1985

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Report Number:

85-565

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CSD-TR-565
December 1985

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Abstract

Using configuration space to plan collision free motion for a single rigid object amongst physical objects, reduces the problem to planning motion for a mathematical point amongst "grown" configuration space obstacles, (the points in configuration space which correspond to the object overlapping one or more obstacles). The problem of collision free motion for a point is simple since a point can now be moved without restriction in any connected region of configuration space. The difficult part of the technique comes in the construction of the configuration space obstacles. In the past object representations have been polyhedral approximations to the real object. However it has progressively become easier for geometric modeling systems to deal with objects that are defined by quadric (degree 2) surfaces. It is in this sophisticated modeling environment that we characterize the surface boundary of the configuration space obstacles arising from the motion of a sphere amongst obstacles whose boundary is defined by patches of quadric surfaces. The problem of generating these configuration space obstacles is also shown to be closely related to the problem of blending quadric surfaces by spherical envelopes.

Index Terms: Robotics, Geometric Modeling, Spatial Planning, Obstacle Avoidance, Envelopes, Blending, Computational Geometry, Parameterization, Computer-aided Design

1. Introduction

Perhaps the most important and universal scheme used in motion planning is that of transforming the problem in such a way that the rigid object to be moved is represented as a *point* in what is known as *configuration space*. For example, the position and orientation of a rigid object in the plane can be represented by a point (x, y, θ) in a 3-dimensional parameter configuration space where x, y represent the position of a reference point of the object and θ represents the orientation of a reference line of the object (say its angle with the positive x-axis). Similarly, a rigid object translating and rotating in 3-dimensional space can be represented as a point moving in a 6-dimensional configuration space. Early examples of a configuration space approach are [5], [11] and more recently [8,9]. Using configuration space to plan collision free motion for a single rigid object amongst physical objects, reduces the problem to planning motion for a mathematical point amongst "grown" configuration space obstacles, (the points in configuration space which correspond to the object overlapping one or more obstacles). The problem of collision free motion for a point is simple since a point can now be moved without restriction in any connected region of configuration space. The difficult part of the technique comes in the construction of the configuration space obstacles, (henceforth $C\text{-}space$ obstacles).

There are few techniques known for computing or approximating the $C\text{-}space$ obstacles, resulting from motion of a rigid object in 3-dimensional space, [7]. These techniques have primarily been confined to the motion of the class of polyhedral (degree 1 surface) objects amongst polyhedral obstacles [3, 4, 9, 10]. We now consider the generation of $C\text{-}space$ obstacles arising from the motion of a spherical object amongst obstacles whose surface boundary consists of patches of quadric surfaces. In § 2 of this paper we characterize the problem of "growing" general $C\text{-}space$ obstacles and show that the $C\text{-}space$ obstacle boundary surface is the envelope surface of the reversed object, (reversed with respect to the reference point of the object), as the reference point moves around the boundary of the obstacle. In § 3 we consider the case of the moving object being a sphere. More general quadric surface moving objects will be explored in a subsequent paper [2]. In § 4 we consider the obstacles whose boundary consists of patches of quadric surfaces. In § 5 we show how generation of these obstacles is closely related to the problem of blending quadric surfaces by spherical envelopes. The problem of blending surfaces, [6], is thus a special case of the general problem of generating $C\text{-}space$ obstacles.

The choice of the sphere in § 3 is advantageous for a number of reasons. A practical methodology that is increasingly gaining ground in robot task planning, is that of hierarchical representations, [7]. The notion of hierarchical representations involves attempting to solve problems concerning physical objects by starting with very simple representations of the properties involved, and introducing more complex representations only as they are required to solve the problem. A system thus could initially approximate all objects by interior and exterior enclosing spheres. If the exterior spheres do not intersect at all during a planned motion, the motion is known to be safe. If the interior spheres intersect during such a motion, a collision free motion is impossible. If neither of these conditions are met the system then could proceed to a finer level of detail. In most industrial applications, the workspace environment of the robot is sparsely cluttered and finding collision free paths for general objects by considering the motion of enclosing spheres of the objects would suffice. Possibly, though at high computational cost, exact high degree surface representations could be used.

There is a further advantage in considering spheres. Approximations of the moving object by the lowest degree (planar) surfaces, i.e., polyhedral objects amongst polyhedral obstacles lead to an immediate computational difficulty. The unrestricted motion of a polyhedral rigid object reduces to the motion of a point in 6-dimensional configuration space where both finding connected regions for collision free motion and characterizing 6-dimensional $C\text{-}space$ obstacles is an arduous task, [4,10]. The unrestricted motion of a sphere on the other hand is transformed to the motion of a point amongst 3-dimensional $C\text{-}space$ obstacles.

2. Characterization of Growing C-space Obstacles

Let A be an object and B be a fixed obstacle. Let a point of A, say a_0, be designated as the reference point of A. Throughout we consider A to be free to move under translation but not rotation. In this case configuration space is 3-dimensional. Let A_p denote the set of points in 3-space covered by A when A is located with a_0 at the point p. A_0 denotes this set of points when a_0 is at the origin. Let B denote the set of points in 3-space occupied by the obstacle B. The $C\text{-}space$ obstacle corresponding to B is the set of configuration space points $\{p \mid A_p \cap B \neq \emptyset\}$. We define $B\text{--}A$ to be the set of points $\{p \mid p = b - a,\ a \in A,\ b \in B\}$ in 3-space, where $b - a$ is the vector difference of a and b. With translation but no rotation the $C\text{-}space$ obstacle is given by the "grown" obstacle, $B\text{--}A_0$ as shown below.

Fact 1: Let B' be the $C\text{-}space$ obstacle corresponding to the obstacle B and the object A, then $B' = B\text{--}A_0 = \{p \mid p = b - a,\ a \in A_0,\ b \in B\}$.

Proof:

$B' = \{ p \in R^3 \mid B \cap A_p \neq \emptyset \}$

$= \{ p \in R^3 \mid \exists\, b \in B \cap A_p \}$

$= \{ p \in R^3 \mid \exists\, b \in B$ such that $b = a + p$ for some $a \in A_0 \}$

$= \{ p \in R^3 \mid p = b - a$ for some $b \in B$ and $a \in A_0 \}$

$= B - A_0$ □

If A and B are convex and have planar (degree 1) boundary surfaces, that is they are convex polyhedral objects then the boundary of $B - A_0$ = Convex Hull (Vertices (B) – Vertices (A)), [9]. The case of A and B being non-convex can be handled by first decomposing A and B into convex components. If A is a convex rigid object with boundary surface of degree ≥ 2 making single point contact with obstacle B, then the boundary of $B - A_0$ = *Envelope* of the boundary of $-A$, (object reversed with respect to the reference point), as the reference point a_0 moves on the boundary of B. This can be seen as follows.

The *C−space* obstacle corresponding to B is the set of all reference points $\bar{p}$ such that $A_{\bar{p}}$, (the object A, located with a_0 at $\bar{p}$), intersects with B, i.e., $\{\bar{p} \mid A_{\bar{p}} \cap B \neq \emptyset\}$. In particular then the boundary of the *C−space* obstacle is the trace or envelope of the reference points $a_0 = \bar{p}$ as $A_{\bar{p}}$ moves on B with the surface boundaries of $A_{\bar{p}}$ and B just touching in a point contact. Let $A_{\bar{p}}$ and B make contact at a common surface point p. Then $p - \bar{p}$ is a point on the boundary of A_0 and $\bar{p} - p = -(p - \bar{p})$ is a point on the boundary of $(-A)_0$. When we place $-A$ with the reference point a_0 at p, $\bar{p}$ is on the boundary of $(-A)_p$ and thus as the reference point a_0 moves on the boundary of B, the boundary of the *C−space* obstacle or the trace of the points $\bar{p}$ is an envelope of the boundary of $-A$.

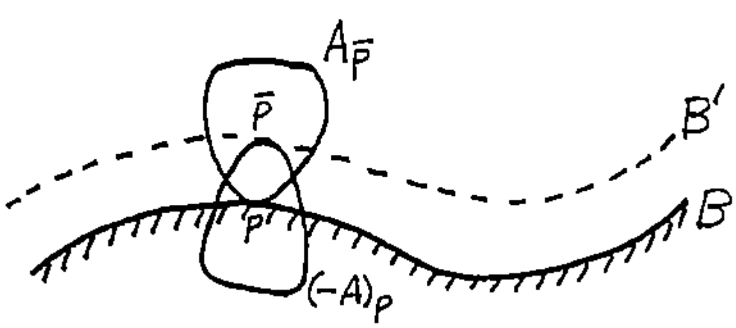

Figure 1

Let the boundary of the obstacle B be defined by a smooth convex surface given by Bdr(B) : $f(x,y,z) = 0$ and the obstacle B itself be defined by $B : f(x,y,z) \leq 0$, and the boundary of the object A_0 be defined by another smooth convex surface given by Bdr(A_0) : $g_0(x,y,z) = 0$ and the object A_0 itself is defined by $A_0 : g_0(x,y,z) \le 0$. If $\bar{p} = (\bar{x},\bar{y},\bar{z})$, then the boundary of $A_{\bar{p}}$ is given by $\mathrm{Bdr}(A_{\bar{p}}) : g_0(x-\bar{x}, y-\bar{y}, z-\bar{z}) = 0$. Next, let $A_{\bar{p}}$ and B make contact at a point $p = (x,y,z)$. The two surfaces $g_0(x-\bar{x},y-\bar{y},z-\bar{z}) = 0$ and $f(x,y,z) = 0$ have a common tangent plane at $p = (x,y,z)$. Hence, for some $\alpha > 0$, $p = (x,y,z) \in \mathrm{Bdr}(B)$, and $p-\bar{p} = (x-\bar{x}, y-\bar{y}, z-\bar{z}) \in \mathrm{Bdr}(A_0)$, we have the following relation between the two normals of these surfaces at the point $p = (x,y,z)$,

$$\nabla f(x,y,z) = -\alpha \nabla g_0(x-\bar{x},y-\bar{y},z-\bar{z})$$

$$\nabla f(p) = -\alpha \nabla g_0(p-\bar{p})$$

Now, the boundary of $(-A)_0$ is defined by $\mathrm{Bdr}((-A)_0) : g(x,y,z) = g_0(-x,-y,-z) = 0$ and $\nabla g(x,y,z) = -\nabla g_0(-x,-y,-z)$. Thus for some $\alpha > 0$, $p = (x,y,z) \in \mathrm{Bdr}(B)$, and $\bar{p}-p = (\bar{x}-x, \bar{y}-y, \bar{z}-z) \in \mathrm{Bdr}((-A)_0)$, we have

$$\nabla f(x,y,z) = \alpha \nabla g(\bar{x}-x, \bar{y}-y, \bar{z}-z)$$

Hence, the solution for the boundary of the $C\text{-}space$ obstacle B' is the set of all the points $\bar{p}$ which satisfies the following partial differential equation

$$\nabla f(p) = \alpha \nabla g_0(\bar{p}-p)$$

for some $\alpha > 0$ and some $p \in \mathrm{Bdr}(B)$. In general, for given f and g, this partial differential equation is difficult to solve directly. We now attempt to characterize the solution for the cases where $f = 0$ is a quadric surface and $g = 0$ is a sphere.

3. Moving a Spherical Object

Consider the object A to be a sphere of radius r with its center as a reference point and suppose the boundary of an obstacle B be given by a smooth surface

$$\mathrm{Bdr}(B): \ f(x,y,z) = 0$$

and the obstacle B itself is given by

$$B : f(x,y,z) \le 0$$

At a point $(x,y,z) \in \mathrm{Bdr}(B)$, $\nabla f(x,y,z)$ is the outward normal direction of f at (x,y,z). Hence,

$$(x,y,z) + r\,\frac{\nabla f(x,y,z)}{||\nabla f(x,y,z)||}$$

is the position of the reference point (center of sphere) when the sphere makes contact with the obstacle B. The boundary surface of the $C\text{-}space$ obstacle B' will be

$$\{\, (\bar{x},\bar{y},\bar{z}) \in R^3 \mid (\bar{x},\bar{y},\bar{z}) = (x,y,z) + r\,\frac{\nabla f(x,y,z)}{||\nabla f(x,y,z)||} \quad \text{with } f(x,y,z)=0 \text{ or } (x,y,z) \in \mathrm{Bdr}(B) \,\}$$

Hence, to get the surface equation for Bdr(B') we need to solve the equation

$$(\bar{x},\bar{y},\bar{z}) = (x,y,z) + r\,\frac{\nabla f(x,y,z)}{||\nabla f(x,y,z)||} \quad \text{for } x,y,z \text{ in terms of } \bar{x},\bar{y},\bar{z}.$$

Even in the case of quadric surfaces of degree 2, $||\nabla f(x,y,z)||$ is not of a simple form. So, the general solution is not easy. But, with special conditions on $||\nabla f(x,y,z)||$, the solution can be obtained easily. First, let us assume that $||\nabla f(x,y,z)|| = \text{constant} = K$, then

$$(\bar{x},\bar{y},\bar{z}) = (x,y,z) + \frac{r}{K}\,\nabla f(x,y,z)$$

There are three cases (plane, cylinder, and sphere) of $||\nabla f||$ = constant = K among the algebraic surfaces of degree ≤ 2.

3.1. Special Cases of $||\nabla f||$ = Constant

3.1.1. Plane :

When $f(x,y,z) = ax + by + cz + d = 0$, we have

$$\nabla f(x,y,z) = (a,b,c), \quad ||\nabla f(x,y,z)|| = \sqrt{a^2+b^2+c^2} = K.$$

By the above result

$$(\bar{x},\bar{y},\bar{z}) = (x,y,z) + \frac{r}{K}\,(a,b,c)$$

$$x = \bar{x} - \frac{r}{K}\,a, \quad y = \bar{y} - \frac{r}{K}\,b, \quad z = \bar{z} - \frac{r}{K}\,c$$

Hence

$$f(\bar{x}-\frac{r}{K}a,\, \bar{y}-\frac{r}{K}b,\, \bar{z}-\frac{r}{K}c) = 0$$

$$a(\bar{x}-\frac{r}{K}a) + b(\bar{y}-\frac{r}{K}b) + c(\bar{z}-\frac{r}{K}c) + d = a\bar{x} + b\bar{y} + c\bar{z} - \frac{r}{K}(a^2+b^2+c^2) + d = 0$$

$$a\bar{x} + b\bar{y} + c\bar{z} - r(\sqrt{a^2+b^2+c^2}) + d = 0$$

Hence, a plane is grown into another plane.

3.1.2. Circular Cylinder :

We may assume $f(x,y,z) = x^2 + y^2 - R^2 = 0, \ R > 0,$ then

$$\nabla f(x,y,z) = (2x, 2y, 0), \quad ||\nabla f(x,y,z)|| = \sqrt{4x^2+4y^2} = 2\sqrt{x^2+y^2} = 2R$$

And so,

$$(\bar{x},\bar{y},\bar{z}) = (x,y,z) + \frac{r}{2R}\,(2x, 2y, 0) = (x+\frac{r}{R}\,x,\, y+\frac{r}{R}\,y,\, z)$$

$$x = \frac{R}{R+r}\,\bar{x}, \quad y = \frac{R}{R+r}\,\bar{y}, \quad z = \bar{z}$$

and

$$f(\frac{R}{R+r}\bar{x},\ \frac{R}{R+r}\bar{y},\ \bar{z}) = 0,$$

$$(\frac{R}{R+r})^2 (\bar{x})^2 + (\frac{R}{R+r})^2 (\bar{y})^2 = R^2,$$

$$(\bar{x})^2 + (\bar{y})^2 = (R+r)^2$$

Hence, a circular cylinder of radius R is grown into another circular cylinder of radius $R+r$.

3.1.3. Sphere :

We may assume $f(x,y,z) = x^2 + y^2 + z^2 - R^2 = 0,\quad R > 0,$ then

$\nabla f(x,y,z) = (\, 2x,\ 2y,\ 2z\,),\quad ||\nabla f(x,y,z)|| = \sqrt{4x^2 + 4y^2 + 4z^2} = 2\sqrt{x^2 + y^2 + z^2} = 2R$

And so,

$$(\bar{x},\bar{y},\bar{z}) = (x,y,z) + \frac{r}{2R}(2x,\ 2y,\ 2z) = (x+\frac{r}{R}x,\ y+\frac{r}{R}y,\ z+\frac{r}{R}z)$$

$$x = \frac{R}{R+r}\bar{x},\ y = \frac{R}{R+r}\bar{y},\ z = \frac{R}{R+r}\bar{z}$$

and

$$f(\frac{R}{R+r}\bar{x},\ \frac{R}{R+r}\bar{y},\ \frac{R}{R+r}\bar{z}) = 0,$$

$$(\frac{R}{R+r})^2 (\bar{x})^2 + (\frac{R}{R+r})^2 (\bar{y})^2 + (\frac{R}{R+r})^2 (\bar{z})^2 = R^2,$$

$$(\bar{x})^2 + (\bar{y})^2 + (\bar{z})^2 = (R+r)^2$$

Hence, a sphere of radius R is grown into another sphere of radius $R+r$.

3.2. Surfaces of Revolution

Besides the above three surfaces, the cone is another very important and useful quadric surface in geometric design. Such surfaces also arise in modeling cutting tools in machining operations. However, a conic surface given by the equation $f(x,y,z) = x^2 + y^2 - z^2 = 0$, and having $||\nabla f(x,y,z)|| = ||(2x,2y,-2z)|| \neq$ constant, requires another technique to compute the $C\text{-}space$ obstacle. We shall first consider a more general case of the surface of revolution and see that the $C\text{-}space$ obstacle of a surface of revolution is again a surface of revolution. Generating the $C\text{-}space$ obstacle of a cone is then a special case of this.

3.2.1. Growing of a Convex-Downwards Generating Curve

When the boundary surface of an obstacle is a surface of revolution of a convex-downwards generating curve, $y = c(x) \geq 0$, (with $c''(x) \leq 0$), $a \leq x \leq b$ around the x-axis, the equation of the surface is given by

$$y^2 + z^2 = [c(x)]^2, \quad a \leq x \leq b.$$

Let

$$f(x,y,z) = -[c(x)]^2 + y^2 + z^2 = 0, \quad a \leq x \leq b.$$

then

$$\nabla f(x,y,z) = (\,-2 \cdot c'(x) \cdot c(x),\, 2y,\, 2z), \quad a \leq x \leq b.$$

Hence, this gradient vector is parallel to the plane containing the x-axis and the point (x,y,z). And so, when we grow the point $p = (x,y,z)$ into another point $\overline{p} = (\overline{x},\overline{y},\overline{z})$, this grown point $\overline{p}$ will be on the same plane determined by the x-axis and the point $p = (x,y,z)$. We can easily see that the *C−space* obstacle of a surface of revolution is again a surface of revolution. To get the *C−space* obstacle of a surface of revolution it is sufficient to get the generating curve of it. We can get the generating curve of the *C−space* obstacle by growing the original generating curve $y = c(x)$ in the xy-plane with respect to the circle of radius r with its center as a reference point.

Let's consider a convex-downwards generating curve

$$y = c(x) \geq 0, \quad (\text{with } c''(x) \leq 0), \quad a \leq x \leq b.$$

At the point $(x,\, c(x))$, $(1,\, c'(x))$ is a tangent vector to the curve $y = c(x)$ at the point $(x,\, c(x))$. $\frac{1}{\sqrt{1 + c'(x)^2}} \cdot (1,\, c'(x))$ is an unit tangent vector to the curve $y = c(x)$ at $(x,\, c(x))$. Now, when we multiply this vector by the rotation matrix by angle $\frac{\pi}{2}$

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} \\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2} \end{bmatrix},$$

since the generating curve is convex-downwards, we get the following outward unit normal vector to the curve $y = c(x)$ at $(x,\, c(x))$

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \dfrac{1}{\sqrt{1 + (c'(x))^2}} \\ \dfrac{c'(x)}{\sqrt{1 + c'(x)^2}} \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{-c'(x)}{\sqrt{1 + c'(x)^2}} \\ \dfrac{1}{\sqrt{1 + c'(x)^2}} \end{bmatrix} = \frac{1}{\sqrt{1 + c'(x)^2}} \begin{bmatrix} -c'(x) \\ 1 \end{bmatrix}$$

Now, the growing of the point $(x, c(x))$ toward this outward normal direction by distance r is

$$\begin{bmatrix} x \\ c(x) \end{bmatrix} + r \cdot \frac{1}{\sqrt{1 + c'(x)^2}} \begin{bmatrix} -c'(x) \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} x + \dfrac{-c'(x) \cdot r}{\sqrt{1 + c'(x)^2}} \\ c(x) + \dfrac{r}{\sqrt{1 + c'(x)^2}} \end{bmatrix}$$

Let

$$\bar{x} = x + \frac{-c'(x) \cdot r}{\sqrt{1 + c'(x)^2}}, \quad \bar{y} = c(x) + \frac{r}{\sqrt{1 + c'(x)^2}}$$

In general, it's very difficult to represent $\bar{y}$ in terms of $\bar{x}$ to get a generating curve for the boundary of the *C−space* obstacle. But in some special cases, x can be represented by $\bar{x}$ in a simple way, and so $\bar{y}$ can be represented by $\bar{x}$ easily. Remarkably, the cone is such an easy case.

3.2.2. Generating curve is a line segment :

When the generating curve $y = c(x) \geq 0$, $a \leq x \leq b$ is a straight line segment $y = \alpha x \geq 0$, $a \leq x \leq b$, its surface of revolution is a piece of a cone. We have

$$c(x) = \alpha x, \; c'(x) = \alpha, \text{ and } \sqrt{1 + c'(x)^2} = \sqrt{1 + \alpha^2}.$$

By using the above result we get

$$\bar{x} = x + \frac{-c'(x) \cdot r}{\sqrt{1 + c'(x)^2}} = x + \frac{-\alpha r}{\sqrt{1 + \alpha^2}}, \text{ and } \; x = \bar{x} + \frac{\alpha r}{\sqrt{1 + \alpha^2}}$$

and so

$$\bar{y} = c(\bar{x} + \frac{\alpha \cdot r}{\sqrt{1 + \alpha^2}}) + \frac{r}{\sqrt{1 + \alpha^2}} = \alpha \bar{x} + \frac{\alpha^2 \cdot r + r}{\sqrt{1 + \alpha^2}} = \alpha \bar{x} + (\sqrt{1 + \alpha^2}) \cdot r$$

Now, since x and $\bar{x}$ differ only by a constant $\dfrac{\alpha \cdot r}{\sqrt{1 + \alpha^2}}$

$$a - \frac{\alpha r}{\sqrt{1 + \alpha^2}} \leq \bar{x} \leq b - \frac{\alpha r}{\sqrt{1 + \alpha^2}} \quad \text{iff} \quad a \leq x \leq b$$

Hence,

$$\bar{y} = \alpha \bar{x} + r\sqrt{1 + \alpha^2}, \quad a - \frac{\alpha r}{\sqrt{1 + \alpha^2}} \leq \bar{x} \leq b - \frac{\alpha r}{\sqrt{1 + \alpha^2}}$$

Hence, the growing of the cone $y^2 + z^2 - \alpha^2 \cdot x^2 = 0$, $a \leq x \leq b$ is the surface of revolution of the following curve about x-axis

$$\bar{y} = \bar{c}(\bar{x}) = \alpha \bar{x} + r\sqrt{1 + \alpha^2}, \quad a - \frac{\alpha r}{\sqrt{1 + \alpha^2}} \leq \bar{x} \leq b - \frac{\alpha r}{\sqrt{1 + \alpha^2}},$$

that is, the surface

$$y^2 + z^2 = (\alpha x + r\sqrt{1+\alpha^2})^2, \quad a - \frac{\alpha r}{\sqrt{1+\alpha^2}} \le x \le b - \frac{\alpha r}{\sqrt{1+\alpha^2}},$$

3.2.3. General convex-downwards generating curve :

Now, let us consider a more general convex-downwards generating curve of the form

$$(t, c(t)), \text{ (with } c(t) \ge 0 \text{ and } c''(t) \le 0) \text{ for } a \le t \le b.$$

When we grow this curve by a circle of radius r, we will get

$$y = c(t) + \frac{r}{\sqrt{1+c'(t)^2}}, \quad x = t + \frac{-c'(t)\cdot r}{\sqrt{1+c'(t)^2}}, \text{ where } a \le t \le b$$

Hence, a new generating curve for the grown curve is

$$\bar{c}(t) = c(t) + \frac{r}{\sqrt{1+c'(t)^2}}, \quad a \le t \le b$$

The corresponding surface of revolution for this grown generating curve can be obtained by eliminating t from the follwing two equations and replacing the bounds for the parameter t by that of x,

$$y^2 + z^2 = [\bar{c}(t)]^2 = \left[c(t) + \frac{r}{\sqrt{1+c'(t)^2}} \right]^2,$$

$$x = t + \frac{-c'(t)r}{\sqrt{1+c'(t)^2}}, \qquad \text{where } a \le t \le b$$

3.2.4. Example : Cone

Cone is a surface of revolution of the following two line segments.

$$y = \alpha x \ge 0, \quad 0 \le x \le b$$

$$x = b, \quad 0 \le y \le \alpha b$$

Since there are two singular points (0,0) and $(b, \alpha b)$, these will be grown into a piece of a circle in the generating xy-plane. Two line segments will be grown into line segments of equal length and slope. Hence, the generating curve of the C*-space* obstacle of a cone consists of 4 pieces of elementary curves.

(1) First of all, $y = \alpha x \ge 0, \ 0 \le x \le b$ will be grown into a line segment

$$y = \alpha x + r\sqrt{1+\alpha^2}, \quad -\frac{\alpha r}{\sqrt{1+\alpha^2}} \le x \le b - \frac{\alpha r}{\sqrt{1+\alpha^2}}$$

(2) $x = b \ (0 \le y \le \alpha b)$ will be grown into a line segment

$$x = b + r, \quad 0 \le y \le \alpha b$$

(3) The singular point (0,0) will be grown into a piece of circle

$$y = \sqrt{r^2 - x^2}, \quad -r \le x \le \frac{-\alpha r}{\sqrt{1+\alpha^2}}$$

(4) The singular point $(b, \alpha b)$ will be grown into a piece of circle

$$y = \sqrt{r^2 - (x-b)^2} + \alpha b, \qquad b - \frac{\alpha r}{\sqrt{1+\alpha^2}} \le x \le b + r$$

Now, when we revolve these elementary line segments about x-axis, we can get the following surfaces of revolution

(1′) The line segment $y = \alpha x + r\sqrt{1+\alpha^2}$, $-\frac{\alpha r}{\sqrt{1+\alpha^2}} \le x \le b - \frac{\alpha r}{\sqrt{1+\alpha^2}}$ will generate

$$y^2 + z^2 = (\alpha x + r\sqrt{1+\alpha^2})^2, \quad -\frac{\alpha r}{\sqrt{1+\alpha^2}} \le x \le b - \frac{\alpha r}{\sqrt{1+\alpha^2}}$$

(2′) The line segment $x = b + r$, $0 \le y \le \alpha b$ will generate a disc of radius αb

$$y^2 + z^2 \le (\alpha b)^2, \quad x = b + r$$

(3′) The piece of a circle (which is a growing of the point (0,0)) will generate a piece of sphere as follows

$$y^2 + z^2 = (\sqrt{r^2 - x^2})^2, \quad -r \le x \le \frac{-\alpha r}{\sqrt{1+\alpha^2}}$$

or equivalently

$$x^2 + y^2 + z^2 = r^2, \quad -r \le x \le \frac{-\alpha r}{\sqrt{1+\alpha^2}}$$

(4′) The piece of circle (which is a growing of the singular point $(b, \alpha b)$) will generate

$$y^2 + z^2 = (\sqrt{r^2 - (x-b)^2} + \alpha b)^2, \quad b - \frac{\alpha r}{\sqrt{1+\alpha^2}} \le x \le b + r$$

or equivalently

$$(x-b)^2 + y^2 + z^2 = r^2 + \alpha^2 b^2 + 2\alpha b\sqrt{r^2 - (x-b)^2}, \quad b - \frac{\alpha r}{\sqrt{1+\alpha^2}} \le x \le b + r$$

It is a piece of a torus (algebraic surface of degree 4) if $\alpha b \ne 0$.

3.2.5. Example : More general surface of revolution

In general, when the generating curve consists of a finite number (say n) of piecewise smooth convex-downwards curves of the form

$$(t, c_i(t)), \ \text{(with } c_i(t) \ge 0 \text{ and } c_i''(t) \le 0) \quad \text{where } a_i \le t \le b_i = a_{i+1},$$
$$\text{for } i = 1, \ldots, n \quad \text{with } a_1 = 0,\ b_n = b,\ c_1(0) = 0, \text{ and } c_n(b) = 0$$

Each of the curve segments $(t, c_i(t))$, $a_i \le t \le b_i$ will be grown into a curve segement

$(x(t), y(t)),\ a_i \le t \le b_i$ given by

$$x(t) = t + \frac{-c_i'(t) r}{\sqrt{1 + (c_i'(t))^2}}, \quad y(t) = c_i(t) + \frac{r}{\sqrt{1 + (c_i'(t))^2}}, \quad \text{where } a_i \le t \le b_i$$

Each of the singular points $(a_1, 0) = (0, 0)$, $(a_2, c_2(a_2))$, ..., $(a_n, c_n(a_n))$, $(b_n, 0) = (b, 0)$ will be grown into a piece of circle as follows

(1) (0, 0) will be grown into

$$y = \sqrt{r^2 - x^2}, \quad -r \le x \le \frac{-c'_1(0)\, r}{\sqrt{1 + (c_1'(0))^2}}$$

(2) $(a_i, c_i(a_i)),\ i = 2, ..., n$ will be grown into

$$y = \sqrt{r^2 - (x - a_i)^2} + c_i(a_i)$$

$$\text{where } a_i - \frac{c_i'(a_i)\, r}{\sqrt{1 + (c_i'(a_i))^2}} \le x \le a_{i+1} - \frac{c_i'(a_{i+1})\, r}{\sqrt{1 + (c_i'(a_{i+1}))^2}}$$

(3) $(b_n, 0) = (b, 0)$ will be grown into

$$y = \sqrt{r^2 - x^2}, \quad b - \frac{c_n'(b)\, r}{\sqrt{1 + (c_n'(b))^2}} \le x \le b+r$$

4. Obstacle consisting of patches of planes, circular cylinders, spheres, and cones

Let an obstacle B be a convex set such that the boundary of B consists of a finite number of surface patches which are pieces of planes, circular cylinders, spheres, or cones. Let the boundary of B have m faces $F_1, ..., F_m$, n edges $E_1, ..., E_n$ (which are intersecting curves of faces), and l vertices $v_1, ..., v_l$ (which are intersecting points of edges or apexes of cones). Each face F_i $(i=1,...,m)$ will be grown into another face F'_i $(i=1,...,m)$. Each planar face will be grown into another planar face at distance r. Each circular cylindrical face (of radius R) will be grown into another circular cylindrical face (of radius $R+r$). Each spherical face (of radius R) will be grown into another spherical face (of radius $R+r$). Each conic face will be grown into another conic face without an apex, because the apex will be grown into a circular edge and a spherical face.

4.1. Growing Edges into Edges and Faces

When a face is bounded by some edges, its grown face will be bounded by the grown edges of the boundary edges of the original face. In the case of common boundary edge E of two different faces F_1 and F_2, as a curve on the face F_1, E will be grown into a boundary edge E_1' of F_1' and at the same time as a curve on the face F_2, it will be grown into another boundary edge E_2' of F_2'. In addition to these two edges E_1' and E_2', E will be grown into a face F_E' which is a piece of the envelope surface of sphere of radius r as the center of the sphere moves along the edge E. This envelope surface has a tubular shape and is a member of the class of surfaces known as *Canal* surfaces. In our case since the obstacle B is convex, we can get the parametric representation of the tubular face $F_E{}'$ easily once we can parametrize the curve E and can get the surface normals of the faces F_1 and F_2 on this curve E.

Suppose the edge E is parametrized by a curve $C : [a, b] \rightarrow E$. By using the surface normals of F_1 and F_2 on this curve C (or edge E) we can parametrize the grown edges E'_1 and E'_2 by the curves $C_1 : [a, b] \rightarrow E$ and $C_2 : [a, b] \rightarrow E$ so that $C_1(t) - C(t)$ is normal to the face F_1 at $C(t) \in E$ and $C_2(t) - C(t)$ is normal to the face F_2 at $C(t) \in E$. Since the obstacle B is convex, the directions between $N_1 = C_1(t) - C(t)$ and $N_2 = C_2(t) - C(t)$ are always outward directions of B. These directions between N_1 and N_2 determine a geodesic curve segment $\gamma_{C(t)}$ from $C_1(t)$ to $C_2(t)$ on the sphere of radius r centered at $C(t) \in E$. We can show this curve segment $\gamma_{C(t)}$ will be on the envelope surface of the sphere of radius r moving with its center along the boundary of B, and when we move its center along the curve C (i.e. the edge E), the trace of the circular curve $\gamma_{C(t)}$ will generate a tubular surface patch and this is the face generated from the edge E by the growing operation. Actually when we locate a sphere of radius r with its center at a point on the circular curve $\gamma_{C(t)}$, this sphere will contact with the obstacle B exactly at the point $C(t) \in E$.

When the face F_1 is given by $F_1 : f_1(x,y,z) = 0$ and the face F_2 is given by $F_2 : f_2(x,y,z) = 0$, the parametric curves C_1 and C_2 for the grown edges E'_1 and E'_2 can be given as follows

$$C_1(t) = C(t) + r \cdot \frac{\nabla f_1(C(t))}{||\nabla f_1(C(t))||}, \quad a \le t \le b$$

$$C_2(t) = C(t) + r \cdot \frac{\nabla f_2(C(t))}{||\nabla f_2(C(t))||}, \quad a \le t \le b$$

and the parametric representation of the grown face $F_E{}'$ can be given as

$$H(s,t) = \gamma_{C(t)}(s) = C(t) + r \; \frac{\frac{1-s}{2}\, C_1(t) + \frac{1+s}{2}\, C_2(t)}{||\frac{1-s}{2}\, C_1(t) + \frac{1+s}{2}\, C_2(t)||}, \quad \text{where } -1 \le s \le 1 \text{ and } a \le t \le b$$

4.2. Growing Vertices into Circular Edges and Spherical Faces

When a vertex v is an apex of a conic surface patch, it will be grown into a circular edge and a spherical patch by the result of growing a surface of revolution (§ 3.2.4). When several faces $F_1, ..., F_k$ are placed counter-clockwise around a common vertex v and v is not an apex of a conic surface, v is a common vertex of k edges E_1 (common edge of F_1 and F_2), E_2 (common edge of F_2 and F_3), ..., E_k (common edge of F_k and F_1) at the same time. As a point on each faces $F_1, ..., F_k$, v has k normal directions $N_1, ..., N_k$. These k normal vectors determine k points $v_1, ..., v_k$ on the sphere of radius r centered at v. By the same way as we did in growing the edges, there are geodesic curves γ_1 from v_1 to v_2, γ_2 from v_2 to v_3, ..., γ_k from v_k to v_1 on the sphere of radius r centered at v and we can show that the circular edges defined by these geodesic curve segments are on the $C\!-\!space$ obstacle B' of B and the convex region bounded by the closed curve $\gamma_1 \rightarrow \gamma_2 \rightarrow ... \rightarrow \gamma_1$ defines a spherical surface patch which is a grown face of the vertex v. In summary v will be grown into k boundary vertices $v_1, ..., v_k$ of $F_1', ..., F_k'$, into k circular edges E_{v_1} (connecting v_1 and v_2), ..., E_{v_k} (connecting v_k and v_1), and into a spherical face F_v' which is bounded by the edges $E_{v_1}, ..., E_{v_k}$.

When we have m faces, n edges, q apexes of cones, and l vertices which is not an apex of a cone (where each vertex is a common vertex of k_i edges, $i = 1, ..., l$), we will have $(m+n+q+l)$ faces, $(2n+q+k_1+...+k_l)$ edges, and $(k_1+...+k_l)$ vertices on the $C\!-\!space$ obstacle B' of B. An apex of a conic face can be a vertex of a non-conic face at the same time. This case is excluded in the above consideration, but we can do the growing of this vertex by the same way as we did in the other cases and we can get quite similar result on the number of faces, edges, and vertices and the $C\!-\!space$ obstacle B' of B.

4.3. Boundary consisting of several planar patches

When the boundary of an obstacle B consists of only planar faces, the edges are all stright line segements. Each face will be grown into another planar face of the same shape and area, each edge will be grown into two line segments and a piece of right circular cylinder, and each vertex will be grown into vertices, circular edges, and a spherical face. The boundary of $C\!-\!space$ obstacle B' will be composed of pieces of planes, circular cylinders, and spheres.

4.3.1. Example : Box

When we have a box bounded by six planar patches

(1) $-R \le x, y \le R,\ z = -R$

(2) $-R \le x, y \le R,\ z = R$

(3) $-R \le y, z \le R,\ x = -R$

(4) $-R \le y, z \le R,\ x = R$

(5) $-R \le x, z \le R,\ y = -R$

(6) $-R \le x, z \le R,\ y = R$

The C-space obstacle of this box will be bounded by 6 planar patches, 12 cylindrical patches, and 8 spherical patches

(P1) $-R \le x, y \le R,\ z = -R-r$

(P2) $-R \le x, y \le R,\ z = R+r$

(P3) $-R \le y, z \le R,\ x = -R-r$

(P4) $-R \le y, z \le R,\ x = R+r$

(P5) $-R \le x, z \le R,\ y = -R-r$

(P6) $-R \le x, z \le R,\ y = R+r$

(C1) $(x-R)^2+(z-R)^2 = r^2, x \ge R, z \ge R, -R \le y \le R$

(C2) $(x-R)^2+(z+R)^2 = r^2, x \ge R, z \le -R, -R \le y \le R$

(C3) $(x+R)^2+(z+R)^2 = r^2, x \le -R, z \le -R, -R \le y \le R$

(C4) $(x+R)^2+(z-R)^2 = r^2, x \le -R, z \ge R, -R \le y \le R$

(C5) $(y-R)^2+(z-R)^2 = r^2, y \ge R, z \ge R, -R \le x \le R$

(C6) $(y-R)^2+(z+R)^2 = r^2, y \ge R, z \le -R, -R \le x \le R$

(C7) $(y+R)^2+(z+R)^2 = r^2, y \le -R, z \le -R, -R \le x \le R$

(C8) $(y+R)^2+(z-R)^2 = r^2, y \le -R, z \ge R, -R \le x \le R$

(C9) $(x-R)^2+(y-R)^2 = r^2, x \ge R, y \ge R, -R \le z \le R$

(C10) $(x-R)^2+(y+R)^2 = r^2, x \ge R, y \le -R, -R \le z \le R$

(C11) $(x+R)^2+(y+R)^2 = r^2, x \le -R, y \le -R, -R \le z \le R$

(C12) $(x+R)^2+(y-R)^2 = r^2, x \le -R, y \ge R, -R \le z \le R$

(S1) $(x-R)^2+(y-R)^2+(z-R)^2 = r^2, x \ge R, y \ge R, z \ge R$

(S2) $(x-R)^2+(y-R)^2+(z+R)^2 = r^2, x \ge R, y \ge R, z \le -R$

(S3) $(x-R)^2+(y+R)^2+(z-R)^2 = r^2, x \ge R, y \le -R, z \ge R$

(S4) $(x-R)^2+(y+R)^2+(z+R)^2 = r^2, x \ge R, y \le -R, z \le -R$

(S5) $(x+R)^2+(y-R)^2+(z-R)^2 = r^2, x \le -R, y \ge R, z \ge R$

(S6) $(x+R)^2+(y-R)^2+(z+R)^2 = r^2, x \le -R, y \ge R, z \le -R$

(S7) $(x+R)^2+(y+R)^2+(z-R)^2 = r^2, x \le -R, y \le -R, z \ge R$

(S8) $(x+R)^2+(y+R)^2+(z+R)^2 = r^2, x \le -R, y \le -R, z \le -R$

4.3.2. Boundary consisting of planar patches and cylindrical patches

The intersecting curves of two cylinderical patches are in general very complicated. Some simple cases of these are considered in § 5 in connection with the surface blending problems. The intersecting curve γ of a planar patch P and a cylindrical patch C can be a straight line, a circle, or an ellipse. When we grow a convex obstacle B, the planar patch P will be grown into another planar patch P' of the same shape and the same area, the cylindrical patch C (of radius R) will be grown into another cylindrical patch C' (of radius $R+r$), and the intersecting curve γ will be grown into a cylindrical patch (if γ is a stright line), a piece of torus (if γ is a circle), or a piece of elliptic torus (if γ is an ellipse). Growing of a vertex is quite similar to the case of several planar patches.

4.3.3. Example : Cylinder

When we have a cylinder bounded by 3 surface patches

(1) $x^2+y^2 = R^2, \ 0 \le z \le R$

(2) $x^2+y^2 \le R^2, \ z = 0$

(3) $x^2+y^2 \le R^2, \ z = R$

The *C−space* obstacle of this cylinder will be bounded by a cylindrical patch, 2 planar patches, and 2 toroidal patches.

(C) $x^2+y^2 = (R+r)^2, \ 0 \le z \le R$

(P1) $x^2+y^2 \le R^2, \; z=-r$

(P2) $x^2+y^2 \le R^2, \; z=R+r$

(T1) $(x^2+y^2+z^2+R^2-r^2)^2=4R^2(x^2+y^2), \; z \le 0, \; x^2+y^2 \ge R^2$

(T2) $(x^2+y^2+(z-R)^2+R^2-r^2)^2=4R^2(x^2+y^2), \; z \ge R, \; x^2+y^2 \ge R^2$

5. Relation to Blending :

When the boundary of a convex obstacle B consists of finite number of smooth surface patches (i.e. piecewise smooth), the *C-space* obstacle B' of B due to a moving sphere of radius r, has a smooth boundary surface. This is because the faces of the *C-space* obstacle B', due to the way they are constructed, fill out the discontinuities of the directions of the surface normals on the edges and vertices of B. The normal directions of the smooth blending tubular faces over edges of B and the blending spherical patches over vertices of B, of the constructed *C-space* obstacle B' give all the missing outward normal directions bewteen two adjacent non-smoothly connecting boundary faces of B.

5.1. Example : Two cylinders of same radius intersecting at right angle

Consider two cylinders of the same radius R intersecting at right angles ($R > r$),

$$F : f(x,y,z) = R^2 - x^2 - z^2 = 0$$
$$G : g(x,y,z) = R^2 - y^2 - z^2 = 0$$

By solving these two equations simultaneously, we get

$$x^2 - y^2 = 0, \quad (x-y)(x+y) = 0, \text{ and } \; y = \pm x$$

Hence, the intersecting curves are on the planes $y=x$ and $y=-x$. The intersecting curves can be divided into 4 pieces and parameterized by t as follows

$$c_1(t) = (t, \; t, \; \sqrt{R^2-t^2}), \quad -R \le t \le R$$
$$c_2(t) = (t, \; t, -\sqrt{R^2-t^2}), \quad -R \le t \le R$$
$$c_3(t) = (t, -t, \; \sqrt{R^2-t^2}), \quad -R \le t \le R$$
$$c_4(t) = (t, -t, -\sqrt{R^2-t^2}), \quad -R \le t \le R$$

When we grow these cylinders inwards (we assume the obstacle is defined by $x^2+z^2 \ge R^2$ and $y^2+z^2 \ge R^2$), the piece of cylinders will shrink toward their relative axes. Since the intersecting curves are on both F and G, as a curve on F, it will shrink toward the y-axis and as a curve on G, it will shrink toward the x-axis. The gap between the boundary edges of the reduced cylinders F' and G' is filled out by the envelope of a sphere of radius r moving with its center on the intersecting curves of F and G, forming the blend of the reduced cylinders.

Now, assume that a sphere of radius r is located with its center at the intersecting curve $c_1(t) = (t, t, \sqrt{R^2 - t^2})$ for some $-R \le t \le R$. At this point, the normal direction of cylinder F is

$$\begin{aligned}
&\nabla f(t, t, \sqrt{R^2 - t^2}) \\
&= (-2x, 0, -2z) \quad \text{with } x = t,\ y = t,\ z = \sqrt{R^2 - t^2} \\
&= (-2t, 0, -2\sqrt{R^2 - t^2})
\end{aligned}$$

and the normal direction of cylinder B is

$$\begin{aligned}
&\nabla g(t, t, \sqrt{R^2 - t^2}) \\
&= (0, -2y, -2z) \quad \text{with } x = t,\ y = t,\ z = \sqrt{R^2 - t^2} \\
&= (0, -2t, -2\sqrt{R^2 - t^2})
\end{aligned}$$

The straight line $L_t(s) = ((-2t)\, s, 0, (-2\sqrt{R^2 - t^2}\,)\, s\,)$, $s \ge 0$ intersects with a sphere of radius r as follows

$$\begin{gathered}
(-2t)^2 s^2 + (-2\sqrt{R^2 - t^2}\,)^2 s^2 = r^2 \\
4t^2 s^2 + 4(R^2 - t^2) s^2 = r^2 \\
4R^2 s^2 = r^2 \\
s = \frac{r}{2R} \ge 0
\end{gathered}$$

Hence, $v = (\frac{-rt}{R}, 0, \frac{-r\sqrt{R^2 - t^2}}{R})$ is a point on the sphere of radius r in the direction of ∇f.

Similarly, $w = (0, \frac{-rt}{R,}\ \frac{-r\sqrt{R^2 - t^2}}{R})$ is another point on the sphere of radius r in the direction of ∇g.

The straight line connecting these two points v and w is given by

$$\begin{aligned}
P_t(s) &= \frac{1-s}{2}\left(\frac{-rt}{R}, 0, \frac{-r\sqrt{R^2 - t^2}}{R}\right) + \frac{1+s}{2}\left(0, \frac{-rt}{R}, \frac{-r\sqrt{R^2 - t^2}}{R}\right) \\
&= \left(\frac{r}{2R}t, 0, \frac{r}{2R}\sqrt{R^2 - t^2}\,\right)(s-1) + \left(0, \frac{-r}{2R}t, \frac{-r}{2R}\sqrt{R^2 - t^2}\,\right)(s+1) \\
&= \left(\frac{r}{2R}t, -\frac{r}{2R}t, 0\right)s + \left(\frac{-r}{2R}t, -\frac{r}{2R}t, -\frac{r}{R}\sqrt{R^2 - t^2}\,\right), \quad \text{for } -1 \le s \le 1
\end{aligned}$$

The geodesic curve segment connecting v and w on the sphere of radius r is

$$Q_t(s) = r \cdot \frac{P_t(s)}{||P_t(s)||}$$

Now, when we translate this curve so that it will be on the sphere of redius r with its center at the curve $c_1(t) = (t, t, \sqrt{R^2 - t^2})$, we have

$$H_1(s,t) = r \cdot \frac{P_t(s)}{||P_t(s)||} + (t, t, \sqrt{R^2 - t^2}) \qquad \text{for } -1 \le s \le 1,\ -R \le t \le R$$

This is a parametrization of the blend connecting the gap (due to the curve c_1) between the reduce cylinders of F and G smoothly. We can do similar work for the other 3 intersecting curves c_2, c_3, and c_4.

5.2. Three cylinders of same radius intersecting at right angles

When three cylinders of the same radius R intersect at right angles to each other, we have

$$F : f(x,y,z) = R^2 - x^2 - y^2 = 0$$
$$G : g(x,y,z) = R^2 - y^2 - z^2 = 0$$
$$H : h(x,y,z) = R^2 - z^2 - x^2 = 0$$

and

$$\nabla f(x,y,z) = (-2x, -2y, 0), \quad \nabla g(x,y,z) = (0, -2y, -2z), \quad \nabla h(x,y,z) = (-2x, 0, -2z).$$

When we solve three equations simultaneously, we have

$$x^2 - z^2 = 0, \quad z^2 + x^2 = R^2, \quad 2x^2 = R^2, \text{ and } \quad x = \pm\frac{R}{\sqrt{2}},$$

and similarly

$$x = \pm\frac{R}{\sqrt{2}}, \quad y = \pm\frac{R}{\sqrt{2}}, \quad z = \pm\frac{R}{\sqrt{2}}.$$

Hence, there are 8 vertices $(\pm\frac{R}{\sqrt{2}}, \pm\frac{R}{\sqrt{2}}, \pm\frac{R}{\sqrt{2}})$ which are common to F, G, and H.

Let us intersect the two cylinders $g(x,y,z) = 0$ & $h(x,y,z) = 0$ first, and then intersect the third cylinder $f(x,y,z) = 0$ with the intersection of the first two. The parts of the intersecting curves of the first two cylinders $g = 0$ & $h = 0$ which lie inside of the third cylinder $f = 0$ will be removed in the later intersection. So, we can construct 4 pieces of parametrized intersecting curves for each pair of cylinders and then take out the parts which are inside the other cylinder. We would still have 4 pieces of parametrized intersecting curves, but now of shorter length. We can construct total 12 pieces of parametrized intersecting curves in this way. These 12 pieces of intersecting lines can be formalized as follows.

$$c_1(t) = (\sqrt{R^2 - t^2}, \ \sqrt{R^2 - t^2}, \ t), \qquad -\frac{R}{\sqrt{2}} \le t \le \frac{R}{\sqrt{2}}$$
$$c_2(t) = (-\sqrt{R^2 - t^2}, \ \sqrt{R^2 - t^2}, \ t), \qquad -\frac{R}{\sqrt{2}} \le t \le \frac{R}{\sqrt{2}}$$
$$c_3(t) = (-\sqrt{R^2 - t^2}, \ -\sqrt{R^2 - t^2}, \ t), \qquad -\frac{R}{\sqrt{2}} \le t \le \frac{R}{\sqrt{2}}$$
$$c_4(t) = (\sqrt{R^2 - t^2}, \ -\sqrt{R^2 - t^2}, \ t), \qquad -\frac{R}{\sqrt{2}} \le t \le \frac{R}{\sqrt{2}}$$
$$c_5(t) = (\sqrt{R^2 - t^2}, \ t, \ \sqrt{R^2 - t^2}), \qquad -\frac{R}{\sqrt{2}} \le t \le \frac{R}{\sqrt{2}}$$

$$c_6(t) = (\ \sqrt{R^2-t^2},\ t,\ -\sqrt{R^2-t^2}), \qquad -\frac{R}{\sqrt{2}} \le t \le \frac{R}{\sqrt{2}}$$

$$c_7(t) = (-\sqrt{R^2-t^2},\ t,\ -\sqrt{R^2-t^2}), \qquad -\frac{R}{\sqrt{2}} \le t \le \frac{R}{\sqrt{2}}$$

$$c_8(t) = (-\sqrt{R^2-t^2},\ t,\ \sqrt{R^2-t^2}), \qquad -\frac{R}{\sqrt{2}} \le t \le \frac{R}{\sqrt{2}}$$

$$c_9(t) = (\ t,\ \sqrt{R^2-t^2},\ \sqrt{R^2-t^2}), \qquad -\frac{R}{\sqrt{2}} \le t \le \frac{R}{\sqrt{2}}$$

$$c_{10}(t) = (\ t,\ -\sqrt{R^2-t^2},\ \sqrt{R^2-t^2}), \qquad -\frac{R}{\sqrt{2}} \le t \le \frac{R}{\sqrt{2}}$$

$$c_{11}(t) = (\ t,\ -\sqrt{R^2-t^2},\ -\sqrt{R^2-t^2}), \qquad -\frac{R}{\sqrt{2}} \le t \le \frac{R}{\sqrt{2}}$$

$$c_{12}(t) = (\ t,\ \sqrt{R^2-t^2},\ -\sqrt{R^2-t^2}), \qquad -\frac{R}{\sqrt{2}} \le t \le \frac{R}{\sqrt{2}}$$

The growing of all these curves into smooth surface patches will be done exactly the same way as we did it in the case of two intersecting cylinders. A new situation in this problem is how to grow the common vertices of the three orthogonal cylinders.

At the point $(\frac{R}{\sqrt{2}}, \frac{R}{\sqrt{2}}, \frac{R}{\sqrt{2}})$, we have three normal directions, one for each cylinder.

$$\nabla f(\frac{R}{\sqrt{2}},\frac{R}{\sqrt{2}},\frac{R}{\sqrt{2}}) = (-\sqrt{2}\,R, -\sqrt{2}\,R, 0)$$

$$\nabla g(\frac{R}{\sqrt{2}},\frac{R}{\sqrt{2}},\frac{R}{\sqrt{2}}) = (0, -\sqrt{2}\,R, -\sqrt{2}\,R)$$

$$\nabla h(\frac{R}{\sqrt{2}},\frac{R}{\sqrt{2}},\frac{R}{\sqrt{2}}) = (-\sqrt{2}\,R, 0, -\sqrt{2}\,R)$$

Obviously, the three intersecting points of the sphere of radius r together with the straight half lines coming out of the origin toward

$$\nabla f(\frac{R}{\sqrt{2}},\frac{R}{\sqrt{2}},\frac{R}{\sqrt{2}}),\ \ \nabla g(\frac{R}{\sqrt{2}},\frac{R}{\sqrt{2}},\frac{R}{\sqrt{2}}),\ \text{ and }\ \nabla h(\frac{R}{\sqrt{2}},\frac{R}{\sqrt{2}},\frac{R}{\sqrt{2}})$$

are

$$(-\frac{r}{\sqrt{2}}, -\frac{r}{\sqrt{2}}, 0),\ \ (0, -\frac{r}{\sqrt{2}}, -\frac{r}{\sqrt{2}}),\ \text{ and }\ (-\frac{r}{\sqrt{2}}, 0, -\frac{r}{\sqrt{2}}).$$

These three points define a spherical face which blends the gap at the corner (due to the vertex $(\frac{R}{\sqrt{2}},\frac{R}{\sqrt{2}},\frac{R}{\sqrt{2}})$) among the reduced cylinders of F, G, and H, and the tubular patches grown from the edge curves c_1, c_5, and c_9. We can do similar work for the other 7 vertices.

5.3. Two cylinders of different radii intersecting at right angles

When we have two cylinders of different radii R_1 and R_2 ($R_1 > R_2$) intersecting at right angle, we may assume these are given by

$$F : f(x,y,z) = R_1^2 - x^2 - z^2 = 0$$
$$G : g(x,y,z) = R_2^2 - y^2 - z^2 = 0$$

and so

$$\nabla f(x,y,z) = (-2x,\ 0,\ -2z\ ),\quad \nabla g(x,y,z) = (0,\ -2y,\ -2z\ )$$

When we solve two equations simultaneously, we have

$$x^2 - y^2 = R_1^2 - R_2^2,\quad x^2 = y^2 + (R_1^2 - R_2^2), \text{ and } \quad x = \pm\sqrt{y^2 + (R_1^2 - R_2^2)}$$

The intersecting curves can be parametrized as follows

$$(\pm\sqrt{t^2 + (R_1^2 - R_2^2)}\,,\ t\,,\pm\sqrt{R_2^2 - t^2}),\quad -R_2 \le t \le R_2$$

There are four intersecting curves

$$\begin{aligned}
c_1(t) &= (\ \sqrt{t^2 + (R_1^2 - R_2^2)},\ t,\ \sqrt{R_2^2 - t^2}), & -R_2 \le t \le R_2 \\
c_2(t) &= (\ \sqrt{t^2 + (R_1^2 - R_2^2)},\ t, -\sqrt{R_2^2 - t^2}), & -R_2 \le t \le R_2 \\
c_3(t) &= (-\sqrt{t^2 + (R_1^2 - R_2^2)},\ t,\ \sqrt{R_2^2 - t^2}), & -R_2 \le t \le R_2 \\
c_4(t) &= (-\sqrt{t^2 + (R_1^2 - R_2^2)},\ t, -\sqrt{R_2^2 - t^2}), & -R_2 \le t \le R_2
\end{aligned}$$

Let's consider the curve

$$c_1(t) = (\sqrt{t^2 + (R_1^2 - R_2^2)},\ t,\ \sqrt{R_2^2 - t^2}),\quad -R_2 \le t \le R_2$$

On this curve, we have two normal directions

$$\nabla f(c_1(t)) = (-2\sqrt{t^2 + (R_1^2 - R_2^2)},\ 0,\ -2\sqrt{R_2^2 - t^2})$$
$$\nabla g(c_1(t)) = (0,\ -2t,\ -2\sqrt{R_2^2 - t^2}),\quad -R_2 \le t \le R_2$$

Now solve the following equation for $s > 0$,

$$\| s \cdot \nabla f \| = r$$
$$4s^2(t^2 + (R_1^2 - R_2^2)) + 4s^2(R_2^2 - t^2) = r^2$$
$$4R_1^2 s^2 = r^2$$
$$s = \frac{r}{2R_1} > 0$$

Similarly,

$$\| s \cdot \nabla g \| = r$$
$$4s^2t^2 + 4s^2(R_2^2 - t^2) = r^2$$
$$s = \frac{r}{2R_2} > 0$$

Hence, a point on the curve $c_1(t)$ will be grown into the geodesic curve connecting two points on the sphere of radius r

$$v = \left(\frac{-r}{R_1}\sqrt{t^2 + (R_1^2 - R_2^2)},\ 0,\ \frac{-r}{R_1}\sqrt{R_2^2 - t^2} \right) \text{ and } w = \left(0, -\frac{r}{R_2}t, -\frac{r}{R_2}\sqrt{R_2^2 - t^2} \right)$$

A straight line connecting these two points is given by

$$\begin{aligned} &P_t(s) \\ &= \frac{1-s}{2}\left(\frac{-r}{R_1}\sqrt{t^2 + (R_1^2 - R_2^2)},\ 0,\ \frac{-r}{R_1}\sqrt{R_2^2 - t^2} \right) + \frac{1+s}{2}\left(0,\ \frac{-r}{R_2}t,\ \frac{-r}{R_2}\sqrt{R_2^2 - t^2} \right) \\ &= \left(\frac{r}{2R_1}\sqrt{t^2 + (R_1^2 - R_2^2)},\ 0,\ \frac{r}{2R_1}\sqrt{R_2^2 - t^2}\right)(s-1) + \left(0,\ \frac{-r}{2R_2}t,\ \frac{-r}{2R_2}\sqrt{R_2^2 - t^2}\right)(s+1) \\ &\qquad \text{for } -1 \le s \le 1 \end{aligned}$$

Hence, the geodesic curve connecting v & w on the sphere of radius r is

$$Q_t(s) = r \cdot \frac{P_t(s)}{||P_t(s)||}$$

Now, when we translate this curve so that it will be on the sphere of radius r with center at

$$c_1(t) = (\sqrt{t^2 + (R_1^2 - R_2^2)},\ t,\ \sqrt{R_2^2 - t^2}), \qquad -R_2 \le t \le R_2$$

we have

$$H_1(s,t) = r \cdot \frac{P_t(s)}{||P_t(s)||} + (\sqrt{t^2 + (R_1^2 - R_2^2)},\ t,\ \sqrt{R_2^2 - t^2})$$

$$\text{for } -1 \le s \le 1,\ -R_2 \le t \le R_2$$

We can do similar work for the other 3 intersecting curves c_2, c_3, and c_4.

5.4. Two cylinders of same radius intersecting at skew-angle

When two cylinders intersect at skew angle, by rotating appropriately we can place one of the cylinders parallel to the y-axis. We may assume one of the cylinder is given by

$$F : f(x,y,z) = R^2 - x^2 - z^2 = 0$$

Another cylinder intersecting at skew angle can be given by rotating the following cylinder by angle ϕ counter-clockwise about z-axis.

$$y^2 + z^2 = R^2$$

The rotated cylinder is

$$G : R^2 - (y')^2 - z^2 = 0$$

When $x = r\cos\theta$ and $y = r\sin\theta$,

$$x' = r\cos(\theta-\phi) = r(\cos\theta\cos\phi + \sin\theta\sin\phi) = x\cos\phi + y\sin\phi$$

$$y' = r\sin(\theta-\phi) = r(\sin\theta\cos\phi - \cos\theta\sin\phi) = y\cos\phi - x\sin\phi$$

Hence, the rotated cylinder is

$$G : g(x,y,z) = R^2 - (y\ \cos\phi - x\ \sin\phi)^2 - z^2 = 0$$

When we solve the following simultaneous equations

$$x^2 + z^2 = R^2,\ \ (x\ \sin\phi - y\ \cos\phi)^2 + z^2 = R^2$$

we have

$$x^2 - (x\ \sin\phi - y\ \cos\phi)^2 = 0$$

$$x^2 - (x^2 \sin^2\phi - 2xy\ \sin\phi\ \cos\phi + y^2\cos^2\phi) = 0$$

$$(1 - \sin^2\phi)x^2 + \sin(2\phi)\ xy\ - \cos^2\phi\ y^2 = 0$$

$$(\cos^2)\ x^2 + (\sin2\phi)\ xy\ - \cos^2\phi\ y^2 = 0$$

We may assume $0 < \phi < \frac{\pi}{2}$ $(\therefore \cos^2\phi \neq 0)$, and so

$$x^2 + \frac{\sin2\phi}{\cos^2\phi}\ xy\ - y^2 = 0$$

$$x^2 + (2\tan\phi)\ xy\ - y^2 = 0$$

$$(x\ + (\tan\phi)y)^2 - (\tan^2\phi + 1)\ y^2 = 0$$

$$(x\ + (\tan\phi)y)^2 - \frac{1}{\cos^2\phi}\ y^2 = 0$$

$$y = \pm \cos\phi\ (x\ + (\tan\phi)\ y)$$

$$y = \pm ((\cos\phi)\ x\ + (\sin\phi)\ y\ )$$

Hence,

$$y = \frac{\cos\phi}{1-\sin\phi}\ x \qquad \text{------(1)}$$

or

$$y = \frac{-\cos\phi}{1+\sin\phi}\ x \qquad \text{------(2)}$$

In case (1), since $z^2 = R^2 - x^2$,

$$z = \pm\sqrt{R^2 - x^2},\ \ y = \frac{\cos\phi}{1-\sin\phi}\ x$$

Similarly, in case (2),

$$z = \pm\sqrt{R^2 - x^2},\ y = -\frac{\cos\phi}{1+\sin\phi}\ x$$

Hence, the intersecting curves can be divided into 4 parameteric curves

$$c_1(t) = (\ t,\ \frac{\cos\phi}{1-\sin\phi}\ t,\ \sqrt{R^2 - t^2}),\quad -R \le t \le R$$

$$c_2(t) = (\ t,\ \frac{\cos\phi}{1-\sin\phi}\ t,\ -\sqrt{R^2 - t^2}),\quad -R \le t \le R$$

$$c_3(t) = (\ t,\ -\frac{\cos\phi}{1+\sin\phi}\ t,\ \sqrt{R^2 - t^2}),\quad -R \le t \le R$$

$$c_4(t) = (\ t,\ -\frac{\cos\phi}{1+\sin\phi}\ t,\ -\sqrt{R^2 - t^2}),\quad -R \le t \le R$$

Now, when we grow the cylinders inwards by a sphere of radius r, the cylinder

$$x^2 + z^2 = R^2$$

will be grown into another cylinder

$$x^2 + z^2 = (R-r)^2$$

which is bounded by the 4 parametric curves

$$\bar{c}_1(t) = (\frac{R-r}{R} t, \frac{\cos\phi}{1-\sin\phi} t, \frac{R-r}{R} \sqrt{R^2-t^2}), \quad -R \le t \le R$$

$$\bar{c}_2(t) = (\frac{R-r}{R} t, \frac{\cos\phi}{1-\sin\phi} t, -\frac{R-r}{R} \sqrt{R^2-t^2}), \quad -R \le t \le R$$

$$\bar{c}_3(t) = (\frac{R-r}{R} t, \frac{-\cos\phi}{1+\sin\phi} t, \frac{R-r}{R} \sqrt{R^2-t^2}), \quad -R \le t \le R$$

$$\bar{c}_4(t) = (\frac{R-r}{R} t, \frac{-\cos\phi}{1+\sin\phi} t, -\frac{R-r}{R} \sqrt{R^2-t^2}), \quad -R \le t \le R$$

And, the other cylinder

$$(y')^2 + z^2 = (R-r)^2$$

or equivalently,

$$(y \ \cos\phi - x \ \sin\phi)^2 + z^2 = R^2$$

and the 4 boundary curves of this cylinder will be grown as follows.

First, rotate these curves by ϕ, clockwise (i.e. $-\phi$, counterclockwise), shrink it towards the x axis, and then rotate it back into the x' axis.

(A) Rotate by $-\phi$ counterclockwise

$$c_1^*(t) = (t \ \cos\phi + \frac{\sin\phi \ \cos\phi}{1-\sin\phi} t, \frac{\cos^2\phi}{1-\sin\phi} t - t \ \sin\phi, \sqrt{R^2-t^2})$$

$$= (\frac{\cos\phi}{1-\sin\phi} t, t, \sqrt{R^2-t^2}), \quad -R \le t \le R$$

$$c_2^*(t) = (\frac{\cos\phi}{1-\sin\phi} t, t, -\sqrt{R^2-t^2}), \quad -R \le t \le R$$

$$c_3^*(t) = (t \ \cos\phi - \frac{\sin\phi \ \cos\phi}{1+\sin\phi} t, -\frac{\cos^2\phi}{1+\sin\phi} t - t \ \sin\phi, \sqrt{R^2-t^2})$$

$$= (\frac{\cos\phi}{1+\sin\phi} t, -t, \sqrt{R^2-t^2}), \quad -R \le t \le R$$

$$c_4^*(t) = (\frac{\cos\phi}{1+\sin\phi} t, -t, -\sqrt{R^2-t^2}), \quad -R \le t \le R$$

(B) Now, when we shrink it towards the x-axis, we will have

$$c_1^{**}(t) = (\frac{\cos\phi}{1-\sin\phi} t, \frac{R-r}{R} t, \frac{R-r}{R} \sqrt{R^2-t^2}), \quad -R \le t \le R$$

$$c_2^{**}(t) = (\frac{\cos\phi}{1-\sin\phi} t, \frac{R-r}{R} t, -\frac{R-r}{R} \sqrt{R^2-t^2}), \quad -R \le t \le R$$

$$c_3^{**}(t) = \left(\frac{\cos\phi}{1+\sin\phi}\, t, -\frac{R-r}{R}\, t, \frac{R-r}{R}\sqrt{R^2-t^2} \right), \quad -R \le t \le R$$

$$c_4^{**}(t) = \left(\frac{\cos\phi}{1+\sin\phi}\, t, -\frac{R-r}{R}\, t, -\frac{R-r}{R}\sqrt{R^2-t^2} \right), \quad -R \le t \le R$$

(C) Finally, we need rotate these curves back into the original angle position. Hence, we have the following 4 curves on the second cylinder which was grown towards x'-axis.

$$\bar{\bar{c}}_1(t) = \left(\frac{\cos^2\phi}{1-\sin\phi}\, t - t\,\frac{R-r}{R}\sin\phi,\ \cos\phi\,\frac{R-r}{R}\, t + \frac{\sin\phi\cos\phi}{1-\sin\phi}\, t,\ \frac{R-r}{R}\sqrt{R^2-t^2} \right)$$

$$= \left((1+\frac{r}{R}\sin\phi)\, t,\ (\tan\phi+sec\phi-\frac{r}{R}\cos\phi)\, t,\ \frac{R-r}{R}\sqrt{R^2-t^2} \right)$$

$$\bar{\bar{c}}_2(t) = \left((1+\frac{r}{R}\sin\phi)\, t,\ (\tan\phi+sec\phi-\frac{r}{R}\cos\phi)\, t,\ -\frac{R-r}{R}\sqrt{R^2-t^2} \right)$$

$$\bar{\bar{c}}_3(t) = \left((1-\frac{r}{R}\sin\phi)\, t,\ (\tan\phi-sec\phi+\frac{r}{R}\cos\phi)\, t,\ \frac{R-r}{R}\sqrt{R^2-t^2} \right)$$

$$\bar{\bar{c}}_4(t) = \left((1-\frac{r}{R}\sin\phi)\, t,\ (\tan\phi-sec\phi+\frac{r}{R}\cos\phi)\, t,\ -\frac{R-r}{R}\sqrt{R^2-t^2} \right)$$

where $-R \le t \le R$

Now, we need to fill in the gaps between the curves $\bar{c}_i$ and $\bar{\bar{c}}_i$ for i = 1,2,3,4 by growing the points on the intersecting curves c_i (i = 1,2,3,4) into pieces of geodesic curves on the sphere. At a point $c_i(t)$ on the curve c_i, we can compute two directions (i.e. one toward the y-axis, one toward x'-axis)

Let's consider the curve

$$c_1(t) = \left(t, \frac{\cos\phi}{1-\sin\phi}\, t, \sqrt{R^2-t^2} \right), \quad -R \le t \le R$$

Since the cylinder along y-axis is given by

$$F : f(x,y,z) = R^2 - x^2 - z^2 = 0$$

we have

$$\nabla f(x,y,z) = (-2x, 0, -2z) \text{ with } x = t,\ z = \sqrt{R^2-t^2}$$
$$= (-2t, 0, -2\sqrt{R^2-t^2})$$

Hence, the unit normal vector on the curve $c_1(t)$ toward the y-axis is

$$n_1(t) = \left(\frac{-x}{\sqrt{x^2+z^2}}, 0, \frac{-z}{\sqrt{x^2+z^2}} \right) = \left(\frac{-t}{R}, 0, -\frac{\sqrt{R^2-t^2}}{R} \right)$$

The cylinder along x'-axis is given by

$$G : g(x,y,z) = R^2 - (x\sin\phi - y\cos\phi)^2 - z^2 = 0$$

and so

$$\nabla g = (-2\sin\phi\,(x\sin\phi - y\cos\phi),\ 2\cos\phi(x\sin\phi - y\cos\phi),\ -2z)$$

$$= \left(-2\sin\phi\,(t\sin\phi - \frac{\cos^2\phi}{1-\sin\phi}t),\ 2\cos\phi\,(t\sin\phi - \frac{\cos^2\phi}{1-\sin\phi}t),\ -2\sqrt{R^2-t^2} \right)$$

$$= ((2\sin\phi)t, (-2\cos\phi)t, -2\sqrt{R^2 - t^2})$$

The unit normal vector toward x'-axis is

$$n_2(t) = \frac{\nabla g}{||\nabla g||}$$
$$= (\frac{\sin\phi}{R} t, -\frac{\cos\phi}{R} t, -\frac{\sqrt{R^2 - t^2}}{R}), \quad -R \le t \le R$$

Now the straight line segment connecting the two vectors $r \cdot n_1(t)$ and $r \cdot n_2(t)$ on the sphere of radius r is

$$P_t(s) = \frac{1-s}{2} \cdot r \cdot n_1(t) + \frac{1+s}{2} \cdot r \cdot n_2(t), \quad -1 \le s \le 1$$

When we project it onto the sphere of radius r, we can get the following geodesic curve on the sphere of radius r

$$Q_t(s) = r \cdot \frac{P_t(s)}{||P_t(s)||}, \quad -1 \le s \le 1$$

Finally, we need to translate this geodesic curve by $c_1(t)$ to place it at a correct position

$$H_1(s,t) = Q_t(s) + c_1(t), \quad \text{where } -R \le t \le R, \ -1 \le s \le 1$$

This is a parametrization of the blend of the gap (due to the edge curve c_1) between the reduced two cylinders.

Similarly, we can construct $H_2(s,t), H_3(s,t), H_4(s,t)$ from $c_2(t), c_3(t), c_4(t)$.

5.5. Two cylinders of different radii intersecting at skew-angle

Let's consider two unequal radius cylinders intersecting at skew-angle. By the same way as before, we may assume that two cylinders are given as follows

$$F : f(x,y,z) = R_1^2 - x^2 - z^2 = 0 \qquad (\text{assume } R_1 > R_2)$$
$$G : g(x,y,z) = R_2^2 - (x\sin\phi - y\cos\phi)^2 - z^2 = 0$$

then

$$x^2 - (x\sin\phi - y\cos\phi)^2 = R_1^2 - R_2^2$$

and

$$(\cos^2\phi)\, x^2 + (\sin 2\phi)\, xy - (\cos^2\phi)\, y^2 = R_1^2 - R_2^2$$

We may assume that $0 < \phi < \frac{\pi}{2}$ (i.e. $\cos^2\phi \ne 0$), then

$$x^2 + (2\tan\phi)\, xy - y^2 = \frac{R_1^2 - R_2^2}{\cos^2\phi}$$
$$x^2 + (2\tan\phi)\, xy - y^2 = (R_1^2 - R_2^2)\sec^2\phi$$
$$y^2 - 2\,(\tan\phi\, x)\, y - x^2 + (R_1^2 - R_2^2)\sec^2\phi = 0$$

$$y = (\tan\phi)\, x \pm \sqrt{\tan^2\phi\, x^2 + x^2 - (R_1^2 - R_2^2)sec^2\phi}$$
$$= (\tan\phi)\, x \pm \sqrt{(1+\tan^2\phi)\, x^2 - (R_1^2 - R_2^2)sec^2\phi}$$
$$= (\tan\phi)\, x \pm \sqrt{sec^2\phi\, x^2 - (R_1^2 - R_2^2)\, sec^2\phi}$$
$$= (\tan\phi)\, x \pm sec\phi \sqrt{x^2 - (R_1^2 - R_2^2)}$$
$$\text{for } -\sqrt{R_1^2 - R_2^2} \le x \le \sqrt{R_1^2 - R_2^2}$$

Hence, we can get 4 parametrized intersecting curves as follows

$$c_1(t) = (\, t,\ (\tan\phi)\, t + (sec\phi) \sqrt{t^2 - (R_1^2 - R_2^2)},\ \sqrt{R_1^2 - t^2})$$
$$c_2(t) = (\, t,\ (\tan\phi)\, t + (sec\phi) \sqrt{t^2 - (R_1^2 - R_2^2)},\ -\sqrt{R_1^2 - t^2})$$
$$c_3(t) = (\, t,\ (\tan\phi)\, t - (sec\phi) \sqrt{t^2 - (R_1^2 - R_2^2)},\ \sqrt{R_1^2 - t^2})$$
$$c_4(t) = (\, t,\ (\tan\phi)\, t - (sec\phi) \sqrt{t^2 - (R_1^2 - R_2^2)},\ -\sqrt{R_1^2 - t^2})$$
$$\text{for } -\sqrt{R_1^2 - R_2^2} \le t \le \sqrt{R_1^2 - R_2^2}$$

The unit normal vector toward y-axis at $c_1(t)$ is

$$n_1(t) = (\, \frac{-x}{\sqrt{x^2 + z^2}},\ 0,\ \frac{-z}{\sqrt{x^2 + z^2}})$$
$$= (\, \frac{-t}{R_1},\ 0,\ \frac{-\sqrt{R_1^2 - t^2}}{R_1}), \quad \text{for } -\sqrt{R_1^2 - R_2^2} \le t \le \sqrt{R_1^2 - R_2^2}$$

And, since the cylinder parallel to x'-axis is given by

$$G : g(x,y,z) = R_2^2 - (x \sin\phi - y \cos\phi)^2 - z^2 = 0,$$

at $c_1(t)$, we have

$$\nabla g = (\, -2 \sin\phi\ (x \sin\phi - y \cos\phi),\ 2 \cos\phi\ (x \sin\phi - y \cos\phi),\ -2z)$$
$$=$$
$$(-2\sin\phi\ (t\sin\phi - t\sin\phi - \sqrt{t^2 - (R_1^2 - R_2^2)}\,)),\ 2\cos\phi\ (t\sin\phi - t\sin\phi - \sqrt{t^2 - (R_1^2 - R_2^2)}),\ -2\sqrt{R_1^2 - t^2}\,)$$
$$= (\, 2 \sin\phi \sqrt{t^2 - (R_1^2 - R_2^2)},\ -2 \cos\phi \sqrt{t^2 - (R_1^2 - R_2^2)},\ -2 \sqrt{R_1^2 - t^2})$$

and the unit normal vector toward x'-axis at $c_1(t)$ is

$$n_2(t) = \frac{\nabla g}{||\nabla g||}$$
$$= (\, \frac{\sin\phi}{R_2} \sqrt{t^2 - (R_1^2 - R_2^2)},\ \frac{-\cos\phi}{R_2} \sqrt{t^2 - (R_1^2 - R_2^2)},\ -\frac{1}{R_2} \sqrt{R_1^2 - t^2}\,)$$

The rest of work is exactly the same as before.

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