Generation of Correlated Rayleigh Fading Envelopes for Spread Spectrum Applications

Balasubramaniam Natarajan, Carl R. Nassar, *Member, IEEE*, and V. Chandrasekhar

***Abstract*—In this letter, a procedure for generating N Rayleigh fading envelopes with *any desired covariance matrix* is given. This method, numerical in nature, enables researchers to simulate correlated fading envelopes, for use in: 1) the study of the impact of correlation on diversity system performance and 2) the study of multicarrier CDMA (MC-CDMA), where the number of carriers notably exceeds the degree of system diversity.**

***Index Terms*—Correlation, diversity method, fading, multipath channels, spread spectrum communications.**

I. Introduction

RECENTLY, multiple carrier spread spectrum schemes, such as MC-CDMA (multicarrier-code division multiple access) have emerged as a powerful alternative to DS-CDMA (direct sequence CDMA) [1]. In MC-CDMA, each data symbol is transmitted simultaneously over N narrowband subcarriers, where each subcarrier is encoded with a +1 or −1 (as determined by a PN code). The processing gain and the number of users supported by MC-CDMA correspond to the number of carriers employed. To support, e.g., $K = 128$ users, MC-CDMA requires $N = 128$ carriers.

CDMA systems spread the transmitted signal over a wide bandwidth, typically resulting in a threefold to fourfold diversity gain. Consequently in MC-CDMA, with, e.g., $N = 128$ carriers in a bandwidth with only fourfold diversity gain, significant correlation exists among subcarrier fades.

Typically, researchers simulating CDMA systems with diversity assume that the signal fades are uncorrelated with one another, due to a lack of the existence of a simple procedure for generating fading envelopes with desired cross correlation. This leads to simulations which do not represent the physical reality of MC-CDMA communication channels.

Recently, in [2], a new method to generate two equal power correlated fades with a specific correlation coefficient was provided. Additionally, Ziegler [3] eloquently addresses the problem of generating correlated Rayleigh random variables in his Ph.D. dissertation. In this paper, we extend the work of [2] and [3] and provide a simple numerical method to generate N $(N \geq 2)$ correlated fading envelopes with an *arbitrary* covariance matrix. This enables researchers studying multicarrier spread spectrum (with N carriers) to accurately simulate the transmission environment. Section II discusses the theory underlying this new method and Section III introduces a stepwise algorithm for generating the Rayleigh fading envelopes.

Manuscript received March 18, 1999. The associate editor coordinating the review of this letter and approving it for publication was Prof. A. Haimovich.

The authors are with the Department of Electrical and Computer Engineering, Colorado State University, Fort Collins, CO 80523-1373 USA (e-mail: nbalsu@engr.colostate.edu).

Publisher Item Identifier S 1089-7798(00)01187-X.

II. Theory

It is well known that Rayleigh random variables are closely related to complex Gaussian random variables. To illustrate this point, consider N complex Gaussian signals $(v_1, v_2, \cdots, v_N)$ where

$$v_i = x_i + jy_i \tag{1}$$

and x_i, y_i are independent zero mean Gaussian random variables with variance $(1/2)\sigma_g^2$. The envelopes of $(v_1, v_2, \cdots, v_N)$, labeled $(r_1, r_2, \cdots, r_N)$, are Rayleigh distributed and correspond to

$$r_i = |v_i| = \sqrt{x_i^2 + y_i^2}. \tag{2}$$

Assume we want to generate N Rayleigh envelopes $(r_1, r_2, \cdots, r_N)$ with a normalized covariance matrix

$$\boldsymbol{K_{r//}} = \begin{pmatrix} 1 & \rho_{r1,2} & \rho_{r1,3} & \cdots & \rho_{r1,N} \\ \rho_{r2,1} & 1 & \rho_{r2,3} & \cdots & \rho_{r2,N} \\ \vdots & & & & \\ \rho_{rN,1} & \rho_{rN,2} & \rho_{rN,3} & \cdots & 1 \end{pmatrix}. \tag{3}$$

The idea underlying this work is to generate N complex Gaussian random variables $(v_1, v_2, \cdots, v_N)$ with a corresponding normalized covariance matrix

$$\boldsymbol{K_g} = \begin{pmatrix} 1 & \rho_{g1,2} & \rho_{g1,3} & \cdots & \rho_{g1,N} \\ \rho_{g2,1} & 1 & \rho_{g2,3} & \cdots & \rho_{g2,N} \\ \vdots & & & & \\ \rho_{gN,1} & \rho_{gN,2} & \rho_{gN,3} & \cdots & 1 \end{pmatrix} \tag{4}$$

such that creation of the desired $(r_1, r_2, \cdots, r_N)$ results by taking absolute values of $(v_1, v_2, \cdots, v_N)$.

The generation of $\boldsymbol{K_g}$ given $\boldsymbol{K_r}$ is based on the following realization: the value $\rho_{ri,j}$, the (i, j)th element of $\boldsymbol{K_r}$ (representing the correlation coefficient between r_i and r_j), is determined exclusively by $|\rho_{gi,j}|$, the absolute value of the (i, j)th element of $\boldsymbol{K_g}$ (the magnitude of the correlation coefficient between n_i and n_j). For simplicity in presentation and algorithm implementation, we assume that the elements of $\boldsymbol{K_g}$ ($\rho_{gi,j}$'s) are real, and consequently the value $\rho_{ri,j}$ is determined exclusively by $\rho_{gi,j}$ (and vice-versa).

TABLE I
CORRELATION OF RAYLEIGH ENVELOPES $\rho_{ri,\,j}$ FOR VALUES OF COMPLEX GAUSSIAN CORRELATION $\rho_{gi,\,j}$

$\rho_{gi,j}$	$\rho_{ri,j}$	$\rho_{gi,j}$	$\rho_{ri,j}$
0.00	0.0000	0.50	0.2227
0.05	0.0047	0.55	0.2752
0.10	0.0056	0.60	0.3327
0.15	0.0243	0.65	0.4133
0.20	0.0337	0.70	0.4562
0.25	0.0559	0.75	0.5410
0.30	0.0737	0.80	0.6073
0.35	0.0965	0.85	0.6974
0.40	0.1494	0.90	0.7913
0.45	0.1836	0.95	0.9005

The exact analytical relationship relating $\rho_{ri,\,j}$ and $\rho_{gi,\,j}$ is given by [4]

$$\rho_{ri,\,j} = \frac{(1+|\rho_{gi,\,j}|)E_i\left(\dfrac{2\sqrt{|\rho_{gi,\,j}|}}{1+|\rho_{gi,\,j}|}\right) - \dfrac{\pi}{2}}{2-\dfrac{\pi}{2}} \tag{5}$$

where $E_i(\eta)$ denotes the complete elliptic integral of the second kind with modulus η. In [2], the lack of a direct closed form solution to the $\rho_{ri,\,j} - \rho_{gi,\,j}$ equation was resolved by the use of numerical methods, namely polynomial approximation (to evaluate the elliptical integrals).

While the results of [2] may be used to relate $\rho_{gi,\,j}$ to $\rho_{ri,\,j}$ (using an intermediate variable λ), a new method is provided here which offers an immediate, simple, one-to-one relationship between $\rho_{gi,\,j}$ and $\rho_{ri,\,j}$. Given $\rho_{gi,\,j}$, create pairs of complex Gaussian samples with correlation $\rho_{gi,\,j}$ from pairs of uncorrelated samples by employing Cholesky decomposition [5]; numerically evaluate the correlation among the envelopes of Gaussian samples with correlation coefficient $\rho_{gi,\,j}$—this provides $\rho_{ri,\,j}$. In this way, a look-up table of values relating $\rho_{gi,\,j}$ and $\rho_{ri,\,j}$ is available. Table I and Fig. 1 show a table and corresponding plot relating $\rho_{gi,\,j}$ and $\rho_{ri,\,j}$. The table and figure may be used as a quick reference to evaluate $\rho_{gi,\,j}$ given $\rho_{ri,\,j}$ (using linear interpolation).

In this way, all elements of $\boldsymbol{K_r}$ can be mapped to corresponding elements in $\boldsymbol{K_g}$. Once $\boldsymbol{K_g}$ is determined, the N correlated Gaussian samples $(v_1,\, v_2,\, \cdots,\, v_N)$ are generated by Cholesky decomposition [5], and the desired $(r_1,\, r_2,\, \cdots,\, r_N)$ are created by evaluating the envelopes of the N complex Gaussian samples.

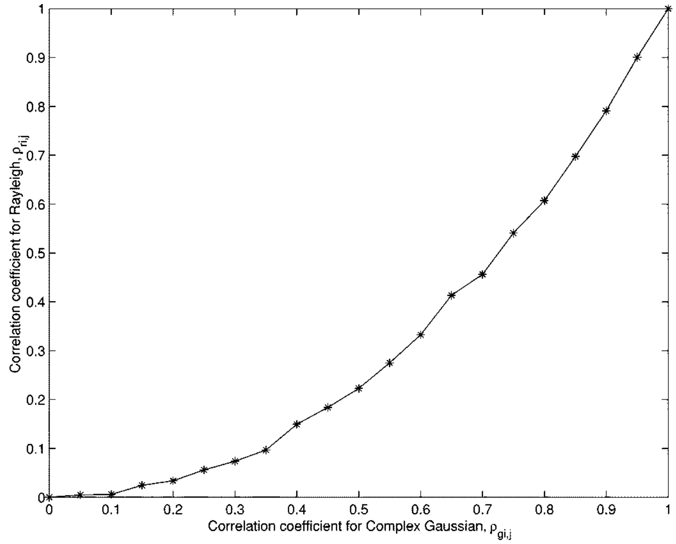

Fig. 1. Relationship between correlation coefficients of complex Gaussian $\rho_{gi,\,j}$ and Rayleigh $\rho_{ri,\,j}$.

III. ALGORITHM

The procedure for generating the N correlated Rayleigh fading signals is summarized as follows.

The starting point is the desired covariance matrix of the Rayleigh envelopes $(r_1,\, r_2,\, \cdots,\, r_N)$, given by

$$\hat{\boldsymbol{K}}_{\boldsymbol{r}} = \begin{pmatrix} \sigma_{r1}^2 & \hat{\rho}_{r1,\,2} & \hat{\rho}_{r1,\,3} & \cdots & \hat{\rho}_{r1,\,N} \\ \hat{\rho}_{r2,\,1} & \sigma_{r2}^2 & \hat{\rho}_{r2,\,3} & \cdots & \hat{\rho}_{r2,\,N} \\ \vdots & & & & \\ \hat{\rho}_{rN,\,1} & \hat{\rho}_{rN,\,2} & \hat{\rho}_{rN,\,3} & \cdots & \sigma_{rN}^2 \end{pmatrix}. \tag{6}$$

1) Normalize this matrix to create the normalized covariance matrix

$$\boldsymbol{K_r} = \begin{pmatrix} 1 & \rho_{r1,\,2} & \rho_{r1,\,3} & \cdots & \rho_{r1,\,N} \\ \rho_{r2,\,1} & 1 & \rho_{r2,\,3} & \cdots & \rho_{r2,\,N} \\ \vdots & & & & \\ \rho_{rN,\,1} & \rho_{rN,\,2} & \rho_{rN,\,3} & \cdots & 1 \end{pmatrix} \tag{7}$$

where $\rho_{ri,\,j} = \hat{\rho}_{ri,\,j} \big/ \sqrt{\sigma_{ri}^2 \sigma_{rj}^2}$.

2) For each cross correlation coefficient $\rho_{ri,\,j}$, compute the corresponding $\rho_{gi,\,j}$ by a) using Table I (and linear interpolation) or b) relating $\rho_{ri,\,j}$'s and $\rho_{gi,\,j}$'s as discussed in Section II.

3) Form the normalized covariance matrix of complex Gaussian samples:

$$\boldsymbol{K_g} = \begin{pmatrix} 1 & \rho_{g1,\,2} & \rho_{g1,\,3} & \cdots & \rho_{g1,\,N} \\ \rho_{g2,\,1} & 1 & \rho_{g2,\,3} & \cdots & \rho_{g1,\,N} \\ \vdots & & & & \\ \rho_{gN,\,1} & \rho_{gN,\,2} & \rho_{gN,\,3} & \cdots & 1 \end{pmatrix}. \tag{8}$$

4) Generate N uncorrelated complex Gaussian samples $\boldsymbol{V} = \{v_1,\, v_2,\, \cdots,\, v_N\}$ each with variance σ_g^2; then, determine the coloring matrix $\boldsymbol{L}$ corresponding to $\boldsymbol{K_g}$ (the coloring matrix $\boldsymbol{L}$ is the lower triangular matrix such that $\boldsymbol{L}\boldsymbol{L}^{\boldsymbol{T}} = \boldsymbol{K_g}$ where $\boldsymbol{L}^{\boldsymbol{T}}$ represents the transpose of

$\boldsymbol{L}$) and generate correlated complex Gaussian samples using $\boldsymbol{W} = \boldsymbol{LV}$.

5) The N envelopes of the Gaussian samples in $\boldsymbol{W}$ correspond to Rayleigh random variables $(r'_1, r'_2, \cdots, r'_N)$ with normalized covariance matrix $\boldsymbol{K_r}$ and equal variance [6]

$$\sigma_r^2 = \left(2 - \frac{\pi}{2}\right)\frac{1}{2}\sigma_g^2. \tag{9}$$

6) Create the desired Rayleigh envelopes $(r_1, r_2, \cdots, r_N)$ from the samples $(r'_1, r'_2, \cdots, r'_N)$, by evaluating $r_i = A_i \cdot r'_i$ where $A_i = \sigma_{ri}/\sigma_r$.

IV. Conclusions

This letter proposes a straightforward procedure for generating N correlated Rayleigh fading envelopes. The method circumvents the difficulties of determining an analytical relationship between the correlation coefficients of complex Gaussian and Rayleigh samples. This procedure will be immediately helpful in the analysis of spread spectrum multiple access schemes such as MC-CDMA.

References

[1] S. Hara and R. Prasad, “Overview of multi-carrier CDMA,” *IEEE Commun. Mag.*, vol. 35, no. 12, pp. 126–133, Dec. 1997.

[2] R. B. Ertel and J. H. Reed, “Generation of two equal power correlated Rayleigh fading envelopes,” *IEEE Commun. Lett.*, vol. 2, pp. 276–278, Oct. 1998.

[3] R. Ziegler, “Digital communication over fading and dispersive channels, with emphasis on mobile radio applications,” Ph.D. dissertation, Stanford Univ., Stanford, CA, 1991.

[4] W. C. Jakes, Ed., *Microwave Mobile Communications*, New York: IEEE Press, 1974.

[5] E. Kreyszig, *Advanced Engineering Mathematics*, 8th ed. New York, NY: Wiley , 1998.

[6] J. Proakis, *Digital Communications*, 3rd ed. New York, NY: McGraw-Hill, 1995.