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Generation of Oriented Matroids—A Graph Theoretical Approach*

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Abstract. We discuss methods for the generation of oriented matroids and of isomorphism classes of oriented matroids. Our methods are based on single element extensions and graph theoretical representations of oriented matroids, and all these methods work in general rank and for non-uniform and uniform oriented matroids as well. We consider two types of graphs, cocircuit graphs and tope graphs, and discuss the single element extensions in terms of localizations which can be viewed as partitions of the vertex sets of the graphs. Whereas localizations of the cocircuit graph are well characterized, there is no graph theoretical characterization known for localizations of the tope graph. In this paper we prove a connectedness property for tope graph localizations and use this for the design of algorithms for the generation of single element extensions by use of tope graphs. Furthermore, we discuss similar algorithms which use the cocircuit graph. The characterization of localizations of cocircuit graphs finally leads to a backtracking algorithm which is a simple and efficient method for the generation of single element extensions. We compare this method with a recent algorithm of Bokowski and Guedes de Oliveira for uniform oriented matroids.

1. Introduction

Oriented matroids (OMs) can be viewed as an axiomatic combinatorial abstraction of geometric structures such as real hyperplane arrangements, convex polytopes, or point configurations in the Euclidean space. The notion of OMs was introduced independently by Bland and Las Vergnas [6] and by Folkman and Lawrence [13]. There are several different (but equivalent) axiom systems and representations of OMs, and the theory of OMs has connections and applications to many areas of mathematics. For a comprehensive introduction to the theory of OMs we refer to the monograph of Björner et al. [4].

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In the following we use *finite sphere arrangements* as an illustration of OMs. A finite sphere arrangement $\mathcal{S} = \{S_e \mid e \in E\}$ is a collection of $(d-1)$-dimensional unit spheres on the d-dimensional unit sphere S^d in the Euclidean space $\mathbb{R}^{d+1}$, where every sphere S_e is oriented (i.e. has a $+$ side and a $-$ side). The sphere arrangement $\mathcal{S}$ induces a cell complex $\mathcal{W}$ on S^d. For every point x in S^d we define a *sign vector* $X \in \{+, 0, -\}^E$ by setting $X_e = 0$ if x is on S_e, otherwise $X_e = +$ (or $X_e = -$) if x is on the $+$ side (or $-$ side, respectively) of S_e; let $\mathcal{F}$ denote the set of all these sign vectors. Obviously there is a one-to-one correspondence between the faces in $\mathcal{W}$ and the sign vectors in $\mathcal{F}$. The pair $\mathcal{M} = (E, \mathcal{F})$ is called *a linear OM*. There are non-linear OMs; however, it is well known that every OM can be represented by an arrangement of topological spheres [4], [13].

In this paper we study the generation problem of OMs, the fundamental question of constructing all oriented matroids of a given rank r and a given size of the ground set E. Often one is not interested in generating OMs which are equivalent in some given sense, therefore only classes of OMs are considered, i.e. the problem is to generate exactly one representative of every equivalence class of OMs. Two types of classes of OMs are important in the following: the *reorientation class* and the *isomorphism class*. A reorientation of an OM is defined when for some elements in E all signs are replaced by their opposites (i.e. the orientation of the corresponding spheres S_e is reversed). Two OMs are called isomorphic if they are equal up to reorientation and renaming of the ground sets. Our main interest is to generate OMs up to isomorphism.

In any generation method, the choice of the underlying OM representation is of great importance. The representations which are discussed in this paper are based on graphs that are defined by the OMs. Consider again a sphere arrangement $\mathcal{S}$ with cell complex $\mathcal{W}$, and let $\mathcal{M} = (E, \mathcal{F})$ be the corresponding OM. Without loss of generality we assume that the normal vectors of the spheres S_e span $\mathbb{R}^{d+1}$. The sign vectors in $\mathcal{F}$ corresponding to 0-faces and d-faces are called *cocircuits* and *topes*, respectively. There are two fundamental graphs which are defined by $\mathcal{M}$, the *cocircuit graph* and the *tope graph* of $\mathcal{M}$, which are named according to their vertex sets: the cocircuit graph is the 1-skeleton of $\mathcal{M}$ (or $\mathcal{W}$), the tope graph is defined by the adjacency relation of the topes in $\mathcal{M}$ (which corresponds to the obvious adjacency relation of the d-faces in $\mathcal{W}$). For precise definitions see Section 2.

In the following we study the cocircuit graph and the tope graph of an OM and their *single element extensions*: Consider again a sphere arrangement with cell complex $\mathcal{W}$ with corresponding OM $\mathcal{M} = (E, \mathcal{F})$. If a new element $f \notin E$ is added, i.e. a new $(d-1)$-dimensional unit sphere S_f is introduced, this defines a new complex $\mathcal{W}'$ and a new OM $\mathcal{M}' = (E \cup \{f\}, \mathcal{F}')$, a single element extension of $\mathcal{M}$. The new element f partitions the set of cocircuits (or topes) into three parts, those which are on the $+$ side of f, those on the $-$ side of f, and those "cut" by f: in the cell complex $\mathcal{W}$ this corresponds to 0-faces (or d-faces) which are on the $+$ or $-$ side of f and 0-faces which are contained in S_f (d-faces which are divided by S_f into two new d-faces, respectively). Hence this single element extension defines a signature on the vertex set of the cocircuit (and tope) graph. If a signature comes from a topological abstraction of the linear extension as discussed above, then it is called a *localization*. As the set of cocircuits (or topes) defines the entire OM, one can prove that a localization determines a single element extension. Therefore we can generate all single element extensions of a given OM by generating all localizations of its cocircuit or tope graph; we use this approach for our methods.

The generation problem is very natural by itself as many researchers are interested in a complete set of problem instances (e.g. for testing conjectures on point configurations in $\mathbb{R}^d$). Furthermore, the study of methods for efficiently generating OMs leads to new results for OM representations. Techniques for listing OMs for small d and $|E|$ were studied, among others, by Bokowski, Sturmfels, and Guedes de Oliveira (e.g. [7]–[9]) using the chirotope axioms of OMs. They also proved by successful applications, e.g. to geometric embeddability problems, the usefulness of OM generation. However, it seems that the methods are designed primarily for the case of uniform OMs. The approach of this paper is based on graph theoretical representations of OMs (tope graphs and cocircuit graphs), and we discuss methods which work for general OMs (especially also non-uniform OMs). One of our methods can be considered as a variant of an algorithm of Bokowski and Guedes de Oliveira [7] in a dual setting; however, our representation leads to implementations which are able to handle easily any single element extension in general rank, for non-uniform and uniform OMs as well.

2. Definitions and Notations

We present the definitions and the notations used in this paper which were not introduced in Section 1. Some notions are defined again, extending their former meaning in the setting of the sphere model to the axiomatic of OMs as presented in the following.

The *zero support* of a sign vector $X \in \{+, 0, -\}^E$ is the set $X^0 := \{e \in E \mid X_e = 0\}$, and *the negative* $-X$ *of* X is defined by $(-X)_e := -X_e$ for $e \in E$; when only the sign of $e \in E$ is reversed, we write $\overline{e}X$. For two sign vectors $X, Y \in \{+, 0, -\}^E$ we say that X *conforms to* Y (denoted by $X \preceq Y$) if $X_e \neq 0$ implies $X_e = Y_e$. The *composition of* X *and* Y (denoted by $X \circ Y$) is the sign vector W with $W_e = Y_e$ for $e \in X^0$ and $W_e = X_e$ otherwise. An element $e \in E$ *separates* X *and* Y if $X_e = -Y_e \neq 0$; we denote the set of separating elements by $D(X, Y)$. For a given set $\mathcal{F}$ of sign vectors, we call $e \in E$ a *loop* if $X_e = Y_e$ for all $X, Y \in \mathcal{F}$, and two elements $e, f \in E$ are called *parallel* if $X_e = X_f$ for all $X \in \mathcal{F}$ or $X_e = -X_f$ for all $X \in \mathcal{F}$; clearly, parallelness is an equivalence relation. Finally, $\mathcal{F}$ is called *simple* if there are no loops and no parallel elements (i.e. all parallel classes have cardinality 1).

An OM $\mathcal{M}$ is a pair $(E, \mathcal{F})$ of a finite set E and a set $\mathcal{F} \subseteq \{+, 0, -\}^E$ of sign vectors (called *covectors*) for which the OM covector axioms (V1)–(V4) are valid:

- (V1) $\mathbf{0} \in \mathcal{F}$.
- (V2) $X \in \mathcal{F} \Rightarrow -X \in \mathcal{F}$.
- (V3) $X, Y \in \mathcal{F} \Rightarrow X \circ Y \in \mathcal{F}$.
- (V4) For all $X, Y \in \mathcal{F}$ and $e \in D(X, Y)$ there exists $Z \in \mathcal{F}$ such that $Z_e = 0$ and, for all $f \in E \backslash D(X, Y)$, $Z_f = (X \circ Y)_f$.

It is not difficult to see that these OM covector axioms hold for any OM $(E, \mathcal{F})$ as defined by sphere arrangements $\mathcal{S}$ in Section 1.

The set $\mathcal{F}$ of covectors ordered by the conformal relation $\preceq$, together with an additional artificial greatest element 1, forms a lattice $\hat{\mathcal{F}}$ which has the Jordan–Dedekind property. The *rank* of a covector X is defined as the height of X in $\hat{\mathcal{F}}$, and we define $\operatorname{rank}(\mathcal{M}) := \max_{X \in \mathcal{F}} \operatorname{rank}(X)$. The covectors in $\mathcal{F} \backslash \{\mathbf{0}\}$ of minimal (maximal) rank, i.e. of rank 1 (rank($\mathcal{M}$), respectively), are called *cocircuits* (*topes*). The set of cocircuits $\mathcal{C}$ determines $\mathcal{F}$ as the closure under composition. The set of topes $\mathcal{T}$ determines $\mathcal{F}$ by $\mathcal{F} = \{X \in \{+, 0, -\}^E \mid X \circ T \in \mathcal{T} \text{ for all } T \in \mathcal{T}\}$, which was first observed by A. Mandel (unpublished). An OM $\mathcal{M}$ is called *uniform* if the set of the zero supports of cocircuits is the set of all (rank($\mathcal{M}$) $- 1$)-subsets of E.

A *graph* $G = (V(G), E(G))$ is a pair of a finite set of *vertices* $V(G)$ and a set of *edges* $E(G)$ that are represented as unordered pairs of vertices. The (combinatorial) distance between two vertices $v, w \in V(G)$ is denoted by $d_G(v, w)$, and the diameter of G by $\text{diam}(G)$. *The cocircuit graph of an OM* $\mathcal{M}$ is a graph G whose vertices can be associated by a bijection $\mathcal{L}\colon V(G) \to \mathcal{C}$ to the cocircuits of $\mathcal{M}$ such that $\{v, w\}$ is an edge in $E(G)$ if and only if, for $V := \mathcal{L}(v)$ and $W := \mathcal{L}(w)$, $V \circ W = W \circ V$ and V and W are the only cocircuits conforming to $V \circ W$. *The tope graph of an OM* $\mathcal{M}$ is a graph G whose vertices can be associated by a bijection $\mathcal{L}\colon V(G) \to \mathcal{T}$ to the topes of $\mathcal{M}$ such that $\{v, w\}$ is an edge in $E(G)$ if and only if, for $V := \mathcal{L}(v)$ and $W := \mathcal{L}(w)$, $D(V, W)$ is a parallel class of E (or, equivalently, there is a covector whose rank is rank($\mathcal{M}$) $- 1$ and which conforms to V and W).

3. Tope Graphs and Single Element Extensions

In this section we consider tope graphs of OMs and their relation to single element extensions of OMs, i.e. we study localizations of tope graphs of OMs (see the Introduction and the definitions in Section 2). We restrict the discussion in this section to simple OMs as we are not interested in generating single element extensions by introducing loops or parallel elements: these extensions can be considered as being trivial.

The tope graph of a (simple) OM determines its isomorphism class [3], and there are efficient algorithms for computing a representative OM from the given tope graph [10]. Furthermore, it is possible to decide for a given graph in polynomial time whether it is a tope graph or not [15], [16]. So, isomorphism classes of simple OMs can be generated if it is possible to generate tope graphs or at least a (not too large) superset of graphs. Unfortunately, the known characterizations of OM tope sets (see, e.g. [17] and [12]) did not lead to a direct and simply checkable characterization of tope graphs. However, there are easily checkable necessary properties for tope graphs of OMs (see Lemma 3.1) and their localizations (see Lemma 3.3) [15]. We strengthen these necessary properties in Theorem 3.5, which will be important in the generation algorithms of OM isomorphism classes (Section 4).

We first rephrase the former definition of localizations of tope graphs: Consider two simple OMs $\mathcal{M} = (E, \mathcal{F})$ and $\mathcal{M}' = (E', \mathcal{F}')$ with tope sets $\mathcal{T}$ and $\mathcal{T}'$, respectively, where $E' = E \cup \{f\}$ for $f \notin E$. Furthermore, assume that $\mathcal{M} = \mathcal{M}' \setminus f$, i.e. $\mathcal{M}'$ is a single element extension of $\mathcal{M}$. Associating the tope graph G of $\mathcal{M}$ to $\mathcal{T}$ by $\mathcal{L}\colon V(G) \to \mathcal{T}$, the above single element extension defines a signature $\sigma\colon V(G) \to \{+, 0, -\}$ on the vertex set of G by

$$\sigma(v) := \begin{cases} + & \text{if, for } T \in \mathcal{T}', \quad (T_E = \mathcal{L}(v) \Rightarrow T_f = +), \\ - & \text{if, for } T \in \mathcal{T}', \quad (T_E = \mathcal{L}(v) \Rightarrow T_f = -), \\ 0 & \text{otherwise} \end{cases}$$

for $v \in V(G)$. As defined above, we call signatures that are defined by single element extensions of OMs *localizations*. A localization σ, together with a tope graph G and $\mathcal{L}\colon V(G) \to \mathcal{T} \subseteq \{+,-\}^E$, determines the extended tope set $\mathcal{T}'$ by

$$\mathcal{T}' := \{T \in \{+,-\}^{E\cup\{f\}} \mid \text{there exists } v \in V(G) \text{ s.t. } T_E = \mathcal{L}(v) \text{ and } \sigma(v) \in \{T_f, 0\}\}$$

for $f \notin E$ being a new element. The rank of the extended OM $\mathcal{M}'$ is the same as the rank of $\mathcal{M}$ unless $\sigma(v) = 0$ for all $v \in V(G)$ (then the rank increases by 1 as f is a coloop in $\mathcal{M}'$). Obviously, for any given OM tope graph G the property of a signature $\sigma\colon V(G) \to \{+,0,-\}$ being a localization of G or not is independent from the choice of the representative in the isomorphism class defined by G.

For a given tope graph G of an OM, a signature σ of G defines a partition on the vertex set $V(G)$ by $V^s := \{v \in V(G) \mid \sigma(v) = s\}$ for $s \in \{+,0,-\}$. Furthermore, set $V^\oplus = V^+ \cup V^0$ and $V^\ominus = V^- \cup V^0$, and let G^+, G^0, G^-, $G^\oplus$, and $G^\ominus$ denote the subgraphs of G induced by V^+, V^0, V^-, $V^\oplus$, and $V^\ominus$, respectively. Then the tope graph corresponding to $\mathcal{T}'$ is determined by G and a localization σ as a graph G' with vertex set

$$V(G') = \{v^+ \mid v \in V^\oplus\} \cup \{v^- \mid v \in V^\ominus\}$$

and edge set

$$\begin{aligned} E(G') \;=\; &\{\{v^+, v^-\} \mid v \in V^0\} \cup \{\{v^+, w^+\} \mid \{v,w\} \in E(G^\oplus)\} \\ &\cup \{\{v^-, w^-\} \mid \{v,w\} \in E(G^\ominus)\}. \end{aligned}$$

We now state some important properties of tope sets and tope graphs [15]:

Lemma 3.1. *Let G be the tope graph of a simple OM $\mathcal{M}$ and let $\mathcal{L}\colon V(G) \to \mathcal{T}$ be an associating bijection between the vertex set of G and the tope set of $\mathcal{M}$. Then*:

- (T1) *For $X, Y \in \mathcal{T}$, $X \neq Y$, there exists $e \in D(X,Y)$ such that ${}_{\overline{e}}X \in \mathcal{T}$.*
- (T2) *The length of any shortest path $x = u^0, \ldots, u^d = y$ in G is $d = |D(\mathcal{L}(x), \mathcal{L}(y))|$, and then $|D(\mathcal{L}(u^{i-1}), \mathcal{L}(u^i))| = 1$ for $i \in \{1, \ldots, d\}$.*
- (T3) *For every vertex $v \in V(G)$ there is a unique vertex $\overline{v} \in V(G)$ such that $d_G(v, \overline{v}) = \mathrm{diam}(G)$, and then $\mathcal{L}(\overline{v}) = -\mathcal{L}(v)$.*

Definition 3.2 (Antipode). Let G be the tope graph of an OM. For a vertex $v \in V(G)$ we call the vertex $\overline{v} \in V(G)$ determined by $d_G(v, \overline{v}) = \mathrm{diam}(G)$ *the antipode of* v.

A characterization of the localizations of a given tope graph is not known, but the following properties necessarily hold (see [15]):

Lemma 3.3. *Let G be the tope graph of an OM and let $\sigma\colon V(G) \to \{+,0,-\}$ be a localization of G. Then the following properties are valid*:

- (L1) $\sigma(\overline{v}) = -\sigma(v)$ *for all* $v \in V(G)$,
- (L2) $E(G) \cap (V^+ \times V^-) = \emptyset$, *and*

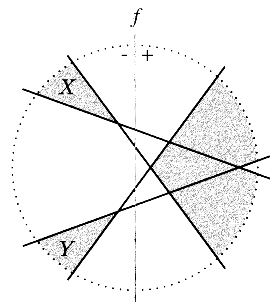


Fig. 1. Example for non-connectedness in the affine case.

(L3) $d_{G^{\oplus}}(v, w) = d_G(v, w)$ *for all* $v, w \in V^{\oplus}$, *and* $d_{G^{\ominus}}(v, w) = d_G(v, w)$ *for all* $v, w \in V^{\ominus}$.

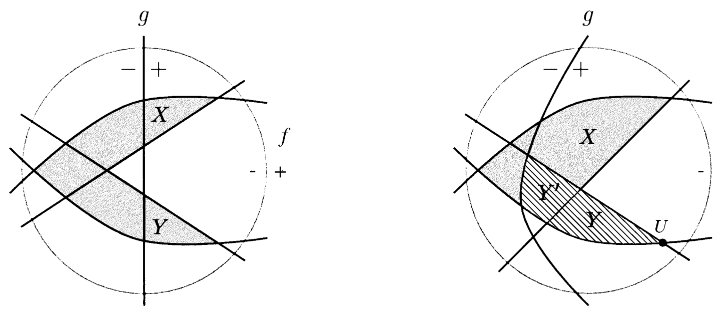
Definition 3.4 (Acycloidal Signature). Let G be the tope graph of an OM. We call a signature σ of G *an acycloidal signature of* G if (L1)–(L3) are satisfied.

We strengthen the necessary properties of localizations:

Theorem 3.5. *Let G be the tope graph of an OM and let* $\sigma\colon V(G) \to \{+, 0, -\}$ *be a localization of* G. *Then*:

(L4) G^+ (*and also* G^-) *is a connected subgraph of* G.

Before we prove this connectedness property (L4), we give some remarks. First, we show in Fig. 1 an example for the analogous *affine* case where the connectedness in the sense of (L4) is not valid (in the example the gray regions are those d-faces not cut by the new hyperplane f, and obviously X and Y are not connected on the $-$ side of f). Second, we give a sketch of the proof. Consider two regions X and Y which are not cut by the new element f and are on the same side of f, say the $-$ side. There exists an element $g \in E\backslash\{f\}$ that bounds X and does not separate X and Y; if we consider g as an *infinity element*, we may call X an unbounded region. There are two cases to consider: (i) Y is also an unbounded region and (ii) Y is not an unbounded region. The two cases are illustrated in Fig. 2 showing the $-$ side of f only; note that case (i), restricted to affine space (i.e. to the $+$ side of g), is exactly the example of Fig. 1. In case (i) we consider the contraction with respect to g and use a non-trivial inductive argument to prove that X and Y are connected in the sense of (L4). In case (ii) we show that Y is connected in the sense of (L4) to an unbounded region Y', which is known to be connected to X because of case (i). The unbounded region Y' is found using an OM program (see also Theorem 3.6 below) which has an optimal solution U. The solution U defines an unbounded cone (hatched with white lines in Fig 2) which contains regions that are all connected in the sense of (L4). We summarize the results on OM programming needed for the proof of Theorem 3.5 in the following theorem, which is essentially the strong duality theorem for OM programming [5], [13], [4]:



Case (i) Case (ii)

Fig. 2. The two cases in the proof of Theorem 3.5.

Theorem 3.6 [14]. *For any OM program $(\mathcal{M}, g, f)$, which is a triple of an OM $\mathcal{M} = (E, \mathcal{F})$ and two distinct elements $f, g \in E$, exactly one of the following three statements is valid*:

(i) *$(\mathcal{M}, g, f)$ is not feasible, i.e. there is no $Y \in \mathcal{F}$ with $Y_{E\setminus\{f,g\}} \geq 0$ and $Y_g = +$ (Y is called feasible for $(\mathcal{M}, g, f)$).*
(ii) *$(\mathcal{M}, g, f)$ is unbounded, i.e. $(\mathcal{M}, g, f)$ is feasible and there exists $Z \in \mathcal{F}$ with $Z_{E\setminus\{f,g\}} \geq 0$, $Z_g = 0$, and $Z_f = +$ (Z is called an unbounded augmenting direction for $(\mathcal{M}, g, f)$).*
(iii) *$(\mathcal{M}, g, f)$ has an optimal solution, i.e. there exists $U \in \mathcal{F}$ which is feasible for $(\mathcal{M}, g, f)$, and there is no $Z' \in \mathcal{F}$ with $Z'_g = 0$, $Z'_f = +$, and $Z'_e \geq 0$ for all $e \in E\setminus\{f, g\}$ with $U_e = 0$ (Z' is called an augmenting direction for U).*

Proof of Theorem 3.5. Let G be the tope graph of a (simple) OM and let $\sigma\colon V(G) \to \{+, 0, -\}$ be a localization of G which defines a single element extension $\mathcal{M}$ with tope set $\mathcal{T} \subseteq \{+, -\}^E$ with new element $f \in E$; $\mathcal{M}$ is then also a simple OM. In order to show that the subgraph G^- of G is connected (then the same claim for G^+ follows by the symmetry of (T3) and (L1) under negation), we prove the following (equivalent) statement:

(∗) For any two topes $X, Y \in \mathcal{T}^- := \{Z \in \mathcal{T} \mid Z_f = - \text{ and } {}_{\overline{f}}Z \notin \mathcal{T}\}$ there exists a sequence $X = Z^0, \ldots, Z^k = Y$ such that $Z^i \in \mathcal{T}^-$ for $i \in \{0, \ldots, k\}$ and $|D(Z^{i-1}, Z^i)| = 1$ for $i \in \{1, \ldots, k\}$.

The proof of (∗) is by induction in the rank of $\mathcal{M}$. For some small rank r, say $r \leq 2$, (∗) is obviously true. We consider $\mathcal{M}$ with $\operatorname{rank}(\mathcal{M}) \geq 3$. If $\mathcal{T}^- = \emptyset$, then the claim is trivially true, so assume $\mathcal{T}^- \neq \emptyset$. Let $X, Y \in \mathcal{T}^-$. Then $X_f = Y_f = -$ implies $X \neq -Y$, and by (T1) there exists $g \in D(X, -Y) = E\setminus D(X, Y)$ such that ${}_{\overline{g}}X \in \mathcal{T}$. $X \in \mathcal{T}^-$ implies $g \neq f$. Obviously $X_g = Y_g \neq 0$, and without loss of generality assume $X_g = Y_g = +$.

(i) If ${}_{\overline{g}}Y \in \mathcal{T}$: Consider the contraction minor $\mathcal{M}/g$ (i.e. the contraction of $\mathcal{M}$ to faces which contain g in the zero support) which is a (not necessarily simple) OM whose rank is $\operatorname{rank}(\mathcal{M}) - 1$. Denote by $\tilde{\mathcal{M}}$ a simplification of $\mathcal{M}/g$ where the parallel class containing f is represented by f. Note that $X_{E\setminus\{g\}} \in \mathcal{M}/g$ and

$Y_{E\setminus\{g\}} \in \mathcal{M}/g$, and denote by $\tilde{X}$ and $\tilde{Y}$ their images in $\tilde{\mathcal{M}}$, then $\tilde{X}, \tilde{Y} \in \tilde{\mathcal{T}}^-$, where $\tilde{\mathcal{T}}^-$ is defined for $\tilde{\mathcal{M}}$ as $\mathcal{T}^-$ for $\mathcal{M}$. By induction, there exists a sequence $\tilde{X} = \tilde{U}^0, \ldots, \tilde{U}^k = \tilde{Y}$ in $\tilde{\mathcal{T}}^-$ such that $|D(\tilde{U}^{i-1}, \tilde{U}^i)| = 1$ for $i \in \{1, \ldots, k\}$. Consider $i \in \{0, \ldots, k\}$: $\tilde{U}^i \in \tilde{\mathcal{T}}^-$ implies that there exist $U^i \in \mathcal{T}$ such that $U^i_g = +$ and ${}_{\overline{g}}U^i \in \mathcal{T}$, where $\tilde{U}^i$ is the image of $U^i_{E\setminus\{g\}}$ in $\tilde{\mathcal{M}}$; furthermore, $U^i_f = -$, and at most one of ${}_{\overline{f}}U^i$ and ${}_{\overline{fg}}U^i$ is in $\mathcal{T}$, i.e. at least one of U^i and ${}_{\overline{g}}U^i$ is in $\mathcal{T}^-$. We define $\hat{U}^i := U^i$ if $U^i \in \mathcal{T}^-$, otherwise $\hat{U}^i := {}_{\overline{g}}U^i \in \mathcal{T}^-$. Since $\hat{U}^0 = X$ and $\hat{U}^k = Y$, it remains to show that $\hat{U}^{i-1}$ and $\hat{U}^i$ are connected within $\mathcal{T}^-$ for all $i \in \{1, \ldots, k\}$ in the sense of $(*)$.

Let be $i \in \{1, \ldots, k\}$. By (T2) there exist two sequences $U^{i-1} = V^0, \ldots, V^d = U^i$ and ${}_{\overline{g}}U^{i-1} = W^0, \ldots, W^d = {}_{\overline{g}}U^i$ with $|D(V^{j-1}, V^j)| = |D(W^{j-1}, W^j)| = 1$ for all $j \in \{1, \ldots, d\}$, where $d = |D(U^{i-1}, U^i)|$. If at least one of the two sequences for $i \in \{1, \ldots, k\}$ lies entirely in $\mathcal{T}^-$, the claim follows by combining all these sequences in $\mathcal{T}^-$. Assume that for some $i \in \{1, \ldots, k\}$ neither of the two sequences is entirely in $\mathcal{T}^-$, i.e. there exist $s, t \in \{0, \ldots, d\}$ such that $V' := {}_{\overline{f}}V^s \in \mathcal{T}$ and $W' := {}_{\overline{f}}W^t \in \mathcal{T}$. The OM axiom (V4) applied to V', W', and g implies that there exists $Z \in \mathcal{F}$ such that $Z_g = 0$ and $Z_e = (V' \circ W')_e$ for $e \notin D(V', W')$, i.e. $Z_e = V'_e = W'_e$ for $e \notin D(V', W')$, especially $Z_f = +$. Note that $D := D(U^{i-1}, U^i)$ is a parallel class of $\mathcal{M}/g$, so $Z_D = 0$, $Z_D = \hat{U}^{i-1}_D$, or $Z_D = \hat{U}^i_D$, and with $D(V', W') \subseteq D \cup \{g\}$ it follows that $Z \circ \hat{U}^{i-1} = {}_{\overline{f}}\hat{U}^{i-1} \in \mathcal{T}$ or $Z \circ \hat{U}^i = {}_{\overline{f}}\hat{U}^i \in \mathcal{T}$, a contradiction.

(ii) If ${}_{\overline{g}}Y \notin \mathcal{T}$: We show that Y is connected within $\mathcal{T}^-$ in the sense of $(*)$ to some $Y' \in \mathcal{T}^-$ for which ${}_{\overline{g}}Y' \in \mathcal{T}$; then the claim follows from (i). Without loss of generality assume $Y_e = +$ for all $e \in E\setminus\{f\}$ (reorientation does not affect connectedness within $\mathcal{T}^-$). Consider the OM program $(\mathcal{M}, g, f)$. Since Y is feasible for $(\mathcal{M}, g, f)$, and since no unbounded augmenting direction $Z \in \mathcal{F}$ exists (otherwise $Z \circ Y = {}_{\overline{f}}Y \in \mathcal{T}$, a contradiction), there exists an optimal solution $U \in \mathcal{F}$ for $(\mathcal{M}, g, f)$; note that $U_{E\setminus\{f\}} \geq 0$, $U_g = +$, and $U_f \leq 0$ (since $U_f = +$ implies $U \circ Y = {}_{\overline{f}}Y \in \mathcal{T}$). Set $V := -U \circ Y \in \mathcal{T}$. By (T2) there exists a sequence $Y = W^0, \ldots, W^d = V \in \mathcal{T}$ such that $|D(W^{i-1}, W^i)| = 1$ for $i \in \{1, \ldots, d\}$, where $d = |D(Y, V)|$. Since $Y_g = +$ and $V_g = -U_g = -$, there exists $k \in \{1, \ldots, d\}$ such that $W^i_g = +$ for $i < k$ and $W^k_g = -$. Set $Y' := W^{k-1}$, then ${}_{\overline{g}}Y' = W^k \in \mathcal{T}$, and it remains to show that $W^i \in \mathcal{T}^-$ for $i \in \{1, \ldots, k-1\}$. Assume $W^i \notin \mathcal{T}^-$ for some $i \in \{1, \ldots, k-1\}$, i.e. there exists $W' \in \mathcal{T}$ such that $W'_{E\setminus\{f\}} = W^i_{E\setminus\{f\}}$ and $W'_f = +$. Apply the OM axiom (V4) to W', $-U$, and g: there exists $Z' \in \mathcal{F}$ such that $Z'_g = 0$ and $Z'_e = (W' \circ -U)_e$ for $e \notin D(W', -U)$, especially $Z'_f = +$, and, for all $e \neq f$ with $U_e = 0$, $V_e = Y_e = +$, so also $W'_e = +$ and $Z'_e = W'_e = +$, i.e. Z' is an augmenting direction for U, in contradiction to the optimality of U. □

Definition 3.7 (Weak and Strong Acycloidal Signature). Let G be the tope graph of an OM and let σ be a signature of G. We call σ *a weak acycloidal signature of* G if (L1), (L2), and (L4) are satisfied and *a strong acycloidal signature of* G if (L1)–(L4) are satisfied.

The new property (L4) which is proved in Theorem 3.5 is independent from (L1)–(L3): there are acycloidal signatures which are not strong acycloidal signatures. However, strong acycloidal signatures are not necessarily localizations (the smallest example is a signature of a tope graph of an OM $\mathcal{M} = (E, \mathcal{F})$ with $\text{rank}(\mathcal{M}) = 3$ and $|E| = 4$).

4. Generation of Tope Graphs

In this section we present an incremental method for the generation of the tope graphs of all OMs. Note again that the generation of tope graphs is equivalent to the generation of OMs up to isomorphism (see beginning of the previous section). We discuss first (in Sections 4.1 and 4.2) two algorithms which find for a given OM tope graph the tope graphs of the corresponding single element extensions. These algorithms can be used in the incremental method for tope graph generation which is described in Section 4.3.

4.1. *Generation of Extensions by Reverse Search*

Let G be the tope graph of some OM $\mathcal{M} = (E, \mathcal{F})$; the goal of this section is to find all tope graphs of single element extensions of $\mathcal{M}$ up to graph isomorphism (which is equivalent to finding all single element extensions up to OM isomorphism). Note that our method is working with graph G and not with $\mathcal{M}$. The main idea is to generate first all weak acycloidal signatures and then to test these signatures for being strong acycloidal signatures, finally for being localizations (again in polynomial time, see [15] and [16]). The tope graphs of the extended OMs are easily obtained from the localizations (see Section 3), and finally graph isomorphism checking leads to a set of representatives up to isomorphism.

The first step in our method is the generation of all weak acycloidal signatures of a given tope graph G. Property (L4) is essential for our method as it makes it possible to generate all weak acycloidal signatures of G without repetition. For this we modify a reverse search method for the generation of all connected subgraphs of a given graph [1]. Enumerate the vertices of the given tope graph G in an arbitrary way such that $V(G) = \{1, \dots, n\}$. Remember that every weak acycloidal signature σ defines a set $V^- := \{v \in V(G) \mid \sigma(v) = -\}$, and the subgraph G^- of G induced by the vertices in V^- is connected.

For the reverse search method we define a directed graph $\mathcal{G}$ as follows (in the language of the original reference [1] the directed edges of $\mathcal{G}$ define a local search function): The vertices of $\mathcal{G}$ are the weak acycloidal signatures of G; there is for every weak acycloidal signature σ with $V^- \neq \emptyset$ exactly one directed edge $(\sigma \to \tau) \in E(\mathcal{G})$, where τ is defined as follows: let V^- be defined by σ, and let $u \in V^-$ be the smallest vertex such that the subgraph of G induced by $V^- \backslash \{u\}$ remains connected (u obviously exists); then let τ be the signature with $\tau(w) = \sigma(w)$ for $w \in V(G) \backslash \{v, \overline{v}\}$ and $\tau(v) = \tau(\overline{v}) = 0$ (then τ is a weak acycloidal signature). There is a unique sink in $\mathcal{G}$, namely the signature with $\sigma(v) = 0$ for all $v \in V(G)$, and every vertex in $\mathcal{G}$ is connected to the sink. The search starts with the sink of $\mathcal{G}$ and exploits $\mathcal{G}$ by traversing the edges in reversed direction: by this all weak acycloidal signatures of G can be found without repetition. A description of

```
Input: A tope graph G.
Output: A list 𝒜 of all weak acycloidal signatures of G.

begin WeakAcycloidalSignaturesReverseSearch(G);
    determine all antipodes in G;
    let σ be the signature with σ(v) = 0 for all v ∈ V(G);
    𝒜 := {σ};  𝒜' := {σ};
    while 𝒜' ≠ ∅ do
        take any τ ∈ 𝒜' and remove τ from 𝒜';
        for all v ∈ V(G) with τ(v) = 0 do
            if there is no w ∈ V(G) which is adjacent to v with σ(w) = + or w = v̄ and
                    there is w ∈ V(G) which is adjacent to v with σ(w) = − then
                σ := τ;  σ(v) := −;  σ(v̄) := +;
                determine V⁻ from σ;
                find smallest u ∈ V⁻ s.t. subgraph induced by V⁻\{u} is connected;
                if u = v then  𝒜 := 𝒜 ∪ {σ};  𝒜' := 𝒜' ∪ {σ}  endif
            endif
        endfor
    endwhile;
    return 𝒜
end WeakAcycloidalSignaturesReverseSearch.
```

Fig. 3. Algorithm WeakAcycloidalSignaturesReverseSearch.

the algorithm WeakAcycloidalSignaturesReverseSearch is given in Fig. 3. Note that (different from the simple presentation here) it is not necessary in the reverse search method to store the output list (here, in $\mathcal{A}$); furthermore, the method is parallelizable.

At the beginning of this section we have described how every weak acycloidal signature can be tested, in polynomial time, whether it is a localization, and every localization defines the tope graph of the corresponding single element extension. It remains to check which of the graphs are isomorphic. For the isomorphism checking of tope graphs the special structure of tope graphs can be exploited heavily; e.g. using (T2) one can express isomorphisms of tope graphs with shortest paths between antipodes, which reduces isomorphism checking to the search for "equivalent" shortest paths between antipodes (we omit the details here).

4.2. *Avoiding Isomorphic Signatures*

In this section we discuss a method which is similar to the method of Section 4.1, we only replace the part of the generation of weak acycloidal signatures. The key observation used in the following is that two signatures σ and τ of a tope graph G lead to isomorphic extensions if there is a graph automorphism $\varphi \in \mathrm{Aut}(G)$ such that $\sigma = \tau \circ \varphi$, i.e. $\sigma(v) = \tau(\varphi(v))$ for all $v \in V(G)$; we call such signatures σ and τ *isomorphic*. Isomorphic signatures can be used for a more efficient isomorphism checking, and—as presented in the following—for a variant of the algorithm WeakAcycloidalSignatures-ReverseSearch which generates weak acycloidal signatures only up to isomorphism (in the sense defined above), i.e. exactly one representative of each isomorphism class is returned from the list of all weak acycloidal signatures. This new algorithm WEAKACYCLOIDALSIGNATURESUPTOISOMORPHISM does not use reverse search, but still can be more efficient than the reverse search method, as isomorphism checking will avoid the generation of many subtrees in the search tree.

As before, the generation of signatures starts with $\sigma\colon V(G) \to 0$, i.e. $V^- = \emptyset$, and then augments V^- by adding single vertices, but now not only with "minimal" vertices as in the reverse search method. We say that a signature σ is an *augmentation* of a weak acycloidal signature τ with respect to $v \in V(G)$ if σ is a weak acycloidal signature and $\sigma(w) = \tau(w)$ for all $w \in V(G)\backslash\{v, \overline{v}\}$, $\sigma(v) = -$, and $\tau(v) = 0$. The augmentations are generated with increasing cardinality $|V^-| = k$, and for every k only one representative of every isomorphism class is kept for further augmentations. This leads to an algorithm WEAKACYCLOIDALSIGNATURESUPTOISOMORPHISM as described in Fig. 4; the correctness follows from the following inductive argument:

Lemma 4.1. *Let G be the tope graph of an OM. Consider the set $\mathcal{A}_k$ of all weak acycloidal signatures of G with $|V^-| = k$, where $k \geq 0$ is an integer. Let $\mathcal{A}_k^*$ be a set containing exactly one representative of every isomorphism class of $\mathcal{A}_k$. Define $\mathcal{A}'_{k+1}$ as the set of all augmentations of signatures in $\mathcal{A}_k^*$, and let $\mathcal{A}_{k+1}^*$ be a set containing exactly one representative of every isomorphism class of $\mathcal{A}'_{k+1}$. Then $\mathcal{A}_{k+1}^*$ contains a representative of every isomorphism class of the set $\mathcal{A}_{k+1}$ of all weak acycloidal signatures of G with $|V^-| = k+1$.*

Proof. Let G, $\mathcal{A}_k$, $\mathcal{A}_k^*$, $\mathcal{A}'_{k+1}$, $\mathcal{A}_{k+1}^*$, and $\mathcal{A}_{k+1}$ be as described above. Consider any $\sigma \in \mathcal{A}_{k+1}$. We have to show that there exists a signature $\sigma^* \in \mathcal{A}_{k+1}^*$ which is isomorphic to σ. Take any $\tau \in \mathcal{A}_k$ such that σ is an augmentation of τ (obviously τ exists) with respect to some vertex $v \in V(G)$. Then there exists $\tau^* \in \mathcal{A}_k^*$ such that $\tau = \tau^* \circ \varphi$ for some $\varphi \in \mathrm{Aut}(G)$. As φ is a graph automorphism and all properties of weak acycloidal

```
Input: A tope graph G.
Output: A list A* of all weak acycloidal signatures of G up to isomorphism.

begin WeakAcycloidalSignaturesUpToIsomorphism(G);
    let σ be the signature with σ(v) = 0 for all v ∈ V(G);
    A* := {σ};   A*_0 := {σ};   k := 0;
    while A*_k ≠ ∅ do
        A'_{k+1} := the set of all augmentations of signatures in A*_k;
        A*_{k+1} := a set of representatives of the isomorphism classes of A'_{k+1};
        A* := A* ∪ A*_{k+1};
        k := k + 1
    endwhile;
    return A*
end WeakAcycloidalSignaturesUpToIsomorphism.
```

Fig. 4. Algorithm WEAKACYCLOIDALSIGNATURESUPTOISOMORPHISM.

signatures are preserved under graph automorphisms, there is $\sigma' \in \mathcal{A}'_{k+1}$, which is an augmentation of τ^* with respect to $\varphi(v)$, therefore $\sigma = \sigma' \circ \varphi$; σ and σ' are isomorphic. Since some signature $\sigma^* \in \mathcal{A}^*_{k+1}$ is isomorphic to σ', the claim follows. □

As stated above, we do not use reverse search for algorithm WeakAcycloidalSignaturesUpToIsomorphism, and the reason may be seen when considering the proof of Lemma 4.1: in a reverse search method the augmenting vertices have to satisfy a minimal property, so in the inductive argument both v and $\varphi(v)$ have to be minimal, which is not true in general. Still it may be possible that WeakAcycloidalSignaturesUpToIsomorphism can be combined with the reverse search method (e.g. using a special choice for the representatives of isomorphism classes).

We conclude this section with a remark on how both methods presented above may be slightly improved. When considering strong acycloidal signatures instead of weak acycloidal signatures, we may add an infeasibility check to the two algorithms WeakAcycloidalSignaturesReverseSearch and WeakAcycloidalSignaturesUpToIsomorphism: when for a signature σ there exist $v, w \in V^-$ such that $d_{G^\ominus}(v, w) > d_G(v, w)$, then neither σ nor any augmentations of σ will satisfy (L3), i.e. we discard such signatures in the algorithms (for the augmentations observe that $d_G(v, w)$ does not change and $d_{G^\ominus}(v, w)$ will not decrease since $V^\ominus$ becomes smaller as V^+ becomes larger).

4.3. *Incremental Generation of Tope Graphs*

The generation of all extended tope graphs of a given tope graph as described above can be used as part of a general method of generating all tope graphs of OMs incrementally, i.e. with increasing cardinality of E and increasing rank. This method starts with some initial tope graph, say the tope graph of a (simple) OM $\mathcal{M} = (E, \mathcal{F})$ with $\text{rank}(\mathcal{M}) = |E| = 2$. From this graph—which is a cycle of length 4—we generate all tope graphs of (simple) OMs with $\ell \geq 2$ elements and rank in $\{2, \ldots, \ell\}$. The graphs are generated with increasing ℓ, and for every ℓ with increasing rank; this can be modified, e.g. if only OMs of some given rank should be considered.

Let $G_{\ell,r}$ denote the set of all tope graphs of (simple) OMs on ℓ elements and of rank r. The set of all single element extensions of a $G \in G_{\ell,r}$ contains one graph in $G_{\ell+1,r+1}$ (this graph is generated by the localization $\sigma\colon V(G) \to 0$), the other extensions belong to $G_{\ell+1,r}$: It is clear from G and the localization to which set, $G_{\ell+1,r+1}$ or $G_{\ell+1,r}$, the extension belongs to. If all extensions of all graphs $G \in G_{\ell,r}$ for all $r \in \{2, \ldots, \ell\}$ are computed, then all the sets $G_{\ell+1,r}$ for all $r \in \{2, \ldots, \ell+1\}$ are found—however, every extension may be found in multiple ways, and we have to check for isomorphic graphs again. In the following we discuss how these multiplicities can be reduced.

In the method described above, every OM $\mathcal{M} = (E, \mathcal{F})$ is obtained as a single element extension of some deletion minor. Usually $\mathcal{M}$ has several (up to $|E|$) non-isomorphic deletion minors, but only one is needed to generate $\mathcal{M}$. We restrict our method to extensions of deletion minors with a minimal number of topes; this will eliminate many but not all multiplicities in the method. Furthermore—we describe this in the following—it can be checked from tope graphs and signatures whether the extension comes from a minor with a minimal number of topes, and this criterion will reduce the amount of enumeration of weak acycloidal signatures.

Consider an OM $\mathcal{M}$ and a deletion minor $\mathcal{M}\setminus f$, which defines a localization σ of the tope graph G of $\mathcal{M}\setminus f$. The number of topes of $\mathcal{M}\setminus f$ is minimal among all deletion minors of $\mathcal{M}$ if and only if the difference of the numbers of topes of $\mathcal{M}\setminus f$ and $\mathcal{M}$ is maximal, in which case we call σ *a maximal localization*. Maximal localizations of G are characterized as follows: The relation $\sim$ defined on $E(G)$ by $\{v, w\} \sim \{v', w'\}$ if $d_G(v, v') < d_G(v, w')$ and $d_G(w, w') < d_G(w, v')$ is an equivalence relation and leads to a partition of $E(G)$ into *element classes* (which correspond to the elements in the ground set of $\mathcal{M}\setminus f$, see also [15]). Then a localization σ of G is maximal if and only if, for every edge class $E^e \subseteq E(G)$,

$$\text{(M)} \qquad |V^0| \geq |E^e| + |E^e \cap (V^0 \times V^0)|.$$

If (M) is not valid for some weak acycloidal signature σ, then (M) is also violated for any augmentation of σ: an augmentation will decrease $|V^0|$ by 2 and $|E^e \cap (V^0 \times V^0)|$ by at most 2 (edges incident to a common vertex belong to different edge classes). Therefore signatures which violate (M) can be discarded in the enumeration algorithms, and by this the amount of enumeration is reduced considerably.

5. Cocircuit Graphs and Single Element Extensions

In this section we consider cocircuit graphs of OMs and their relation to single element extensions of OMs, i.e. we study localizations of cocircuit graphs of OMs (see the Introduction and the definitions in Section 2). Again we are interested in simple OMs and may restrict our discussion accordingly. In contrast to tope graphs, a cocircuit graph of an OM $\mathcal{M}$ does not characterize the isomorphism class of $\mathcal{M}$ [11], and there is also no characterization of cocircuit graphs known. Nevertheless, cocircuit graphs will be helpful, as shown in the following; a major benefit comes from a characterization of localizations of cocircuit graphs (see Theorem 5.2).

Localizations of cocircuit graphs were introduced in Section 1; we give here a more detailed discussion. Consider two OMs $\mathcal{M} = (E, \mathcal{F})$ and $\mathcal{M}' = (E', \mathcal{F}')$ with cocircuit sets $\mathcal{C}$ and $\mathcal{C}'$, respectively, where $E' = E \cup \{f\}$ for $f \notin E$. Furthermore, assume that $\mathcal{M} = \mathcal{M}'\setminus f$, i.e. $\mathcal{M}'$ is a single element extension of $\mathcal{M}$. Associating the cocircuit graph G of $\mathcal{M}$ to $\mathcal{C}$ by $\mathcal{L}\colon V(G) \to \mathcal{C}$, the above single element extension defines a signature $\sigma\colon V(G) \to \{+, 0, -\}$ on the vertex set of G by $\sigma(v) := X'_f$ for $v \in V(G)$, where $X' \in \mathcal{C}'$ is uniquely determined by $X'_E = \mathcal{L}(v) \in \mathcal{C}$. We call σ *a localization of G with respect to $\mathcal{M}$*. On the other hand consider the cocircuit graph G of an OM $\mathcal{M}$, again associated to the set of cocircuits $\mathcal{C}$ by $\mathcal{L}\colon V(G) \to \mathcal{C}$. Then a localization σ of G with respect to $\mathcal{M}$ determines the extended cocircuit set $\mathcal{C}'$ as the set of all sign vectors $X' \in \{+, 0, -\}^{E\cup\{f\}}$ for which either

- $X'_E = \mathcal{L}(v)$ and $X_f = \sigma(v)$ for some vertex $v \in V(G)$, or
- $X'_E = \mathcal{L}(v)\circ\mathcal{L}(w)$ and $X_f = 0$ for some edge $\{v, w\} \in E(G)$ with $\{\sigma(v), \sigma(w)\} = \{+, -\}$,

where $f \notin E$ is a new element. The rank of the extended OM $\mathcal{M}'$ is the same as the rank of $\mathcal{M}$. If $\sigma(v) = 0$ for all $v \in V(G)$, then f is a loop in $\mathcal{M}'$. If $\mathcal{M}$ is a simple OM, then $\mathcal{M}'$ is also simple unless

- $\sigma(v) = 0$ for all $v \in V(G)$ or
- there exists $e \in E$ s.t. $\sigma(v) = \mathcal{L}(v)_e$ for all $v \in V(G)$ or
- there exists $e \in E$ s.t. $\sigma(v) = -\mathcal{L}(v)_e$ for all $v \in V(G)$.

The cocircuit graph of the single element extension $\mathcal{M}'$ is determined by $\mathcal{C}'$, as any set of cocircuits determines the corresponding cocircuit graph. We briefly describe an algorithm which computes the cocircuit graph G for a given set of cocircuits $\mathcal{C} \subseteq \{+, 0, -\}^E$ in $O(|\mathcal{C}|^3|E|)$ elementary arithmetic steps as follows: The vertex set of G is a set $V(G)$ associated by a bijection $\mathcal{L}$ to $\mathcal{C}$. For every vertex $v \in V(G)$ consider the set

$$S(v) := \{(\mathcal{L}(v) \circ \mathcal{L}(w))^0 \subseteq E \mid \text{there exists } w \in V(G)\backslash\{v\} \text{ s.t. } D(\mathcal{L}(v), \mathcal{L}(w)) = \emptyset\},$$

then $\{v, w\} \subseteq V(G)$ is an edge of G if and only if $(\mathcal{L}(v) \circ \mathcal{L}(w))^0$ is maximal in $S(v)$.

The vertex set $V(G)$ of a cocircuit graph G is partitioned by a signature σ into V^+, V^0, and V^-, where $V^s := \{v \in V(G) \mid \sigma(v) = s\}$ for $s \in \{+, 0, -\}$; let G^+, G^0, and G^- denote the subgraphs of G induced by V^+, V^0, and V^-, respectively.

For the following discussion we have to introduce the notion of coline cycles:

Definition 5.1 (Coline Cycle). Let $\mathcal{M} = (E, \mathcal{F})$ be an OM with $\text{rank}(\mathcal{M}) \geq 2$ and let G be the cocircuit graph of $\mathcal{M}$ with associating bijection $\mathcal{L}\colon V(G) \to \mathcal{C}$. Let $\{v, w\} \in E(G)$ be an edge, then we call $(\mathcal{L}(v) \circ \mathcal{L}(w))^0 \subseteq E$ *the coline of* $\{v, w\}$. Let $U \subseteq E$ be a coline. The edges in $E(G)$ whose coline is U form a cycle $c(U)$ in G which we call *the coline cycle of* U.

The following characterization of localizations of cocircuit graphs is due to Las Vergnas [18], [4]:

Theorem 5.2. *Let G be the cocircuit graph of an OM $\mathcal{M}$, given with the set of all coline cycles of G, and let $\sigma\colon V(G) \to \{+, 0, -\}$ be a signature of G. Then σ is a localization of G with respect to $\mathcal{M}$ if and only if for every coline cycle c in G one of the following is valid*:

(I) $\sigma(v) = 0$ *for every vertex v in c.*
(II) *There are two vertices v and v' in c with $\sigma(v) = \sigma(v') = 0$ such that v and v' divide c into two paths c^+ and c^- of the same length which connect v and v', and, for every vertex w in c different from v and v', $\sigma(w) = +$ if w is in c^+ and $\sigma(w) = -$ if w is in c^-.*
(III) *Same as* (II) *except that $\sigma(v) = +$ and $\sigma(v') = -$.*

We refer to I–III as the three possible *types* of a coline cycle (see Fig. 5 for an illustration). It is not difficult to see that a single element extension of a uniform OM is again uniform if and only if the corresponding localization of the cocircuit graph has $V^0 = \emptyset$, i.e. every coline cycle has type III.

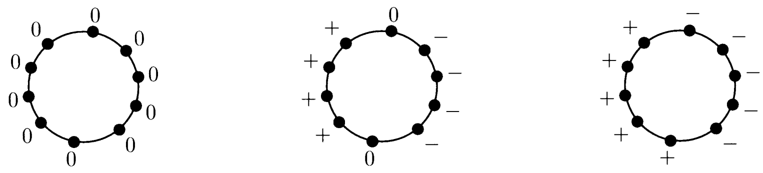


Fig. 5. The three possible types of a coline cycle.

We conclude this section with the following lemma which is important for some of the algorithms in the next section:

Lemma 5.3. *Let G be the cocircuit graph of some OM and let $\sigma\colon V(G) \to \{+, 0, -\}$ be a localization of G. Then G^+ (and also G^-) is a connected subgraph of G.*

For the proof of the lemma we need the following result from [11] and [2]:

Lemma 5.4. *Let $\mathcal{M} = (E, \mathcal{F})$ be an OM with cocircuit graph G and associating bijection $\mathcal{L}\colon V(G) \to \mathcal{C}$. For an arbitrary element $e \in E$ let V_e^+ denote the set of vertices with $\mathcal{L}(v)_e = +$. Then the subgraph of G induced by V_e^+ is connected.*

Proof of Lemma 5.3. Let G be the cocircuit graph of an OM $\mathcal{M} = (E, \mathcal{F})$ with associating bijection $\mathcal{L}\colon V(G) \to \mathcal{C}$, and let $\sigma\colon V(G) \to \{+, 0, -\}$ be a localization of G and V^+ corresponding to σ. The localization σ defines a single element extension $\mathcal{M}' = (E \cup \{f\}, \mathcal{F}')$ with new element f. Consider the cocircuit graph G' of $\mathcal{M}'$ with associating bijection $\mathcal{L}'\colon V(G') \to \mathcal{C}'$. Let V_f^+ denote the set of vertices with $\mathcal{L}'(v)_f = +$. By Lemma 5.4, the subgraph of G' induced by V_f^+ is connected. For any $v, w \in V^+$ there are uniquely determined vertices $v', w' \in V(G')$ such that $\mathcal{L}'(v')_E = \mathcal{L}(v)$ and $\mathcal{L}'(w')_E = \mathcal{L}(w)$, and then $\mathcal{L}'(v')_f = \mathcal{L}'(w')_f = +$, i.e. $v', w' \in V_f^+$. Hence there exists a path $v' = u_0', \ldots, u_k' = w'$ in G' connecting v' and w' with $u_i' \in V_f^+$ for $i \in \{0, \ldots, k\}$. For every u_i' there is a uniquely determined $u_i \in V(G)$ such that $\mathcal{L}(u_i) = \mathcal{L}'(u_i')_E$, and then $\sigma(u_i) = +$; furthermore, $\{u_{i-1}', u_i'\} \in E(G')$ implies $\{u_{i-1}, u_i\} \in E(G)$ for $i \in \{1, \ldots, k\}$: v, w are connected within V^+, hence G^+ is connected. The connectedness of G^- follows by symmetry. □

Definition 5.5 (Weak Localization). Let G be the cocircuit graph of an OM $\mathcal{M}$ with associating bijection $\mathcal{L}\colon V(G) \to \mathcal{C}$. For every vertex $v \in V(G)$ we call the vertex $\overline{v}$ determined by $\mathcal{L}(\overline{v}) = -\mathcal{L}(v)$ *the antipode of* v. We call a signature σ of G *a weak localization of* G if $\sigma(\overline{v}) = -\sigma(v)$ for every vertex $v \in V(G)$ and G^+ (and by symmetry also G^-) is connected.

It is clear from Theorem 5.2 and Lemma 5.3 that every localization of a cocircuit graph is also a weak localization, but obviously not every weak localization is a localization (fails already for rank 2 and a ground set of three elements).

6. Generation of Localizations of Cocircuit Graphs

In this section we describe how localizations of cocircuit graphs can be generated. This can be used as part of an incremental method similar to that described in Section 4. A difference in the methods comes from the fact that cocircuit graphs—in contrast to tope graphs—are not sufficient to characterize the isomorphism classes of OMs [11]; we discuss a possible solution of this problem below. Then we restrict further discussions to algorithms for the generation of all localizations in a given cocircuit graph, presenting two algorithms which are similar to the algorithms in Section 4 and a third method which is quite different.

We first address the question of how OMs, represented by sets of cocircuits, can be checked for being isomorphic. Consider an OM $\mathcal{M}$ with cocircuit graph G, then the associating bijection $\mathcal{L}\colon V(G) \to \mathcal{C}$ induces a *matroid label* $L\colon V(G) \to 2^E$ on the vertex set of G by $L(v) := \mathcal{L}(v)^0$. It is known [11], [2] that the cocircuit graph of an OM together with its matroid label determines the reorientation class of the OM, and therefore we know that two OMs $\mathcal{M} = (E, \mathcal{F})$ and $\tilde{\mathcal{M}} = (\tilde{E}, \tilde{\mathcal{F}})$ are isomorphic if and only if the corresponding cocircuit graphs G and $\tilde{G}$ with matroid labels L and $\tilde{L}$ are isomorphic in the following sense: there exist a graph isomorphism $\varphi\colon V(G) \to V(\tilde{G})$ and a bijection $\pi\colon E \to \tilde{E}$ such that $\pi(L(v)) = \tilde{L}(\varphi(v))$ for all $v \in V(G)$. This leads to the corresponding algorithmic solutions.

The rest of this section discusses three methods for the generation of localizations in a given cocircuit graph. The main idea of the first two methods is to generate first all weak localizations and then to test these signatures for being localizations using the characterization of Theorem 5.2. As for the generation of weak acycloidal signatures in tope graphs the property that the subgraphs G^+ and G^- are connected graphs is essential, and in fact this leads to algorithms WeakLocalizationsReverseSearch and WeakLocalizationsUpToIsomorphism which are similar to the algorithms discussed in Section 4, therefore we do not discuss these algorithms in detail. However, the characterization of localizations of cocircuit graphs as formulated in Theorem 5.2 offers a more structured approach to localizations than was possible for tope graphs: we may try to assign to every coline cycle in a given cocircuit graph a sign pattern of type I, II, or III in a consistent way. We do this using a simple backtracking method, which leads to a third algorithm LocalizationsPatternBacktrack as discussed in the following.

Let $\mathcal{C}$ be a set of cocircuits of an OM $\mathcal{M} = (E, \mathcal{F})$. As described in the previous section, we can compute in $O(n^3\ell)$ elementary arithmetic steps its cocircuit graph G and an associating bijection $\mathcal{L}\colon V(G) \to \mathcal{C}$, when setting $\ell := |E|$, $m := E(G)$, and $n := V(G) = |\mathcal{C}|$. Then compute the set $\{c_1, \ldots, c_s\}$ of all coline cycles of G, where every cycle c_i is represented as a list of vertices $\{v_1^i, \ldots, v_{m_i}^i\}$ which is ordered such that $\{v_{j-1}^i, v_j^i\}$ is an edge for all $j \in \{2, \ldots, m_i\}$, where m_i is the length of coline cycle c_i. This computation costs at most $O(mn\ell)$, i.e. not more than $O(n^3\ell)$ (note that $\sum m_i = m$ and $s \le m \le n^2$). For a signature σ of G let σ_i denote the restriction of σ to the vertex set of cycle c_i. Theorem 5.2 implies that σ_i has one of three types, more precisely one of $2m_i + 1$ *patterns*, which we encode in a number $p_i \in \{0, \ldots, 2m_i\}$ as follows (set $v_{m_i+1}^i := v_1^i$):

$p_i = 0$	σ_i is of type I
$p_i = 2j$ for $j \in \{1, \dots, m_i\}$	σ_i is of type II and $\sigma(v^i_j) = 0$, $\sigma(v^i_{j+1}) = +$
$p_i = 2j - 1$ for $j \in \{1, \dots, m_i\}$	σ_i is of type III and $\sigma(v^i_j) = -$, $\sigma(v^i_{j+1}) = +$

Our algorithm will set all p_i (and by this σ_i) for $i \in \{1, \dots, s\}$ in a consistent way, i.e. such that for every vertex $v \in V(G)$ the sign of $\sigma_i(v)$ is the same for all colines i which contain v. Assume that for a set of indices $I \subseteq \{1, \dots, s\}$ we have chosen p_i for all $i \in I$ (in a consistent way), and it remains to choose p_i for $i \notin I$. Obviously, the patterns p_i for $i \in I$ restrict the possibilities for the remaining choices. Consider $i \notin I$: For some of the vertices in the coline cycle c_i the signs may be determined by previously fixed patterns of coline cycles which intersect c_i, and therefore only some (or possibly none) of the $2m_i + 1$ patterns remain. We call these directly computable restrictions *the first-order consequences implied by p_i for $i \in I$*. These first-order consequences will usually determine the signs of vertices on cycles c_i with $i \notin I$ which were not set before, and these new signs imply further restrictions for the p_i for $i \notin I$, and so on. The computation of implied restrictions can be continued recursively and will finally lead to what we call *the second-order consequences implied by p_i for $i \in I$*. Although the second-order consequences are important in practice, we simplify the following discussion of our algorithm by restricting to first-order consequences.

We describe in the following an algorithm LOCALIZATIONSPATTERNBACKTRACK which serves as a concrete variant of our method. This algorithm is quite simple and rather efficient, but it can be improved (e.g. using second-order consequences or more sophisticated data structures). We assume that all coline cycles c_i of the given cocircuit graph G have been computed as described above. The goal is to enumerate all localizations of G by enumerating all consistent choices $(p_1, \dots, p_s)$ with $p_i \in \{0, \dots, 2m_i\}$. Consider $I \subseteq \{1, \dots, s\}$ as a set of indices for which the corresponding p_i have been fixed (in the beginning $I = \emptyset$). The first-order consequences implied by p_i for $i \in I$ restrict the possible choices of every p_i with $i \notin I$ to one of the following cases:

- (P1) All $2m_i + 1$ possibilities.
- (P2) A range $[p, p'] \subseteq \{1, \dots, 2m_i\}$ of possibilities for $p, p' \in \{1, \dots, 2m_i\}$ with p, p' odd, where $[p, p'] := \{p, \dots, p'\}$ if $p \leq p'$ and $[p, p'] := \{p', \dots, 2m_i, 1, \dots p\}$ otherwise.
- (P3) The choice is one of $0, 2j, 2j + m_i$ for $j \in \{1, \dots, m_i/2\}$.
- (P4) The only choice is $2j$ for $j \in \{1, \dots, m_i\}$.
- (P5) The only choice is $p_i = 0$.
- (P6) There is no feasible choice.

An important element in the following algorithm is the augmentation of the set I of fixed patterns by an additional element i^*; then the information of the possible choices has to be updated. For this a matrix A of size $s \times s$ is computed (once at the beginning of the algorithm, which will cost at most $O(ns^2)$ operations) such that $A_{ii^*} = 0$ if c_i and c_{i^*} have no vertex in common or $i = i^*$, otherwise $A_{ii^*} = j > 0$ such that v^i_j is on c_{i^*}. We call *A a coline adjacency matrix*. It is not difficult to see that then an update of the first-order consequences from I to $I \cup \{i^*\}$ needs for every $i \in \{1, \dots, s\}$ only a

Global Data: $s, m_1, \ldots, m_s$, a coline adjacency matrix A (see above).
Input: $I \subseteq \{1, \ldots, s\}$; $\tilde{p}_1, \ldots, \tilde{p}_s$ such that $\tilde{p}_i = p_i$ for $i \in I$ and $\tilde{p}_i$ for $i \notin I$ contains the information which patterns are possible with respect to the first-order consequences implied by p_i for $i \in I$.
Output: is generated whenever a new localization is found.

begin LocalizationsPatternBacktrack(I; $\tilde{p}_1, \ldots, \tilde{p}_s$);
  if some $\tilde{p}_i$ indicates that no choice p_i is feasible **then return**
  else if $I = \{1, \ldots, s\}$ **then**
    output the localization defined by $\tilde{p}_1, \ldots, \tilde{p}_s$;
    return
  else if $I = \emptyset$ **then** choose i^* such that m_{i^*} is maximal
  else
    $I^* := \{i \notin I \mid A_{ii'} > 0 \text{ for some } i' \in I\}$;
    choose $i^* \in I^*$ such that $\tilde{p}_{i^*}$ has a minimal number of possible choices
  endif;
  for all possible choices of p_{i^*} **do**
    LocalizationsPatternBacktrack($I \cup \{i^*\}$; $\tilde{p}_1, \ldots, \tilde{p}_s$ updated w.r.t. choice p_{i^*});
  endfor;
  return
end LocalizationsPatternBacktrack.

Fig. 6. Algorithm LocalizationsPatternBacktrack.

constant number of operations. It can be seen that for the enumeration of coline cycle patterns a coline adjacency matrix A and a list giving all the lengths m_i of the coline cycles is sufficient (we do not need an explicit description of the cocircuit graph or the coline cycles).

It remains to discuss the order in which we fix the patterns p_i, and this is of great importance with respect to the efficiency of the algorithm. If $I = \emptyset$ we choose any $i \in \{1, \ldots, s\}$ with maximal m_i (i.e. a longest cycle). If $\emptyset \neq I \subsetneqq \{1, \ldots, s\}$, let I^* denote the set of all $i \notin I$ for which c_i intersects at least one coline cycle from I (I^* is not empty [11]); then we choose $i^* \in I^*$ such that p_{i^*} has a minimal number of possible choices with respect to the first-order consequences implied by I. We call this a *dynamic ordering*. This finally leads to the algorithm LocalizationsPatternBacktrack which is summarized in Fig. 6.

The algorithm LocalizationsPatternBacktrack is much more efficient than all the previous algorithms for the generation of OMs described in this article, which is also observed from the performance of implementations. With the current implementations[1] all isomorphism classes of oriented matroids up to $|E| = 8$ have been computed in any rank, up to $|E| = 9$ in any rank except $r = 4, 5, 6$, and the two longest times for these computations were 4.1 hours for $|E| = 8$, $r = 4$ and 5.6 hours for $|E| = 9$, $r = 3$ on a Sun Sparc Ultra-60 using one processor at 360 MHz. Also considering only first-order consequences instead of second-order consequences did not cause many infeasible situations in the backtracking method, at least for a small size of ground sets (e.g. for $|E| \leq 6$ and any rank the number of infeasible cases was always less than 10% of the number of localizations, and for larger instances this also increases only a little).

[1] For details and access to data see http://www.om.math.ethz.ch.

Whereas the first four algorithms described in this article do not seem to be similar to previously known methods for the generation of OMs, the algorithm LOCALIZATIONSPATTERNBACKTRACK turned out to be related to an algorithm of Bokowski and Guedes de Oliveira [7]. At first, the two algorithms appear to be rather different. While we use cocircuits and cocircuit graphs, the OM representation in [7] is based on the chirotope axioms and concentrates on uniform OMs. This leads to different data structures in the algorithms (see below). Nevertheless, the two algorithms are closely related when interpreted as algorithms in dual settings, namely hyperplane arrangements versus point configurations: Localizations of cocircuit graphs correspond to hyperline configurations in [7] as we can consider (halves of) coline cycles and hyperlines (or lines) as being equivalent under dualization. Then patterns of coline cycles as introduced for algorithm LOCALIZATIONSPATTERNBACKTRACK and gap positions as used in the algorithm in [7] coincide. Also the basic idea of how the patterns (or gap positions) are fixed is similar.

Comparison of the algorithms shows that both are based on similar algorithmic concepts, however, there are also some important differences. In particular, while the algorithm in [7] stores both the set of colines and that of bases signatures, our algorithm carries the colines only. Furthermore, the colines are represented by their bases in [7] that are not unique in non-uniform OMs, our algorithm stores the colines directly. For generating non-uniform OMs, these differences can be substantial. Our algorithm LOCALIZATIONSPATTERNBACKTRACK is designed for the general case and the implementation is straightforward, independent from rank or uniformity. Furthermore we can, if we want, easily restrict to the uniform case: we simply do not consider patterns of type I or II, i.e. the only change in algorithm LOCALIZATIONSPATTERNBACKTRACK is that only odd values of p_i are allowed for patterns.

Another remarkable difference is the order in which the fixing of the patterns (or gap positions) is done: the algorithm in [7] uses a fixed order of hyperlines, our algorithm chooses the next coline $i^* \notin I$ according to the first-order consequences of the choices in I, thus reducing the amount of enumeration and the number of infeasibilities. This is possibly the reason why the use of second-order consequences was a crucial improvement from earlier algorithms in [7]. We agree that second-order consequences are important as they reduce infeasible cases considerably; in the case of rank 3 OMs they even eliminate all infeasibilities (which was already noted in [7]). However, our experience shows that even without second-order consequences the performance can be good because the dynamic ordering tends to eliminate infeasible cases efficiently.

A comparison of the efficiency of the two algorithms would be very tentative at the moment, as the implementations are too different to be compared directly. More detailed comparisons of the OM generation algorithms will be a basis for further investigations and improvements.

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