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Generation Results for B-bounded Semigroups (*).

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Abstract. – *In* [3] *A. Bellini-Morante defined and analysed a new one-parameter family of bounded operators which he called a B-bounded semigroup. The definition was motivated by an example from the transport theory where the evolution generated by an operator A was in a certain sense controlled by another operator B. In this paper we show that a given pair* (A, B) *generates a B-bounded semigroup if and only if in a certain extrapolation space related to the operator B, the closure of A generates a semigroup and we also address some related topics.*

1. – Introduction and motivation.

It is well known (see e.g. [6]) that if $(\exp(tA))_{t\geq 0}$ is a semigroup generated in a Banach space X by an operator A, then there are constants $M>0$ and ω such that

$$\|\exp tA\| \leq Me^{\omega t}, \qquad t\geq 0 .$$

In such a case we write $A \in \mathcal{G}(M, \omega, X)$. If the existence of the semigroup generated by A is proved directly by the Hille-Yoshida estimates, then the constants M and ω appear directly in the process. However, a number of techniques have been developed recently which give the existence of a semigroup in a non-constructive way, in particular when the positivity properties are employed, see e.g. [1]. Then it is of interest to derive other inequalities involving $(\exp(tA))_{t\geq 0}$. In [10,3] the authors proved the existence of a semigroup solving certain problem from transport theory using Arendt's method for resolvent positive operators (Theorem 2.5 of [1]). In such a case the constant M is difficult to determine but the authors noticed that there exists another operator, say B with norm not larger than 1, such that the solution u of the original problem satisfies $\|Bu(t)\| \leq \|B\mathring{u}\|$, for all $t\geq 0$, where $\mathring{u}$ is the initial value for the problem. This example shows that it may be advantageous to consider evolution families of operators which behave well if looked at through the "lens" of another operator. The first definition of such a family appeared in [3]. The final version reads as follows.

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DEFINITION 1.1. – *Let $(A, D(A))$ and $(B, D(B))$ be two linear operators in a Banach space X such that $D(A) \subset D(B)$ and for some $\omega \in \mathbb{R}$ the resolvent set of A satisfies $\varrho(A) \supset]\omega, \infty[$.*

A one-parameter family of operators $(Y(t))_{t \geq 0}$, which satisfies

1) *$D(Y(t)) =: \Omega \supseteq D(B)$, and for any $t \geq 0$ and $f \in D(B)$*

$$\|Y(t) f\| \leq M \exp(\omega t) \|Bf\|,$$

2) *The function $t \to Y(t) f \in C([0, \infty[, X)$ for any $f \in \Omega$.*

3) *For any $f \in \Omega_0 := \{f \in D(A);\ Af \in \Omega\} \subset D(A) \subset D(B)$*

$$Y(t) f = Bf + \int_0^t Y(s) Af ds, \qquad t \geq 0,$$

is called a B-quasi bounded semigroup generated by A and B.

Note that if B is bounded, then the definition of $(Y(t))_{t \geq 0}$ is much simpler as it is then a family of bounded operators defined on the whole space ($\Omega = X$), and $\Omega_0 = D(A)$.

REMARK 1.1. – The definition of B-quasi bounded semigroup admits another straightforward generalization which will be useful in the sequel. Namely, we can assume that $B : X \to Z$, where Z is another Banach space; then also $Y(t) : X \to Z$. The results quoted below will remain valid without any changes (except for replacing X by Z as the target space). ■

We note that despite the fact that the constructions and final formulas for B-bounded semigroups are similar to those appearing in the theory of C-semigroups and interpolation of semigroups (comapre [4, 5, 2]), these two families are different. Indeed, from the definition of C-semigroup it follows that C must commute with the C-semigroup, whereas for B-bounded semigroups no commutativity is assumed or needed. Moreover, we treat B which may be neither invertible, nor bounded. Thus, in some sense B-bounded semigroups prove to be a generalization of the C-semigroups, though the starting points of both definitions were different. On the other hand, the lack of commutativity makes many applications of C-semigroups unavailable for B-bounded semigroups. In fact, there are rather indications that B-bounded semigroups are related to B-evolutions and the empathy theory introduced and developed in [7, 8, 9], and may be used in the theory of Sobolev equations. This area is a subject of present research and the results will be published later.

We provide, for a quick reference, the main results proved in [3]. It follows that for any $f \in \Omega$, $\lambda > \omega$ and $n \in \mathbb{N}$

$$B(\lambda I - A)^{-n} f = \int_0^\infty t^{n-1} \exp(-\lambda t)\, Y(t) f dt, \tag{1.1}$$

and, if $f \in D(B)$, then

$$\|B(\lambda I - A)^{-n} f\| \leq \frac{M}{(\lambda - \omega)^n} \|Bf\| . \tag{1.2}$$

These relations are used to prove that a given pair (A, B) of operators generate at most one family satisfying conditions 1-3, hence the definition is correct. For such a pair we use the notation $A \in B - \mathcal{G}(M, \omega, X)$ (or $A \in B - \mathcal{G}(M, \omega, X, Z)$ in the case described in Remark 1.1).

For bounded operator B the author proved in [3] the following counterpart of the exponential formula for semigroups ([6], pp. 33-34.)

THEOREM 1.1. – *If the space X is weakly complete, $D(A)$ is dense in X and (A, B) generate a B-quasi bounded semigroup $(Y(t))_{t \geq 0}$, then for each $f \in X$ there exists a subsequence of $(B(I - (t/n) A)^{-n} f)_{n \in \mathbb{N}}$ which is weakly convergent to $Y(t) f$ for any $t \geq 0$.*

The main aim of this paper is to give a full characterization of generators of B-quasi bounded semigroups. We prove that the necessary and sufficient condition for A to belong to $B - \mathcal{G}(M, \omega, X, Z)$ is that $\overline{A}^{X_B} \in \mathcal{G}(M, \omega, X_B)$ where X_B is the completion of X with respect to the (semi)norm $\|\cdot\|_B = \|B \cdot\|_Z$ and $\overline{A}^{X_B}$ is the closure of a suitably understood extension of A to X_B, provided $B[D(A)]$ is dense in X (if X is reflexive, then the latter condition can be dropped). This result yields as an easy corollary an improvement of Theorem 1.1, that the sequence $(B(I - (t/n) A)^{-n} f)_{n \in \mathbb{N}}$ converges in norm to $Y(t)$ uniformly in t on finite intervals.

In Section 5 we discuss the adjoint problem under assumptions that B is bounded, and X and Z are reflexive spaces. The main result of this section is that $A \in B - \mathcal{G}(M, \omega, X, Z)$ is equivalent to $A^* \in \mathcal{G}(M, \omega, \overline{R(B^*)})$ where $R(B^*)$ is the range of B^* equipped with the graph norm. The importance of this result lies in the fact that X_B is very often a complicated abstract space (formally a space of equivalence classes of Cauchy sequences), whereas $R(B^*)$ is a subspace of X^*. This is a significant simplification in particular when X is a Hilbert space and A, B are self-adjoint operators.

2. – Construction of the extrapolation space X_B.

In this section we introduce the space X_B. Most results here belong to mathematical folklore, so we focus rather on explaining the necessary notation but provide some proofs where, in our opinion, the results are not quite standard.

Throughout this section we assume that $(B, D(B))$ is a possibly unbounded operator with $D(B) \subset X$ and $R(B) \subset Z$, where X and Z are Banach spaces. If B is bounded, then we understand that $D(B) = X$.

In general, if we have any equivalence relation $\sim$, then by $[x]$ we denote the equivalence class generated by x. Since sometimes we will be working with several equivalence relations, we shall later introduce special notation, but $[\cdot]$ will serve as a universal symbol.

DEFINITION 2.1. – *Let us consider the set $\mathcal{X}$ of sequences $(x_n)_{n\in\mathbb{N}}$ such that $x_n \in D(B)$ for $n \in \mathbb{N}$ and $(Bx_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. We define X_B to be the space of all classes of equivalence of sequences $(x_n)_{n\in\mathbb{N}} \in \mathcal{X}$ with respect to the following relation:*

$$(x_n)_{n\in\mathbb{N}} \sim (y_n)_{n\in\mathbb{N}} \quad \textit{if and only if} \quad \lim_{n\to\infty} \|Bx_n - By_n\|_Z = 0\,. \tag{2.1}$$

The space X_B is a normed space with the norm

$$\|[(x_n)_{n\in\mathbb{N}}]\|_{X_B} = \lim_{n\to\infty} \|Bx_n\|_Z. \tag{2.2}$$

This follows as in the standard proof of the theorem on completion of a normed space (see e.g., [12], pp. 95-97) the only difference being that the functional $p_B(\cdot) = \|B\cdot\|_Z$ is in general a seminorm and not a norm. This is taken care, however, by incorporating the null-space of B into the class of elements equivalent to the zero sequence.

To make the presentation more clear, and also having in mind applications, it is advantageous to distinguish two cases: with invertible B and with (not necessarily) invertible B. We emphasise, however, that the discussion of these two cases is rather parallel; the results for B invertible can be obtained as a particular case of the general ones.

2.1. *The case of invertible operator B.*

If B is an invertible operator, then X_B coincides with the completion of $D(B)$ in the norm $\|\cdot\|_B = \|B\cdot\|_Z$. By construction, $(D(B), \|\cdot\|_B)$ is isometric to a dense subspace of X_B, denoted by $\widetilde{D(B)}$. The class $[(x, x, x \ldots)] \in \widetilde{D(B)}$ is denoted by $\tilde{x}$ and the canonical isometry by j:

$$j : \widetilde{D(B)} \to D(B), \qquad j\tilde{x} = x\,. \tag{2.3}$$

Equation (2.2) shows that on $\widetilde{D(B)}$ we have $\|\tilde{x}\|_{X_B} = \|Bx\|_Z$. The operator $B \circ j \in \mathcal{L}(\widetilde{D(B)}, Z)$ is an isometry since $\|(B\circ j)\,\tilde{x}\|_Z = \|Bx\|_Z = \|\tilde{x}\|_{X_B}$, and has a unique extension by continuity to an operator $\mathcal{B} \in \mathcal{L}(X_B, Z)$. The following result is standard.

LEMMA 2.1. – *The operator $\mathcal{B}$ is an isometric isomorphism of X_B onto $Z_B = \overline{R(B)}^Z$: for any $\phi \in X_B$ we have*

$$\|\mathcal{B}\phi\|_Z = \|\phi\|_{X_B}. \tag{2.4}$$

It is worthwhile to note that if $\|Bf\|_Z \geq c\|f\|_X$ for some constant c, then X_B can be identified with a subspace of X (compare with the construction of the Friedrichs extension); in fact, it is $D(\overline{B})$. Indeed, B is a closeable operator and each class $[(x_n)_{n\in\mathbb{N}}]$ can be identified with $x = \overline{B}^{-1}y$ where $y = \lim_{n\to\infty} Bx_n$ is independent of the choice of the sequence $(x_n)_{n\in\mathbb{N}}$ in the class by Definition 2.1. Conversely, each element of $D(\overline{B})$ determines a unique class consisting of all sequences $(x_n)_{n\in\mathbb{N}}$ converging to x such that $(Bx_n)_{n\in\mathbb{N}}$ is convergent in Z.

Clearly, if $B \in \mathcal{L}(X, Z)$ (the set of all bounded linear operators) is such that $B^{-1} \in \mathcal{L}(Z, X)$, then X and X_B are isometrically isomorphic.

2.2. *The case of non-invertible B.*

Let us assume now that $N(B) \neq \{0\}$. We consider the quotient space $\underline{D(B)} = D(B)/N(B)$ and define $\underline{x} = [x]$ where $u \in [x]$ if and only if $y - x \in N(B)$. The canonical quotient operator $\pi: D(B) \to \underline{D(B)}$ is given by $\pi x = \underline{x}$, and we define

$$\underline{B}\underline{x} = Bx . \tag{2.5}$$

This reduces the problem to that discussed in the previous subsection with B replaced by $\underline{B}$ and $D(B)$ replaced by $\underline{D(B)}$. In fact, $\|\cdot\|_{\underline{B}} = \|\underline{B}\cdot\|_Z$ is a norm on $\underline{D(B)}$, therefore the completion, denoted by $\underline{X_B}$, is a Banach space and the corresponding extension by continuity of $\underline{B}$, denoted by $\underline{\mathcal{B}}$, is an isometric isomorphism of $\underline{X_B}$ onto $Z_B = \overline{R(\underline{B})}^Z = \overline{R(B)}^Z$. Keeping the rules of notation as in the previous subsection, we have in particular that the subspace $\underline{D(\tilde{B})}$, consisting of classes generated by constant sequences of elements of $\underline{D(B)}$, is a dense subspace of X_B isometrically isomorphic to the space $(\underline{D(B)}, \|\cdot\|_{\underline{B}})$; by $\underline{j}$ we denote the canonical embedding $\underline{j}: \underline{D(\tilde{B})} \to \underline{D(B)}$ defined for $\tilde{\underline{x}} = [(\underline{x}, \underline{x}, \dots)]$ by

$$\underline{j}\tilde{\underline{x}} = \underline{x} . \tag{2.6}$$

The following proposition relates this construction to the space X_B. The proof consists in a careful manipulation of equivalence classes and therefore it is omitted.

PROPOSITION 2.1. – *The space $\underline{X_B}$ is isometrically isomorphic to X_B and the isomorphisms $\mathfrak{J}: \underline{X_B} \to X_B$ is defined as follows: let $(\underline{x}_n)_{n \in \mathbb{N}} \in \phi$, where $\underline{x}_n \in \underline{D(B)}$ for $n = 1, 2, \cdots$, then*

$$\mathfrak{J}\phi = [(x_n)_{n \in \mathbb{N}}], \tag{2.7}$$

where $[(x_n)_{n \in \mathbb{N}}]$ is the class in X_B determined by an arbitrary sequence $(x_n)_{n \in \mathbb{N}}$ such that $x_n \in \underline{x}_n$. Thus, in particular, X_B is a Banach space.

The space $(\underline{D(B)}, \|\cdot\|_{\underline{B}})$ is isometrically isomorphic to a dense subspace of X_B and the isomorphism is given by

$$J\underline{x} = [(x, x, \dots)] \tag{2.8}$$

where x is an arbitrary element of $\underline{x}$.

Moreover, the operator $\mathfrak{p}: D(B) \to X_B$ defined by

$$\mathfrak{p}x = J\pi x = [(x, x, \dots)], \tag{2.9}$$

maps continuously $D(B)$ onto a dense subspace of X_B, denoted by $D(\tilde{B})$ and $D(\tilde{B}) = J(\underline{D(B)})$.

Using this proposition and Lemma 2.1 we obtain that the operator $\mathcal{B}$ defined for $\phi \in X_B$ by

$$\mathcal{B}\phi = \underline{\mathcal{B}}\mathfrak{J}^{-1}\phi \tag{2.10}$$

is an isometric isomorphism of X_B onto Z_B. Proposition 2.1 allows to give the intrinsic characterization of $\mathcal{B}$. We have

$$\mathcal{B}\phi = \tilde{B}\tilde{x} = Bx\,, \tag{2.11}$$

if $\phi = [(x, x, \ldots)] = \tilde{x} \in \mathfrak{p}(D(B))$, and

$$\mathcal{B}\phi = \lim_{n\to\infty} \tilde{B}\tilde{x}_n = \lim_{n\to\infty} Bx_n \tag{2.12}$$

if $\phi = [(x_n)_{n\in\mathbb{N}}]$. Indeed, both definitions (2.10) and (2.12) coincide on $\mathfrak{p}(D(B))$ and equality on X_B follows by density.

3. – Operators on X_B.

In this section we shall discuss extensions of operators A and $Y(t)$ to X_B. As in the previous section it is advantageous to consider cases of invertible and non-invertible B separately.

From condition 3) of Definition 1.1 it follows that it is necessary to restrict the operator A to a smaller domain and the requirement of the definition seems to be insufficient. It turns advantageous to start from the part of A in $D(B)$, that is $(A, D_B(A))$, where $D_B(A) = \{x \in D(A); Ax \in D(B)\}$.

If B is invertible, then the definition of the shift of A to X_B is straightforward. We define

$$\begin{cases} D_B(\tilde{A}) = j^{-1} D_B(A)\,, \\ \tilde{A}\tilde{x} = j^{-1} A j \tilde{x} = [(Ax, Ax, \ldots)]\,, \end{cases} \tag{3.1}$$

where we preserved the notation of Subsection 2.1. Note that in this case $\mathfrak{p} = j^{-1}$.

To define a counterpart of A in X_B for a non-invertible B we first prove the following lemma.

LEMMA 3.1. – *If $A \in B - \mathcal{G}(M, \omega, X, Z)$, then the following relations hold:*

(i) *For any $x \in N(B)$ and $t \geq 0$*

$$Y(t)\, x = 0\,. \tag{3.2}$$

(ii) *For any $x \in X$ and any $t \geq 0$*

$$Y(t)\, x \in \overline{R(B)}\,. \tag{3.3}$$

(iii) *For any $\lambda > \omega$ the space $N(B)$ reduces $(\lambda I - A)^{-1}$. In other words*

$$(\lambda I - A)^{-1} N(B) = D(A) \cap N(B) = D_B(A) \cap N(B)\,. \tag{3.4}$$

PROOF. – Item (i) follows from the condition 1) of Definition 1.1, as for $x \in N(B)$ we have

$$\|Y(t)\, x\|_Z \leq M \exp(\omega t) \|Bx\|_Z = 0\,.$$

To prove (ii) we use eq. (1.1) with $n = 1$. Let $Y(t_0)\,x \neq 0$ for some $x \in X$ and $t_0 \geq 0$. Then there is $f \in Z^*$ satisfying $f(Y(t_0)\,x) \neq 0$ and $f(\overline{R(B)}) = 0$. Using the continuity of f we have for all $\lambda > \omega$

$$0 = f(B(\lambda I - A)^{-1} x) = \int_0^{+\infty} \exp(-\lambda)\, f(Y(t)\, x)\, dt\,,$$

which, by the continuity of Y, yields $f(Y(t)\,x) = 0$ for all $t \geq 0$.

To prove (iii) we use eq. (1.2) with $n = 1$, which shows that if $x \in N(B)$, then $B(\lambda I - A)^{-1} x = 0$, hence $(\lambda I - A)^{-1} N(B) \subset N(B)$. On the other hand, let $f \in N(B) \cap D(A) = N(B) \cap R((\lambda I - A)^{-1})$; then for some $x \in X$ and all $\lambda > \omega$ we have:

$$0 = Bf = B(\lambda I - A)^{-1} x = \int_0^{+\infty} e^{-\lambda t}\, Y(t)\, x\, dt\,,$$

hence $Y(t)\,x = 0$ for all $t \geq 0$ by condition 2) of Definition 1.1. But $Y(0)\,x = Bx = 0$, hence $x \in N(B)$ and $f \in (\lambda I - A)^{-1} N(B)$. ■

Hence, we have natural definitions of operators in $\underline{D(B)} = D(B)/N(B)$ (see Subsection 2.2)

$$\underline{Y}(t): \underline{D(B)} \to Z_B, \qquad t > 0\,,$$

$$\underline{A_\lambda}: \underline{D(B)} \to \underline{D(B)}\,,$$

where, for any $x \in \underline{x}$,

$$\underline{Y}(t)\,\underline{x} = Y(t)\,x\,, \tag{3.5}$$

$$\underline{A_\lambda}\,\underline{x} = \underline{(\lambda I - A)^{-1} x}\,. \tag{3.6}$$

The fact that the range of $\underline{A_\lambda}$ is in $\underline{D(B)}$ follows from $D(A) \subset D(B)$ (conditions of Definition 1.1). We have the following lemma.

LEMMA 3.2. – *The operator* $\underline{A}$: $D_B(\underline{A}) \to \underline{D(B)}$, *where* $D_B(\underline{A}) = \{\underline{x} \in \underline{X};\ \underline{x} \cap D_B(A) \neq \emptyset\}$ *and for* $\underline{x} \in D_B(\underline{A})$

$$\underline{A}\,\underline{x} = \underline{Ax}$$

for arbitrary $x \in \underline{x} \cap D_B(A)$, *is well-defined. Moreover,* $(\omega, +\infty) \in \varrho(\underline{A})$ *and*

$$\underline{A_\lambda}\,\underline{x} = (\lambda \underline{I} - \underline{A})^{-1}\,\underline{x}\,. \tag{3.7}$$

PROOF. – Let $y \in D_B(A) \cap N(B)$, then by Lemma 3.1 we obtain that for each $\lambda > \omega$ there is $x_\lambda \in N(B)$ such that $y = (\lambda I - A)^{-1} x_\lambda$, which gives $Ay = \lambda y - x_\lambda \in N(B)$, hence $A(D_B(A) \cap N(B)) \subset N(B)$. Clearly, the definition then makes sense, as if we have $x_1, x_2 \in D_B(A)$ such that $x_1 - x_2 \in N(B)$ (that is, $x_1 \in \underline{x_2}$), then $A(x_1 - x_2) \in N(B)$ and consequently $\underline{Ax_1} = \underline{Ax_2}$.

To prove eq. (3.7) we note that by the surjectivity of $\lambda I - A$ and by the definition of $\underline{A}$, the operator $\lambda \underline{I} - \underline{A}$ is also surjective and therefore for a given $\underline{x} \in \underline{D(B)}$ there is $f \in D_B(\underline{A})$ such that $(\lambda \underline{I} - \underline{A}) f = \underline{x}$. Let us take $g = \underline{A_\lambda}\, \underline{x}$. Hence, there is $\overline{f_1} \in D_B(A)$ such that $(\lambda I - A) f_1 = x + b$ for some $b \in N(B)$. On the other hand, since $g = \underline{(\lambda I - A)^{-1} x}$, there exists $g_1 \in D_B(A)$ such that $(\lambda I - A) g_1 = x$. This shows that $f_1 = \overline{g_1} + \overline{(\lambda I - A)^{-1} b}$, and, since $b \in N(B)$, Lemma 3.1 yields $(\lambda I - A)^{-1} b \in N(B)$, and consequently $\underline{f} = f_1 = \underline{g_1} = \underline{g}$. ■

This lemma allows to use directly the definition (3.1) to shift $\underline{A}$ to $\underline{X_B}$; in this way we obtain

$$\begin{cases} D_B(\underline{\tilde{A}}) = \underline{j}^{-1} D_B(\underline{A}), \\ \underline{\tilde{A}}\underline{\tilde{x}} = \underline{j}^{-1} \underline{A} \underline{j} \underline{\tilde{x}} = [(\underline{Ax}, \underline{Ax}, \ldots)], \end{cases} \tag{3.8}$$

where $\underline{\tilde{x}} = \underline{j}^{-1} \pi x \in \underline{D(B)}$.

Our main interest is to define the extension of A to X_B. Using Lemma 3.2 we can define the shift of A to $\widetilde{D(B)}$ in the following way:

$$\begin{cases} D_B(\tilde{A}) = \mathfrak{p} D_B(A), \\ \tilde{A}\tilde{x} = \mathfrak{p} A x = [(Ax, Ax, \ldots)], \end{cases} \tag{3.9}$$

where $x \in D_B(A)$ and $\tilde{x} = \mathfrak{p} x$. From Proposition 2.1 we obtain the following identities:

$$D_B(\tilde{A}) = J D_B(\underline{A}) = \mathfrak{J} D_B(\underline{\tilde{A}}), \tag{3.10}$$

$$\tilde{A}\tilde{x} = J \underline{A} J^{-1} \tilde{x} = \mathfrak{J} \underline{\tilde{A}} \mathfrak{J}^{-1} \tilde{x}\,. \tag{3.11}$$

We shall need the following result.

PROPOSITION 3.1. – *The operator $(\tilde{A}, D_B(\tilde{A}))$ is closable in X_B if and only if the operator $(\underline{\tilde{A}}, D_B(\underline{\tilde{A}}))$ is closable in $\underline{X_B}$. If we denote*

$$\mathfrak{A} = \overline{\tilde{A}}^{X_B}, \tag{3.12}$$

then we have

$$\mathfrak{J}^{-1} \mathfrak{A}\, \mathfrak{J} = \overline{\underline{\tilde{A}}}^{\underline{X_B}}. \tag{3.13}$$

Moreover, if $\phi \in D(\mathfrak{A})$, then there is $(x_n)_{n \in \mathbb{N}} \in \phi$ such that $x_n \in D_B(A)$ for $n \in \mathbb{N}$ and the sequences $(Bx_n)_{n \in \mathbb{N}}$ and $(BAx_n)_{n \in \mathbb{N}}$ are fundamental in Z. Conversely, if $(x_n)_{n \in \mathbb{N}}$ is a sequence of elements of $D_B(A)$ such that the sequences $(Bx_n)_{n \in \mathbb{N}}$ and $(BAx_n)_{n \in \mathbb{N}}$ are fundamental in Z, then $\phi = [(x_n)_{n \in \mathbb{N}}] \in D(\mathfrak{A})$. Then, putting $\lim_{n \to \infty} Bx_n = y$ and

$\lim_{n \to \infty} BAx_n = z$, *we obtain*

$$\phi = \mathcal{B}^{-1} y\,, \qquad \textit{and} \qquad \mathfrak{A}\,\phi = \mathcal{B}^{-1} z\,, \tag{3.14}$$

where $\mathcal{B}$ *is the extension of* $\tilde{B}$ *defined by eq.* (2.11).

PROOF. – Equation (3.13) is a straigtforward consequence of Proposition 2.1. To prove the other part of the proposition we note that $\phi \in D(\mathfrak{A})$ if and only if

$$\lim_{n \to \infty} \phi_n = \phi\,, \qquad \text{and} \qquad \lim_{n \to \infty} \mathfrak{A}\,\phi_n = \psi = \mathfrak{A}\,\phi$$

in X_B, where $\phi_n \in D_B(\tilde{A}) = J(D_B(\underline{A}))$. This means that $\phi_n = [(x_n, x_n, \ldots)]$ for some $x_n \in \underline{x_n} \cap D_B(A)$. Then also $(Bx_n)_{n \in \mathrm{N}}$ is fundamental in Z, and similarly, the sequence $(BA\underline{x_n})_{n \in \mathrm{N}}$ is fundamental in Z. Hence both sequences determine classes of equivalence in X_B. It follows now that $\phi = [(x_1, x_2, \ldots)]$. The proof of this statement is analogous to the standard proof that the completion of a normed space is a Banach space. Similarly we obtain that $\mathfrak{A}\,\phi = [(Ax_1, Ax_2, \ldots)]$.

Consider now a sequence $(x_n)_{n \in \mathrm{N}}$ such that $x_n \in D_B(A)$ for $n = 1, 2, \ldots$ and such that $(Bx_n)_{n \in \mathrm{N}}$ and $(BAx_n)_{n \in \mathrm{N}}$ are fundamental in Z. Then $\phi = [(x_n)_{n \in \mathrm{N}}]$ and $\psi = [(Ax_n)_{n \in \mathrm{N}}]$ are well-defined.

Consider now the sequence of classes $\phi_n = [(x_n, x_n \ldots)]$. As before, for each n, $\phi_n \in D_B(\tilde{A})$ and $\lim_{n \to \infty} \phi_n = \phi$. Moreover, $\mathfrak{A}\,\phi_n = [(Ax_n, Ax_n \ldots)]$ and

$$\|\psi - \tilde{A}\phi_n\|_{X_B} = \lim_{m \to \infty} \|BAx_m - BAx_n\|_Z,$$

hence $\lim_{n \to \infty} \mathfrak{A}\,\phi_n = \psi$ and the proof is complete. ■

4. – The main generation theorem.

In this section we shall prove the main theorem of this paper.

THEOREM 4.1. – *If* $A \in B - \mathcal{G}(M, \omega, X, Z)$ *and* $B[D_B(A)]$ *is dense in* Z_B, *then the operator* $\mathfrak{A}$, *defined as in Proposition* 3.1, *satisfies* $\mathfrak{A} \in \mathcal{G}(M, \omega, X_B)$. *Conversely, if there is an extension* $\mathfrak{a}$ *of* $\tilde{A}$ *such that* $\mathfrak{a} \in \mathcal{G}(M, \omega, X_B)$, *then* $\mathfrak{a} = \mathfrak{A}$ *and* $A \in B - \mathcal{G}(M, \omega, X, Z)$.

The B-quasi bounded semigroup $(Y(t))_{t \geq 0}$ *is given for* $x \in D(B)$ *by*

$$Y(t)\,x = \exp(t\mathcal{B}\mathfrak{A}\,\mathcal{B}^{-1})\,Bx = \mathcal{B}\exp(t\mathfrak{A}\,)\,\mathfrak{p}x\,, \tag{4.1}$$

where $\mathcal{B}$ *is defined as in eq.* (2.10).

The proof of the theorem is lengthy and will be split into several steps. The first step, stated as an independent lemma, is a variation of Kato's results on pseudoresolvents (see e.g. [6]), specified for our particular situation.

LEMMA 4.1. – *Assume that the operator* $(K, D(K))$ *with* $\overline{D(K)} = X$ *has the following properties*:

1) *for* $\lambda > \omega$ *the operators* $\lambda I - K$ *are invertible,*

2) *for each* $\lambda > \omega$, *the range* $R(\lambda I - K) = R$ *is indepdnent of* λ *and dense in* X,

3) *for each* $y \in R$

$$\sup_{\lambda > \omega} \|\lambda(\lambda I - K)^{-1} y\| \le M\|y\| < +\infty . \tag{4.2}$$

Then $(K, D(K))$ *is closable,* $]\omega, \infty[\subset \varrho(\overline{K})$ *and* $(\lambda I - \overline{K})^{-1}$ *is the unique extension by continuity of* $(\lambda I - K)^{-1}$.

PROOF. – By [6], pp. 36-38, (4.2) implies that the extension by continuity of $(\lambda I - K)^{-1}$, denoted by H_λ, is the resolvent of adensely defined closed operator, say $\widehat{K}$. Since $((\lambda I - \widehat{K})^{-1})^{-1} x = ((\lambda I - K)^{-1})^{-1} x$ for $x \in D(K)$ and

$$\widehat{K} = \lambda I - ((\lambda I - \widehat{K})^{-1})^{-1},$$

we obtain that $\widehat{K}$ is an extension of K, and since it is closed, K is closable and $\overline{K} \subset \widehat{K}$. Let now $x \in D(\widehat{K})$; then $x = (\lambda I - \widehat{K})^{-1} y$ for some $y \in X$. This means that

$$x = \lim_{n \to \infty} (\lambda I - K)^{-1} y_n$$

for $y_n \in R$ and $y_n \to y$. In other words, $x_n = (\lambda I - K)^{-1} y_n \in D(K)$ converges to x. Solving this equation we get $Kx_n = \lambda x_n - y_n$ and $(Kx_n)_{n \in \mathbb{N}}$ converges, hence $x \in D(\overline{K})$ and $\overline{K}x = \lambda x - y = \lambda x - ((\lambda I - \widehat{K})^{-1})^{-1} x = \widehat{K}$. ∎

PROOF OF THEOREM 4.1. – *Step 1.* (Case of invertible B and $R(B)$ dense in Z).
Note that in this case $\mathfrak{p} = j^{-1}$, where j is defined by eq. (2.3). We define the operator

$$\widehat{A}x = BAB^{-1}x \tag{4.3}$$

for $x \in D(\widehat{A}) = B[D_B(A)]$, hence $D(\widehat{A})$ is dense in Z by assumption. We have $(\lambda I - \widehat{A}) = B(\lambda I - A) B^{-1}$ and thanks to the assumption (ii) of Definition 1.1 we have for $\lambda > \omega$:

$$(\lambda I - \widehat{A})^{-1} g = B(\lambda I - A)^{-1} B^{-1} g \tag{4.4}$$

for $g \in R(B) \supset B[D_B(A)]$ and the latter set is dense in Z. From eq. (1.2) we obtain for any $g \in R(B)$ and $\lambda > \omega$:

$$\|(\lambda I - \widehat{A})^{-1} g\|_Z = \|B(\lambda I - A)^{-1} B^{-1} g\|_Z \le \frac{M\|BB^{-1}g\|_Z}{\lambda - \omega} = \frac{M\|g\|_Z}{\lambda - \omega}. \tag{4.5}$$

We see now that the assumptions of Lemma 4.1 are satisfied. Hence, $(\lambda I - \widehat{A})^{-1}$ has a unique extension to the resolvent of the closure of $\widehat{A}$ in Z, $\overline{\widehat{A}}^{Z}$. Using again the properties of B-bounded semigroups we obtain further:

$$\|(\lambda I - \widehat{A})^{-n} g\|_Z = \|B(\lambda I - A)^{-n} B^{-1} g\|_Z \le \frac{M\|g\|_Z}{(\lambda - \omega)^n}. \tag{4.6}$$

Hence, for each n the operator $(\lambda I - \widehat{A})^{-n}$ has a unique extension onto Z and by uniqueness, this extension must be equal to $(\lambda I - \overline{\widehat{A}})^{-n}$. We conclude that $\overline{\widehat{A}}^{Z} = \overline{BAB^{-1}}^{Z}$ generates a semigroup $(\exp(t\overline{\widehat{A}}))_{t \geq 0}$ on Z. As a consequence, we obtain

$$\exp(t\overline{\widehat{A}})\, f = f + \int_0^t \exp(s\overline{\widehat{A}})\, \overline{\widehat{A}} f ds$$

for any $f \in D(\overline{\widehat{A}})$, or

$$\exp(t\overline{\widehat{A}})\, f = f + \int_0^t \exp(s\overline{\widehat{A}})\, \widehat{A} f ds\,,$$

if $f \in D(\widehat{A}) = B[D_B(A)]$. Then, for arbitrary $x \in D_B(A)$ there is $f = Bx \in B[D_B(A)]$ and

$$\exp(t\overline{\widehat{A}})\, Bx = Bx + \int_0^t \exp(s\overline{\widehat{A}})\, Bx\, ds\,.$$

Since $Y(t)$ exists and is uniquely defined we conclude that

$$Y(t)\, x = \exp(t\overline{\widehat{A}})\, Bx\,. \tag{4.7}$$

The equation (4.5) can be interpreted in another way. Let $f = B^{-1}g$ for $g \in R(B)$:

$$\|(\lambda I - \widehat{A})^{-1} f\|_Z = \|B(\lambda I - A)^{-1} f\|_Z \|(\lambda I - A)^{-1} f\|_B \leq \frac{M\|Bf\|_Z}{\lambda - m} = \frac{M\|f\|_B}{\lambda - m}\,, \tag{4.8}$$

where $\|\cdot\|_B$ was defined in Section 2.1. By Lemma 4.1 we conclude from this inequality that the operator $\widetilde{A}$, defined on $D_B(\widetilde{A}) = j^{-1}D_B(A)$, is closeable in X_B. Denoting $\mathfrak{A} = \overline{\widetilde{A}}^{X_B}$, its resolvent $(\lambda I - \mathfrak{A})^{-1}$ is a unique extension by continuity in X_B of $(\lambda I - \widetilde{A})^{-1} = j^{-1}(\lambda I - A)^{-1} j$, defined originally on $\widetilde{D(B)}$. As before we obtain for any $\phi \in X_B$:

$$\|(\lambda I - \mathfrak{A})^{-n} \phi\|_B \leq \frac{M\|\phi\|_{X_B}}{(\lambda - \omega)^n}\,.$$

Let us take $\phi \in X_B$; then there is $y \in Z$ such that $y = \mathcal{B}\phi$, and, since $B[D_B(A)]$ is dense in Z, $y = \lim_{n \to \infty} Bx_n$ for some sequence $(x_n)_{n \in \mathbb{N}}$ of elements of $D_B(A)$. By continuity of $\mathcal{B}^{-1}$ we obtain

$$\phi = \lim_{n \to \infty} \mathcal{B}^{-1} Bx_n = \lim_{n \to \infty} \mathcal{B}^{-1}\mathcal{B} j^{-1} x_n = \lim_{n \to \infty} j^{-1} x_n$$

and since $j^{-1}x_n \in j^{-1}D_B(A) \subset D(\mathfrak{A})$, $D(\mathfrak{A})$ is dense in X_B and consequently $\mathfrak{A} \in \mathcal{G}(M, \omega, X_B)$.

Now we clarify the relation between $\mathfrak{A}$ and $\overline{\widehat{A}}^{Z} = \overline{(BAB^{-1})}^{Z}$. Consider $\mathcal{B}\mathfrak{A}\mathcal{B}^{-1}$ defined on $\mathcal{B}[D(\mathfrak{A})]$. Let $x \in B[D_B(A)]$. Then, in particular, $x \in R(B)$ and $\mathcal{B}^{-1}x = j^{-1}B^{-1}x$ by the construction of $\mathcal{B}$, and hence $\mathcal{B}^{-1}x \in j^{-1}D(A)$. Therefore $\mathfrak{A}\mathcal{B}^{-1}x = j^{-1}AB^{-1}x$. This shows that $\mathfrak{A}\mathcal{B}^{-1}x \in \widetilde{X}$ and therefore $\mathcal{B}j^{-1} = \widetilde{B}j^{-1} = B$ and we obtain $\mathcal{B}\mathfrak{A}\mathcal{B}^{-1}x = BAB^{-1}x$, thus $\mathcal{B}\mathfrak{A}\mathcal{B}^{-1}$ is an extension of BAB^{-1}, $\widehat{A} \subset \mathcal{B}\mathfrak{A}\mathcal{B}^{-1}$. By $\mathcal{B}(\lambda I -$

$-\mathfrak{A})^{-1}\mathcal{B}^{-1} = (\lambda I - \mathcal{B}\mathfrak{A}\mathcal{B}^{-1})^{-1}$, we see that $\mathcal{B}\mathfrak{A}\mathcal{B}^{-1}$ is a closed operator, therefore we must have $\overline{\widetilde{A}} \subset \mathcal{B}\mathfrak{A}\mathcal{B}^{-1}$ and since both operators have resolvents for $\lambda > \omega$, they must coincide:

$$\mathcal{B}\mathfrak{A}\mathcal{B}^{-1} = \overline{(BAB^{-1})}^{Z}. \tag{4.9}$$

Let $\mathcal{A}$ be any extension of $\widetilde{A}$ which generates a semigroup on X_B. Firstly, we show that it must be the closure of $\widetilde{A}$. Clearly, we have $(\lambda I - \widetilde{A})^{-1} \subset (\lambda I - \mathcal{A})^{-1}$ and hence

$$\|(\lambda I - \widetilde{A})^{-1}\phi\|_{X_B} \leqslant \frac{M\|\phi\|_{X_B}}{\lambda - \omega}. \tag{4.10}$$

Since $R((\lambda I - \widetilde{A})^{-1} = D(\widetilde{B})$ by the assumption (ii) of Definition 1.1, we can use Lemma 4.1 to obtain that the resolvent of $\overline{\widetilde{A}}^{X_B}$, $(\lambda I - \overline{\widetilde{A}}^{X_B})^{-1}$, is well-defined and satisfies inequality (4.10). Since $\mathfrak{A}$ is closed, we must have $\overline{\widetilde{A}}^{X_B} \subset \mathfrak{A}$ and as above we see that we have the equality.

Next, let us define for $x \in D(B)$ (remember that now $j^{-1} = \mathfrak{p}$)

$$Y(t)\,x = \mathcal{B}\exp(t\mathfrak{A})\,j^{-1}x.$$

We have by Lemma 2.1

$$\|Y(t)\|_Z = \|\mathcal{B}\exp(t\mathfrak{A})\,j^{-1}x\|_Z = \|\exp(t\mathfrak{A})\,j^{-1}x\|_{X_B} \leqslant \frac{M\|j^{-1}x\|_{X_B}}{\lambda - \omega} \leqslant \frac{M\|Bx\|_Z}{\lambda - \omega}.$$

Moreover, for $\phi \in D(\mathfrak{A})$ we have

$$\exp(t\mathfrak{A})\,\phi = \phi + \int_0^t \exp(s\mathfrak{A})\,\mathfrak{A}\,\phi\,ds.$$

If $\phi = j^{-1}x$ where $x \in D_B(A)$ we obtain $\mathfrak{A}\,\phi = \widetilde{A}(j^{-1}x)$, and $\mathcal{B}j^{-1}x = \widetilde{B}j^{-1}x = Bx$, hence operating on both sides with $\mathcal{B}$ we get

$$\mathcal{B}\exp(t\mathfrak{A})\,j^{-1}x = Bx + \int_0^t \mathcal{B}\exp(s\mathfrak{A})\,j^{-1}Ax\,ds$$

where the operator $\mathcal{B}$ can be moved inside the integral due to its continuity. This is the B-bounded semigroup identity:

$$Y(t)\,x = Bx + \int_0^t Y(t)\,Ax\,ds,$$

and, since $Y(t)\,x$ is continuous, we see that $Y(t)$ is a B-bounded semigroup generated by A.

Step 2. (Case of non-invertible B).

We can apply Step 1 to the quotient space $\underline{D(B)}$ with Z replaced by Z_B using definitions (3.5), (3.6) and Lemma 3.2. Firstly, the family $(\underline{Y}(t))_{t\geq 0}$ is a B-quasi bounded semigroup acting from $\underline{D(B)}$ into Z_B and $\underline{A}$ is its $\underline{B}$-generator. Indeed, for any $x \in \underline{X}$, where $\underline{x} \in \underline{D(B)}$ we have

$$\|\underline{Y}(t)\,\underline{x}\|_{Z_B} = \|\underline{Y}(t)\,\underline{x}\|_Z = \|Y(t)\,x\|_Z \leq M \exp(\omega t)\|Bx\|_Z = M \exp(\omega t)\|\underline{B}\,\underline{x}\|_{Z_B} \tag{4.11}$$

for the same M and ω as in the definition of $(Y(t))_{t\geq 0}$. Clearly, the continuity condition 2 of Definition 1.1 is satisfied. Finally, for $\underline{x} \in D(\underline{A})$ we chose $x_1 \in D_B(A)$ and obtain

$$\underline{Y}(t)\,\underline{x} = Y(t)\,x_1 = Bx_1 + \int_0^t Y(s)\,Ax_1\,ds = \underline{Bx} + \int_0^t \underline{Y}(s)\,\underline{Ax}\,ds$$

since we have $Y(s)\,Ax_1 = \underline{Y}(s)\,\underline{Ax_1} = \underline{Y}(s)\,\underline{Ax}$.

We note that if $B[D_B(A)]$ is dense in $Z_B = \overline{R(B)}$, then also $\underline{B}[D_B(\underline{A})]$ is dense in $R(B)$. Indeed, let $\overline{R(B)} \ni y = \lim_{n\to\infty} BAx_n$ for some sequence $(x_n)_{n\in\mathbb{N}}$ of elements of $D_B(A)$. But $\underline{B}\underline{A}x_n = \underline{B}\underline{Ax_n} = \overline{BAx_n}$ and the statement is proved.

Defining now

$$Y(t)\,x = \underline{Y}(t)\,\underline{x} = \exp(t\overline{\underline{B}\underline{A}\underline{B}^{-1}}^{Z})\,\underline{Bx} = \underline{\mathcal{B}}\exp(t\overline{\underline{\tilde{A}}})\,\underline{j}^{-1}\underline{x} \tag{4.12}$$

we see that $A \in B - \mathcal{G}(M, m, X, Z)$ if and only if there is an extension $\underline{\mathcal{A}}$ of $\underline{\tilde{A}}$ such that $\underline{\mathcal{A}} \in \mathcal{G}(M, m, \underline{X_B})$. Moreover, we must have $\underline{\mathcal{A}} = \underline{\tilde{A}}^{\underline{X_B}}$ and the B-quasi bounded semigroup is given then by

$$Y(t)\,x = \exp(t\overline{\underline{B}\underline{A}\underline{B}^{-1}}^{Z})\,Bx = \mathcal{B}\exp(t\overline{\underline{\tilde{A}}})\,\underline{j}^{-1}\underline{x} \tag{4.13}$$

for $x \in D(B)$, where $\underline{\mathcal{B}}$ is defined as in Lemma 2.1.

The theorem follows now from Propositions 2.1 and 3.1. By eq. (3.13), $\mathfrak{J}^{-1}\mathfrak{A}\,\mathfrak{J} = \underline{\tilde{A}}^{\underline{X_B}}$, and the exponential formula for semigroups ([6], pp. 33-35) gives $\mathfrak{J}^{-1}\exp(t\mathfrak{A})\,\mathfrak{J} = \exp(t(\underline{\tilde{A}}))$, hence eq. (2.10) yields

$$\underline{\mathcal{B}}\exp(t(\overline{\underline{\tilde{A}}}))\,\underline{j}^{-1}\underline{x} = \mathcal{B}\mathfrak{J}^{-1}\exp(t\mathfrak{A})\,\mathfrak{J}\underline{x} = \mathcal{B}\exp(t\mathfrak{A})\,\mathfrak{p}x\,.$$

The other equality can be proved in the same way. ∎

It follows from this theorem that if $B[D_B(A)]$ is dense in Z_B, then $D(\mathfrak{A})$ is dense in X_B, since $\mathfrak{A}$ generates a strongly continuous semigroup. The converse is also true: if $D(\mathfrak{A})$ is dense in X_B, then $B[D_B(A)]$ is dense in Z_B. Indeed, in this case also $D(\tilde{A})$ is dense in X_B since by Proposition 3.1, $D_B(\tilde{A}) = \mathfrak{p}(D_B(A))$ is a core for $D(\mathfrak{A})$. Let now $z \in Z_B$; then $z = \mathcal{B}\phi$ for some $\phi \in X_B$ by eq. (2.10). If $(\tilde{x}_n)_{n\in\mathbb{N}}$, $x_n \in D_B(A)$, is an approximating sequence for ϕ, then by eq. (2.11) we have

$$\|z - Bx_n\|_Z = \|\mathcal{B}\phi - \mathcal{B}\tilde{x}_n\|_Z = \|\phi - \tilde{x}_n\|_{X_B} \to 0,$$

as $n \to \infty$, and the statement is proved.

This observation allows to discard the assumption that $B[D_B(A)]$ is dense in Z_B if Z (and consequently Z_B) is a reflexive Banach space. We have the following corollary.

COROLLARY 4.1. – *If Z is a reflexive space, then the thesis of Theorem 4.1 is valid without the assumptions that $B[D_B(A)]$ is dense in Z_B.*

A stronger version of the exponential formula given in Theorem 1.1 follows as a corollary from Theorem 4.1. Namely, we have the following result.

THEOREM 4.2. – *If $A \in B - \mathcal{G}(M, \omega, X, Z)$, then for any $x \in D(B)$*

$$\lim_{n \to \infty} B\left(I - \frac{t}{n}A\right)^{-n} x = Y(t)\,x. \tag{4.14}$$

PROOF. – Firstly, we prove that for $x \in D(B)$

$$B\left(I - \frac{t}{n}A\right)^{-n} x = \mathcal{B}\left(I - \frac{t}{n}\mathfrak{A}\right)^{-n} \mathfrak{p}x. \tag{4.15}$$

Let $g = (I - (t/n)A)^{-1}x$. Then the layer $\mathfrak{p}g = [(g, g, \ldots)] \in D(\mathfrak{A})$ and by eqs. (3.9) and (2.9) $\mathfrak{A}\,\mathfrak{p}g = \mathfrak{p}Ag = [(Ag, Ag, \ldots)]$. This shows that

$$\mathfrak{p}g = \mathfrak{p}\left(\left(I - \frac{t}{n}A\right)^{-1} x\right) = \left(I - \frac{t}{n}\mathfrak{A}\right)^{-1} \mathfrak{p}x.$$

Iterating, we obtain

$$\left(I - \frac{t}{n}\mathfrak{A}\right)^{-n} \mathfrak{p}x = \mathfrak{p}\left(\left(I - \frac{t}{n}A\right)^{-n} x\right),$$

and eq. (2.11) gives eq. (4.15). This yields

$$\left\|B\left(I - \frac{t}{n}A\right)^{-n} x - Y(t)\,x\right\|_Z = \left\|\left(I - \frac{t}{n}\mathfrak{A}\right)^{-n} \mathfrak{p}x - \exp(t\mathfrak{A})\,\mathfrak{p}x\right\|_{X_B} \tag{4.16}$$

and the theorem follows from the exponential formula for semigroups [6], pp. 33-35. ■

Similarly we get a characterization of B-geneators in terms of operators A and B.

THEOREM 4.3. – *Let operators A and B satisfy the conditions* (i) *and* (ii) *of Definition* 1.1. *Then $A \in B - \mathcal{G}(M, \omega, X, Z)$ if and only if the following conditions hold:*

1) $B[D_B(A)]$ *is dense in* Z_B,

2) $A(N(B) \cap D(A)) \subset N(B)$,

3) *there exists $M > 0$ and $\omega \in \mathbb{R}$ such that for any $x \in D(B)$ and $n \in \mathbb{N}$:*

$$\|B(I - \lambda A)^{-n} x\|_Z \leq \frac{M}{(\lambda - \omega)^n} \|Bx\|_Z. \tag{4.17}$$

PROOF. – We prove only the sufficient condition, as the necessary one follows from Theorem 4.1. From (4.17) we have

$$(\lambda I - A)^{-1} N(B) \subset N(B) \cap D(A).$$

Assumption 2) yields that for $x \in D(A) \cap N(B)$ we have $\lambda x - Ax \in N(B)$, hence $(\lambda I - - A)^{-1} N(B) \subset D(A) \cap N(B)$. Thus $N(B) \cap D(A) = N(B) \cap D_B(A)$ and we can follow the prof of Lemma 3.2 to see that A can be shifted to $\tilde{A}$ acting on $\mathfrak{p} D_B(A)$.

Then arguments as in Theorem 4.2 show that eq. (4.17) can be rewritten as

$$\|(I - \lambda \tilde{A})^{-n} \tilde{x}\|_{X_B} \leq \frac{M}{(\lambda - \omega)^n} \|\tilde{x}\|_{X_B}.$$

valid on a dense subset $D(\tilde{B})$ of X_B. Lemma 4.1 implies that this equation is valid for the closure of $\tilde{A}$, $\mathfrak{A}$, on the whole X_B.

Now, assumption 1) implies that $D(\tilde{B})$ is dense in X_B and since it is a core for $\mathfrak{A}$, $D(\mathfrak{A})$ is also dense in X_B and the Hille-Yoshida theorem gives the assertion of the theorem. ∎

5. – Adjoint semigroups.

The space X_B introduced in Section 2 is very often difficult to handle as it is in some sense an enlargement of the original space. In this section we give a counterpart of the previous results in terms of adjoint operators. Roughly speaking, we show that the adjoint space to X_B can be identified with the range of the adjoint operator of B. Therefore, if e.g., $X = Z = L_2$, then the range of B^* lies again in L_2 and we can work in a subspace of the original space rather than in some abstract, larger space.

Throughout this section we assume that the operators A and B satisfy the conditions (i)-(ii) of Definition 1.1. Moreover, we assume that B is a bounded operator and the Banach spaces X and Definition 1.1. Moreover, we assume that B is a bounded operator and the Banach spaces X and Z are reflexive. Some of the results of this section can be obtained without these assumptions (e.g. for unbounded but closeable B one can replace X by $D(B)$ with the graph norm as only the part of A in $D(B)$ is used) but in our opinion the adopted conditions allow to present the theory in a closed and elegant form.

The space Z is reflexive, hence by Corollary 4.1 the assumption of the density of $B[D(A)] = B[D_B(A)]$ of Theorem 4.1 can be dropped.

5.1. *The case of B invertible with dense range.*

In this subsection we assume that:

(B) $B \in \mathcal{L}(X, Z)$ is an injective operator satisfying $\overline{R(B)} = Z$.

Let X^* be the adjoint space to X and $\langle \cdot, \cdot \rangle_{X^* \times X}$ denote the corresponding duality pairing. The same convention will be applied to other spaces.

Since X is reflexive, from [12], pp. 98 and 265 we obtain the following result: the operator $(j^{-1})^*$, where j is defined by eq. (2.3), is a canonical embedding of $(X_B)^*$ onto a dense subspace of X^*; moreover, if $x \in X$ and $x^* = (j^{-1})^* \tilde{x}^* \in (j^{-1})^* X_B^*$, then

$$\langle x^*, x \rangle_{X^* \times X} = \langle \tilde{x}^*, \tilde{x} \rangle_{X_B^* \times X_B} \tag{5.1}$$

where, as in Subsection 2.1, $\tilde{x} = j^{-1} x$ is the class in X_B determined by $(x, x, \ldots)$.

Let $B^* \in \mathcal{L}(Z^*, X^*)$ be the adjoint of B. By the assumption (B), the operator B^{-1} exists, is densely defined and closed, hence its adjoint is also a densely defined and lcosed operator satisfying $(B^*)^{-1} x^* = (B^{-1})^* x^*$, for all $x^* \in R(B^*) = D((B^*)^{-1})$.

By X_{B^*} we denote the range of B^* equipped with the graph norm

$$\|x^*\|_{X_{B^*}} = \|(B^*)^{-1} x^*\|_{Z^*}. \tag{5.2}$$

Since $(B^*)^{-1}$ is injective and closed, X_{B^*} is a Banach space densely and continuously embedded in X^*. The following lemma is crucial.

LEMMA 5.1.

$$X_{B^*} = (j^{-1})^* X_B^* \tag{5.3}$$

and $(j^{-1})^*$ *is an isometric isomorphism.*

PROOF. – Recall that $\mathcal{B} \in \mathcal{L}(X_B, Z)$ is an isometric isomorphism such that $\mathcal{B} j^{-1} x = Bx$ for $x \in X$. Therefore $\mathcal{B}^* \in \mathcal{L}(Z^*, X_B^*)$ is also an isometric isomorphism. Let $y^* \in Z^*$ and $x \in X$; then

$$\langle (j^{-1})^* \mathcal{B}^* y^*, x \rangle_{X^* \times X} = \langle \mathcal{B}^* y^*, j^{-1} x \rangle_{X_B^* \times X_B} = \langle y^*, \mathcal{B} j^{-1} x \rangle_{Z^* \times Z} = \langle B^* y^*, x \rangle_{X^* \times X},$$

hence for any $y^* \in Z^*$ we have $(j^{-1})^* \mathcal{B}^* y^* = B^* y^*$ and the surjectivity of $\mathcal{B}$ yields $X_{B^*} = R(B^*) = (j^{-1})^* X_B^*$. Furthermore, for $x^* \in X_B^*$ we obtain

$$\|(j^{-1})^* x^*\|_{X_{B^*}} = \sup_{\|y\|_Z = 1} |\langle (B^*)^{-1} (j^{-1})^* x^*, y \rangle_{Z^* \times Z}| = \sup_{\|y\|_Z = 1} |\langle (\mathcal{B}^{-1})^* x^*, y \rangle_{Z^* \times Z}| =$$

$$= \sup_{\|y\|_Z = 1} |\langle x^*, \mathcal{B}^{-1} y \rangle_{X_B^* \times X_B}| = \|x^*\|_{X_B^*}$$

as $\mathcal{B}$ is an isometric isomorphism of X_B onto Z. ∎

Since X_B^* is a reflexive Banach space as the dual of the reflexive space X_B, then also X_{B^*} is reflexive by eq. (5.3). We have the following theorem.

THEOREM 5.1. – *Assume that B satisfies the assumption* (B) *and in addition $D(A)$ is dense in X. Then $A \in B - \mathcal{G}(M, \omega, X, Z)$ if and only if $(A^*, D(A^*) \cap X_{B^*}) \in \in \mathcal{G}(M, \omega, X_{B^*})$.*

PROOF. – By Theorem 4.1, $A \in B - \mathcal{G}(M, \omega, X, Z)$ if and only if $\mathfrak{A} = \overline{\tilde{A}}^{X_B}$ generates a semigroup in X_B. Thanks to the reflexivity of X_B, by [6], p. 41, we obtain that this is equivalent to the condition that $\mathfrak{A}^*$ generaties a semigroup (of the same type) in X_B^* and by isometry this is equivalent to the fact that $j^* \mathfrak{A}^* (j^{-1})^*$ generates a semigroup in X_{B^*}. However, for any $x^* \in X^*$ satisfying $j^* x^* \in D(\mathfrak{A}^*) \subset X_B^*$ and $x \in D(A)$

$$\langle x^*, Ax \rangle_{X^* \times X} = \langle x^*, j \mathfrak{A} j^{-1} x \rangle_{X^* \times X} = \langle (j^{-1})^* \tilde{A} j^* x^*, x \rangle_{X^* \times X},$$

and since $(j^{-1})^* \mathfrak{A}^* j^* x^* \in X^*$, we see that $x^* \in D(A^*)$, in other words, that $(j^{-1})^* \mathfrak{A}^* j^* \subset A^*$. On the other hand, let $x^* \in D(A^*) \cap X_{B^*}$, then $x^* \in D(j^*)$ and for any $x \in D(A)$ we obtain

$$\langle j^* A^* x^*, j^{-1} x \rangle_{X_B^* \times X_B} = \langle j^* x^*, \mathfrak{A} j^{-1} x \rangle_{X^* \times X},$$

which shows that there is $w^* = j^* A^* x^* \in X_B^*$ such that $\langle w^*, \phi \rangle_{X_B^* \times X_B} = \langle \psi^*, \mathfrak{A} \phi \rangle_{X^* \times X}$ for any $\phi \in j^{-1} D(A)$, where $\psi^* = j^* x^* \in X_B^*$. Since $\mathfrak{A}$ is the closure of $(j^{-1} A j, j^{-1} D(A))$, this equality holds for any $\phi \in D(\mathfrak{A})$ and $\psi \in D(\mathfrak{A}^*)$. Consequently, $x^* \in D((j^{-1})^* \mathfrak{A}^* j^*)$. Thus $((j^{-1})^* \mathfrak{A}^* j^*, (j^{-1})^* D(\mathfrak{A}^*)) = (A^*, X_{B^*} \cap D(A^*))$ and the theorem is proved. ■

5.2. *Non-invertible B.*

Let us turn our attention to the case when $N(B) \neq \{0\}$ and/or $\overline{R(B)} \neq Z$. As in Subsection 2.2, we introduce the factor space $\underline{X} = X/N(B)$ and the space $Z_B = R(B)^Z$. If X and Z are reflexive, then so are $\underline{X}$ and Z_B (see e.g. [11], pp. 220-221). Clearly, the operator $\underline{B}: \underline{X} \to Z_B$ defined by eq. (2.5) satisfies the assumption (B).

The domain $D(\underline{A})$ of the operator $\underline{A}$, defined in Lemma 3.2, is dense in $\underline{X}$, provided $D(A)$ is dense in X. Indeed, let $\underline{x} \in \underline{X}$; take any representative $x \in \underline{x}$ and the sequence $(x_n)_{n \in \mathbb{N}}$ of elements of $D(A)$ such that $\lim_{n \to \infty} x_n = x$. Then $\lim_{n \to \infty} \underline{x_n} = \underline{x}$ as follows from $\|\underline{x_n} - \underline{x}\|_{\underline{X}} \leq \|x - x_n\|_X$. Hence we can apply considerations of Step 2 of the proof of Theorem 4.1 to the effect that $A \in B - \mathcal{G}(M, \omega, X, Z)$ if and only if $(A^*, D(\underline{A}^* \cap R(\underline{B}^*)) \in \in \mathcal{G}(M, \omega, \underline{X}_{B^*})$, where $\underline{X}_{B^*} = R(\underline{B}^*)$ equipped with the graph norm $\|(\underline{B}^*)^{-1} \cdot\|_{Z_B^*}$. This result is, however, of limited practical value as the operators and spaces involved are quite complicated. In what follows we obtain a simpler formulation.

It is known, [11], pp. 220-221, that the space $\underline{X}^* = (X/N(B))^*$ is isometrically isomorphic with $N(B)^\perp$ (the annihilator of $N(B)$) and the isomorphism $\mathfrak{J}: N(B)^\perp \to \to (X/N(B))^*$ is given by

$$\langle \mathfrak{J} x^*, \underline{x} \rangle_{\underline{X}^* \times \underline{X}} = \langle x^*, x \rangle_{X^* \times X} \tag{5.4}$$

We note the following lemma.

LEMMA 5.2 *The operator $(A^*, D(A^*) \cap N(B)^{\perp})$ is an extension of $(\mathfrak{J}^{-1}\underline{A}^*\mathfrak{J}, \mathfrak{J}^{-1}D(\underline{A}^*))$. In fact, if we define*

$$D^* = \{x^* \in D(A^*) \cap N(B)^{\perp}, A^*x^* \in N(B)^{\perp}\}, \tag{5.5}$$

then

$$(A^*, D^*) = (\mathfrak{J}^{-1}\underline{A}^*\mathfrak{J}, \mathfrak{J}^{-1}D(\underline{A}^*)). \tag{5.6}$$

Moreover, if $D(A) \cap N(B)$ is dense in $N(B)$, then $D^* = D(A^*) \cap N(B)^{\perp}$ and

$$A^*|_{D(A^*) \cap N(B)^{\perp}} = \mathfrak{J}^{-1}\underline{A}^*\mathfrak{J}. \tag{5.7}$$

PROOF. – Let $x^* \in \mathfrak{J}^{-1}D(\underline{A}^*)$ and $x \in D(A)$; then we have by eq. (5.4)

$$\langle x^*, Ax\rangle_{X^* \times X} = \langle \mathfrak{J}x^*, \underline{A}\underline{x}\rangle_{\underline{X}^* \times \underline{X}} = \langle \mathfrak{J}^{-1}\underline{A}^*\mathfrak{J}x^*, x\rangle_{X^* \times X}.$$

This shows that $x^* \in D(A^*)$ and consequently the first part of the theorem is proved. Let now $x^* \in D^*$, then $A^*x^* \in N(B)^{\perp}$ and by eq. (5.4) we obtain for any $x \in D(A)$

$$\langle A^*x^*, x\rangle_{X^* \times X} = \langle \mathfrak{J}^{-1}\mathfrak{J}A^*x^*, x\rangle_{X^* \times X} = \langle \mathfrak{J}A^*x^*, \underline{x}\rangle_{\underline{X}^* \times \underline{X}}.$$

On the other hand

$$\langle A^*x^*, x\rangle_{X^* \times X} = \langle x^*, Ax\rangle_{X^* \times X} = \langle \mathfrak{J}x^*, \underline{A}\underline{x}\rangle_{\underline{X}^* \times \underline{X}}$$

for arbitrary $\underline{x} \in D(\underline{A})$. This shows that $\mathfrak{J}x^* \in D(\underline{A}^*)$ and we have the equality (5.6). Let $\overline{D(A) \cap N(B)} = N(B)$. For $x^* \in D(A^*) \cap N(B)^{\perp}$ and $x \in D(A) \cap N(B)$ we obtain

$$\langle A^*x^*, x\rangle_{X^* \times X} = \langle x^*, Ax\rangle_{X^* \times X} = 0,$$

and by density we obtain that $A^*(D(A^*) \cap N(B)^{\perp}) \subset N(B)^{\perp}$, hence $D(A^*) \cap N(B)^{\perp} = D^*$ and eq. (5.7) holds. ■

Let $\mathcal{X}_{B^*}$ be the space defined by $\mathcal{X}_{B^*} = R(B^*) \subset X^*$ with the norm given by

$$\|x^*\|_{\mathcal{X}_{B^*}} = \inf_{x^* = B^*y^*} \|y^*\|_{Z^*}.$$

Note that if $\overline{(R(B))} = Z$, then B^* is injective and the norm simplifies to $\|x^*\|_{\mathcal{X}_{B^*}} = \|(B^*)^{-1}x^*\|_{Z^*}$. We need the following result.

LEMMA 5.3. – *The operator $\mathfrak{J}$: $\mathcal{X}_{B^*} \to \underline{X}_{B^*}$, defined by eq.* (5.4) *is an isometric isomorphism.*

PROOF. – We know that Z_B^* is isometrically isomorphic to $Z^*/(\overline{R(B)}^{\perp})$ and this isomorphism is given by the following formula: $\iota y_B^* = [y^*]$, where y^* is an arbitrary extension of y_B^* to a functional from Z^* via Hahn-Banach theorem. Let $y_B^* \in Z_B^*$ and $x \in X$. Then

$$\langle \mathfrak{J}^{-1}(\underline{B})^* y_B^*, x\rangle_{X^* \times X} = \langle (\underline{B})^* y_B^*, \underline{x}\rangle_{\underline{X}^* \times \underline{X}} =$$

$$= \langle y_B^*, \underline{B}\underline{x}\rangle_{Z_B^* \times Z_B} = \langle y^*, Bx\rangle_{Z^* \times Z} = \langle B^* y^*, x\rangle_{X^* \times X}$$

which shows that $(\underline{B})^* = \mathfrak{J}B^*$ and consequently $R(B^*) = R(\mathfrak{J}^{-1}(\underline{B})^*)$. For the norms we have

$$\|\mathfrak{J}x^*\|_{\underline{X}_{B^*}} = \inf\{\|y^*\|_{Z^*};\ y^* \in [y^*],\ \mathfrak{J}x^* = \underline{B}^*[y^*]\} =$$

$$= \inf\{\|y^*\|_{Z^*};\ y^* \in [y^*],\ x^* = \mathfrak{J}^{-1}\underline{B}^*[y^*]\} = \inf\{\|y^*\|_{Z^*};\ x^* = B^*y^*\} = \|x^*\|_{\mathcal{X}_{B^*}},$$

where in the last equality we used $(\underline{B})^* = \mathfrak{J}B^*$. ■

With these two results we can write the following generalization of Theorem 5.1.

THEOREM 5.2 *Assume that* $B \in \mathcal{L}(X, Z)$ *and* $D(A)$ *is dense in* X. *Then* $A \in B-$ $-\mathcal{G}(M, m, X, Z)$ *if and only if* $(A^*, D^*) \in \mathcal{G}(M, \omega, \mathcal{X}_{B^*})$, *where* D^* *was defined in eq.* (5.5). *If, moreover,* $\overline{D(A) \cap N(B)} = N(B)$, *then* (A^*, D^*) *can be replaced by* $A^* |_{D(A^*) \cap N(B)^\perp}$.

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