

On the Generation of a Scale-Invariant Spectrum of Adiabatic Fluctuations in Cosmological Models with a Contracting Phase

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In Pre-Big-Bang and in Ekpyrotic Cosmology, perturbations on cosmological scales today are generated from quantum vacuum fluctuations during a phase when the Universe is contracting (viewed in the Einstein frame). The backgrounds studied to date do not yield a scale invariant spectrum of adiabatic fluctuations. Here, we present a new contracting background model (neither of Pre-Big-Bang nor of the Ekpyrotic form) involving a single scalar field coupled to gravity in which a scale-invariant spectrum of curvature fluctuations and gravitational waves results. The equation of state of this scalar field corresponds to cold matter. We demonstrate that if this contracting phase can be matched via a nonsingular bounce to an expanding Friedmann cosmology, the scale-invariance of the curvature fluctuations is maintained. We also find new background solutions for Pre-Big-Bang and for Ekpyrotic cosmology, involving two scalar fields with exponential potentials and whose background values are evolving in time, for which a scale-invariant spectrum of adiabatic fluctuations is generated. We comment on the difficulty of obtaining a scale-invariant spectrum of adiabatic fluctuations with background solutions which have been studied in the past.

I. INTRODUCTION

Both Pre-Big-Bang and Ekpyrotic cosmology are attempts to construct alternatives to inflationary cosmology by introducing ideas of string theory to cosmology. The Pre-Big-Bang scenario [1] is based on considering the dilaton at an equal footing to the gravitational field, motivated by the fact that these are the important low energy degrees of freedom. The action is given (in the string frame) by

$$S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} e^{-\varphi} (R - (\partial\varphi)^2), \quad (1)$$

where φ denotes the dilaton and $\kappa^2 \equiv M_{\text{pl}}^{-2} \equiv 8\pi G$. The Universe is assumed to begin in a cold empty state with an accelerating dilaton. The initial evolution is dominated by the effects of the dilaton, and in the string frame yields super-exponential expansion. In the Einstein frame, this corresponds to a contracting phase with a scale factor $a(t)$ given by $a(t) \sim (-t)^{1/3}$ (the time t is negative in this phase). By a duality transformation, this solution is related to an expanding Friedmann-Robertson-Walker cosmology. Without corrections to the action (1), however, the initial dilaton-dominated (pre BB) branch and the late time expanding (post BB) branch are separated by a singularity. Since the Hubble radius decreases faster than the physical wavelength corresponding to fixed comoving scales, quantum fluctuations on microscopic scales during the pre BB branch can be stretched to scales which are cosmological at the

present time, as in the case of inflationary cosmology. See e.g. [2,3] for recent reviews of Pre-Big-Bang cosmology.

The Ekpyrotic scenario [4] (see also [5]) assumes that the visible Universe is a boundary brane in five dimensional bulk space-time, and that the heating event which corresponds to the Big Bang of Standard cosmology resulted from the collision of this brane with a parallel one which is attracted to it by an inter-brane potential $V(\varphi)$. The dynamics is described by a four-dimensional toy model in which the separation of the branes in the extra dimension is modeled as a scalar field φ . The effective action is taken to be

$$S = \int d^4x \sqrt{-g} \left(\frac{1}{2\kappa^2} R + \frac{1}{2} (\partial\varphi)^2 - V(\varphi) \right), \quad (2)$$

with a potential which for values of φ relevant to the generation of cosmological fluctuations is given by

$$V(\varphi) = -V_0 e^{-\sqrt{\frac{2}{p}} (m_p)^{-1} \varphi}, \quad (3)$$

where $0 < p \ll 1$ and m_p denotes the 4-d Planck mass (using the notation of [6]). The branes are assumed to start out widely separated and at rest. In this case, the energy is negative and the scale factor associated with the action (2) is contracting with $a(t) \sim (-t)^p$ (the time t is again negative in this phase). The time $t = 0$ corresponds to a singularity of the four dimensional model (2), as in the case of Pre-Big-Bang cosmology*. As in

*See, however, [7-10] for criticism of the scenario.

the case of Pre-Big-Bang cosmology, comoving scales contract less fast than the Hubble radius during this phase, and thus it is again possible that microscopic sub-Hubble scale fluctuations during the phase of contraction produce perturbations on cosmological scales today.

Neither for Pre-Big-Bang cosmology nor in the Ekpyrotic scenario [†] is a scale-invariant spectrum of adiabatic fluctuations generated at the level of the single field actions described above. A heuristic way to understand this is to note that the initial values of the fluctuations when they exit the Hubble radius are set by the Hubble constant. The Hubble constant is increasing rapidly as a function of time in both scenarios, and thus a deeply blue spectrum of initial fluctuations will result (spectral index $n = 4$ in the case of Pre-Big-Bang cosmology, $n = 3$ in the case of the Ekpyrotic scenario). Careful studies taking into account the gravitational dynamics on super-Hubble scales confirmed this result both for Pre-Big-Bang cosmology [12] and in the Ekpyrotic scenario [6,13–16] [‡].

The idea of obtaining the “big bang” of our Universe from a previous phase of cosmological contraction is, however, very interesting. In Section 2 we discuss a model consisting of scalar field matter with an equation of state $P = 0$, P denoting pressure, in which the quantum vacuum fluctuations of the field during the phase of contraction, matched to an expanding Friedmann cosmology at a nonsingular bounce, yield a scale-invariant spectrum of curvature fluctuations [§]. Such a model is obtained here by considering an exponential potential for the scalar field. In the PBB scenario exponential potentials for the dilaton may be generated by non-perturbative effects or by considering non-critical string theory (a cosmological constant in the string frame can generate an exponential potential in the Einstein frame).

In order to connect the contracting phase to an expanding phase, it is necessary to assume that at sufficiently high curvatures corrections to Einstein gravity become important, yielding a nonsingular bounce. Similar ideas are invoked to achieve a graceful exit in Pre-Big-Bang cosmology. In Section 3 we apply matching conditions [20,21] corresponding to continuity of the induced metric and of the extrinsic curvature for the infrared modes to calculate the induced curvature fluctuations in the ex-

panding phase. We find that the dominant mode of the curvature perturbation ζ in the expanding phase inherits the scale-invariance of the growing mode of the contracting phase. A new aspect of this matching problem is that the dominant mode of ζ increases on super-Hubble scales in the contracting phase, in contrast to what occurs in inflationary cosmology, where it is constant.

In Sections 4 and 5 we study backgrounds with two scalar matter fields. Note that both for Pre-Big-Bang cosmology and in the Ekpyrotic scenario, there are other light fields which should be included in the respective actions (1) and (2). In both cases there are axion and moduli fields which could play an important role. These fields can be dynamical during the collapse phase [22].

In the case of Pre-Big-Bang cosmology, it was realized in [23] that, in the presence of moving extra dimensions, axion fluctuations are amplified, and that the motion of the extra dimensions can be chosen such that a scale invariant spectrum of isocurvature perturbations results (for more work along these lines see e.g. [24,25]). However, such a primordial spectrum of inhomogeneities seeded by axion fluctuations are ruled out by the latest CMB anisotropy results (see e.g. [25] for an analysis of this issue).

In Sections 4 and 5 of this paper, we present a new two scalar field background which generates a scale-invariant spectrum of adiabatic fluctuations. This background has an interpretation both in the case of the Pre-Big-Bang scenario and in Ekpyrotic cosmology. In the former case it corresponds to a dilaton-axion background with exponential dilaton potential, in the latter case it involves the scalar field φ representing the separation of the two branes and a rolling axion. We also comment on the possibility of transforming a scale-invariant spectrum of isocurvature modes on super-Hubble scales into a scale-invariant spectrum of adiabatic fluctuations.

II. FLUCTUATIONS IN SINGLE FIELD BACKGROUND MODELS

We begin with a brief review of the analysis of the spectrum of cosmological fluctuations in single field background models.

To linear order in fluctuations (and neglecting gravitational waves and vector modes), the metric can be written as (see e.g. [26] for a comprehensive review)

$$ds^2 = a^2(\eta)[(1 + 2\Phi)d\eta^2 - (1 - 2\Phi)dx^i dx_i], \quad (4)$$

where η is conformal time. We have used the fact that if matter consists of scalar fields there is to linear order no anisotropic stress.

Via the Einstein constraint equations, the linear gravitational fluctuations (described by Φ) are coupled to the matter field fluctuations. In the single matter field case, with matter fluctuations denoted by $\delta\varphi$, a convenient and gauge-invariant variable is [27] (see also [28])

[†]Note that the fluctuation generation mechanism in the recently proposed cyclic model [11] is the same as in the Ekpyrotic scenario, and hence also does not yield a scale-invariant spectrum of fluctuations.

[‡]In the case of the Ekpyrotic scenario, the calculation of [17] yields a different result, but at least in our opinion is flawed because it is based on an ad hoc matching condition at the bounce which (as already mentioned in [13] does not yield the correct result when applied to power-law inflation (see also [18] and [19] for recent work on this issue.)

[§]Note that we are assuming the absence of any initial classical fluctuations, as is done in the Ekpyrotic scenario.

$$v = a(\delta\varphi + \frac{\dot{\Phi}}{H}\Phi) \equiv aQ, \quad (5)$$

with an overdot denoting the derivative with respect to physical time t . In particular, in the action for joint metric and matter fluctuations, v is a canonically normalized field, and hence it is useful to quantize the fluctuations in terms of it. We will be studying the linear perturbation equations for this variable in momentum space, with k standing for comoving momentum.

In an expanding Universe described by the Einstein equations, a convenient variable to use to track the amplitude of the fluctuations on super-Hubble scales is ζ , the curvature perturbation in comoving gauge [29,30]. In variable ζ is related to the metric fluctuations via

$$\zeta = \frac{2}{3} \frac{\Phi + H^{-1}\dot{\Phi}}{1+w} + \Phi. \quad (6)$$

and it is related to Q and v as follows:

$$\zeta = \frac{H}{\dot{\phi}} Q = \frac{v}{z}, \quad (7)$$

with

$$z = a \frac{\dot{\phi}}{H}. \quad (8)$$

In an expanding Universe and in the absence of entropy fluctuations, ζ is constant on scales larger than the Hubble radius, as can be seen from its equation of motion which is

$$\dot{\zeta} = -\frac{H}{\dot{H}} \frac{k^2}{a^2} \Phi. \quad (9)$$

In inflationary cosmology, it is thus useful to calculate the magnitude of ζ at the time when the fluctuation scale becomes larger than the Hubble radius, to use the constancy of ζ to evolve until the time when the scale re-enters the Hubble radius, and to infer the values of Φ and $\dot{\Phi}$ (which determine, for example, the spectrum of CMB anisotropies) at that time.

However, in the case of a contracting Universe one must (even in the absence of entropy fluctuations) be more careful, since the term in (9) proportional to k^2 may grow. In the case of Pre-Big-Bang cosmology, the growth is only logarithmic in η , and for the potential used in the original version of the Ekpyrotic model there is no growth of ζ at all. Hence, in these models ζ remains a good variable to follow the magnitude of the density fluctuations on super-Hubble scales.

In the Einstein frame, the equations of motion for cosmological perturbations reduce to the following equation for the Fourier mode of the variable v defined in (5) with comoving wavenumber k (we suppress the index k on v)

$$v'' + (k^2 - \frac{z''}{z})v = 0, \quad (10)$$

where a prime denotes the derivative with respect to conformal time η .

Making use of the background equations of motion, this equation becomes

$$v'' + (k^2 - \frac{a''}{a} + a^2 V'' + 2a^2 [\frac{\dot{H}}{H} + 3H])v = 0, \quad (11)$$

where V is the potential of the matter field φ . From this equation it is clear that on scales much smaller than the Hubble radius, v is oscillating with frequency given by k . The vacuum state normalization of v on these scales is

$$v = \frac{1}{\sqrt{2k}} e^{-ik\eta}. \quad (12)$$

We consider backgrounds which correspond in the Einstein frame to power law contraction

$$a(t) \propto (-t)^p. \quad (13)$$

In this case, the last two terms within the parentheses multiplying v in (11) cancel, and the equation reduces to

$$v'' + (k^2 - \frac{p(2p-1)}{(p-1)^2} \frac{1}{\eta^2})v = 0. \quad (14)$$

Its solution can be expressed in terms of Bessel functions Z_ν :

$$v = \sqrt{-\eta} Z_{|\nu|}(-k\eta) \quad (15)$$

where the index ν is related to the index p by

$$\frac{\nu^2 - 1}{4} = \frac{p(2p-1)}{(p-1)^2} \quad (16)$$

and therefore

$$\nu = \frac{1}{2} \frac{1-3p}{1-p}. \quad (17)$$

As can be seen from the large argument expansion of the Bessel functions, this solution automatically has the correct vacuum normalization.

Making use of the long wavelength limit of the Bessel functions, and of the fact that for our class of backgrounds $\dot{\phi}/H$ is independent of time, we obtain the following power spectrum of the variable ζ :

$$P_\zeta(k) = \frac{k^3}{2\pi^2} |\zeta|^2 \sim p k^{3-2\nu}. \quad (18)$$

Thus, to obtain a scale-invariant spectrum of adiabatic fluctuations, we require $\nu = 3/2$, whereas in the single field Ekpyrotic scenario with $p \sim 0$ one obtains $\nu \sim 1/2$ and thus spectral index $n \sim 3$. In the single field Pre-Big-Bang scenario the resulting values are $p = 1/3$, $\nu = 0$ and therefore $n = 4$.

An interesting background is obtained if $p = 2/3$. In this case, $\nu = 3/2$ and hence a scale-invariant spectrum of adiabatic curvature fluctuations is generated in the collapsing phase^{**}. This background corresponds to a contracting Universe dominated by cold matter with equation of state $P = 0$ (with P denoting pressure). The cold matter in this case is modeled by a scalar field with an exponential potential. Note that the fluctuations in this model behave differently than the fluctuations in a hydrodynamical model with $P = c_s^2 = 0$, where c_s^2 is the speed of sound. This can be seen by comparing the equation of motion (11) for scalar-field-induced fluctuations with the corresponding equation for hydrodynamical matter (see Chapter 5 of [26]). Note that modeling cold matter by a scalar field with nontrivial potential will avoid the Jeans instability problem on small scales which plagues $P = 0$ hydrodynamical matter. Note also that such a contraction ($a(t) \sim (-t)^{2/3}$) solve the horizon problem.

III. SPECTRUM OF FLUCTUATIONS AFTER THE BOUNCE

Let us consider the contracting model with $p = 2/3$ introduced above and imagine that it is connected via a nonsingular bounce to an expanding Friedmann-Robertson-Walker cosmology. In the following we will argue that in such a model the scale-invariant spectrum of ζ connects through the bounce to a late time scale-invariant spectrum in the expanding phase. Thus, our model yields an alternative to cosmological inflation in providing a mechanism for producing a scale-invariant spectrum of adiabatic fluctuations on scales which could, provided that the phase of contraction lasts sufficiently long, be of cosmological interest today.

The graceful exit problem faced in order to obtain such a bounce is similar to what is required to obtain a graceful exit in Pre-Big-Bang cosmology. In the latter case, there are indications that higher curvature and string loop effects can yield a nonsingular bounce [31–35] (see also [36,37]). Back-reaction effects also can play an important role [38]. There is also a construction [39,40] based on a Lagrangian with higher order corrections of the nonsingular Universe construction [41,42] which yields such a regular bounce. Since the corrections to Einstein gravity will be important only very close to the bounce, and since we are interested in scales which at the bounce are much larger than the Hubble radius, it appears reasonable to model the bounce (for the purpose of matching the pre-bounce and post-bounce fluctuations) as a gluing of two Einstein Universes at a fixed surface specified by

^{**}This result was initially noted in [13] and can also be seen from Fig. 3 of [16]

some physical criterion (see, however, the concerns raised in [15,18] on this issue).

The naive expectation is that the dominant mode of the curvature fluctuation ζ after the bounce (which is constant in time) is given by the dominant mode prior to the bounce, the mode which in our background has a scale-invariant spectrum. This is what happens in inflationary cosmology with reheating modeled as a discontinuous change in the equation of state [43,44], Pre-Big-Bang cosmology [21] and in the Ekpyrotic scenario with a nonsingular bounce [13]. Note however, that a similar “plausibility” argument applied to the variable Φ fails in both of the above cases. In both Pre-Big-Bang and Ekpyrotic cosmology the pre-bounce growing mode of Φ does not contribute to leading order in the wavenumber k to the dominant post-bounce mode of Φ . Thus, in order to be able to draw conclusions about the post-bounce spectrum, it is necessary to carefully match the fluctuation variables at the bounce.

The general relativistic matching conditions demand that at the boundary surface the induced surface metric and the extrinsic curvature be continuous [20,21]. As matching surface one can choose either a constant energy density surface (from the point of view of longitudinal gauge) or a constant scalar field surface^{††}. For super-Hubble scale fluctuations, the difference between these two surfaces is of order k^2 and does not effect the results to leading order in k^2 . The matching on a constant energy density surface implies the continuity of ζ and Φ across the surface. In Pre-Big-Bang cosmology [21] and in the Ekpyrotic scenario [13] this leads to the conclusion that the growing mode of Φ during the collapse phase (which in the case of the Ekpyrotic scenario has a scale-invariant spectrum) does not couple to leading order in k^2 to the dominant (constant) mode of the post-bounce phase. In contrast, the late time value of ζ is the same as at the bounce. These results are confirmed in studies of cosmological perturbations in generalized Einstein theories [16,45,46] which yield a bounce (see also [47,48]).

At the end of Section II we have shown that, for our new background, the growing mode of ζ in the contracting phase obtains a scale-invariant spectrum. We need to show that to leading order in k^2 there is a non-vanishing coupling between the pre-bounce growing mode of ζ and the post-bounce dominant mode. Note that the rapid growth of ζ is very different from what occurs in the inflationary Universe, in Pre-Big-Bang cosmology and in the Ekpyrotic scenario, and that the analysis of the matching conditions needs to be reconsidered.

We begin with the following general solution (to leading order in k^2) for ζ (see e.g. Eq. (12.27) in [26]):

^{††}See [18] for a justification of this choice of the matching surface.

$$\zeta = D + S \int \frac{d\eta}{z^2}, \quad (19)$$

where D and S are constant coefficients. In our background, the S mode is the growing one, whereas in inflationary cosmology it is decaying and thus subdominant. Making use of (9), we can find the corresponding form of Φ :

$$\Phi = -\frac{1}{2m_p^2} \frac{\mathcal{H}}{a^2} \left(\frac{S}{k^2} \right), \quad (20)$$

where \mathcal{H} is the Hubble constant in conformal time. This gives the mode of Φ which is decaying in inflationary cosmology. It also shows that this ‘‘decaying’’ mode of Φ effects the value of ζ to order k^2 (to leading order it cancels out). There is also a constant mode of Φ which determines the constant mode (D mode) of ζ . The coefficients of the constant modes of Φ and ζ are related by a function of the equation of state of the background which contains no k -dependence. Thus, in order to perform the matching analysis we consider the following forms for Φ and ζ :

$$\begin{aligned} \zeta &= D + S f(\eta), \\ \Phi &= \alpha D + \beta \frac{S}{k^2} \frac{\mathcal{H}}{a^2}, \end{aligned} \quad (21)$$

where $f(\eta) = \int \frac{d\eta}{z^2}$ and where the coefficients α and β depend only on the equation of state.

It is now straightforward to calculate the consequences of the continuous matching of Φ and ζ across the bounce for the coefficients of the two modes of ζ before and after the bounce. Quantities before the bounce will be denoted by a superscript $-$, those after the bounce by a superscript $+$. Simple algebra yields

$$\begin{aligned} D^+ \left(1 - \frac{\alpha^+ f^+ a^2 k^2}{\beta^+ \mathcal{H}^+} \right) &= D^- \left(1 - \frac{\alpha^- f^- a^2 k^2}{\beta^- \mathcal{H}^-} \right) \\ &+ S^- \left(f^- - \frac{\beta^- \mathcal{H}^-}{\beta^+ \mathcal{H}^+} f^+ \right). \end{aligned} \quad (22)$$

In the Ekpyrotic scenario, it follows from (15) and from the fact that $\nu = 1/2$ that $D^- \sim k^{-1/2}$ and $S^- \sim k^{1/2}$. Hence, the D-mode of ζ after the bounce has spectral index $n = 3$, in agreement with our earlier matching results [13]. However, for our present background we have $\nu = 3/2$ and hence $D^- \sim k^{3/2}$ and $S^- \sim k^{-3/2}$. Since the matching of the S^- mode to the D^+ mode is not suppressed by factors of k^2 , the post-bounce D^+ mode inherits the scale-invariance of the pre-bounce S^- mode. Thus, we have shown that our background yields a scale-invariant spectrum of fluctuations at late times.

Let us now return to the analysis of Section II and determine the amplitude of the scale-invariant spectrum of curvature perturbations at (and, as we have shown above, therefore also after) the bounce in our model with $p = 2/3$. As follows from Eq. (7) and from the background values of $\dot{\varphi}$ and H :

$$\zeta = \frac{H}{\dot{\varphi}} Q = \sqrt{\frac{p}{2}} \frac{Q}{M_{\text{pl}}} \quad (23)$$

According to (15), the normalized solution for Q is:

$$Q = e^{[i(\tilde{\nu}+1/2)\pi/2]} \frac{\sqrt{-\pi\eta}}{2a} H_{\tilde{\nu}}(-k\eta) \quad (24)$$

where H denotes the Hankel function, $\tilde{\nu} = |\nu|$, and where the scale factor is expressed in terms of conformal time as

$$a(\eta) = (-(1-p)M_{\text{pl}}\eta)^{\frac{p}{1-p}}. \quad (25)$$

By inserting the long-wavelength limit ($-k\eta \ll 1$) of the Hankel function one gets:

$$P_\zeta(k) = \frac{k^3}{2\pi^2} |\zeta|^2 = p 2^{2\tilde{\nu}-3} \frac{\Gamma(\tilde{\nu})^2}{2\pi^3} \left[\frac{\mathcal{H}(1-p)}{M_{\text{pl}}p} \right]^{4\tilde{\nu}} \frac{M_{\text{pl}}^{2\tilde{\nu}-3}}{k^{2\tilde{\nu}-3}}. \quad (26)$$

For $p = 2/3$ one has an amplitude in agreement with observations for $\mathcal{H}_*/M_{\text{pl}} \sim 10^{-1}$, where \mathcal{H}_* indicates the absolute value of the Hubble rate at the bounce (when the contraction stops).

The model with $p = 2/3$ which we are considering also generates a scale-invariant spectrum of gravitational waves. This can be seen immediately since $\dot{h} = ah$, where h is the amplitude of the tensor perturbation of the metric, obeys the same equation of motion as the scalar fluctuation variable v . This is true in all models with an exponential potential for the scalar field [49]. In order to compare the amplitudes of the power spectra P_ζ for scalar metric fluctuations and P_h for gravitational waves in the collapsing phase, we first need to divide v by M_{pl} (for dimensional reasons). Then, using the relation (7) between ζ and v and the background values of H and $\dot{\varphi}$ we immediately obtain

$$\frac{P_\zeta}{P_h} = \frac{p}{2}. \quad (27)$$

This is a definite prediction resulting from the analysis during the contracting phase.

IV. A BACKGROUND WITH TWO EVOLVING FIELDS

We now turn to the question of whether it is possible to generate a scale-invariant spectrum of adiabatic fluctuations in Pre-Big-Bang cosmology and in the Ekpyrotic scenario by turning on nontrivial background time-dependence of a suitably chosen second scalar field. We will consider the action

$$\begin{aligned} S &= \int d^4x \sqrt{-g} \left(\frac{1}{2\kappa^2} R + \frac{e^{\alpha\varphi/M_{\text{pl}}}}{2} (\partial\sigma)^2 \right. \\ &\quad \left. + \frac{1}{2} (\partial\varphi)^2 - V(\varphi) \right), \end{aligned} \quad (28)$$

with an exponential potential for φ :

$$V(\varphi) = -V_0 e^{-\beta\varphi/M_{\text{pl}}}. \quad (29)$$

In the Ekpyrotic scenario, the interpretation of this action is that we add to the single field action (2) (with φ representing the separation of the branes) an axion field σ with non-minimal coupling to gravity. In Pre-Big-Bang cosmology, the interpretation of (28) is to add an exponential potential for the dilaton φ and an axion-like field (non-minimally coupled in the Einstein frame) to the original action of (1). For $\alpha = 0$ the field σ should be interpreted as a modulus field.

We now look for self-consistent analytical solutions of the background equations:

$$\mathcal{H}^2 = \frac{1}{3M_{\text{pl}}^2} \left(\frac{\varphi'^2}{2} + a^2 V(\varphi) + e^{\alpha\varphi/M_{\text{pl}}} \frac{\sigma'^2}{2} \right) \quad (30)$$

$$\varphi'' + 2\mathcal{H}\varphi' + a^2 V_\varphi = \frac{\alpha}{2M_{\text{pl}}} e^{\alpha\varphi/M_{\text{pl}}} \sigma'^2 \quad (31)$$

$$\sigma'' + (2\mathcal{H} + \alpha \frac{\varphi'}{M_{\text{pl}}})\sigma' = 0. \quad (32)$$

We consider the following ansatz for $a(\eta)$ and $\varphi(\eta)$:

$$\varphi(\eta) = A \log(-M_{\text{pl}}(1-p)\eta) \quad (33)$$

$$a(\eta) = (-M_{\text{pl}}(1-p)\eta)^{\frac{p}{1-p}} \quad (34)$$

Eq. (32) can be immediately integrated giving:

$$\sigma' = C \frac{e^{-\alpha\varphi/M_{\text{pl}}}}{a^2} \quad (35)$$

where C is an integration constant. We have the following parameters: α, β, p, A, C . By imposing that all the terms in the equation for φ (31) have the same time dependence we obtain:

$$\beta \frac{A}{M_{\text{pl}}} - \frac{2p}{1-p} = 2 \quad (36)$$

$$\alpha \frac{A}{M_{\text{pl}}} + \frac{4p}{1-p} = 2 \quad (37)$$

and these relations lead to the constraint

$$\alpha/\beta = 1 - 3p. \quad (38)$$

Note that this relation implies that for the case of Pre-Big-Bang cosmology with $p = 1/3$ our action describes a modulus field σ minimally coupled in the Einstein frame, whereas in the case of the Ekpyrotic scenario, σ corresponds to an axion.

From the equation for \mathcal{H}' ^{‡‡}:

^{‡‡}It is not an independent equation, but is obtained from the time derivative of the Hamiltonian constraint (30), making use of the equations of motion for φ and σ . From this equation it is easy to derive another constraint among the parameters.

$$\mathcal{H}' - \mathcal{H}^2 = -\frac{1}{2M_{\text{pl}}^2} \left[\varphi'^2 + \sigma'^2 e^{\alpha\varphi/M_{\text{pl}}} \right] \quad (39)$$

we get:

$$\left(\frac{A}{M_{\text{pl}}} \right)^2 + \left(\frac{C}{M_{\text{pl}}^2(1-p)} \right)^2 = \frac{2p}{(1-p)^2}. \quad (40)$$

The interpretation of this result is that A/M_{pl} and $C/[M_{\text{pl}}^2(1-p)]$ are constrained to be on a circle of radius proportional to \sqrt{p} . From the energy constraint (30) we obtain:

$$-\frac{V_0}{M_{\text{pl}}^4} = p(3p-1). \quad (41)$$

This concludes all of the independent relations between the parameters and integration constants for the exact solution for the background obtained from the ansatz (33-34). This solution generalizes the PBB axion-dilaton solution of [23] with vanishing potential ($p = 1/3$, see Eq. (41)) to the case of generic exponential potentials.

Let us compare the dependence of a and $V(\varphi)$ on p and on time in our two field solution with the corresponding scalings in the single field model of Section II. Comparing (34) with (25) it follows that the dependence of a on p and on time is the same. Inserting (33) into (29), making use of (41), and comparing with the result in the single field case (see e.g. Eq. (12) of [13]) it follows that the dependence of $V(\varphi)$ on p and on time is also identical. This implies that the dependence on p of the spectrum and amplitude of gravitational waves is the same in the single field and multi field models. Eq. (40) shows that the sum of the kinetic terms in the multi field model with the axion is the same of the single field model.

A final remark for the case $\alpha = 0$, in which σ corresponds to a modulus field. The solution constructed in this section has a meaning for $\alpha = 0$, only in the case $p = 1/3$, i.e. vanishing potential for φ . Only in this case analytic solutions exist, since both φ and σ are massless, and the global equation of state is stiff matter (pressure density equal to energy density). In the case of an exponential potential as in Eq. (29), the modulus σ dominates the energy density for early times if $p < 1/3$ and for late times if $p > 1/3$.

V. FLUCTUATIONS IN THE TWO FIELD BACKGROUND

To analyze the spectrum of fluctuations in the two field models introduced in the previous section, we start with the expression for the joint metric and matter fluctuations expressed in terms of the generalization of the gauge-invariant variable Q of Section II to the case of multiple scalar fields φ_i :

$$Q_i = \delta\varphi + \frac{\dot{\varphi}_i}{H} \Phi, \quad (42)$$

where $\dot{\varphi}_i$ denotes the time derivative of the background value of φ_i . These variables describe the scalar field fluctuations in uniform spatial curvature gauge.

If all of the scalar fields φ_i are minimally coupled, they then satisfy the following set of coupled differential equations (see e.g. [50])

$$\ddot{Q}_i + 3H\dot{Q}_i + \frac{k^2}{a^2}Q_i + \sum_{j=1}^n [V_{,ij} - \frac{8\pi G}{a^3}(\frac{a^3}{H}\dot{\phi}_i\dot{\phi}_j)]Q_j = 0. \quad (43)$$

We would like to apply this set of equations to the action (28) for the two scalar fields φ and σ . In our case, however, σ is not minimally coupled to gravity. Hence, we cannot directly use this system of equations. We here suggest an approximate analysis of the induced spectrum of fluctuations by neglecting the coupling between the Q s, and hope to give a more rigorous analysis in a subsequent publication. We note that in an axion-free dilaton background, the same approximation leads to the correct result for the spectrum of adiabatic and isocurvature fluctuations [23].

We will first study the fluctuations of φ , assuming that the effects of the fluctuations in σ are negligible. Based on a comparison of the coupling term in (43), which is proportional to $\dot{\varphi}\dot{\sigma}$, with the ‘‘direct’’ term proportional to $\dot{\varphi}^2$, it appears that our approximation will be good if $\dot{\sigma} \ll \dot{\varphi}$ which will be the case for our background if $\alpha\varphi/M_{pl} \gg 1$ or if $A \gg C/M_{pl}$. In this case, we propose to use the single field version of (43) to describe the evolution of $\delta\varphi$, keeping in mind that in this case the kinetic term of σ contributes (due to its non-minimal coupling to gravity) to the potential V for φ . The resulting equation of motion is

$$Q''_{\varphi} + 2\mathcal{H}Q'_{\varphi} + \left[k^2 + a^2V_{\varphi\varphi} - \frac{\alpha^2}{2M_{pl}^2}e^{\alpha\varphi/M_{pl}}\sigma'^2 \right. \\ \left. - \frac{1}{M_{pl}^2}\frac{1}{a^2}\left(\frac{a^2}{\mathcal{H}}\varphi'^2\right)' \right] Q_{\varphi} = 0. \quad (44)$$

With the help of the background equations, the effective frequency for Q_{φ} becomes:

$$\Omega_{\varphi}^2 = k^2 + a^2V_{\varphi\varphi} - \frac{\alpha^2}{2M_{pl}^2}e^{\alpha\varphi/M_{pl}}\sigma'^2 - \frac{1}{M_{pl}^2}\frac{1}{a^2}\left(\frac{a^2}{\mathcal{H}}\varphi'^2\right)' \\ = k^2 - \left[\beta^2\frac{(1-3p)^2}{2(1-p)^2}\frac{C^2}{M_{pl}^4} + \frac{(1-3p)}{(1-p)^2}\beta^2p \right. \\ \left. - \frac{(1-3p)}{p}\frac{4}{(1-p)^2}\beta^{-2} \right] \eta^{-2}. \quad (45)$$

This relation satisfies two consistency checks. First, for $p = 1/3$ (in which case α vanishes and one has minimal coupling of σ) one recovers the result, known in Pre-Big-Bang cosmology, that only the term proportional to k^2 remains. As a second check, in the case of the Ekpyrotic

scenario, when $\beta = \sqrt{2/p}$ and $C = 0$, one again obtains the result that only the k^2 term remains.

In the equation for the aQ_{φ} , the multi-field generalization of the variable v of Section II, the damping term vanishes, but there is an additional term a''/a in the expression for the effective frequency which is, as in Eq. (14), given by

$$\frac{a''}{a} = p\frac{2p-1}{(p-1)^2\eta^2}. \quad (46)$$

We can now determine for which value of β one obtains a scale invariant spectrum of fluctuations in φ . If we define the new index n by:

$$\beta = \sqrt{\frac{n}{p}}, \quad (47)$$

then the index ν of the Hankel function solution for aQ_{φ} is determined by the sum of the coefficients of the η^{-2} terms in (45) and (46). Comparing with (16), and making use of the constraints (36) and (40) to simplify the coefficients in (45) we get

$$\nu_{\varphi}^2 = \frac{1}{4} + \frac{p(2p-1)}{(1-p)^2} \\ + \frac{(1-3p)[(1-3p)(n-2) + n - 4/n]}{(1-p)^2}. \quad (48)$$

As follows from the discussion of Section II, to obtain a scale-invariant spectrum, the index must be $|\nu| = 3/2$. In standard Pre-Big-Bang cosmology ($p = 1/3$) it is not possible to obtain this value. Since in this case σ is minimally coupled to gravity and thus describes a modulus field, we reach the conclusion that the addition of a modulus field with nontrivial background dynamics cannot be used to obtain a scale-invariant spectrum of adiabatic fluctuations. However, if we assume that the dilaton has an exponential potential, there are solutions with $p \neq 1/3$. For these solutions we find that it is possible for the addition of an axion-like field with suitably chosen coupling to gravity to yield a scale-invariant spectrum of adiabatic fluctuations.

In the case of Ekpyrotic cosmology ($p \sim 0$), however, one obtains a scale invariant spectrum for the choice ^{§§}

$$n = 1 + \sqrt{3}. \quad (49)$$

In this case, σ corresponds to an axion-like field. We reach the conclusion that the addition of an axion-like field σ to the effective four-dimensional action which has been used to describe Ekpyrotic cosmology can yield a

^{§§}Note that for this value of n , the condition $\alpha\varphi/M_{pl} \gg 1$ which ensures that the coupling term in the fluctuation equation is negligible, is easily satisfied.

scale-invariant spectrum provided that the coefficients α and β which describe the non-minimal coupling of σ and the exponent of the potential of φ , respectively, are chosen such that they obey the constraint of (38). A side result is that the addition of moduli fields to the usual action for Ekpyrotic cosmology does not help in obtaining a scale-invariant spectrum.

Note that in the previous two paragraphs, we are implicitly assuming that both the fluctuations in σ and the background energy density in σ are negligible. Our solutions can be viewed as a generalization of the construction of Copeland et al. [23].

To conclude, let us also consider the case in which the background and fluctuations in the axion σ dominate over the contributions of φ . This will be the case, for example, if $A \ll C/M_{\text{pl}}$ and $\alpha\varphi/M_{\text{pl}} < 1$. In this case, a reasonable approximation is to study fluctuations in σ neglecting the coupling to the fluctuations in φ . The resulting equation is

$$\ddot{Q}_\sigma + (3H + \frac{\alpha}{M_{\text{pl}}}\dot{\varphi})\dot{Q}_\sigma + \left[\frac{k^2}{a^2} + \frac{e^{\frac{\alpha\varphi}{M_{\text{pl}}}}}{M_{\text{pl}}^2}\sigma'^2 \left(3 + \frac{\dot{H}}{H^2} \right) \right] Q_\sigma = 0. \quad (50)$$

After transforming to conformal time, and in terms of the rescaled variable corresponding to v one has:

$$(ae^{\frac{\alpha\varphi}{2M_{\text{pl}}}}Q_\sigma)'' + [k^2 + \omega^2] (ae^{\frac{\alpha\varphi}{2M_{\text{pl}}}}Q_\sigma) = 0 \quad (51)$$

with

$$\omega^2 = \frac{e^{\frac{\alpha\varphi}{M_{\text{pl}}}}}{M_{\text{pl}}^2}\sigma'^2 \left(3 - \frac{1}{p} \right) - \frac{a''}{a} \quad (52)$$

where in the frequency the contribution from the non trivial damping term cancels out because of Eq.(36). The main difference with respect to the case of a modulus field is that:

$$\frac{e^{\frac{\alpha\varphi}{M_{\text{pl}}}}}{M_{\text{pl}}^2}\sigma'^2 = \frac{C^2}{M_{\text{pl}}^4\eta^2(1-p)^2}. \quad (53)$$

Thus, the index of the Bessel function is

$$\nu_\sigma^2 = \frac{1}{4} + \frac{p(2p-1)}{(1-p)^2} + \left(\frac{1}{p} - 3 \right) \frac{C^2}{(1-p)^2 M_{\text{pl}}^4}. \quad (54)$$

Therefore for $p \sim 0$ with

$$\frac{C^2}{M_{\text{pl}}^4} \sim 2p \quad (55)$$

one gets a scale invariant spectrum for the axion. Because of the constraint (40) it is natural that $C^2/M_{\text{pl}}^4 \sim \mathcal{O}(p)$, and since the value (55) implies that $A \sim 0$, the consistency condition to be able to neglect the effects of $\delta\varphi$ is satisfied. Note that once again, for $p = 1/3$ it is not

possible to obtain a scale-invariant spectrum for the axion. Indeed in the PBB scenario considered in [23], a scale invariant spectrum is obtained for the axion only in presence of other moduli fields (interpreted as time-dependent extra dimensions before their compactification [25]).

We note that for $n = 1 + \sqrt{3}$ $\nu_\sigma^2 = 1/4 + 2(3 - \sqrt{3})$, and therefore the axion fluctuations are amplified with a blue spectrum. However, we note that a background dominated by the axion, as in Eq. (55), would imply a very red spectrum for Q_φ in the approximation in which we neglect the couplings between the Q s. A careful study is needed in this case.

As a final remark in this section: Another way to obtain a scale-invariant spectrum of adiabatic curvature fluctuations at late times is to start with a scale-invariant spectrum of isocurvature fluctuations (like in the work of [23]), and to have the axion field which is responsible for the scale-invariant spectrum decay at some late time when it dominates the background energy density [51,52].

VI. DISCUSSION

In this paper we have discussed ways of obtaining a scale-invariant spectrum of adiabatic fluctuations in models in which a contracting Universe is matched via a nonsingular bounce to an expanding Friedmann-Robertson-Walker cosmology. We have assumed that new physics at high curvatures leads to a short period in which the background evolution is not described by the Einstein equations, thus enabling a transition from a contracting to an expanding phase (from the point of view of the Einstein frame scale factor).

Our first result is that a contracting Universe dominated by cold matter modeled as a scalar field with an exponential potential with appropriately chosen index can yield a scale-invariant spectrum of curvature fluctuations (fluctuations in the variable ζ) in the contracting phase, which is matched at the bounce to a scale-invariant spectrum during the expanding phase.

Our second result is that it is possible in the context of Ekpyrotic cosmology (and also of other models with a phase of power law contraction with power $p \neq 1/3$) to obtain a scenario yielding a scale-invariant spectrum of adiabatic fluctuations by considering a model with two scalar fields φ and σ , with exponential potential for φ and non-minimal coupling for σ , provided that the coefficients in the exponents of the potential and describing the non-minimal coupling satisfy a particular relation. In the context of Ekpyrotic cosmology, this corresponds to adding to the usual four-dimensional effective action which contains a scalar field φ describing the separation of the branes an axionic field σ . It is important that both fields have time-dependent backgrounds. In the context of the Pre-Big-Bang scenario, our model can be interpreted as adding to the dilaton-gravity action an axion

field, and assuming that there is an exponential potential for the dilaton.

A third result is that in the two field models considered it seems also possible to obtain a scale-invariant spectrum of axionic fluctuations. It is not clear if in this case the isocurvature and adiabatic fluctuations are coupled for since the trajectory in the background field space is a straight line [53]. If the axion in these models later dominates the background, this will correspond to a spectrum of adiabatic fluctuations.

Note that neither in Pre-Big-Bang cosmology nor in the Ekpyrotic scenario the addition of a minimally coupled modulus field can generate a primordial scale-invariant spectrum of adiabatic fluctuations.

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