and Thermalization of Plasma Waves

By T. H. Stix *

A fully ionized gas immersed in an axial magnetic field may exhibit low-frequency oscillations where the electric field and macroscopic velocity perturbations are perpendicular to the axis. The ordinary and extraordinary hydromagnetic waves of Alfvén [1] and Åström [2] are examples of these transverse oscillations. Ion cyclotron waves [3] are another example. The physical environment appropriate to these low-frequency transverse waves is present in several types of proposed thermonuclear reactors, including the stellarator [4] and the pyrotron.[5]

For several reasons, the thermonuclear experimentist may be interested in generating these waves. First, they are a hydromagnetic phenomenon that can be produced in the laboratory and experimental data can be checked against hydromagnetic theory. Second, generating and observing these waves at low power may be a useful diagnostic technique for determining plasma characteristics which are important parameters of these waves such as magnetic field strength, plasma density and density distribution, plasma temperature and composition, and the state of ionization of the plasma. Finally, generating these waves at high power is a possible scheme for rapid heating of the plasma. Calculations on ion cyclotron wave generation and thermalization indicate an efficiency, in a typical example, of more than 65% for the transfer of power from an rf power source into effective random ion motion.

An effect which is termed "cyclotron damping" is found to be extremely important in the theory of ion cyclotron waves. Cyclotron damping causes free ion cyclotron waves to decay exponentially with time, or propagated ion cyclotron waves to decay exponentially with axial distance. Furthermore, cyclotron damping converts the energy of the wave motion into effectively random transverse ion motion, which makes ion cyclotron heating especially useful for a pyrotron-type device.

Cyclotron damping occurs for transverse oscillations which are periodic both in time and in axial distance. Ions moving along lines of force will "feel" the oscillations of the transverse electric field at a frequency which differs from the plasma rest frame frequency by the Doppler shift. Some ions will "feel" the oscillations at their own cyclotron frequency, and Lenard [6] has pointed out that these ions will absorb energy at a constant rate from the electric fields. Dawson [6] has shown that the presence of classes of such ions leads to the damping with distance of propagated oscillations. In the excellent paper of Gershman [7] the effects of cyclotron damping on ordinary and extraordinary hydromagnetic waves have been computed.

The kinematic effects of cyclotron damping appear in the off-diagonal components,

$$n_i m \langle w_x w_z \rangle_{\mathrm{Av}} \text{ and } n_i m \langle w_y w_z \rangle_{\mathrm{Av}},$$

of the stress tensor. The axial ion velocity w_z is a constant of the motion through first order in these transverse oscillations, and is a zero-order quantity. The off-diagonal stress tensor terms are then first order terms, and must be included in a theory of small-amplitude oscillations. Although the stress tensor does not appear explicitly, it is the calculation of these momentum transport quantities which occupies the first section of this paper. A dispersion relation is obtained at the end of this section.

Then the physical effects of cyclotron damping are discussed in terms of phase relations, power absorption and damping rates. The reduction of the dispersion relation and the phase relations to the undamped results previously obtained (Ref. 3) is shown for frequencies slightly removed from the ion cyclotron frequency.

After that we calculate the energy input from an inducing coil to a cylindrical plasma. The circumstances include excitation of the plasma in the immediate neighborhood of the ion cyclotron frequency and excitation of ion cyclotron waves and hydromagnetic waves. Estimates are made for the width of the power absorption resonances, and the power transferred to the plasma is compared to the ohmic losses in the induction coil. For a plasma of appreciable density, the power transfer is efficient only in the cases of wave generation. A plasma heating scheme proposed in the last section is to generate ion cyclotron waves at a long wave-length, in order to achieve efficient transfer from the rf power source, and allow the waves to propagate through a region of magnetic field which decreases with axial distance

* Project Matterhorn, Princeton University, Princeton, New Jersey.

from the coil The wavelength of the ion cyclotron waves becomes shorter and shorter, and finally local cyclotron damping starts to occur In this local region, the waves damp out and the wave energy is converted into energy of effectively random transverse ion motion

DISPERSION RELATION FOR CYCLOTRON DAMPING

Small amplitude oscillations perpendicular to the lines of force are considered for a fully ionized gas immersed in an axial magnetic field Particle collisions are infrequent and the temperature of the gas corresponding to ion motions perpendicular to the lines of force is assumed to be zero The temperature of ions corresponding to motions parallel to the lines of force is taken to be finite We consider oscillations which are periodic in axial distance and in time, and neglect effects of resistivity gravity viscosity and electron inertia

For transverse oscillations the velocity of an individual ion along a line of force is approximately constant We divide up the plasma into a massless electron fluid and into constituent streams of ions, where each stream is populated by ions with the same zero-order axial velocity We solve the equation of motion for each constituent stream separately in terms of the transverse electric field along a line of force Adding the streams together gives a charge and current distribution due to the ion motions in terms of the electric field Inserting this ion charge and current distribution into Ohm's Law and Maxwell's equations yields homogeneous equations in the transverse electric fields The solution of these equations determines the x and y dependence of the electric fields for a given z and t dependence, and the elimination of the electric field variables from the equation yields a dispersion relation for the system

The coefficients in the dispersion relation are integrals over the distribution of zero-order parallel ion velocities Following a suggestion made by Ira Bernstein [8] we can avoid divergences in these integrals at resonance by considering an initial value problem in the ion motion

A list of symbols used in this paper is given in Table 1, together with their definitions Dimensionless quantities will be used extensively

The first-order linearized equation for the macroscopic motion of ions in one of the constituent streams is

$$\frac{\partial \mathbf{u}}{\partial t} + w_z \frac{\partial \mathbf{u}}{\partial z} = \frac{Z_1 e}{m} [\mathfrak{E} + \mathbf{u} \times \boldsymbol{\epsilon}_z B_0 + w_z \boldsymbol{\epsilon}_z \times \mathfrak{B}] \qquad (1)$$

We assume that the first-order electric and magnetic fields vary as the real parts of $\tilde{\mathfrak{E}}(x, y) \exp\,(ikz + i\omega t)$ and $\tilde{\mathfrak{B}}(x\ y) \exp\,(ikz + i\omega t)$, where the components of $\tilde{\mathfrak{E}}$ and $\tilde{\mathfrak{B}}$ may be complex A consequence of the massless electron motion is that $\mathfrak{E}_z$ is zero Using this fact and Maxwell's equations, $\mathfrak{B}$ in equation (1) can be expressed in terms of $\tilde{\mathfrak{E}}_x$ and $\tilde{\mathfrak{E}}_y$

A solution to (1) will then express the transverse velocity in terms of the transverse electric fields We choose that particular solution $\mathbf{u}_\perp$ which fits the initial values $u_x = u_y = 0$ at $T = T_0$ and sum $\mathbf{u}_\perp$ over all the constituent streams, to obtain $\mathbf{v}_\perp$ In dimensionless form, we have

$$\begin{aligned}\mathbf{v}_\perp = \mathrm{Re}\{&(1/\Omega)[-A_+ + A_- + iD_+ + iD_-] \\ &\times \tilde{\mathbf{E}}_\perp e^{i\varkappa Z + i\Omega T} + (i/\Omega)[-A_+ - A_- + iD_+ - iD_-] \\ &\times \tilde{\mathbf{E}} \times \boldsymbol{\epsilon}_z e^{i\varkappa Z + i\Omega T}\}\end{aligned} \qquad (2)$$

where

$$A_\pm = \mp \frac{1}{2|\varkappa|} \int_{-\infty}^{\infty} g\left(\frac{\Lambda - \Omega}{\varkappa}\right) \frac{\Lambda}{1 \pm \Lambda} \times \sin\,[(T - T_0)(1 \pm \Lambda]d\Lambda, \qquad (3)$$

$$D_\pm = \mp \frac{1}{2|\varkappa|} \int_{-\infty}^{\infty} g\left(\frac{\Lambda - \Omega}{\varkappa}\right) \frac{\Lambda}{1 \pm \Lambda} \times \{1 - \cos\,[(T - T_0)(1 \pm \Lambda)]\}d\Lambda \qquad (4)$$

The integrals in (3) and (4) approach long-time asymptotic values For values of $T - T_0 \gg |\varkappa s_0|^{-1}$, which means a time long compared with the time it takes ions with the rms velocity to travel an axial distance equal to $\lambda/9$, the integrals may be reasonably approximated by their asymptotic values

For a Maxwellian distribution of axial ion velocities characterized by a mean axial ion energy of $kT_1/2$, $A_\pm$ has the long-time asymptotic form

$$A_\pm = \frac{\pi}{2|\varkappa|\,s_0} e^{-p_\pm^2} = \frac{\lambda B_0 Z_1}{9\ 47 A^{\frac{1}{2}} T_1^{\frac{1}{2}}} e^{-p_\pm^2} \qquad (5)$$

where $p_\pm \equiv (1 \pm \Omega)(\varkappa s_0)^{-1}$.

For large values of $T - T_0$ we may approximate $D_\pm$ by dropping the cosine term and taking the principal part of the remaining integral For a Maxwellian distribution, the long-time asymptotic form is conveniently given by $D_\pm \to I_\pm - \frac{1}{2}$ A graph of $\varkappa s_0 I_\pm$ versus $p_\pm$ is shown in Fig. 1 Also shown on the graph is $|\varkappa| s_0 A_\pm$ $I_\pm$ reaches peak values of $0\ 54 |\varkappa s_0|^{-1}$ at $|p_\pm| \approx 1\ 0$ Useful approximations to $I_\pm$ are $I_\pm \approx (2\varkappa s_0 p_\pm)^{-1}$ for $|p_\pm| \gg 1$ and $I_\pm \approx p_\pm (\varkappa s_0)^{-1}$ for $|p_\pm| \ll 1$

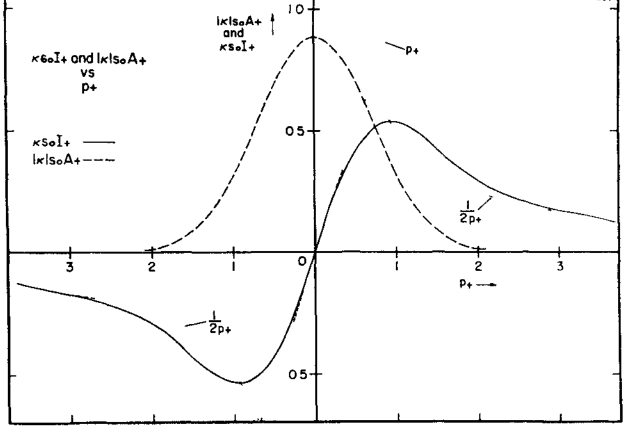

Figure 1 Graphs of $\varkappa s_0 I \pm$ and $|\varkappa| s_0 A \pm$

Table 1

Integers and Operators

A	atomic weight	$\mathbf{Q}$	$\boldsymbol{\epsilon}_x Q_x + \boldsymbol{\epsilon}_y Q_y + \boldsymbol{\epsilon}_z Q_z$
Z_i	ionic atomic number	$\mathbf{Q}_\perp$	$\boldsymbol{\epsilon}_x Q_x + \boldsymbol{\epsilon}_y Q_y$
$\boldsymbol{\nabla}$	$\boldsymbol{\epsilon}_x \frac{\partial}{\partial X} + \boldsymbol{\epsilon}_y \frac{\partial}{\partial Y} + \boldsymbol{\epsilon}_x \frac{\partial}{\partial Z}$	$\tilde{Q}$	$Q = \mathrm{Re}\,\{\tilde{Q} e^{i\varkappa Z + i\Omega T}\}$
Re	real part of		$\boldsymbol{\epsilon}_x, \boldsymbol{\epsilon}_y, \boldsymbol{\epsilon}_z, \boldsymbol{\epsilon}_r, \boldsymbol{\epsilon}_\theta$ unit vectors

cgs-Gaussian Units

$\mathfrak{B}$	first-order magnetic field
B_0	zero-order axial magnetic field strength
c	velocity of light
$\mathfrak{E}$	first-order electric field
e	electronic charge
$\mathfrak{J}^*$	sheet current density in " ideal coil "
k	axial wave-number, $\pm 2\pi/\lambda$
$\mathfrak{k}$	Boltzmann's constant
l_{coil}	over-all length of induction coil
m	ion mass
n_e, n_i	number of electrons, ions per unit volume
$n(w_z)dw_z$	number of ions per unit volume with w_z in range dw_z
r_c	radius of induction coil
r_0	radius of plasma boundary
$r, \theta, z; x, y, z; t$	cylindrical and Cartesian coordinates; time
T_i	ion temperature in °K
$\mathbf{u}$	first order macroscopic velocity for a single constituent stream
$\mathfrak{v}$	macroscopic plasma velocity
$\mathbf{w}$	individual ion velocity
$\lambda = \lvert\lambda\rvert$	axial wave-length
ω	angular frequency
ω_{ci}	angular ion cyclotron frequency, $Z_i e B_0/mc$

Dimensionless Units

$A_\pm$	Eq. (3), (5)	R_0	$r_0\omega_{ci}/c$
B	$\mathfrak{B}/B_0$	S	Eq. (23)
$D_\pm$	Eq. (4)	s_0	$[2\mathfrak{k}T_i/mc^2]^{\frac{1}{2}}$
$\mathbf{E}$	$\mathfrak{E}/B_0$	T	ω_{ci}/t
$F_\pm$	Eq. (6)	$\mathbf{v}$	$\mathfrak{v}/c$
G	Eq. (21)	X, Y, Z	$\frac{\omega_{ci}}{c}[x, y, z]$
$g\left(\frac{w_z}{c}\right) d\left(\frac{w_z}{c}\right)$	$\frac{n(w_z)dw_z}{n_i}$	α	$4\pi n_i mc^2 B_0^{-2}$
I_1, J_1, K_1	Ref. 9	β_m	Eq. (20)
$I_\pm$	Fig. 1	γ	Eq. (9)
$\mathbf{j}^*$	$4\pi\mathfrak{J}^*/B_0$	η	Eq. (18)
L	$\omega_{ci}\, l_{coil}/c$	$\varkappa$	kc/ω_{ci}
M, N	Eq. (14)	Λ	$\Omega + \varkappa w_z c^{-1}$
P_m	Eq. (22)	ν	Eq. (8)
$p_\pm$	$(1 \pm \Omega)(\varkappa s_0)^{-1}$	ξ	Eq. (23)
Q, U, V, W	Section 3	σ_1, σ_2	*cf* Eq. (7)
R	$r\omega_{ci}/c$	Ω	ω/ω_{ci}

We look now at particle energies rather than velocities. Let $m\langle u_x^2 + u_y^2\rangle/2$ denote the average over an oscillatory cycle of the transverse energy of an ion in a single constituent stream. Summing this average energy over all the constituent streams, we obtain

$$\tfrac{1}{2}\, m\langle\langle u_x^2 + u_y^2\rangle\rangle = mc^2\{F_+ \lvert \tilde{E}_x + i\tilde{E}_y\rvert^2 + F_- \lvert E_x - iE_y\rvert^2\} \quad (6)$$

where F_+ has the long-time asymptotic form for a Maxwellian distribution, $F_\pm \to F_0 + (T - T_0)A_\pm/2$. F_0 is the contribution to the sum from ions which do not " feel " their ion cyclotron frequency, and represents the kinetic energy associated with the non-resonant part of the forced ion oscillations. The second term is the contribution to the sum from ions which do " feel " their ion cyclotron frequency. The energy associated with a single-constituent stream at exact resonance increases as the square of the elapsed time. However, because of the spread in axial velocities, fewer and fewer streams remain in resonance as time goes on, so that the total energy associated with all the streams increases only linearly with time. Similarly, the velocity $\mathbf{u}_\perp$ of a single-constituent stream at exact resonance increases linearly with elapsed time, but the summed velocity $\mathbf{v}_\perp$ asymptotically approaches a constant magnitude.

The physical situation we wish to approximate in a self-consistent manner is a plasma with steady-state oscillations. A low but finite collision rate in the plasma creates a quasi-equilibrium situation. (There may be a heating of the plasma due to the oscillations, which we can make arbitrarily slow by reducing the amplitude of oscillation.) To help understand this quasi-equilibrium, we consider, for simplicity, those collisions which interchange the axial velocities of the colliding ions but leave the transverse velocities of each ion unchanged. Each colliding ion will be removed from one constituent stream and put into another. The transverse velocity an ion carries with it from the first stream may be different from the transverse velocity at that point associated with the second stream. If so, this difference will be random. In particular, an ion moving from a resonant stream to a non-resonant stream will bring with it a large transverse velocity, almost all of which will be random with respect to the non-resonant stream. A large number of such transitions will increase the temperature of the transverse motion of the non-resonant ions. Conversely, an ion moving from a non-resonant stream to a resonant stream will bring with it almost no transverse velocity. It will then be accelerated by the transverse electric fields from an initial state of almost zero transverse velocity. A large number of such transitions will cause energy to be absorbed by the resonant ions from the electric fields in the plasma. These two processes are the thermalization and energy absorption mechanisms of cyclotron damping.

With these processes as guides, we can construct a simplified model of a quasi-equilibrium plasma to which our ion orbit calulations would accurately apply. We consider a plasma in which each ion undergoes a catastrophe at a time τ after its previous catastrophe. Each catastrophe causes the ion's transverse velocity to be reduced to zero. The macroscopic velocity $\langle\mathbf{v}_\perp\rangle$ for this plasma is obtained from (2) by averaging over catastrophe times,

$$\langle\mathbf{v}_\perp\rangle = \tau^{-1}\int_{-\tau}^{0} \mathbf{v}_\perp dT_0.$$

This average has the same asymptotic form for large τ that $\mathbf{v}_\perp$ has for large $T - T_0$. $\langle\mathbf{v}_\perp\rangle$ gives us the plasma ion current density, which is all we require. Plasma electron current is obtained from Ohm's Law (Eq. 5*a*, Ref. 3) which states that electrons move freely along magnetic lines of force and with a velocity $c\mathbf{E}\times\boldsymbol{\epsilon}_z$ across magnetic lines of force. In Maxwell's equations (Eq. 6*a*–9*a*, Ref. 3) we can neglect displacement current in both the plasma and vacuum for the wave-lengths and frequencies of interest to us. We obtain the differential equation for the transverse electric field in the plasma

$$[\nabla\times(\nabla\times\tilde{\mathbf{E}})]_\perp = \sigma_1\tilde{\mathbf{E}} + i\sigma_2\tilde{\mathbf{E}}\times\boldsymbol{\epsilon}_z \tag{7}$$

where $\sigma_1 \equiv \alpha(iA_+ - iA_- - 1 + I_+ + I_-)$, and $\sigma_2 \equiv \alpha(iA_+ + iA_- + \Omega + I_+ - I_-)$, and the long-time asymptotic expressions are to be used for $A_\pm$.

In cylindrical coordinates, with no dependence on θ, and when σ_1 and σ_2 are independent of position, the r and θ components of Eq. (7) may be combined to give Bessel's equation. The solution which is finite at the origin [9] is $\tilde{E}_r \sim \tilde{E}_\theta \sim J_1(\nu R)$, where

$$\nu^2 = \frac{\sigma_1{}^2 - 2\sigma_1\varkappa^2 + \varkappa^4 - \sigma_2{}^2}{\sigma_1 - \varkappa^2}. \tag{8}$$

ν is the radial wave-number. By going to the limit of large radii, Eq. (8) may be thought of as the dispersion relation for a Cartesian coordinate system, where ν and $\varkappa$ are wave-numbers in the appropriate Cartesian directions.

PHASE RELATIONS AND CYCLOTRON DAMPING RATES

We distinguish four regions, determined by frequency, wavelength, density, and temperature, where the oscillations described by the dispersion relation (8) may be rather easily characterized. The spectrum also contains other regions, where the plasma behavior is not so easily characterized. The kinematics in the other regions can, however, also be obtained from the equations of the preceding section. The four regions are labelled:

Case I: Neighborhood of ion cyclotron frequency

Case II: Ion cyclotron waves

Case III: Torsional hydromagnetic waves

Case IV: Compressional hydromagnetic waves.

Table 2. Approximate Dispersion Relations, σ_1 and σ_2

	Dispersion relation	σ_1	σ_2
I.	$\nu^2 = -\varkappa^2\left[\frac{1-2i\gamma}{1-i\gamma}\right]$	$\pm\,\sigma_2$	$i\alpha A_\pm$
II.	$\frac{1}{\sigma_1} = \frac{1}{\varkappa^2} + \frac{1}{\varkappa^2+\nu^2}$	$\pm\,\sigma_2$	$\alpha\left(iA_\pm \pm \frac{1}{2(1\pm\Omega)}\right)$
III.	$\Omega^2 = \frac{\varkappa^2}{\alpha}$	$-\frac{\sigma_2}{\Omega}$	$-\frac{\alpha\Omega^3}{1-\Omega^2}$
IV.	$\Omega^2 = \frac{\varkappa^2+\nu^2}{\alpha}$	$-\frac{\sigma_2}{\Omega}$	$-\frac{\alpha\Omega^3}{1-\Omega^2}$

Four parameters are particularly important in characterizing these regions, $|\varkappa|s_0$, $p_\pm$, γ, and [10] $\varkappa^2/\alpha$, where

$$\gamma \equiv \left(\frac{\alpha A_\pm}{\varkappa^2}\right)_{\Omega=\mp1} = 5.20\times10^{-18}\,\frac{Z_i{}^3B_0n_i\lambda^3}{A^{3/2}\,T_i{}^{1/2}}, \tag{9}$$

$$\frac{\varkappa^2}{\alpha} = \frac{mk^2c^2}{4\pi n_i Z_i{}^2e^2} = 2.03\times10^{16}\,\frac{A}{Z_i{}^2n_i\lambda^2}. \tag{10}$$

$|\varkappa|s_0$ is the ratio of the velocity of an ion of energy $\frac{1}{2}T_i$ to the phase velocity of the oscillation, and is small in cases of practical interest. It may be evaluated numerically by inspection of equation (5). $p_\pm$ measures the proximity of the oscillation frequency to the ion cyclotron frequency. γ large or small will determine whether or not the electric field in the plasma is appreciably affected by the induced field due to plasma currents. $\varkappa^2/\alpha$ large or small was the principal criterion found in Ref. (3) separating ion cyclotron waves from hydromagnetic waves. In detail, the conditions in the four regions are:

$$\begin{aligned}
&\text{Case I:} && |p_+| \ll 1 \text{ or } |p_-| \ll 1,\ |\varkappa|s_0 \ll 1,\ \varkappa^2/\alpha \gg 1\\
&\text{Case II:} && |p_+| \gg 1,\ |p_-| \gg 1,\ \varkappa^2/\alpha \gg 1,\ \Omega \approx \pm 1\\
&\text{Case III:} && |p_+| \gg 1,\ |p_-| \gg 1,\ \varkappa^2/\alpha \ll 1\\
&\text{Case IV:} && |p_+| \gg 1,\ |p_-| \gg 1,\ \varkappa^2/\alpha \ll 1.
\end{aligned} \tag{11}$$

Using (2), the radial component of (7), and approximations based on (11), we obtain the approximate dispersion relations and phase relations for each of the four cases given in Tables 2 and 3. Some additional comments may be made on the separate cases. References will be made to (11), Table 2 and Table 3 without specific mention.

Case I

The ions rotate in circular orbits with the same sense of rotation as that for a free ion in a magnetic field. The ion velocity vector is thus circularly polarized. An electric field with the same circular polarization and phase exists in the plasma, so that energy is being put into the ions. Taking the time average of $\mathbf{v}\cdot\mathbf{E}$, and using the phase relations of Table 3 to obtain $\mathbf{v}$ in terms of $\mathbf{E}$, we obtain a power

Table 3. Phase Relations

	$\frac{\tilde{E}_r}{i\tilde{E}_\theta}$	$\frac{\tilde{v}_r}{i\tilde{v}_\theta}$	$\frac{\tilde{v}_r \pm i\tilde{v}_\theta}{E_r \pm i\tilde{E}_\theta}$
I.	$\frac{i\gamma}{1\mp i\gamma}$	$\frac{\Omega}{\lvert\Omega\rvert}$	$2A_\pm$
II.	$-\left(1+\frac{\nu^2}{\varkappa^2}\right)\frac{\Omega}{\lvert\Omega\rvert}$	$-\,\Omega$	$\frac{2i\sigma_1}{\alpha\Omega}$
III.[a]	$\frac{-\,\nu^2}{\Omega\varkappa^2}$	$-\,\Omega\left(1+\frac{\varkappa^2}{\nu^2}\right)$	$\mp\,i$
IV.[a]	$\Omega\left(1+\frac{\varkappa^2}{\nu^2}\right)$	$\frac{\nu^2}{\Omega\varkappa^2}$	$\pm\,i$

[a] Valid for $\alpha\nu^4 \gg \varkappa^6$.

input for Cases I and II which is the same as given by (6), where the energy in each constituent stream was summed over all the streams. The same power input for Case I was also obtained by Kulsrud and Lenard [6] through quite a different calculation.

For Case I, E_θ varies as I_1 (constant $\varkappa R$), where the absolute value of the constant lies between 1 and $\sqrt{2}$. For $|\varkappa|R_0 \ll 1$, $E_\theta \sim R$ for $R \leq R_0$, and is equal to the E_θ that would exist in the region in the absence of plasma due to the induction coil, E_θ^{vacuum}. This E_θ cannot match the boundary condition for a free oscillation (see Ref. 3), so the oscillations can only be forced or driven.

For sufficiently low density or short wavelength, $\gamma \ll 1$ and E_r is very small compared to E_θ. The total vector electric field in the plasma is approximately equal to $\boldsymbol{\epsilon}_\theta E_\theta^{\text{vacuum}}$. E_θ is in phase with v_θ, and in this circumstance cyclotron damping is a strong effect. For $\gamma \gg 1$, corresponding to higher density or longer wavelength, E_r is approximately equal in magnitude to E_θ. The physical explanation for this set of circumstances was offered first by Kulsrud. [6] The currents due to the radial ion motion and the associated longitudinal electron motion induce a fluctuating magnetic field and in turn a radial electric field, E_r. It turns out that this radial electric field is phased so that the resultant E_r and E_θ combination have principally a circular polarization which is opposite to that of the ion motion, v_r and v_θ. The wrongly polarized electric field does not put energy into the ions. There is a small electric field of the correct polarization which supplies enough energy to the ions to overcome the now much reduced cyclotron damping effect. It will turn out that at appreciable densities this inductive effect greatly reduces the efficiency of ion cyclotron heating in the Case I region.

Cases II, III and IV

These are possible oscillatory modes of the plasma. The simple criterion, $|p_+| \gg 1$ and $|p_-| \gg 1$ is sufficient to reduce the formalism in this paper to the simpler approach of Ref. (3), where the stress tensor was neglected in the equation of motion.

In Cases III and IV and in the undamped form of Case II, the electric field in the plasma is in quadrature with the ion velocity. No energy is transferred to or from the plasma wave in the steady state. However, when these waves are externally generated by an induction coil, the electric field due to the coil will appear in the plasma in phase with the ion velocity and in quadrature with the coil current. The electric field due to plasma currents associated with the generated wave will be in quadrature with the ion velocity but in phase with the coil current. This field is the " back emf ". Power flows from the coil to the plasma wave.

Of particular interest is the case for ion cyclotron waves which are slightly damped by a small amount of cyclotron damping. This situation occurs for values of $p_\pm$ such that $A_\pm$ is small but not negligibly small compared to $|I_\pm|$. (See Fig. 1.) In the constituent stream derivation, $\varkappa$ and Ω were assumed real. The dispersion relation is satisfied by complex values of ν. Physically, the cyclotron damping power is furnished by a divergence in the radial (or x-direction) flow of energy. Writing $\nu = \nu_0 + i\nu_1$ where ν_0 and ν_1 are real, $(\nu_1)^{-1}$ is the x-direction e-folding distance for the damping of velocity and field amplitudes. We are also interested in two other cases.

When Ω and ν are real, but $\varkappa$ is complex, the cyclotron damping energy is supplied by a divergence in the axial flow of power. Writing $\varkappa = \varkappa_0 + i\varkappa_1$, $(\varkappa_1)^{-1}$ is the axial damping distance for the wave amplitudes. Finally, when ν and $\varkappa$ are real but $\Omega = \Omega_0 + i\Omega_1$ is complex, the cyclotron damping energy is supplied by the wave itself, causing it to damp out with time at a rate Ω_1.

For slightly damped waves, the damping rate and distances are linearly related to one another with coefficients determined by the undamped waves. Using such linearized perturbations of the dispersion relation, we obtain the damping rate and reciprocal damping distance for slightly damped ion cyclotron waves,

$$\Omega_1 = 2A_\pm(1 \pm \Omega)^2 = \frac{A_\pm}{2}\left(\frac{\alpha}{\varkappa^2} + \frac{\alpha}{\varkappa^2 + \nu^2}\right)^2, \quad (12)$$

$$\varkappa_1 = \pm \frac{\alpha A_\pm}{\varkappa} \frac{(\nu^4 + 4\nu^2\varkappa^2 + 4\varkappa^4)}{(2\nu^4 + 4\nu^2\varkappa^2 + 4\varkappa^4)}. \quad (13)$$

Equation (13), for the case $\nu = 0$, has also been obtained by Dawson.[6]

ENERGY INPUT TO A PLASMA FROM AN INDUCTION COIL OF FINITE LENGTH

The relations given in the preceding sections apply to steady-state oscillations in a cylinder of plasma of infinite length. In practice, one has a plasma of finite length and an induction coil considerably shorter than the plasma. The true resonant modes of such a plasma are periodic in the axial length. However, we shall assume that wave energy which is propagated axially away from the coil is thermalized and absorbed without reflection. Assuming rapid thermalization, the rate of temperature rise of a plasma in a finite size reactor (e.g., stellarator or pyrotron) may be found by dividing the net power input to the plasma by the heat capacity of the plasma. For practical cases, it is useful to have a ratio of power inputs rather than an absolute level. We choose to compare the power input to the plasma with the rf power which is wasted, in a practical system, in ohmic losses in the induction coil. This ohmic loss is the principal source of inefficiency in a practical plasma heating system. And in a diagnostic experiment, power absorption by the plasma must be measured above the background of this ohmic loss. Fortunately, in both plasma heating and plasma diagnostic experiments, this power ratio is convenient to measure.

We call this power ratio W, and break W up into a product of three more ratios, such that $W = QVU$. We discuss these ratios in turn.

W

W is the average rf power input to the plasma divided by the average rf ohmic power loss in the coil system. The coil system includes not only the exciting coil, but also the connecting leads and the resonating elements, such as capacitors.

Q

Q is the figure of merit for the coil system. It is defined as the peak rf magnetic energy stored in the coil system multiplied by the angular frequency, ω, and divided by the average rf ohmic power loss in the coil system. Typical Q values in the megacycle range would run from 200 to 400.

V

A practical coil system stores magnetic energy in regions other than inside the volume normally occupied by plasma. To allow room for the vacuum chamber wall and for electrical insulation, the radius of the plasma boundary may be only one half the coil radius. Magnetic energy will also be stored outside the coil volume in the stray inductances of the resonating elements and connecting leads. We therefore introduce a storage ratio, V, defined under no-plasma conditions. V is the peak rf magnetic energy stored in an "ideal coil" divided by the peak rf magnetic energy stored in the coil system. The "ideal coil" is defined to have the following characteristics: (*a*) its radius is the same as the radius of the plasma boundary, (*b*) it carries a sheet current proportional to $\cos(kz + \omega t)$, (*c*) it has zero stray inductance, and (*d*) internal to the ideal coil, the electromagnetic field produced by the sheet current of the ideal coil is the same as the electromagnetic field produced by the actual coil system.

If the current distribution in a physical coil system is best approximated by standing waves, we would superimpose two running-wave ideal coils. The numerator of V should be the sum of the magnetic energies for both ideal coils. For a practical case, we consider an induction coil constructed of individual sections which are spaced axially every half wavelength and which are electrically connected in series. The azimuthal direction of current flow alternates from section to section. We denote by $\mathscr{L}_f$ that part of the total coil system inductance, $\mathscr{L}_{cs}$, which is associated with the fundamental ($\cos kz \cos \omega t$) component of the induction-coil current. If each section contains N turns spread evenly over a $1/M$th part of a wavelength, we obtain, neglecting end effects, the end-to-end $\mathscr{L}_f$,

$$\mathscr{L}_f = \frac{16}{\pi} \times 10^{-9} \left[MNkr_c \sin\left(\frac{\pi}{M}\right)\right]^2 l_{coil} \times [2I_1(kr_c)K_1(kr_c)] \quad (14)$$

and the storage ratio is

$$V = \frac{I_1(kr_0)K_1(kr_c)\,\mathscr{L}_f}{I_1(kr_c)\,K_1(kr_0)\,\mathscr{L}_{cs}} \quad (15)$$

where $\mathscr{L}_f$ and $\mathscr{L}_{cs}$ are in henries, and l_{coil} is in cm. Evaluating V in (14) for typical cases will reveal the high penalty for short wavelengths and small r_0 to r_c ratios. In practice, values of V may run from 0.05 to 0.2.

U

The numerator of U is the average power input to the plasma for a given ideal-coil current. The denominator of U is the product of the angular frequency, ω, and the peak rf magnetic energy stored in the ideal coil under a no-plasma condition with the same ideal-coil current.

Long-coil Approximation

We consider now an induction coil of finite length. We divide the plasma into three sections: to the left of, underneath, and to the right of the coil. We consider the plasma section underneath the coil as though it were a plasma section underneath an induction coil of infinite length. End effects are neglected.

A coil of finite length will excite a band of wavelengths. If the band is too broad, it may be broken into groups of narrower band-widths. A calculation may be made for each group, and the results superimposed. In treating Case I, we shall assume that the band of excited wavelengths lies entirely within the range allowed by the criteria in (11), and we shall neglect the change of plasma kinematics within the band. In Cases II, III and IV, the effect of the wide band-width for a short coil will appear in the shape and amplitude of the power absorption curve.

In Case I, the power absorption calculation will be based on the cyclotron damping integrals (4). In discussing these integrals, it was mentioned that ions would reach their asymptotic velocity after being subjected to the electric fields for times long compared with the time it would take an ion with energy $\mathfrak{k}T_i/2$ to travel the distance $\lambda/9$. For these integrals to apply to ions drifting through the coil section, the coil length must be long compared to $\lambda/9$.

Case I

We consider the case $|\varkappa|R_0 \ll 1$ so that $E_\theta \approx E_\theta^{\text{vacuum}}$. Underneath the induction coil, the power input per unit volume may be obtained from (6). We use the phase relation in Table 3 to obtain $\tilde{E}_x \pm i\tilde{E}_y$ in terms of $\tilde{E}_\theta^{\text{vacuum}}$. To obtain the power ratio U_I, we integrate the power input per unit volume over the plasma volume, and divide by the reactive power stored magnetically in the ideal coil. For $|\varkappa|R_0 \ll 1$, we may use a solenoid approximation. We obtain, at $\Omega = -1$ or 1,

$$U_I = \frac{\gamma}{1+\gamma^2} \cdot \frac{\pi^2 r_0^2}{2\lambda^2}. \quad (16)$$

U_I is a maximum for $\gamma = 1$. At higher values of γ (higher density, longer wavelength), the plasma-current inductive effect reduces the efficiency. The shape and width of the resonance depend on the magnitude of γ. We derive an improved version of $\tilde{E}_r/i\tilde{E}_\theta$ in Table 3, Case I, using $\sigma_1 = \sigma_2 = \alpha(iA_+ + I_+)$, and obtain by numerical calculation the following half-power frequencies.

$$|\Omega_{\frac{1}{2}}| = 1 \pm 7.0\,\eta \qquad \text{for } \gamma \ll 1,$$
$$|\Omega_{\frac{1}{2}}| = 1 \pm 11.5\,\eta \qquad \text{for } \gamma \gg 1,$$
$$|\Omega_{\frac{1}{2}}| = \begin{Bmatrix} 1 - 1.7\,\eta \\ 1 - 11.4\,\eta \end{Bmatrix} \quad \text{for } \gamma = 1, \tag{17}$$

where

$$\eta \equiv \frac{A^{\frac{1}{2}} T_i^{\frac{1}{2}}}{\lambda B_0 Z_i}. \tag{18}$$

The resonant peak in the first two cases occurs at $|\Omega| = 1$. In the $\gamma = 1$ case, the peak occurs at $|\Omega| = 1 - 8.0\,\eta$, and the calculated power absorption at the peak is 2.6 times the absorption at $|\Omega| = 1$ (Eq. 16). For larger values of γ, this peak moves into the region of ion cyclotron waves. If one evaluates $p\pm$, which measures when damping can occur (see Fig. 1), at the resonant frequency for undamped ion cyclotron waves, its numerical value turns out to be approximately equal to γ. For $\gamma \ll 1$ then, ion cyclotron waves cannot exist because they are overdamped. $\gamma = 1$ is an intermediate case. For $\gamma \gg 1$, cyclotron damping will be small, and there is a weak power absorption peak at $|\Omega| = 1$ and a strong peak associated with wave generation. The calculation of power absorption at the strong peak is done in the following section.

Cases II, III and IV

We consider now the cases where we have a resonance of the total plasma. The power which the exciting coil puts into the plasma in the coil section is propagated or radiated, axially out each end of the section as a hydromagnetic or ion cyclotron resonance wave. (For Case II, a small amount of power will also go into cyclotron damping inside the coil section.)

We expect running waves to squirt out each end of the coil, and to run axially down the plasma to infinity in each direction. We divide the plasma into three axial sections, and join the solutions for each section to one another by making the wave amplitudes continuous at the joint. Further details of end effects at the joints are not considered. For a left-moving running wave, for instance, we assume that everywhere to the left of the left end of the exciting coil the wave motion is a natural mode of oscillation of the system, and that the wave amplitude is constant. To the right of the right end of the exciting coil, the wave amplitude for this left moving wave is assumed to be zero. And underneath the exciting coil, we assume that the plasma behaves as though it were underneath an exciting coil of infinite length.

The solution underneath the coil must satisfy the boundary conditions at the plasma-vacuum interface $R = R_0$ and at the induction coil radius $R = R_c$. Without loss of generality, we can replace the induction coil by an ideal coil at $R = R_0$ carrying a current $\mathbf{j}^* = \boldsymbol{\epsilon}_\theta \,\mathrm{Re}[\tilde{j}^* \exp(i\varkappa Z + i\Omega T)]$. We use the boundary conditions in Ref. (3), Eqs. (4b)–(9b), and expand $J_1(\nu R)$, where it occurs, in an infinite series of $J_1(\nu_m R)$, where ν_m is the radial wave number for the mth radial mode of oscillation of the free cylindrical plasma. The various ν_m are the various solutions of Eq. (12), Ref. (3).

For each mode m we choose particular solutions $E_\theta{}^m$ for the left moving wave which are zero at the right end of the coil, $Z = Z_0$. We obtain, underneath the coil and for $R \leq R_0$,

$$\tilde{E}_\theta{}^m = \frac{\beta_m J_1(\nu_m R)}{\nu^2 - \nu_m{}^2}[1 - \cos\{(\varkappa - \varkappa_m)(Z - Z_0)\} + i\sin\{(\varkappa - \varkappa_m)(Z - Z_0)\}] \tag{19}$$

where

$$\beta_m = \frac{2i\Omega\tilde{j}^*}{R_0[1 + (G^2 - 1)(\nu_m R_0)^{-2}]J_1(\nu_m R_0)}, \tag{20}$$

and

$$G = -\frac{\varkappa R_0 K_1'(\varkappa R_0)}{K_1(\varkappa R_0)}. \tag{21}$$

$\varkappa_m$ is the axial wave-number for the mth radial mode of free oscillation at the frequency Ω. Other amplitudes in the plasma beside $\tilde{E}_\theta$ may be obtained from $\tilde{E}_\theta$ via the phase relations of Table 3.

The power input to the plasma from the ideal coil is found by integrating over the surface of the ideal coil the product of the coil current density with the electric field due to plasma currents. The average power going into the mth radial mode is

$$P_m = \left|\left(\frac{B_0{}^2 c^3}{8\omega_{ci}{}^2}\right)\frac{\beta_m \tilde{j}^* L^2 R_0(\varkappa - \varkappa_m) J_1(\nu_m R_0)}{\nu^2 - \nu_m{}^2}\right| S(\xi), \tag{22}$$

where

$$S(\xi) \equiv \frac{1 - \cos 2\xi + i(2\xi - \sin 2\xi)}{2\xi^2} \tag{23}$$

and $\xi \equiv (\varkappa - \varkappa_m)L/2$, and where we use the real term in $S(\xi)$ to represent real power and the imaginary term to represent reactive power. $S(\xi)$ approaches 1 as $\xi \to 0$ so that P_m is real at resonance.

The power ratio U_{wave} is the ratio of the real part of P_m to the reactive power in an ideal coil. Evaluating this ratio at resonance, $\nu = \nu_m$ and $\varkappa = \varkappa_m$, we obtain

$$U_{\text{wave}} = \left[(2I_1(\varkappa R_0)K_1(\varkappa R_0)) \times \left\{1 + \frac{G^2 - 1}{(\nu R_0)^2}\right\}\frac{\varkappa R_0{}^2}{L}\frac{\partial \nu^2}{\partial \varkappa^2}\right]^{-1}. \tag{24}$$

A useful approximation to (24) is the $|\varkappa| R_0 \ll 1$ limit. Then

$$U_{\text{wave}} \approx \left[\frac{\varkappa R_0^2}{L} \frac{\partial \nu^2}{\partial \varkappa^2}\right]^{-1}. \tag{25}$$

The quantity $\partial \nu^2/\partial \varkappa^2$ is obtained by differentiating the relevant dispersion relation, (8) or Table 2. In the low $\varkappa R_0$ limit, $\nu_m R_0$ are the roots of $J_0(\nu_m R_0)$, and are given approximately by $\nu_m R_0 \approx \pi(m - \frac{1}{4})$ for $m \geq 1$. (Compare Ref. 3.) In this limit, $\nu^2 \gg \varkappa^2$. It is frequently useful to drop the $\varkappa^2/\alpha \gg 1$ criterion on ion cyclotron waves. For the undamped waves, it is sufficient to require only that $\nu^2 \gg \varkappa^2$ and $\nu^2/\alpha \gg \Omega^2$ to obtain the approximate dispersion relation

$$\Omega^2 = [1 + \alpha/\varkappa^2]^{-1}. \tag{26}$$

Using the $\nu^2/\alpha \gg \Omega^2$ and $\nu^2 \gg \varkappa^2$ approximation for ion cyclotron waves, we obtain the $|\varkappa| R_0 \ll 1$ approximation to (24) for cases II, III, IV:

$$U_{\text{II}} = \frac{8 l_{\text{coil}}\, r_0^2}{\pi \lambda^3 (m - \frac{1}{4})^4} (1 + \alpha/\varkappa^2)^{-1} \tag{27}$$

$$U_{\text{III}} = \frac{8 l_{\text{coil}}\, r_0^2}{\pi \lambda^3 (m - \frac{1}{4})^4} (\varkappa^2/\alpha) \tag{28}$$

$$U_{\text{IV}} = \frac{l_{\text{coil}}\, \lambda}{2\pi r_0^2}. \tag{29}$$

The resonance curve of plasma load versus frequency is given by (22). Since the quantity in absolute value signs in Eq. (22) varies only slowly in the vicinity of a resonance, the shape of the resonance curve is essentially determined by $S(\xi)$, Eq. (23). One may substitute for ξ, $\xi = (L/2)(\Omega - \Omega_m)(\partial\Omega/\partial\varkappa)^{-1}$. The half-power points occur at $\xi = \pm 1.4$. $\partial\Omega/\partial\varkappa$ is the group velocity of the wave, and may be determined from the dispersion relation. For our various cases, we obtain the half-power frequencies

$$\frac{\Omega_{\frac{1}{2}} - \Omega_m}{\Omega_m} = \begin{cases} \pm \dfrac{0.45\lambda}{l_{\text{coil}}} [1 + \varkappa^2/\alpha]^{-1} & \text{Case II} \\[2ex] \pm \dfrac{0.45\lambda}{l_{\text{coil}}} & \text{Case III} \\[2ex] \pm \dfrac{1.8 r_0^2}{l_{\text{coil}}\, \lambda\, (m - \frac{1}{4})^2} & \text{Case IV.} \end{cases} \tag{30}$$

THERMALIZATION OF PLASMA WAVES

We can list some of the processes which will act to thermalize the energy of plasma wave motion and which are amenable to calculation. In addition to these known processes, there may occur in an actual plasma a rapid thermalization of wave energy through not yet understood processes similar to those observed by Langmuir [11] and Gabor.[12]

In all three types of plasma waves, ion-electron co lisions will transform the wave energy slowly into ohmic heat. A faster thermalization will take place in hydromagnetic compressional waves through the heating processes of magnetic pumping,[6] which will act on both ions and electrons.

An interesting situation occurs for ion cyclotron waves in a gas containing ions with different charge to mass ratios. Only the resonant ions move with the wave. One might consider heating a deuterium-tritium mixture in this manner. Even if the relative average velocities of deuterium and tritium were insufficient for nuclear reaction without wave thermalization, the ion-ion collisions will cause the wave to be thermalized fairly rapidly.

Cyclotron damping offers a possibility for thermalizing ion cyclotron waves extremely rapidly. It will generally be highly inefficient to put power into a plasma of appreciable density at the short wavelengths required for strong cyclotron damping (Case I). At longer wavelengths, however, one can couple energy into ion cyclotron waves, and the power transfer can be extremely efficient (Case II). For a plasma confinement device, such as a stellarator or a pyrotron, one could cause the steady-state confining field to decrease with the axial distance from each end of the coil section. If the rate of change with distance of the confining field is sufficiently small, the ion cyclotron waves which squirt out each end of the coil section will change in an adiabatic fashion. As they move through the decreasing B_0, their wavelength will gradually decrease, and when it becomes sufficiently short, cyclotron damping will occur locally. The ion cyclotron waves will damp out (Eq. (13)), and the wave energy will be transformed into energy of individual ion motions which are still perpendicular to the lines of force, but with effectively random amplitudes and phases. One might say that the actual heating of the plasma occurs in this region of cyclotron damping.

The author wishes to acknowledge the many useful ideas which came from discussions with Ira Bernstein, John Dawson, Russell Kulsrud, and Andrew Lenard, some of which have been indicated in this text. In addition, the author is profoundly grateful to Professor Lyman Spitzer for his constant encouragement and productive interest.

Addendum

The author also wishes to thank Dr. J. Neufeld, who has recently brought to his attention the paper of Gershman.[7] It may be noted that the damping rate computed from the dispersion relation (8) in the present paper will be equal to Ω^{-1} times the damping rate computed from the dispersion relation (31) in Gershman's work. This difference has its origin in the assumption here that the temperature of transverse ion motion was zero, while Gershman assumed an isotropic distribution.

REFERENCES

1 H Alfven, Arkıv Mat Astron Fysık *29B* No 2 (1942)

2 E Åstrom, Arkıv Fysık, *2* 443 (1951)

3 T H Stıx, Phys Rev, *106* 1146 (1957)

4 L Spıtzer, *The Stellarator Program*, P/2170, Vol 32, these Proceedıngs

5 R F Post, *Summary of UCRL Pyrotron (Mırror Machıne) Program*, P/377, Vol 32, *Pyrotron Hıgh Energy Experıments*, P/379, Vol 32, these Proceedıngs

6. J M. Berger, W A Newcomb, J Dawson, E A Frıeman, R M Kulsrud and A Lenard, *Heatıng of a Confined Plasma by Oscıllatıng Electromagnetıc Fıelds*, P/357, thıs Volume, these Proceedıngs

7 B N Gershman, Zhur Eksptl Teoret Fız *24*, 453 (1953).

8 I B Bernsteın, prıvate communıcatıon

9 G N Watson, *Theory of Bessel Functıons*, Cambrıdge Unıversıty Press, New York (1922)

10 An ıncorrect numerıcal evaluatıon of $\varkappa^2/\alpha$ was gıven ın Ref 3

11 I Langmuır, Phys Rev, *26*, 585 (1925), Z Phys, *46*, 271 (1928)

12 D Gabor, E A Ash and D Dracott, Nature, *176*, 916 (1955).