1986

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Report Number:

86-586

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CSD-TR-586
March 1986

Generation of Configuration Space Obstacles: The Case of Moving Algebraic Surfaces†

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Abstract: We present algebraic algorithms to generate the boundary of configuration space obstacles arising from the translatory motion of convex objects amongst convex obstacles. Both the boundaries of the objects and obstacles are given by patches of algebraic surfaces.

1. Introduction

Using configuration space (*C−Space*) to plan motion for a single rigid object amongst physical obstacles, reduces the problem to planning motion for a mathematical point amongst "grown" configuration space obstacles, (the points in *C−Space* which correspond to the object overlapping one or more obstacles), Lozano-Perez (1983). For example, a rigid polyhedral object in compliant motion, viz., in continuous contact with the boundary of obstacles in 3-Dimensions can be represented as a point constrained to move on the three (or higher) dimension boundaries of grown obstacles embedded in 6-Dimension *C−Space*, Donald (1984). The technique thus relies, (and this is in general the more difficult part), in efficiently generating the boundary of *C−Space obstacles*. Numerous applications such as robot motion in workcells, automated assembly, numerical machining, part tolerancing, etc., exist where gross and fine motion planning in *C−space* have been used, Lozano-Perez, Mason and Taylor (1984), Tiller and Hanson (1984).

Early uses of the configuration space approach were, Freeman (1975), Adamowicz and Albano (1976), Udupa (1977), and more recently, Lozano-Perez and Wesley (1979), Lozano-Perez (1983), Lozano-Perez, Mason and Taylor (1984), Schwartz and Sharir (1983), Sharir and Schorr (1984), Franklin and Akman (1984), Canny (1984), Donald (1984), Yap (1985), Bajaj and Kim (1987a, b). The only efficient algorithms known for generating *C−Space* obstacles have been for polyhedral (degree 1) surface objects and obstacles, using methods for efficiently computing convex hulls, Lozano-Perez (1983), and recently efficient convolution algorithms for Minkowski addition, Guibas and Seidel (1986). However it has progressively become easier for geometric modeling systems to deal with objects that are defined by quadrics (degree 2) and higher degree surfaces, Requicha and Voelcker (1983). Further, motion planning in these sophisticated modeling environments, for example for process simulation, Hopcroft and Krafft (1985), suggests the need to characterize and efficiently generate the surface boundary of *C−Space obstacles* arising from the motion of objects amongst obstacles with curved surface boundaries. The methods based on generating a cylindrical cell decomposition of free *C−Space*, though applicable for general objects and obstacles defined by semi-algebraic sets, are computationally too restrictive, Schwartz and Sharir (1983), Yap (1985).

The main contributions of this paper are as follows. In §3 we show that the boundary of *C−Space obstacles* for general curved objects moving with only translation can be viewed as either the convolution between the obstacle boundary and the reversed object boundary (reversed with respect to a reference point on the object) or as certain *envelopes* of boundary surfaces of the moving reversed object with the reference point moving on the physical obstacle. Next in §4 we give algebraic algorithms to

† Research supported in part by NSF grant DCI-85 21356 and a David Ross Fellowship.

generate the curves and surfaces which make up the boundary of the three dimensional *C−Space obstacles*. Here we only consider objects and obstacles which are convex. These objects and obstacles are represented by a general algebraic boundary representation model discussed in §2. Crucial too here is the internal representation of curves and surfaces, i.e., whether they are parametrically or implicitly defined†. We present algorithms for both these internal representations. Further in §5 we show how to construct the topology of the *C−space* obstacle boundary. Use is made of a Gaussian (spherical) model discussed in §2.

2. Geometric Models

2.1. Solid Algebraic Model

In a boundary representation an object with general algebraic surfaces consists of the following:

(1) A finite set of vertices usually specified by Cartesian coordinates.

(2) A finite set of directed edges, where each edge is incident to two vertices. Typically, an edge is specified by the intersection of two faces, one on the left and one on the right. Here left and right are defined relative to the edge direction as seen from the exterior of the object. Further an interior point is also provided on each edge which helps remove any geometric ambiguity in the representation for high degree algebraic curves, Requicha (1980). Geometric disambiguation may also be achieved by adding tangent and higher derivative information at singular vertices, Hoffmann and Hopcroft (1986).

(3) A finite set of faces, where each face is bounded by a single oriented cycle of edges. Each face also has a surface equation, represented either in implicit or in parametric form. The surface equation has been chosen such that the gradient vector points to the exterior of the object.

In addition edge and face adjacency information is provided. Additional conventional assumptions are also made, e.g., edges and faces are non-singular, two distinct faces intersect only in edges, an auxiliary surface is specified for each edge where adjacent faces meet tangentially, etc. The objects and obstacles that we consider are *solids* and are assumed to enclose non-zero finite volume. Hence non-regularities such as dangling edges and dangling faces which depending on one's viewpoint enclose zero or infinite volume, are not permitted. The *C−spaces* that we construct are also regularized in this fashion and assumed to be solids enclosing non-zero finite volume.

2.2. Gaussian Model

Let S^2 be the unit sphere in R^3, and $Bdr(S)$ be the boundary surface of a convex set $S \subset R^3$. $Bdr(S)$ is homeomorphic to S^2. The *Gaussian Map* of S is defined as follows. For any set $K \subset Bdr(S)$, we shall define a set $N(S, K) \subset S^2$ as follows. A point $e \in S^2$ belongs to $N(S, K)$ if there exists a point $p \in K$ and a supporting plane L_p at p such that the exterior normal to L_p translated to the center of S^2 has e as its end point. This set $N(S, K)$ is called the *Gaussian Image* of K. The function $N(S, \cdot) : P(Bdr(S)) \rightarrow P(S^2)$ is called the *Gaussian Map* of S, where $P(Bdr(S))$ and $P(S^2)$ are the power sets of $Bdr(S)$ and S^2. It is a bijective map and its inverse $N^{-1}(S, \cdot) : P(S^2) \rightarrow P(Bdr(S))$ is called the *Inverse Gaussian Map* of S. For any $G \subset S^2$, the *Inverse Gaussian Image* of G is defined as $N^{-1}(S, G)$. The *Gaussian Curvature* of $p \in Bdr(S)$ is the limit of the ratio (Area of $N(S, K)$) / (Area of K) as K shrinks to the point p, see Pogorelov (1978), Horn (1986).

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† A unit sphere is implicitly given as $x^2+y^2+z^2-1=0$ and in rational parametric form as $x=(1-s^2-t^2)/(1+s^2+t^2)$, $y=2s/(1+s^2+t^2)$ and $z=2t/(1+s^2+t^2)$.

Gaussian Image of Faces, Edges and Vertices

Since all faces are patches of algebraic surfaces, we may assume that each face of a convex object is either a strictly convex face (*Gaussian Curvature* is positive on each point), a convex ruled surface patch, or a planar patch. The Gaussian Model of a curved object then consists of a finite set of vertices, edges and faces on the surface of a unit sphere as follows.

(1) For a strictly convex face F, the Gaussian Image $N(S, F)$ is a patch of S^2 with its boundary curves determined by the normals to the tangent planes of F at the boundary. That is, the boundary of $N(S, F)$ consists of the set of points $\nabla f(p) / \|\nabla f(p)\|$ for $p \in \bigcup_{E \in \Gamma} E$, where Γ is the set of boundary edges of F. For a ruled surface patch F, $N(S, F)$ is a degenerate curve on S^2. And for a planar patch F, $N(S, F)$ is a degenerate point on S^2.

(2) For an edge E, there are two faces F and G intersecting in E. By subdividing E if necessary, we may assume that F and G meet either transversally or tangentially along E. When F and G meet transversally along E, each point $p \in E$ determines two different points n_F and n_G on S^2 determined by the exterior normals of the tangent planes of F and G at p. $N(S, p)$ is the geodesic arc γ_p connecting n_F and n_G on S^2 and $N(S, E) = \bigcup_{p \in E} \gamma_p$ is a patch of S^2. $N(S, E)$ has 4 boundary curves, one is the set of points $\nabla f(p) / \|\nabla f(p)\|$ for $p \in E$, one is the set of points $\nabla g(p) / \|\nabla g(p)\|$ for $p \in E$, and the others are the geodesic arcs γ_{p_1} and γ_{p_2}, where $f = 0$ and $g = 0$ are the surface equations of F and G, and p_1 and p_2 are vertices of E. When F and G meet tangentially along E, $N(S, E)$ is a degenerate curve on S^2. $N(S, E)$ is the common boundary curve of $N(S, F)$ and $N(S, G)$. That is, it is the set of points $\nabla f(p) / \|\nabla f(p)\| = \nabla g(p) / \|\nabla g(p)\|$ for $p \in E$, When F and G are planar patches, E is a linear edge and $N(S, E)$ is a degenerate geodesic arc γ connecting n_F and n_G on S^2, where n_F and n_G are the exterior normals of F and G.

(3) For a vertex p, suppose that there are k adjacent faces (ordered in a counter-clockwise direction) F_1, F_2, ..., F_k intersecting at p. Each face F_i determines a point n_i on S^2 determined by the normal of F_i at p. Let γ_i ($i = 1, ..., k$) be the geodesic arc (greatest circle) on S^2 connecting n_i and n_{i+1} where $n_{k+1} = n_1$. Then $N(S, p)$ is the convex patch on S^2 bounded by the cycle of geodesic arcs $\gamma_1, \gamma_2, ..., \gamma_k$. When F_i and F_{i+1} is tangent on p, γ_i is a degenerate point. In the special case of all k faces being tangent at p, $N(S, p)$ is a degenerate point. $N(S, p)$ can also be a degenerate geodesic arc on S^2 when $Bdr(S)$ is locally non–smooth only along a curve which is tangent at p. Otherwise, $N(S, p)$ is a patch on S^2.

Topology of Gaussian Model

The Gaussian Image of $Bdr(S)$ covers S^2 completely and subdivides S^2 into faces, edges and vertices as described above. We shall fudge the physical distinctions of face, edge and vertex of S^2 a little bit and deal with the degenerate edges and vertices in the same way as with the faces. Let us assume the Gaussian Image of each face, edge and vertex is a generic face of S^2. If any of these Gaussian Images are not faces, we can represent this fact by tagging it as degenerate curves or degenerate points and consider it as faces. By using the connectivity graph of $Bdr(S)$ we can connect these generic faces with the correct topology. We can further include the edges and vertices determined by these faces into the connectivity graph of the Gaussian Image. The edge equations and vertex coordinates are given by the face boundary equations described above. Doing it in this way, we construct a graph on S^2 with degenerate curves and points considered as generic faces tagged appropriately.

Figure 1 (b) and (d) show the Gaussian Models for the convex objects in Figure 1 (a) and (c). In Figure 1 (a), all the faces are strictly convex, and all the edges and vertices are defined by transversally intersecting faces. The $Bdr(S)$ is non–smooth on each edge and vertex and only on the edges and vertices.

Hence, the Gaussian Images for faces, edges and vertices are all patches of S^2. In Figure 1 (c), the face F_3 is a ruled surface and the face F_2 is a planar patch. The corresponding Gaussian Images are a degenerate curve and a degenerate point. Further since faces F_1 and F_3 are tangent to each other along E_2, the Gaussian Image of E_2 is a degenerate curve.

3. C-space Obstacles, Convolution and Envelopes

Let A be a moving object with its reference point at the origin and B be a fixed obstacle in the 3-dimensional real Euclidean plane R^3. Both A and B are modeled by the above boundary representations. For the sake of notation and preciseness in our usage we make the following definitions. For S, P and $Q \subset R^3$, we denote $Int(S)$ as the interior of S, $Bdr(S)$ as the boundary of S, and $Cl(S) = Int(S) \cup Bdr(S)$ as the closure of S. Note that $A = Cl(A)$ and $B = Cl(B)$ by regularity. Further, the exterior of S is denoted by $Ext(S) = Cl(S)^c$ (the complement of $Cl(S)$) $= R^3 - Cl(S)$, where the set difference $P - Q = \{p \in R^3 \mid p \in P$ and p *nomem* $Q\}$. Note that $Int(S)$ and $Ext(S)$ are open sets. We also have $d(p, q)$ as the Euclidean distance between p and q; $NB_\varepsilon(p) = \{q \in R^3 \mid d(p, q) < \varepsilon\}$ = ε–neighborhood around a point p; $-S = \{-p \mid p \in S\}$ = Minkowski inverse, $P \pm Q = \{p \pm q \mid p \in P$ and $q \in Q\}$ = Minkowski sum and difference.

Throughout we consider object A to be free to move with fixed orientation. In this case configuration space is also 3-dimensional. We denote A_p to be $A + \{p\}$ where $p \in R^3$. One also needs the following definitions (1) $A_{\bar{p}}$ is *free* from B <=> $A_{\bar{p}} \cap B = empty$. (2) $A_{\bar{p}}$ *collides* with B <=> $Int(A_{\bar{p}}) \cap Int(B) \neq empty$ (3) $A_{\bar{p}}$ *contacts* with B <=> $A_{\bar{p}} \cap B \neq empty$ and $Int(A_{\bar{p}}) \cap Int(B) = empty$ (Note that these conditions imply $Bdr(A_{\bar{p}}) \cap Bdr(B) \neq empty$.) (4) $CO(A, B)$ = C–*space* obstacle due to A and B = $\{\bar{p} \in R^3 \mid A_{\bar{p}} \cap B \neq empty\}$. (5) O–*Envelope*$(-A, B)$ = Outer envelope due to $-A$ and B = $\{\bar{p} \in R^3 \mid \bar{p} \in Bdr((-A)_p)$ for some $p \in Bdr(B)$, and $\bar{p}$ *nomem* $Int((-A)_q)$ for any $q \in B\}$ (Having $q \in B$ as opposed to $q \in Bdr(B)$ implies that only the outer envelope is considered.) (6) *Convolution*$(Bdr(-A), Bdr(B))$ = Convolution of $Bdr(-A)$ and $Bdr(B)$ = $\{\bar{p} \in R^3 \mid \bar{p} = p - q$ where $p \in Bdr(B)$ and $q \in Bdr(A)$ and B has an outward normal direction at p exactly opposite to an outward normal A has at $q\}$.

We now note the following.

Theorem 3.1 : $CO(A, B) = B - A$

Proof : Lozano-Perez and Wesley (1979). □

>From the above Theorem and our prior definitions we obtain,

Corollary 3.2 : (1) $CO(Int(A), Int(B)) = Int(B) - Int(A) = B - Int(A)$ (This is an open set)

(2) $A_{\bar{p}}$ is *free* from B <=> $\bar{p} \in Ext(CO(Int(A), Int(B)))$

(3) $A_{\bar{p}}$ *collides* with B <=> $\bar{p} \in Int(CO(Int(A), Int(B)))$

(4) $A_{\bar{p}}$ *contacts* with B <=> $\bar{p} \in Bdr(CO(Int(A), Int(B)))$

We next obtain the following important characterizations,

Lemma 3.3 : $Bdr(CO(Int(A), Int(B))) = O$–*Envelope*$(-A, B)$

Proof : ($\subseteq$) : Let $\bar{p} \in Bdr(CO(Int(A), Int(B)))$, then $A_{\bar{p}}$ *contacts* with B, (*Corallary* 3.2 (4)), and $\exists\, p \in Bdr(A_{\bar{p}}) \cap Bdr(B)$. Since $p - \bar{p} \in Bdr(A)$, we have $\bar{p} - p \in Bdr(-A)$ and $\bar{p} \in Bdr((-A)_p)$ for $p \in Bdr(B)$. Further $\bar{p}$ *nomem* $Int((-A)_q)$ for any $q \in B$. Assuming the contrary, if $\bar{p} \in Int((-A)_q)$ for some $q \in B$, then $\bar{p} \in B - Int(A) = Int(B) - Int(A) = Int(CO(Int(A), Int(B)))$, (contradiction).

($\supseteq$) : Let $\bar{p} \in O$–*Envelope*$(-A, B)$, then $\bar{p} \in Bdr((-A)_p)$ for some $p \in Bdr(B)$, and $\bar{p}$ *nomem* $Int((-A)_q)$ for any $q \in B$. Equivalently, $p \in Bdr(A_{\bar{p}}) \cap Bdr(B)$ and q *nomem* $Int(A_{\bar{p}})$ for any $q \in B$. This implies $A_{\bar{p}} \cap B \neq empty$ and $Int(A_{\bar{p}}) \cap Int(B) = empty$. Hence, $A_{\bar{p}}$ contacts with B. □

Theorem 3.4 : $Bdr(CO(A, B)) \subset O\text{-}Envelope(-A, B) \subset Convolution(Bdr(-A), Bdr(B))$

Proof : (1) Using Theorem 3.3 we show $Bdr(CO(A, B)) \subset Bdr(CO(Int(A), Int(B)))$: For any $\bar{p} \in CO(A, B)$, $A_{\bar{p}} \cap B \neq empty$, equivalently $\bar{p} \in Cl(CO(Int(A), Int(B)))$, (*Corollary* 3.2 (2)). Hence, $CO(Int(A), Int(B)) \subset CO(A, B) \subset Cl(CO(Int(A), Int(B)))$ and $Cl(CO(A, B)) = Cl(CO(Int(A), Int(B)))$. Since $Int(CO(Int(A), Int(B))) \subset Int(CO(A, B))$, we have $Bdr(CO(A, B)) \subset Bdr(CO(Int(A), Int(B)))$.

(2) $O\text{-}Envelope(-A, B) \subset Convolution(Bdr(-A), Bdr(B))$: For any $\bar{p} \in O\text{-}Envelope(-A, B) = Bdr(CO(Int(A), Int(B)))$, since $A_{\bar{p}}$ contacts with B at some $p \in Bdr(B)$, $A_{\bar{p}}$ has an outward normal direction at p which is opposite to an outward normal direction B has at p. For $q = p - \bar{p} \in Bdr(A)$, we have $\bar{p} = p - q$ and B has an outward normal direction at p exactly opposite to an outward normal A has at q. Thus $\bar{p} \in Convolution(Bdr(-A), Bdr(B))$. Also see Guibas, Ramshaw, and Stolfi (1983). □

In the special case when both A and B are convex, both the set containments of Theorem 3.4 become equalities. This follows from the properties of convexity. In particular we use the following simple fact. For convex A and B, if $A_{\bar{p}}$ and B have opposite outward normal directions at $p \in Bdr(A_{\bar{p}}) \cap Bdr(B)$, then there is a common supporting plane P_p such that $A_{\bar{p}}$ and B are on opposite sides of the plane P_p, Kelly and Weiss (1979).

Theorem 3.5 : For convex A and B, we have $Bdr(CO(A, B)) = O\text{-}Envelope(-A, B) = Convolution(Bdr(-A), Bdr(B))$.

Proof : Using Theorem 3.4, all we need to show is $Convolution(Bdr(-A), Bdr(B)) \subset Bdr(CO(A, B))$ for convex A and B. Suppose $\bar{p} \in Convolution(Bdr(-A), Bdr(B))$. We first show $\bar{p}$ *nomem* $Ext(CO(A, B))$. If $\bar{p} \in Ext(CO(A, B))$, then $\exists\, \varepsilon > 0$ such that $(A_{\bar{p}} + NB_{\varepsilon}(0)) \cap B = empty$ and $Cl(A_{\bar{p}}) \cap Cl(B) = empty$. Hence, $\bar{p}$ *nomem* $Bdr((-A)_p)$ for any $p \in Bdr(B)$, (contradiction), and so $\bar{p}$ *nomem* $Ext(CO(A, B))$. Now, we show $\bar{p}$ *nomem* $Int(CO(A, B))$. Since $\exists\, p \in Bdr(A_{\bar{p}}) \cap Bdr(B)$ such that $A_{\bar{p}}$ and B have opposite outward normal directions at p, a common supporting plane P_p separates $A_{\bar{p}}$ and B. For any $\varepsilon > 0$, let e be an outward normal vector to B at p such that $\|e\| = \varepsilon$ and e is orthogonal to P_p, then $A_{(\bar{p}+e)}$ and B are separated by the banded region bounded by $P_{(\bar{p}+e)}$ and P_p, and so $A_{(\bar{p}+e)} \cap B = empty$. Hence, $\bar{p}$ *nomem* $Int(CO(A, B))$. Thus $\bar{p}$ *nomem* $Int(CO(A, B)) \cup Ext(CO(A, B))$ implies $\bar{p} \in Bdr(CO(A, B))$. □

This may then suggest a natural method for handling *non-convex* object and obstacle shapes. One first obtains a convex decomposition consisting of the union of convex pieces and then generates the *C-space* obstacle as the union of *C-space* obstacles for convex object and obstacle pairs. Such convex decompositions are possible for polyhedral objects, see Chazelle (1984). However not all objects with algebraic curve and surface boundaries permit decompositions consisting of the union of convex pieces, Bajaj and Kim (1987c). For example a complete toroidal surface cannot be decomposed into the union of convex pieces. To obtain convex decomposition of general curved solid objects (say in terms of union, intersection and difference) is a difficult and as yet unsolved problem, see Requicha and Voelcker (1983). Direct methods of computing *C-space* obstacle boundary of objects with non-convex boundary are computationally quite involved and intricate, and further research needs to be done. Thus for the time being one is restricted to considering convex shaped objects and obstacles.

4. Generating the Boundary of *C-space* Obstacles

Suppose S be $-A$ or B, $p \in Bdr(S)$ be a boundary point, $E \subset Bdr(S)$ be an edge, and $F \subset Bdr(S)$ be a face. Let (F_S, N_{F_s}) be a pair such that $F_S \subset Bdr(S)$ is a face and $N_{F_s} = N(S, F_S)$, where $N(S, \cdot)$ is the

Gaussian Map of S. (E_S, N_{E_s}) be a pair such that $E_S \subset Bdr(S)$ is an edge and $N_{E_s} \subset N(S, E_S)$ with $N_{E_s} \cap N(S, p) \neq$ *empty* for all $p \in E_S$. (p_S, N_{p_s}) be a pair such that $p_S \in Bdr(S)$ is a vertex and $N_{p_s} \subset N(S, p_S)$ with $N_{p_s} \neq$ *empty*. Further let K_B be F_B, E_B or p_B, and let G_{-A} be F_{-A}, E_{-A} or p_{-A}. There are nine (K_B, G_{-A}) pairs. We define *sub−compatible* and *compatible* pairs as follows.

(1) K_B and G_{-A} are *sub−compatible* $\Longleftrightarrow N(B, K_B) \cap N(-A, G_{-A}) \neq$ *empty*

(2) (K_B, N_{K_s}) and $(G_{-A}, N_{G_{-A}})$ are *compatible* $\Longleftrightarrow N_{K_s} = N_{G_{-A}}$

Further denote by $K_B \infty G_{-A}$ that K_B and G_{-A} are *sub−compatible*. Since only sub-compatible pairs can contribute to the *Convolution*, one can show that $Convolution(Bdr(-A), Bdr(B)) = \bigcup_{K_s \infty G_{-A}} Convolution(G_{-A}, K_B)$, where $Convolution(G_{-A}, K_B)$ = Convolution of G_{-A} and $K_B = \{ \bar{p} \in R^3 \mid \bar{p} = p + q$ where $p \in K_B$ and $q \in G_{-A}$, and B has an outward normal direction at p in the same direction as an outward normal A has at $q\}$. We can further refine the right-hand side to be a union of only the *compatible* pairs as follows. For a sub-compatible (K_B, G_{-A}) pair, let $N(K_B, G_{-A}) = N(B, K_B) \cap N(-A, G_{-A})$ be the nonempty intersection of two Gaussian Images of K_B and G_{-A}. $K(K_B, G_{-A}) = N^{-1}(B, N(K_B, G_{-A})) \subset K_B$ and $G(K_B, G_{-A}) = N^{-1}(-A, N(K_B, G_{-A})) \subset G_{-A}$ be the Inverse Gaussian Images of $N(K_B, G_{-A})$. Then $(K(K_B, G_{-A}), N(K_B, G_{-A}))$ and $(G(K_B, G_{-A}), N(K_B, G_{-A}))$ are *compatible*. One can easily show that $Convolution(Bdr(-A), Bdr(B)) = \bigcup_{K_s \infty G_{-A}} Convolution(G(K_B, G_{-A}), K(K_B, G_{-A}))$. Hence, we only need to consider *compatible* pairs to generate the *Convolution*.

When (K_B, N_{K_s}) and $(G_{-A}, N_{G_{-A}})$ are *compatible* with at least one of K_B or G_{-A} being a vertex, the *Convolution* generation is especially easy, i.e. $Convolution(G_{-A}, K_B) = K_B + G_{-A}$. Let $Ch(p)$ = the *characteristic set* of $p = \{\bar{p} = p + q \mid N(B, p) \cap N(-A, q) \neq empty\}$. $Ch(E) = \bigcup_{p \in E} Ch(p)$ is called the *characteristic set* of E, and $Ch(F) = \bigcup_{p \in F} Ch(p)$ is called the *characteristic set* of F. One can easily show that $Convolution(Bdr(-A), Bdr(B)) = (\bigcup_{F \in \Gamma_1} Ch(F)) \cup (\bigcup_{E \in \Gamma_2} Ch(E)) \cup (\bigcup_{p \in \Gamma_3} Ch(p))$, where Γ_1 is the set of all faces of $Bdr(B)$, Γ_2 is the set of all edges of $Bdr(B)$, and Γ_3 is the set of all vertices of $Bdr(B)$.

Growing Faces

For a face $F \subset Bdr(B)$, one can easily show that $Ch(F) = (\bigcup_{F' \infty F} Convolution(G(F, F'), K(F, F'))) \cup (\bigcup_{E \infty F} Convolution(G(F, E), K(F, E))) \cup (\bigcup_{q \infty F} Convolution(q, K(F, q)))$. One can use §4.1 to compute $Convolution(G(F, F'), K(F, F'))$ and §4.2 to compute $Convolution(G(F, E), K(F, E))$, while directly computing $Convolution(G(F, q), K(F, q)) = K(F, q) + \{q\}$ as a simply translated surface patch.

Growing Edges

For an edge $E \in Bdr(B)$, one can easily show that $Ch(E) = (\bigcup_{F \infty E} Convolution(G(E, F), K(E, F))) \cup (\bigcup_{E' \infty E} Convolution(G(E, E'), K(E, E'))) \cup (\bigcup_{q \infty E} Convolution(q, K(E, q)))$. One can use §4.2 to compute $Convolution(G(E, F), K(E, F))$, and §4.3 to compute $Convolution(G(E, E'), K(E, E'))$, while directly computing $Convolution(q, K(E, q)) = \{q\} + K(E, q)$ as a simply translated edge segment.

Growing Vertices

For a vertex $p \in Bdr(B)$, one can easily show that $Ch(p) = (\bigcup_{F \leftrightarrow p} Convolution(G(p, F), p)) \cup (\bigcup_{E \leftrightarrow p} Convolution(G(p, E), p)) \cup (\bigcup_{q \leftrightarrow p} Convolution(q, p))$. Since one has $Convolution(G(p, F), p) = G(p, F) + \{p\}$, $Convolution(G(p, E), p) = G(p, E) + \{p\}$, and $Convolution(q, p) = \{q + p\}$, computing $Ch(p)$ is easy.

Note: (1) For a non–smooth edge E and a non–smooth vertex p the convolution edge $Convolution(G(p, E), p) = G(p, E) + \{p\}$ is a non–smooth edge, and (2) for non–smooth vertices p and q the convolution vertex $Convolution(q, p) = \{q + p\}$ is a non–smooth vertex. As we will see in §4.3, (3) we can also have a non–smooth convolution edge $Convolution(E_{-A}, E_B)$ for *parallel* edge pair E_{-A} and E_B. These are all the non–smooth edges and vertices we can have on the *C–space* obstacle boundary. As we see in this classification, all the non–smooth edges and vertices on the *C–space* obstacle boundary result from very special orientations between the non–smooth edges and vertices of A and B. Most of the non–smoothness of A and B are removed while generating the *C–space* boundary. This smoothing effect of convolution generation raises another question of how to compute and specify the boundary of a convolution surface patch. Since most of adjacent convolution faces meet tangentially to each other, computation of the intersecting edge may be quite unstable. Auxiliary surfaces need to be determined which intersect transversally with the convolution surfaces and thereby boundary curves of the convolution faces.

In §4.1–4.3 we consider both the implicit and rational parametric representation of surface patches since not all algebraic curves and surfaces have rational parametrization, see Walker (1978). For the class of *rational* algebraic curves and surfaces (which have a rational parametric form), algebraic algorithms also exist for converting between the implicit and parametric representations. However their efficiency are limited to curves and surfaces of low degree, see Abhyankar and Bajaj (1987a, b, c).

4.1. Generating $Convolution(F_{-A}, F_B)$

In this section, we consider how to generate the algebraic surface equation, edges and vertices of a convolution surface patch $Convolution(F_{-A}, F_B)$. We can use Theorem 4.1 for the case of F_{-A} and F_B being implicitly defined algebraic surfaces. Corollary 4.1 is useful when F_{-A} is implicit and F_B is parametric, or the other way around. Corollary 4.2 is useful when both F_{-A} and F_B are parametrically defined. For *sub-compatible* F_B and F_{-A}, we are using the notations $N(F_B, F_{-A}) = N(B, F_B) \cap N(-A, F_{-A})$, $K(F_B, F_{-A}) = N^{-1}(B, N(F_B, F_{-A})) \subset F_B$, and $G(F_B, F_{-A}) = N^{-1}(-A, N(F_B, F_{-A})) \subset F_{-A}$.

Theorem 4.1 : Let $F_B \subset Bdr(B)$ be a patch of an algebraic surface $f = 0$ with gradients ∇f. Further let $F_{-A} \subset Bdr(-A)$ be a patch of an algebraic surface $g = 0$ with gradients ∇g, and suppose that F_B and F_{-A} are *sub-compatible*. Then $Convolution(F_{-A}, F_B) = Convolution(G(F_B, F_{-A}), K(F_B, F_{-A}))$ is the set of points $\overline{p} = (\overline{x}, \overline{y}, \overline{z}) = p + q = (x + \alpha, y + \beta, z + \gamma)$ such that

$$\begin{cases} f(x, y, z) = 0 \text{ and } p = (x, y, z) \in K(F_B, F_{-A}) & (1) \\ g(\alpha, \beta, \gamma) = 0 \text{ and } q = (\alpha, \beta, \gamma) \in G(F_B, F_{-A}) & (2) \\ \nabla f \times \nabla g = 0 & (3) \\ \nabla f \cdot \nabla g > 0 & (4) \end{cases}$$

Proof : Since (3)–(4) imply ∇f and ∇g are in the same direction, (3)–(4) are equivalent to the outward normal direction of B at p to be the same as that of $-A$ at q. □

We use Theorem 4.1 as follows. First substitute $x = \overline{x} - \alpha$, $y = \overline{y} - \beta$ and $z = \overline{z} - \gamma$ in the above equations (1) and (3). Then one can obtain the implicit algebraic equation of the $Convolution(F_{-A}, F_B)$ in terms of $\overline{x}$, $\overline{y}$ and $\overline{z}$ by eliminating α, β and γ from the equations (1)–(3). The vector equation $\nabla f \times \nabla g = 0$ gives 3 scalar equations. Since one of these equations is redundant, we can have 2 independent scalar equations from (3). Hence, we have 4 equations and eliminate 3 variables α, β, γ to get an implicit equation in terms of $\bar{x}$, $\bar{y}$, $\bar{z}$. Elimination of variables can be performed by computing resultants on pairs of equations, however this in general leads to extraneous factors and special care needs to be taken in performing this step, van der Waerden (1950). Systematic elimination of variables, a process also known as implicitization, based on Kronecker's elimination method can be adopted to avoid extraneous factors but is limited by its exceedingly high computation time, Bajaj (1987). A closed form resultant for simultaneous elimination of $n-1$ variables from n equations is as yet unknown for $n \geq 3$ and is a major unsolved problem of algebraic geometry, see Abhyankar (1976). For special types of surfaces (called bi-*mic* parametric surfaces) however a closed form Cayley resultant proves sufficient in simultaneously eliminating 2 variables from 3 equations, Dixon (1908).

In computing the implicit equation of the *C-space* surfaces a time complexity analysis may be done as follows. Let $Res_j(d)$ = time complexity of computing the resultant of two j-variate polynomials of maximum degree d. The best known time complexity of $Res_j(d) = O(d^{2j+1}\log d + d^{2j}\log^2 d)$, Collins (1971). On substituting $x = \bar{x} - \alpha$, $y = \bar{y} - \beta$ and $z = \bar{z} - \gamma$ in equations (1) and (3) one has to expand each term $c_{ijk} \cdot x^{d_i} \cdot y^{d_j} \cdot z^{d_k} = c_{ijk} \cdot (\bar{x} - \alpha)^{d_i} \cdot (\bar{y} - \beta)^{d_j} \cdot (\bar{z} - \gamma)^{d_k}$ where $d_i + d_j + d_k \leq d$. This is necessary because in computing resultants to eliminate, say α, one needs to simplify the equations to be polynomials in α with coefficients in $\bar{x}, \bar{y}, \bar{z}, \beta, \gamma$. f, f_x, f_y and f_z have $O(d^3)$ terms of this form, and expansion of each term takes $O(d^3)$ multiplications. Hence, the overall time complexity for expansion and simplification is $O(d^6)$. By Bezout theorem, when we take a resultant of a degree d_1 equation and a degree d_2 equation, the degree of the resulting equation is $d_1 \cdot d_2$. If we are eliminating α, β and γ pairwise, the total time complexity is bound by $O(Res_6(d) + Res_5(d^2) + Res_4(d^4) + d^6) \approx O(d^{36}\log d)$. Further the degree of the convolution faces may be as high as $O(d^8)$ where original faces of A and B were with maximal degree d.

Corollary 4.1 : Let $F_B \subset Bdr(B)$ be a patch of an algebraic surface $f = 0$ with gradients ∇f. Further let $F_{-A} \subset Bdr(-A)$ be a parametric surface patch $G(u,v) = (\alpha(u,v), \beta(u,v), \gamma(u,v))$ with gradients $G_u \times G_v$, and suppose that F_B and F_{-A} are *sub-compatible*. Then $Convolution(F_{-A}, F_B) = Convolution(G(F_B, F_{-A}), K(F_B, F_{-A}))$ is the set of points $\bar{p} = (\bar{x}, \bar{y}, \bar{z}) = p + q = (x + \alpha(u,v),\ y + \beta(u,v),\ z + \gamma(u,v))$ such that

$$
\begin{cases}
f(x,y,z) = 0 \text{ and } p = (x,y,z) \in K(F_B, F_{-A}) & (1) \\
q = (\alpha(u,v), \beta(u,v), \gamma(u,v)) \in G(F_B, F_{-A}) & (2) \\
\nabla f \times (G_u \times G_v) = 0 & (3) \\
\nabla f \cdot (G_u \times G_v) > 0 & (4)
\end{cases}
$$

First substitute $x = \bar{x} - \alpha(u,v)$, $y = \bar{y} - \beta(u,v)$ and $z = \bar{z} - \gamma(u,v)$ in the above equations (1) and (3). Then one can obtain the implicit algebraic equation of the $Convolution(F_{-A}, F_B)$ in terms of $\bar{x}$, $\bar{y}$ and $\bar{z}$ by eliminating u and v from the equations (1) and (3) by computing resultants. Since (3) gives 2 independent scalar equations, we have 3 equations and eliminate 2 variables u, v to get an implicit equation.

Since $G(u,v)$ is a rational parametric surface, we have $\alpha(u,v) = p(u,v) / w(u,v)$, $\beta(u,v) = q(u,v) / w(u,v)$ and $\gamma(u,v) = r(u,v) / w(u,v)$ for polynomials $p(u,v)$, $q(u,v)$, $r(u,v)$ and $w(u,v)$ of maximum degree d. At this time the expansion of each term $c_{ijk} \cdot x^{d_i} \cdot y^{d_j} \cdot z^{d_k} = c_{ijk} \cdot w(u,v)^{d-d_i-d_j-d_k} \cdot (w(u,v) \cdot \bar{x} - p(u,v))^{d_i} \cdot (w(u,v) \cdot \bar{y} - q(u,v))^{d_j} \cdot (w(u,v) \cdot \bar{z} - r(u,v))^{d_k} / w(u,v)^d$ is harder than the case of Theorem 4.1. Again f, f_x, f_y and f_z have $O(d^3)$ terms of this form, and expansion of each term takes $O(d^9)$ multiplications. Hence, the overall time complexity for expansion and simplification prior to elimination is $O(d^{12})$.

The case of F_B being a parametric surface and F_{-A} being an algebraic surface is similar to Corollary 4.1.

Corollary 4.2 : Let $F_B \subset Bdr(B)$ be a parametric surface patch $F(s,t)=(x(s,t), y(s,t), z(s,t))$ with gradients $F_s \times F_t$. Further let $F_{-A} \subset Bdr(-A)$ be a parametric surface patch $G(u,v)=(\alpha(u,v), \beta(u,v), \gamma(u,v))$ with gradients $G_u \times G_v$, and suppose that F_B and F_{-A} are *sub-compatible*. Then $Convolution(F_{-A}, F_B) = Convolution(G(F_B, F_{-A}), K(F_B, F_{-A}))$ is the set of points $\bar{p} = (\bar{x}, \bar{y}, \bar{z}) = p + q = (x(s,t) + \alpha(u,v), y(s,t) + \beta(u,v), z(s,t) + \gamma(u,v))$ such that

$$
\begin{cases}
p=(x(s,t),\ y(s,t),\ z(s,t)) \in K(F_B, F_{-A}) & (1) \\
q=(\alpha(u,v),\ \beta(u,v),\ \gamma(u,v)) \in G(F_B, F_{-A}) & (2) \\
(F_s \times F_t) \times (G_u \times G_v) = 0 & (3) \\
(F_s \times F_t) \cdot (G_u \times G_v) > 0 & (4)
\end{cases}
$$

One can obtain the implicit algebraic equation of the $Convolution(F_{-A}, F_B)$ by eliminating s, t, u and v from the equations $\bar{x} = x(s,t) + \alpha(u,v)$, $\bar{y} = y(s,t) + \beta(u,v)$, $\bar{z} = z(s,t) + \gamma(u,v)$ and the above equation (3). Since (3) gives 2 independent scalar equations, we have 5 equations and need to eliminate 4 variables s, t, u, v to get an implicit equation.

Boundary Edges of $Convolution(F_{-A}, F_B)$

For *sub-compatible* face pairs F_B and F_{-A} which are relatively open with respect to $Bdr(B)$ and $Bdr(-A)$, each boundary edge E_N of $N(F_B, F_{-A})$ $(= N(B, F_B) \cap N(-A, F_{-A}))$ is either a segment of a boundary edge of $N(B, F_B)$ or a segment of a boundary edge of $N(-A, F_{-A})$. Further E_N is either (a) a segment of the common boundary edge of $N(B, F_B)$ and $N(B, E_B)$ for some edge E_B of F_B, or (b) a segment of the common boundary edge of $N(-A, F_{-A})$ and $N(-A, E_{-A})$ for some edge E_{-A} of F_{-A}. Similarly, each boundary edge $E_{CO(A,B)}$ of the surface patch $Convolution(F_{-A}, F_B)$ is either (a) a segment of the common boundary edge of $Convolution(F_{-A}, F_B)$ and $Convolution(Cl(F_{-A}), E_B)$, or (b) a segment of the common boundary edge of $Convolution(F_{-A}, F_B)$ and $Convolution(E_{-A}, Cl(F_B))$, where $Cl(F_B)$ and $Cl(F_{-A})$ are the closures of F_B and F_{-A} with respect to $Bdr(B)$ and $Bdr(-A)$. Edges of type (a) are described in Theorem 4.2, edges of type (b) can be described similarly. Let $sub\text{-}Convolution_{T_N}(G_{-A}, K_B)$ = sub-Convolution of G_{-A} and K_B restricted to the normal directions $T_N = \{ \bar{p} \in R^3 \mid \bar{p} = p + q$ where $p \in K_B$ and $q \in G_{-A}$, and B has a unit outward normal direction n_p at p which is the same as a unit outward normal A has at q where $n_p \in T_N\}$. Since the Gaussian Image of $E_{CO(A,B)}$ is some edge E_N of $N(F_B, F_{-A})$, one can easily show $E_{CO(A,B)} = sub\text{-}Convolution_{E_N}(Cl(F_{-A}), Cl(F_B))$.

Theorem 4.2 : Let F_B and F_{-A} be a *sub-compatible* face pair, E_B be an edge of F_B and E_N be a boundary edge of $N(F_B, F_{-A})$ such that E_N is a segment of the common edge $N(B, E_B) \cap Cl(N(B, F_B))$. Suppose E_B is the common edge of two surface patches F_B and $\hat{F}_B$, where F_B is a patch of an algebraic surface $f = 0$ with gradients ∇f, and $\hat{F}_B$ is a patch of an algebraic surface $\hat{f} = 0$ with gradients $\nabla \hat{f}$. Then

(A) the convolution edge $E_{CO(A,B)} = sub\text{-}Convolution_{E_N}(Cl(F_{-A}), Cl(F_B))$ due to the normal directions E_N is the set of points $\bar{p} = (\bar{x}, \bar{y}, \bar{z}) = p + q = (x + \alpha, y + \beta, z + \gamma)$ such that

$$
\begin{cases}
f(x,y,z)=0 \text{ and } p=(x,y,z) \in Cl(K(F_B, F_{-A})) & (1) \\
\hat{f}(x,y,z)=0 \text{ and } p=(x,y,z) \in Cl(\hat{F}_B) & (2) \\
g(\alpha,\beta,\gamma)=0 \text{ and } q=(\alpha,\beta,\gamma) \in Cl(G(F_B, F_{-A})) & (3) \\
\nabla f \times \nabla g = 0 & (4) \\
\nabla f \cdot \nabla g > 0 & (5)
\end{cases}
$$

(B) The surface patch defined by (1) and (3)–(5) and the surface patch defined by (2)–(5) intersect along the convolution edge $E_{CO(A,B)}$.

Proof : (A) The surface patch defined by (1) and (3)–(5) is the face *Convolution* (F_{-A}, F_B) and all its boundary edges and vertices. Since (1)–(2) restrict the set of points p to the subsegment E'_B of E_B such that $E'_B = N^{-1}(B, E_N)$, (1)–(5) define the convolution edge $E_{CO(A,B)}$.
(B) Since $E_{CO(A,B)}$ is the common solution of (1)–(5), $E_{CO(A,B)}$ is the common edge of the surface patch defined by (1) and (3)–(5) and the surface patch defined by (2)–(5). □

By using an auxiliary surface if necessary we may assume each edge E_B is the common edge of two transversally intersecting surface patches F_B and $\hat{F}_B$. Then in most of the cases $E_{CO(A,B)}$ can also be represented as a common edge of two transversally intersecting surface patches. When these two surface patches intersect tangentially, one may use different auxiliary surface patch $\hat{F}_B$. For two surfaces defined implicitly by $h(x,y,z) = 0$ and $\hat{h}(x,y,z) = 0$ which meet tangentially along the curve C, an auxiliary surface which intersects h and $\hat{h}$ transversally may also be obtained by considering surfaces $k = \alpha h + \beta \hat{h} = 0$ where α and β are arbitrary polynominals in three variables x, y and z. These additional surfaces k also intersect both h and $\hat{h}$ along the curve C and are said to belong to the ideal of the curve C. For suitable α and β auxiliary surfaces which meet h and $\hat{h}$ transversally may be constructed.

The case of a boundary edge E_{-A} of F_{-A} being defined by two transversally intersecting surface patches gives a similar result. Further the cases of F_B, $\hat{F}_B$, F_{-A}, or $\hat{F}_{-A}$ being parametric surfaces give similar results. Also the time and degree complexity analyses are similar to those of Theorem 4.1 and Corollaries 4.1–4.2.

Boundary Vertices of *Convolution* (F_{-A}, F_B)

For a *sub-compatible* face pair F_B and F_{-A} which are relatively open with respect to $Bdr(B)$ and $Bdr(-A)$, each boundary vertex e_N of $N(F_B, F_{-A})$ $(= N(B, F_B) \cap N(-A, F_{-A}))$ is either (a) a boundary vertex of $N(B, F_B)$, (b) a boundary vertex of $N(-A, F_{-A})$, or (c) the intersection of one edge of $N(B, F_B)$ with another edge of $N(-A, F_{-A})$. In the case of (a), suppose p is the vertex of F_B and q is a point of F_{-A} such that $p \in N(B, e_N)$ and $q \in N^{-1}(-A, e_N)$, then the point $p+q$ is the vertex of *Convolution* (F_{-A}, F_B) such that $p+q \in N^{-1}(CO(A, B), e_N)$. $q \in F_{-A}$ can be computed by solving $g = 0$ and $\nabla g \,/\, ||\nabla g|| = e_N$. The case of (b) is similar to the case of (a). In the case of (c), the intersection e_N of one edge of $N(B, F_B)$ with another edge of $N(-A, F_{-A})$ can be computed by Theorems 5.1–5.3. Suppose $p \in Bdr(F_B)$ and $q \in Bdr(F_{-A})$ be such that $p \in N^{-1}(B, e_N)$ and $q \in N^{-1}(-A, e_N)$ where $Bdr(F_B)$ and $Bdr(F_{-A})$ are the boundaries of F_B and F_{-A} with respect to $Bdr(B)$ and $Bdr(-A)$, then $p+q$ is the vertex of *Convolution* (F_{-A}, F_B) such that $p+q \in N^{-1}(CO(A, B), e_N)$. $p \in F_B$ can be computed by solving $f = 0$ and $\nabla f \,/\, ||\nabla f|| = e_N$ and $q \in F_{-A}$ can be computed by solving $g = 0$ and $\nabla g \,/\, ||\nabla g|| = e_N$.

4.2. Generating *Convolution* (F_{-A}, E_B) and *Convolution* (E_{-A}, F_B)

In this section, we consider how to generate the algebraic surface equations, edges and vertices of convolution surface patches *Convolution* (F_{-A}, E_B) and *Convolution* (E_{-A}, F_B). We can use Theorem 4.3 for the case of E_B being defined by the intersection of two implicit algebraic surfaces and F_{-A} being an implicit algebraic surface. The other combinations of implicit and parametric surfaces defining E_B and F_{-A} have similar results as easy Corollaries of Theorem 4.3. Similar results hold for generating *Convolution* (E_{-A}, F_B).

Theorem 4.3 : Let $E_B \subset Bdr(B)$ be the common edge of two faces F_B and $\hat{F}_B$, where F_B and $\hat{F}_B \subset Bdr(B)$ are patches of algebraic surfaces $f = 0$ with gradients ∇f and $\hat{f} = 0$ with gradients $\nabla \hat{f}$. Further let $F_{-A} \subset Bdr(-A)$ be a patch of an algebraic surface $g = 0$ with gradients ∇g. Suppose that E_B and

F_{-A} are *sub-compatible*. Then $Convolution(F_{-A}, E_B) = Convolution(G(E_B, F_{-A}), K(E_B, F_{-A}))$ is the set of points $\bar{p} = (\bar{x}, \bar{y}, \bar{z}) = p + q = (x+\alpha,\ y+\beta,\ z+\gamma)$ such that

$$\begin{cases} f(x, y, z) = \hat{f}(x, y, z) = 0 \text{ and } p = (x, y, z) \in K(E_B, F_{-A}) & (1) \\ g(\alpha, \beta, \gamma) = 0 \text{ and } q = (\alpha, \beta, \gamma) \in G(E_B, F_{-A}) & (2) \\ \nabla g \cdot (\nabla f \times \nabla \hat{f}) = 0 \text{ and } \dfrac{\nabla g}{||\nabla g||} \in N_{E_s} & (3) \end{cases}$$

Proof : (3) is equivalent to an outward normal direction of B at p to be the same as one of the outward normal directions of $-A$ at q. □

One can obtain the implicit algebraic equation of the $Convolution(F_{-A}, E_B)$ in a similar way as in Theorem 4.1. When the face F_{-A} is a parametric surface patch $G(u,v) = (\alpha(u,v), \beta(u,v), \gamma(u,v))$ with gradients $G_u \times G_v$, one can obtain the corresponding Corollary by changing every ∇g into $G_u \times G_v$ and the statement "$g(\alpha, \beta, \gamma) = 0$ and $q = (\alpha, \beta, \gamma) \in G(E_B, F_{-A})$" into "$q = (\alpha(u,v), \beta(u,v), \gamma(u,v)) \in G(E_B, F_{-A})$" in the above Theorem. One can make similar changes to get corresponding Corollaries for the case of F_B and/or $\hat{F}_B$ being parametric surface patches.

When two faces F_B and $\hat{F}_B$ are tangent to each other along E_B, $Convolution(F_{-A}, E_B)$ is a degenerate curve on the $C-space$ obstacle boundary. Actually, it is a common edge of two convolution faces generated in § 4.1.

Boundary Edges of $Convolution(F_{-A}, E_B)$

For a *sub-compatible* edge–face pair E_B and F_{-A} where F_{-A} is relatively open with respect to $Bdr(-A)$ and E_B is relatively open with respect to the intersection curve of two algebraic surfaces $f = 0$ and $\hat{f} = 0$ defining faces F_B and $\hat{F}_B$, each boundary edge E_N of $N(E_B, F_{-A})$ ($= N(B, E_B) \cap N(-A, F_{-A})$) is either a segment of a boundary edge of $N(B, E_B)$ or a segment of a boundary edge of $N(-A, F_{-A})$. Further E_N is either (a) a segment of the common edge of $N(B, E_B)$ and $N(B, F_B)$ for some face F_B adjacent to E_B, (b) a segment of the common edge of $N(-A, F_{-A})$ and $N(-A, E_{-A})$ for some edge E_{-A} of F_{-A}, or (c) a segment of the common edge of $N(B, E_B)$ and $N(B, p_B)$ for a vertex p_B of E_B. Similarly, each boundary edge $E_{CO(A,B)}$ of the surface patch $Convolution(F_{-A}, E_B)$ is either (a) a segment of the common edge of $Convolution(F_{-A}, E_B)$ and $sub-Convolution_{Cl(N(B,F_s))}(Cl(F_{-A}), Cl(F_B))$, (b) a segment of the common edge of $Convolution(F_{-A}, E_B)$ and $Convolution(E_{-A}, Cl(E_B))$, (c) a segment of the common edge of $Convolution(F_{-A}, E_B)$ and $Convolution(Cl(F_{-A}), p_B)$ where $Cl(F_{-A})$ is the closure of F_{-A} with respect to $Bdr(-A)$ and $Cl(E_B)$ is the closure of E_B with respect to the intersection curve of two algebraic surfaces $f = 0$ and $\hat{f} = 0$ defining faces F_B and $\hat{F}_B$. Edges of type (a) have been described in Theorem 4.2, edges of type (b) are described in Theorem 4.4, and edges of type (c) are described in Theorem 4.5. The proofs of Theorems 4.4–4.5 are similar to that of Theorem 4.2.

Theorem 4.4 : Let E_B and F_{-A} be a *sub-compatible* edge–face pair, E_{-A} be an edge of F_{-A} and E_N be an edge of $N(E_B, F_{-A})$ such that E_N is a segment of the common edge $N(-A, E_{-A}) \cap Cl(N(-A, F_{-A}))$. Suppose E_{-A} is the common edge of two surface patches F_{-A} and $\hat{F}_{-A}$, where F_{-A} is a patch of an algebraic surface $g = 0$ with gradients ∇g, and $\hat{F}_{-A}$ is a patch of an algebraic surface $\hat{g} = 0$ with gradients $\nabla \hat{g}$. Then
(A) the convolution edge $E_{CO(A,B)} = sub-Convolution_{E_s}(Cl(F_{-A}), Cl(E_B))$ due to the normal directions E_N is the set of points $\bar{p} = (\bar{x}, \bar{y}, \bar{z}) = p + q = (x+\alpha,\ y+\beta,\ z+\gamma)$ such that

$$\begin{cases} g(\alpha, \beta, \gamma) = 0 \text{ and } q = (\alpha, \beta, \gamma) \in Cl(G(E_B, F_{-A})) & (1) \\ \hat{g}(\alpha, \beta, \gamma) = 0 \text{ and } q = (\alpha, \beta, \gamma) \in Cl(\hat{F}_{-A}) & (2) \\ f(x, y, z) = \hat{f}(x, y, z) = 0 \text{ and } p = (x, y, z) \in Cl(K(E_B, F_{-A})) & (3) \\ \nabla g \cdot (\nabla f \times \nabla \hat{f}) = 0 \text{ and } \dfrac{\nabla g}{||\nabla g||} \in N(E_B, F_{-A}) & (4) \end{cases}$$

(B) The surface patch defined by (1) and (3)–(4) and the surface patch defined by (2)–(4) intersect along the convolution edge $E_{CO(A,B)}$.

When the surface patches of (B) intersect tangentially, one may use different auxiliary surface patch $\hat{F}_{-A}$. One may also select an auxiliary surface patch intersecting transversally to the surface patches of (B) from the ideal of the curve C defining the edge $E_{CO(A,B)}$.

Theorem 4.5 : Let E_B and F_{-A} be a *sub-compatible* edge–face pair, $p_B = (x, y, z)$ be a vertex of E_B and E_N be a boundary edge of $N(E_B, F_{-A})$ such that E_N is a segment of the common edge $Cl(N(B, E_B)) \cap N(B, p_B)$. Suppose E_B is the common edge of two transversally intersecting surface patches F_B and $\hat{F}_B$, where F_B is a patch of an algebraic surface $f = 0$ with gradients ∇f, and $\hat{F}_B$ is a patch of an algebraic surface $\hat{f} = 0$ with gradients $\nabla\hat{f}$. Further let $n = \nabla f(p_B)$ and $\hat{n} = \nabla\hat{f}(p_B)$. Then

(A) the convolution edge $E_{CO(A,B)} = sub\text{–}Convolution_{E_N}(F_{-A}, Cl(E_B))$ due to the normal directions E_N is the set of all the points $\bar{p} = (\bar{x}, \bar{y}, \bar{z}) = p_B + q = (x+\alpha, y+\beta, z+\gamma)$ such that

$$\begin{cases} g(\alpha,\beta,\gamma)=0 \text{ and } q=(\alpha,\beta,\gamma)\in Cl(G(p_B,F_{-A})) & (1)\\ \nabla g\cdot(n\times\hat{n})=0 & (2)\\ \nabla g\cdot(n-(n\cdot\hat{n})\hat{n})\ge 0 & (3)\\ \nabla g\cdot(\hat{n}-(n\cdot\hat{n})n)\ge 0 & (4)\end{cases}$$

(B) The surface patch defined by (1) and the surface patch defined by (2)–(4) intersect along the convolution edge $E_{CO(A,B)}$

When the surface patches of (B) intersect tangentially, one may select an auxiliary surface patch intersecting transversally to the surface patches of (B) from the ideal of the curve C defining the edge $E_{CO(A,B)}$.

Boundary Vertices of $Convolution(F_{-A}, E_B)$

Each vertex of $Convolution(F_{-A}, E_B)$ is a vertex of $Convolution(F_{-A}, F_B)$ for some adjacent face F_B of E_B. Hence, one can use the same methods as in §4.1.

4.3. Generating $Convolution(E_{-A}, E_B)$

In this section, we consider how to generate the algebraic surface equation, edges and vertices of a convolution surface patch $Convolution(E_{-A}, E_B)$. We can use Theorem 4.6 for the case of both E_{-A} and E_B being defined by two implicit algebraic surfaces. The other combinations of implicit and parametric surfaces defining E_{-A} and E_B have similar results as easy Corollaries of Theorem 4.6.

Theorem 4.6 : Let $E_B \subset Bdr(B)$ be a segment of the common edge of two faces F_B and $\hat{F}_B$, where $F_B \subset Bdr(B)$ is a patch of an algebraic surface $f=0$ with gradients ∇f and $\hat{F}_B \subset Bdr(B)$ is a patch of an algebraic surface $\hat{f}=0$ with gradients $\nabla\hat{f}$. Further let $E_{-A} \subset Bdr(-A)$ be a segment of the common edge of two faces F_{-A} and $\hat{F}_{-A}$, where $F_{-A} \subset Bdr(-A)$ is a patch of an algebraic surface $g=0$ with gradients ∇g and $\hat{F}_{-A} \subset Bdr(-A)$ is a patch of an algebraic surface $\hat{g}=0$ with gradients $\nabla\hat{g}$. Suppose that E_B and E_{-A} are *sub-compatible*. Then $Convolution(E_{-A}, E_B) = Convolution(G(E_B, E_{-A}), K(E_B, E_{-A}))$ is the set of points $\bar{p} = (\bar{x}, \bar{y}, \bar{z}) = p + q = (x+\alpha, y+\beta, z+\gamma)$ such that

$$\begin{cases} f(x,y,z)=\hat{f}(x,y,z)=0 \text{ and } p=(x,y,z)\in K(E_B,E_{-A}) & (1)\\ g(\alpha,\beta,\gamma)=\hat{g}(\alpha,\beta,\gamma)=0 \text{ and } q=(\alpha,\beta,\gamma)\in G(E_B,E_{-A}) & (2)\\ \dfrac{\lambda\cdot\nabla f+(1-\lambda)\cdot\nabla\hat{f}}{||\lambda\cdot\nabla f+(1-\lambda)\cdot\nabla\hat{f}||}\in N(E_B,E_{-A}) \text{ and} & \\ \dfrac{\mu\cdot\nabla g+(1-\mu)\cdot\nabla\hat{g}}{\cdot}\in N(E_B,E_{-A}) \text{ for } some\ 0\le\lambda,\mu\le 1 & (3)\end{cases}$$

Proof : (3) is equivalent to an outward normal direction of B at p to be the same as an outward normal direction of $-A$ at q. □

One can obtain the implicit algebraic equation of the *Convolution* (E_{-A}, E_B) in a similar way as in Theorem 4.1. When the face F_B is a parametric surface patch $F(s,t)=(x(s,t), y(s,t), z(s,t))$ with gradients $F_s \times F_t$, one can obtain the corresponding Corollary by changing every ∇f into $F_s \times F_t$ and the statement "$f(x, y, z)=0$ and $p=(x, y, z) \in K(E_B, E_{-A})$" into "$p=(x(s,t), y(s,t), z(s,t)) \in K(E_B, E_{-A})$" in the above Theorem. One can make similar changes to get corresponding Corollaries for the case of F_B, $\hat{F}_B$, F_{-A} and/or $\hat{F}_{-A}$ being parametric surface patches.

When F_B and G_B are tangent to each other along E_B, or H_{-A} and K_{-A} are tangent to each other along E_{-A}, *Convolution* (E_{-A}, E_B) is a degenerate curve on the *C-space* obstacle boundary and is a common edge of two convolution faces generated in § 4.2. In the special case of F_B and G_B being tangent along E_B, and also H_{-A} and K_{-A} being tangent along E_{-A}, *Convolution* (E_{-A}, E_B) is either a degenerate curve or a degenerate point.

Let $N_{E_s}(p) = N_{E_s} \cap N(S, p)$, then $N_{E_s}(p)$ is a geodesic arc on S^2. When two line segments in a plane intersects, either there is a unique intersection point or they overlap entirely on the same line. One can show a similar fact for minimal geodesic arcs on S^2 as follows.

Fact 4.1 : If $N_{E_s}(p) \cap N_{E_{-A}}(q) \neq$ *empty*, either (1) $N_{E_s}(p) \cap N_{E_{-A}}(q)$ is a point or (2) $N_{E_s}(p) = N_{E_{-A}}(q)$.

By subdividing E_B and E_{-A} if necessary, we may assume only one of the conditions (1) or (2) holds for the whole edges E_B and E_{-A}. We call E_B and E_{-A} to be *parallel* if the condition (2) holds on the whole edges E_B and E_{-A}. If E_B and E_{-A} is a *parallel* edge pair, the *Convolution* (E_{-A}, E_B) generated in Theorem 4.5 is a degenerate curve on the *C-space* obstacle. Otherwise it is a surface patch.

Boundary Edges of *Convolution* (E_{-A}, E_B)

For a *sub-compatible* edge pair E_B and E_{-A} where E_B (resp. E_{-A}) is relatively open with respect to the intersection curve of two algebraic surfaces $f=0$ and $\hat{f}=0$ defining faces F_B and $\hat{F}_B$ (resp. $g=0$ and $\hat{g}=0$ defining faces F_{-A} and $\hat{F}_{-A}$), each edge E_N of $N(E_B, E_{-A})$ is either a segment of an edge of $N(B, E_B)$ or a segment of an edge of $N(-A, E_{-A})$. Further E_N is either (a) a segment of the common edge of $N(B, E_B)$ and $N(B, F_B)$ for some face F_B adjacent to E_B, (b) a segment of the common edge of $N(-A, E_{-A})$ and $N(-A, F_{-A})$ for some face F_{-A} adjacent to E_{-A}, (c) a segment of the common edge of $N(B, E_B)$ and $N(B, p_B)$ for some vertex p_B of E_B, or (d) a segment of the common edge of $N(-A, E_{-A})$ and $N(-A, p_{-A})$ for some vertex p_{-A} of E_{-A}. Similarly, each boundary edge $E_{CO(A,B)}$ of the surface patch *Convolution* (E_{-A}, E_B) is either (a) a segment of the common edge of *Convolution* (E_{-A}, E_B) and *Convolution* $(Cl(E_{-A}), F_B)$, (b) a segment of the common edge of *Convolution* (E_{-A}, E_B) and *Convolution* $(F_{-A}, Cl(E_B))$, (c) a segment of the common edge of *Convolution* (E_{-A}, E_B) and *Convolution* $(Cl(E_{-A}), p_B)$, or (d) a segment of the common edge of *Convolution* (E_{-A}, E_B) and *Convolution* $(p_{-A}, Cl(E_B))$. Edges of type (a)–(b) have been described in Theorem 4.4. In the case of (c), *Convolution* $(Cl(E_{-A}), p_B)$ is a degenerate curve segment which is non-smooth on $Bdr(CO(A, B))$ and also equals to the edge $E_{CO(A,B)}$. Hence, $E_{CO(A,B)}$ is the common edge of the face *Convolution* (E_{-A}, E_B) with the face *Convolution* $(E_{-A}, \hat{E}_B)$ for some edge $\hat{E}_B$ adjacent to p_B (or with the face *Convolution* (F_{-A}, p_B) for some face F_{-A} adjacent to E_{-A}). Since $Bdr(CO(A, B))$ is non-smooth on $E_{CO(A,B)}$, $E_{CO(A,B)}$ can be represented as a common edge of two transversally intersecting convolution surface patches. The case of (d) is similar to the case of (c).

Boundary Vertices of *Convolution* (E_{-A}, E_B)

For *sub-compatible* edge pairs E_B and E_{-A}, each vertex e_N of $N(E_B, E_{-A})$ (= $N(B, E_B) \cap N(-A, E_{-A})$) is either (a) a vertex of $N(B, E_B)$, (b) a vertex of $N(-A, E_{-A})$, or (c) the intersection of one edge of $N(B, E_B)$ with another edge of $N(-A, E_{-A})$. In the case of (a), suppose p is a vertex of E_B and q is a point of E_{-A} such that $p \in N(B, e_N)$ and $q \in N(-A, e_N)$, then the point $p+q$ is the vertex of *Convolution* (E_{-A}, E_B) such that $p+q \in N^{-1}(CO(A, B), e_N)$. Further suppose that E_{-A} is the common edge of two faces F_{-A} and $\hat{F}_{-A}$ defined by $g=0$ and $\hat{g}=0$ respectively, then the point $q = (\alpha, \beta, \gamma) \in E_{-A}$ can be computed by solving $g = \hat{g} = 0$ and $(\nabla g \times \nabla \hat{g}) \cdot e_N = 0$. The case of (b) is similar to the case of (a). In the case (c), this intersection is also the intersection of one edge of $N(B, F_B)$ with another edge of $N(-A, F_{-A})$ where F_B is a face adjacent to E_B and F_{-A} is a face adjacent to F_{-A}. This case has been considered in §4.1.

5. Obtaining Gaussian Model of *C−space* Obstacles

We now show how to construct the Gaussian (Spherical) Model of $CO(A, B)$, see Figures 2 (a)–(c). Let $S^2{}_B$ and $S^2{}_{-A}$ be the Gaussian Models of B and $-A$. These define graphs on S^2 with degeneracies tagged appropriately. Let a new graph $S^2{}_{CO(A,B)}$ on S^2 be defined as the overlay of $S^2{}_B$ and $S^2{}_{-A}$. Then $S^2{}_{CO(A,B)}$ is the Gaussian Model of $CO(A, B)$ and determines all *sub-compatible* face, edge and vertex pairs between $Bdr(B)$ and $Bdr(-A)$. Further the topology of the faces, edges and vertices of $Bdr(CO(A, B))$ is given by the topology of the faces, edges and vertices of $S^2{}_{CO(A,B)}$. Construction of $S^2{}_{CO(A,B)}$ requires computing the intersections of edges of $S^2{}_B$ with edges of $S^2{}_{-A}$. These intersections can be computed by using Theorems 5.1–5.3. Edges of $S^2{}_B$ or $S^2{}_{-A}$ are either minimal geodesic arcs on the unit sphere or curve segments of the form $\nabla f(p) / \| \nabla f(p) \|$ for $p \in E$ where $f=0$ is a face equation and E is an edge of this face. Note this curve segment is well defined since we are assuming the nonsingularity of each face on its boundaries. By the regularity and convexity of the object we may assume that the end points of each minimal geodesic arc are not antipodal points of each other. Hence, for two end points n_1 and n_2 of a minimal geodesic arc one has $\lambda \cdot n_1 + (1-\lambda) \cdot n_2 \neq 0$ and $(\lambda \cdot n_1 + (1-\lambda) \cdot n_2) / \| \lambda \cdot n_1 + (1-\lambda) \cdot n_2 \|$ is well defined. The intersection of two minimal geodesic arcs can be computed by Theorem 5.1. The intersection of one general curve segment and one minimal geodesic arc can be computed by Theorem 5.2. The intersection of two general curve segments can be computed by Theorem 5.3.

Next by using a spherical sweep algorithm where one can move a great circle around the sphere and amongst the edge segments, it is possible to compute all the overlay curve intersections. The details are somewhat intricate but a generalization of moving a line in a plane–sweep algorithm.

Theorem 5.1 : Let γ be a minimal geodesic arc connecting n_1 to n_2 on $S^2{}_B$ and γ' be a minimal geodesic arc connecting n'_1 to n'_2 on $S^2{}_{-A}$. Then γ and γ' intersect at $(\lambda \cdot n_1 + (1-\lambda) \cdot n_2) / \| \lambda \cdot n_1 + (1-\lambda) \cdot n_2 \|$ if and only if

$$\begin{cases} (\lambda \cdot n_1 + (1-\lambda) \cdot n_2) \times (\mu \cdot n'_1 + (1-\mu) \cdot n'_2) = 0 & (1) \\ (\lambda \cdot n_1 + (1-\lambda) \cdot n_2) \cdot (\mu \cdot n'_1 + (1-\mu) \cdot n'_2) > 0 & (2) \end{cases}$$

for some $0 \le \lambda, \mu \le 1$.

Proof : (1)–(2) are equivalent to that $\lambda \cdot n_1 + (1-\lambda) \cdot n_2$ is in the same direction as $\mu \cdot n'_1 + (1-\mu) \cdot n'_2$ for some $0 \le \lambda, \mu \le 1$. □

Since the vector equation (1) gives two independent scalar equations in two variables λ, μ, one can solve this system of polynomial equations either numerically or symbolically, Buchberger, Collins, and Loos (1982).

Theorem 5.2 : Let γ be a curve segment on $S^2{}_B$ given by the set of points $\nabla f(p) / ||\nabla f(p)||$ for $p \in E_B$, where $E_B \subset Bdr(B)$ is the common edge of two faces F_B and $\hat{F}_B$, F_B is a patch of an algebraic surface $f = 0$ with gradients ∇f and $\hat{F}_B$ is a patch of an algebraic surface $\hat{f} = 0$ with gradients $\nabla \hat{f}$. And, let γ' be a minimal geodesic arc connecting n_1 to n_2 on $S^2{}_{-A}$. Then γ and γ' intersect at $\nabla f(p) / ||\nabla f(p)||$ if and only if

$$\begin{cases} f(x, y, z) = \hat{f}(x, y, z) = 0 \text{ and } p = (x, y, z) \in E_B & (1) \\ \nabla f \cdot (n_1 \times n_2) = 0 & (2) \\ \nabla f \cdot (n_1 - (n_1 \cdot n_2) n_2) \geq 0 & (3) \\ \nabla f \cdot (n_2 - (n_1 \cdot n_2) n_1) \geq 0 & (4) \end{cases}$$

Proof : (2)–(4) are equivalent to that ∇f is in the same direction as $\lambda \cdot n_1 + (1-\lambda) \cdot n_2$ for some $0 \leq \lambda \leq 1$. (1) restricts the solution for p to the edge E_B. □

Since (1)–(2) give three equations in three variables x, y, z, one can solve this system of polynomial equations. The case of γ being a minimal geodesic arc on $S^2{}_B$ and γ' being a general curve segment on $S^2{}_{-A}$ is similar to Theorem 5.2.

Theorem 5.3 : Let γ be a curve segment on $S^2{}_B$ given by the set of points $\nabla f(p) / ||\nabla f(p)||$ for $p \in E_B$, where $E_B \subset Bdr(B)$ is the common edge of two faces F_B and $\hat{F}_B$, F_B is a patch of an algebraic surface $f = 0$ with gradients ∇f and $\hat{F}_B$ is a patch of an algebraic surface $\hat{f} = 0$ with gradients $\nabla \hat{f}$. And, let γ' be a curve segment on $S^2{}_{-A}$ given by the set of points $\nabla g(q) / ||\nabla g(q)||$ for $q \in E_{-A}$, where $E_{-A} \subset Bdr(-A)$ is the common edge of two faces G_{-A} and $\hat{G}_{-A}$, G_{-A} is a patch of an algebraic surface $g = 0$ with gradients ∇g and $\hat{G}_{-A}$ is a patch of an algebraic surface $\hat{g} = 0$ with gradients $\nabla \hat{g}$. Then γ and γ' intersect at $\nabla f(p) / ||\nabla f(p)||$ if and only if

$$\begin{cases} f(x, y, z) = \hat{f}(x, y, z) = 0 \text{ and } p = (x, y, z) \in E_B & (1) \\ g(\alpha, \beta, \gamma) = \hat{g}(\alpha, \beta, \gamma) = 0 \text{ and } q = (\alpha, \beta, \gamma) \in E_{-A} & (2) \\ \nabla f \times \nabla g = 0 & (3) \\ \nabla f \cdot \nabla g > 0 & (4) \end{cases}$$

Proof : (3)–(4) are equivalent to that ∇f is in the same direction as ∇g. (1) restricts the solution for p to the edge E_B and (2) restricts the solution for q to the edge E_{-A}. □

Since the vector equation (3) gives two independent scalar equations, one has six scalar equations in six variables from (1)–(3) and can solve this system of polynomial equations.

Each face of the overlay graph $S^2{}_{CO(A,B)}$ corresponds to a compatible pair $((K_B, N_{K_B}), (G_{-A}, N_{G_{-A}}))$ of faces, edges and vertices of $Bdr(B)$ and $Bdr(-A)$. Note that we consider the degenerate curves and degenerate points as generic faces of $S^2{}_{CO(A,B)}$. Using the formula defining K_B and G_{-A} one can compute the equation for *Convolution* (G_{-A}, K_B). The edges and vertices of each face *Convolution* (G_{-A}, K_B) can be computed by using the boundary informations of K_B and G_{-A}.

6. Conclusion

We have described algebraic algorithms for computing *C–space* obstacles using boundary representations and Gaussian Image geometric models. The numerical information defining the faces, edges and vertices of the *C–space* obstacle boundary were obtained by solving systems of multivariate polynomial equations. The symbolic solution by means of resultants, though computationally extensive, yields the implicit algebraic equations of the curves and surfaces on the *C–space* obstacle boundary. The topological information defining the adjacency relationalships of faces, edges and vertices of the *C−space* obstacle boundary were obtained by constructing and merging (or overlaying) the Gaussian Image models of the individual moving objects and obstacles.

In comparison with the algorithms for obtaining the *C−space* obstacle boundary for planar case, Bajaj and Kim (1987a), one notes for the *C−space* obstacle generations in space an extensively large increase in complexity both in obtaining the numerical and topological information. A significant problem that arises in the *C−space* generation for curved objects is the analysis of singularities. While all types of point singularities that arise in planar curves can be completely analyzed by the quadratic transformations of Abhyankar (1983), the singularities in algebraic surfaces are considerably harder to deal with. The complete analysis of singularities in plane curves also allows one to deal with the topological constructions of *C−space* obstacles for non−convex algebraic curved moving objects and obstacles as well, see Bajaj and Kim (1987a). Analysis of the possible point and curve singularities that may arise in *C−space* obstacle surfaces may be achieved by a canonical (algorithmic) procedure of mapping the singular surface to a non−singular algebraic variety (a process also termed as "blowing up" the singularity) and recently given by Abhyankar (1982, 86). This is an area for important future research, for its solution would also lead to obtaining *C−space* obstacles for non−convex curved solid moving objects and obstacles – the currently immediate open problem.

7. Acknowledgements

We are grateful to the anonymous referees for their comments and suggestions.

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