

GENERATORS AND MAXIMAL IDEALS IN SOME
ALGEBRAS OF HOLOMORPHIC FUNCTIONS

NIRO YANAGIHARA

(Received October 2, 1973)

1. Introduction. Let D be the unit disk $\{|z| < 1\}$. A holomorphic function $f(z)$ in D is said to belong to the class N of functions of bounded characteristic if

$$(1.1) \quad T(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta = O(1) \text{ as } r \rightarrow 1.$$

A function $f(z) \in N$ is said to belong to the class N^+ [2, p. 25] if

$$(1.2) \quad \lim_{r \rightarrow 1} \int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta = \int_0^{2\pi} \log^+ |f(e^{i\theta})| d\theta.$$

The class N^+ can be considered as an F -space in the sense of Banach [1, p. 51], with the metric [9]

$$(1.3) \quad \rho(f, g) = \frac{1}{2\pi} \int_0^{2\pi} \log(1 + |f(e^{i\theta}) - g(e^{i\theta})|) d\theta \text{ for } f, g \in N^+.$$

N^+ is easily seen to be a topological algebra with respect to this metric (1.3). N^+ is neither locally convex nor locally bounded, but has sufficiently many continuous linear functionals to form a dual system $\langle (N^+)^*, N^+ \rangle$ in the sense of Dieudonné and Mackey [7, p. 88].

On the other hand, we defined a Fréchet space F^+ which contains N^+ [9]. We say that a holomorphic function $f(z)$ in D belongs to the class F^+ if

$$(1.4) \quad M(r, f) = \max_{|z|=r} |f(z)| \leq K_f \exp [\omega_f(r)/(1-r)]$$

with a constant $K_f > 0$ and a continuous function $\omega_f(r)$, $0 \leq r < 1$, depending on $f \in F^+$, such that $\omega_f(r) \downarrow 0$ as $r \rightarrow 1$. A holomorphic function $f(z) = \sum a_n z^n$ belongs to F^+ if and only if

$$(1.5) \quad \|f\|_{F^c} = \int_0^1 \exp \left[\frac{-c}{1-r} \right] M(r, f) dr < \infty$$

for each $c > 0$. (1.4) is equivalent to

$$(1.6) \quad a_n = O(\exp[o(\sqrt{n})]) \quad \text{as } n \rightarrow \infty.$$

F^+ is a countably normed (locally convex) Fréchet space with the system of (semi-)norms $\{\|\cdot\|_{F^e}\}_{e>0}$. F^+ is the second dual space for the space N^+ [11], and is a nuclear as well as a Montel space [11]. We can easily see that F^+ is a topological algebra.

In this note, we will characterize generators of the algebra F^+ , following to the methods of Hörmander [3], Kelleher and Taylor [5], [6]. Although they treat mainly with several variables, we confine here ourselves only to one variable case. Generalizations to several variables are concerns of our further study.

In §§ 5-6, we will determine closed and other maximal ideals in F^+ .

2. Generators for F^+ . Let $f_1, \dots, f_N \in F^+$. The ideal in F^+ generated by $\bar{f} = (f_1, \dots, f_N)$ is denoted as $I(f_1, \dots, f_N)$. We write

$$(2.1) \quad \|\bar{f}(z)\|^2 = |f_1(z)|^2 + \dots + |f_N(z)|^2, \quad z \in D.$$

If $u \in F^+$ belongs to the ideal $I(f_1, \dots, f_N)$, then it is easily seen that there exist a constant $K > 0$ and a continuous function $\omega(r)$, $\omega(r) \downarrow 0$ as $r \rightarrow 1$, such that

$$(2.2) \quad |u(z)| \leq K \|\bar{f}(z)\| \exp [\omega(r)/(1-r)], \quad |z| = r.$$

THEOREM 1. *If $u \in F^+$ satisfies (2.2), then we have*

$$u^2 \in I(f_1, \dots, f_N).$$

As a corollary of Theorem 1, we have

THEOREM 2. *Let $f_1, \dots, f_N \in F^+$. In order that there exist $g_1, \dots, g_N \in F^+$ such that*

$$(2.3) \quad f_1 g_1 + \dots + f_N g_N = 1,$$

it is necessary and sufficient that

$$(2.4) \quad |f_1(z)| + \dots + |f_N(z)| \geq \delta \exp [-\omega(r)/(1-r)]$$

($r = |z|$) for some constant $\delta > 0$ and for some continuous function $\omega(r)$, $\omega(r) \downarrow 0$ as $r \rightarrow 1$.

In contrast to Theorem 1, we have

THEOREM 1*. *(2.2) does not imply that $u \in I(f_1, \dots, f_N)$ for $u \in F^+$.*

For the proof, we follow to the method of Rao [8].

In connection with Theorem 2, we have

THEOREM 2*. *Let $f_1, \dots, f_N \in N^+$. Then, (2.4) is not sufficient for f_1, \dots, f_N to be generators of N^+ . That is, (2.4) does not imply (2.3) in N^+ .*

In contrast to the case of Banach algebras, we have

THEOREM 3. *Maximal ideals in F^+ are not necessarily closed.*

Hence, in § 6, we will use somewhat strange method for compactifying in order to put it in a one-to-one correspondence with the maximal ideal space of F^+ .

3. Proof of Theorem 1.

LEMMA 1. *Let $\omega_1(r)$ be a continuous function, $\omega_1(r) \downarrow 0$ as $r \rightarrow 1$. Then we can find a continuous function $\omega(r)$ such that $\omega(r) \geq \omega_1(r)$ and $\omega(r) \geq \sqrt{1-r}$,*

$$(3.1) \quad \omega(r) \downarrow 0, \quad \omega(r)/(1-r) \uparrow \infty \quad \text{as } r \rightarrow 1$$

$\omega(r)/(1-r)$ is convex.

PROOF. We can suppose that $\omega_1(r)$ is continuously differentiable and $\omega_1'(r) < 0$ for $0 \leq r < 1$.

Let $r_0 = 0$. Let r_1 be a number, $1/2 < r_1 < 1$, and put

$$a_1 = \omega_1(r_0)/(1-r_1)^2, \\ b_1 = -a_1 r_1 + \omega_1(r_0)/(1-r_1).$$

Let r_2 be such that $r_2 > r_1$ and

$$a_1 r_2 + b_1 = \omega_1(r_1)/(1-r_2).$$

Then $r_2 < 1$. Suppose $\{r_k\}_{k=0}^n$, $r_k < r_{k+1}$, and $\{a_k\}_{k=1}^{n-1}$, $\{b_k\}_{k=1}^{n-1}$ be determined. Then, put

$$(3.2) \quad a_n = \omega_1(r_{n-1})/(1-r_n)^2,$$

$$(3.2') \quad b_n = -a_n r_n + \omega_1(r_{n-1})/(1-r_n),$$

and let r_{n+1} be such that $r_{n+1} > r_n$ and

$$(3.2'') \quad a_n r_{n+1} + b_n = \omega_1(r_n)/(1-r_{n+1}),$$

then $r_{n+1} < 1$. We will show that $r_n \uparrow 1$. For that purpose, we put

$$(3.3) \quad \rho = \lim_{n \rightarrow \infty} r_n.$$

We have

$$(3.4) \quad a_n = (r_{n+1} - r_n)^{-1} (\omega_1(r_n)/(1-r_{n+1}) - \omega_1(r_{n-1})/(1-r_n)) \\ = \frac{1}{1-r_{n+1}} \frac{\omega_1(r_n) - \omega_1(r_{n-1})}{r_n - r_{n-1}} \frac{r_n - r_{n-1}}{r_{n+1} - r_n} + \frac{\omega_1(r_{n-1})}{(1-r_{n+1})(1-r_n)}.$$

If ρ in (3.3) would be $\rho < 1$, we would have, letting $n \rightarrow \infty$ in (3.4),

$$\omega_1(\rho)/(1-\rho)^2 = (1-\rho)^{-1}\omega'_1(\rho) \lim_{n \rightarrow \infty} \frac{r_n - r_{n-1}}{r_{n+1} - r_n} + \omega_1(\rho)/(1-\rho)^2$$

hence

$$\lim_{n \rightarrow \infty} ((r_n - r_{n-1})/(r_{n+1} - r_n)) = 0$$

since $\omega'_1(\rho) < 0$. Then, for $\varepsilon < 1$, we would have

$$r_{n+1} - r_n > (1/\varepsilon)^{n-n_0}(r_{n_0} - r_{n_0-1}), \quad n \geq n_0 \quad \text{for an } n_0.$$

Letting $n \rightarrow \infty$, we obtain a contradiction. Hence we must have

$$\lim_{n \rightarrow \infty} r_n = 1.$$

Having proved that $r_n \uparrow 1$, we define

$$(3.5) \quad \omega(r) = (1-r)(a_n r + b_n) \quad \text{for } r_n \leq r \leq r_{n+1}, \quad n = 0, 1, \dots$$

Then, since

$$a_{n+1} > a_n \quad \text{and} \quad \omega(r) > \sqrt{1-r}$$

we have

$$\omega(r)/(1-r) \text{ is convex and } \omega(r)/(1-r) \uparrow \infty.$$

Further,

$$\omega(r_n) = \omega_1(r_{n-1}), \quad \omega(r_{n+1}) = \omega_1(r_n) \downarrow 0$$

and

$$\omega(r) = v_n(r) \quad \text{for } r_n \leq r \leq r_{n+1},$$

where

$$v_n(r) = -a_n r^2 + (a_n - b_n)r + b_n.$$

Since

$$v'_n(r_n) = -2a_n r_n + (a_n - b_n) = 0,$$

we get that

$$\omega(r) \text{ is concave and monotone decreasing for } r_n \leq r \leq r_{n+1}.$$

Thus $\omega(r) \downarrow 0$, and our Lemma 1 is proved.

q.e.d.

By the Lemma 1, functions $\omega(r)$ in the below may be supposed to satisfy the condition (3.1).

LEMMA 2. *There exists a constant K such that for any $z \in D$, $|z - \zeta| \leq K \exp[-\omega(r)/(1-r)]$, $r = |z|$, implies $\zeta \in D$ and moreover*

$$\omega(\rho)/(1 - \rho) \leq 2\omega(r)/(1 - r), \quad \rho = |\zeta|.$$

PROOF. Since $\exp[-\omega(r)/(1 - r)] \leq \exp[-1/\sqrt{1 - r}]$, we have

$$\exp[-\omega(r)/(1 - r)] \leq (1 - r)/2, \quad r \geq R$$

with an $R < 1$. Put $K = (1 - R)/2$. Then we have that, if

$$z \in D, \quad |z - \zeta| \leq K \exp[-\omega(r)/(1 - r)],$$

we get

$$(3.6_1) \quad \rho = |\zeta| \leq (1 + r)/2 < 1.$$

If $\rho \geq r$, we have, as $1 - \rho \geq (1 - r)/2$,

$$(3.6_2) \quad \omega(\rho)/(1 - \rho) \leq 2\omega(r)/(1 - r).$$

If $\rho < r$, we have by (3.1)

$$(3.6_3) \quad \omega(\rho)/(1 - \rho) \leq \omega(r)/(1 - r).$$

(3.6₁₋₃) give the lemma. q.e.d.

We note that $f \in F^+$ implies $f' \in F^+$ [9, Theorem 6].

LEMMA 3. *If f is holomorphic in D , then f belongs to F^+ if and only if for some $\omega(r)$ satisfying (3.1)*

$$(3.7) \quad (\|f\|_\omega)^2 = \frac{1}{\pi} \iint_D |f(re^{i\theta})|^2 \exp\left[\frac{-\omega(r)}{1 - r}\right] r dr d\theta < \infty.$$

PROOF. $f \in F^+$ obviously satisfies (3.7) with some $\omega(r)$. On the other hand, it follows that the mean value of $|f|$ over the disk with center at $z \in D$:

$$\{\zeta; |\zeta - z| \leq K \exp[-\omega(r)/(1 - r)]\} \subset D$$

is bounded by

$$(1/K) \|f\|_\omega \exp[2\omega(r)/(1 - r)].$$

By the subharmonicity of $|f|$, this gives also a bound for $|f(z)|$, $|z| = r$, which shows that $f \in F^+$ by (1.4).

LEMMA 4. *Let g be a form of type (0, 1) in D with locally square summable coefficient $g(r, \theta)$, and let $\phi(r, \theta)$ be a subharmonic function in D such that*

$$\iint_D |g(r, \theta)|^2 e^{-\phi(r, \theta)} r dr d\theta < \infty.$$

It follows that there is a function f (a form of type (0, 0)) with $\bar{\partial}f = g$, and

$$\begin{aligned} & \iint_D |f(r, \theta)|^2 e^{-\phi(r, \theta)} (1+r^2)^{-2} r dr d\theta \\ & \leq \iint_D |g(r, \theta)|^2 e^{-\phi(r, \theta)} r dr d\theta . \end{aligned}$$

Proof is found in [3, p. 945, Lemma 4].

For non-negative integers p and q , we shall denote by L_q^p the set of all differential forms h of type $(0, q)$ with values in $L^p C^N$, such that for some function $\omega(r)$ satisfying (3.1),

$$(3.8) \quad \iint_D |h(r, \theta)|^2 \exp \left[\frac{-\omega(r)}{1-r} \right] r dr d\theta < \infty .$$

In other words, for each p -tuple $S = (i_1, \dots, i_p)$, $1 \leq i_1, \dots, i_p \leq N$, h has a component h_S which is a differential form of type $(0, q)$ such that h_S is skew symmetric in S and

$$\iint_D |h_S(r, \theta)|^2 \exp \left[\frac{-\omega(r)}{1-r} \right] r dr d\theta < \infty .$$

Note that $L_q^p = 0$ if $p > N$ or $q > 1$.

Now $\bar{\partial}$ -operator acts componentwise on the elements of L_q^p and yields a linear mapping $\bar{\partial}: L_q^p \rightarrow \{(0, q+1)\text{-forms with values in } L^p C^N\}$, such that $\bar{\partial}^2 = 0$. Furthermore, the interior product P_f by $\bar{f} = (f_1, \dots, f_N)$ maps L_q^{p+1} into L_q^p : If $h \in L_q^{p+1}$ then

$$(3.9) \quad (P_f h)_S = \sum_{j=1}^N h_{S_j} f_j \quad \text{for } S = (i_1, \dots, i_p) .$$

We define $P_f L_q^0 = 0$. Clearly $P_f^2 = 0$ and P_f commutes with $\bar{\partial}$ since f_1, \dots, f_N are holomorphic. So, we have a double complex.

LEMMA 5. For every $h \in L_q^p$, the equation $\bar{\partial}g = h$ has a solution $g \in L_q^p$.

Proof. This follows immediately from the Lemma 4.

LEMMA 6. For any $v \in C^2(D)$ we have for $0 \leq r < 1$,

$$\int_0^r t^{-1} S(t) dt = \frac{1}{2\pi} \int_0^{2\pi} v(r, \theta) d\theta - v(0) ,$$

where

$$S(t) = \frac{1}{2\pi} \iint_{|z| \leq t} \Delta v dx dy , \quad \Delta v = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} .$$

Proof is a simple consequence of Green's formula. See [4, p. 231, Lemma 3.3].

LEMMA 7. Let $f_1, \dots, f_N \in F^+$. Then, if we put

$$w_{ij}(z) = (f_i(z)f'_j(z) - f_j(z)f'_i(z))/\|\bar{f}(z)\|^2, \quad i, j = 1, \dots, N,$$

we have

$$\iint_D |w_{ij}(z)|^2 \exp\left[\frac{-\omega(r)}{1-r}\right] r dr d\theta < \infty$$

for a function $\omega(r)$ satisfying (3.1).

PROOF. At first we suppose that f_1, \dots, f_N have no common zeros. Then

$$(3.10) \quad v(z) = 2 \log \|\bar{f}(z)\| \in C^2(D),$$

and, if we write

$$w(z) = \frac{1}{4} \Delta v = \frac{1}{4} \sum_{i,j=1}^N |f_i(z)f'_j(z) - f_j(z)f'_i(z)|^2 / \|\bar{f}(z)\|^4,$$

it suffices to prove that

$$\iint_D w(z) \exp\left[\frac{-\omega(r)}{1-r}\right] r dr d\theta < \infty.$$

We apply Lemma 6 for the function v in (3.10). Then

$$\int_0^r t^{-1} S(t) dt \leq \frac{\lambda(r)}{1-r}$$

for a continuous function $\lambda(r)$ satisfying (3.1). Now $S(t)$ is non-negative and increasing, since $v(z)$ in (3.10) is subharmonic, so

$$\int_0^r \frac{S(t)}{t} dt \geq \int_{r^2}^r \frac{S(t)}{t} dt \geq \frac{S(r^2)}{r} r(1-r),$$

thus

$$S(r^2) \leq \frac{\lambda(r)}{(1-r)^2} \leq \frac{4\lambda(r)}{(1-r^2)^2} \text{ hence } S(r) \leq \left(\frac{\omega(r)}{1-r}\right)^2$$

with a continuous function $\omega(r)$ satisfying (3.1).

Then, writing $\omega(r)/(1-r) = p(r)$,

$$\begin{aligned} \iint_D w(z) \exp\left[\frac{-\omega(r)}{1-r}\right] r dr d\theta &\leq \iint_{p(r) \leq 2} + \sum_{n=1}^{\infty} \iint_{2^n \leq p(r) \leq 2^{n+1}} \\ &\leq K_0 + \sum_{n=1}^{\infty} e^{-2^n} \iint_{p(r) \leq 2^{n+1}} \Delta v dx dy \\ &\leq K_0 + \sum_{n=1}^{\infty} 2^{2n+2} \exp[-2^n] < \infty. \end{aligned}$$

For the case where f_1, \dots, f_N have common zeros, the desired conclusion may be deduced via a standard argument by considering $v_i(z) = \log(\|\bar{f}(z)\|^2 + \varepsilon^2)$ and letting $\varepsilon \rightarrow 0$. q.e.d.

PROOF OF THEOREM 1. Suppose $u \in F^+$ satisfies (2.2). Let $\alpha = (\alpha_1, \dots, \alpha_N) \in L_0^1$ be such that

$$\alpha_i = u^2 \bar{f}_i / \|\bar{f}\|^2, \quad i = 1, \dots, N.$$

Then

$$\bar{\partial} \alpha_i = \|\bar{f}\|^{-4} u^2 \sum_{j=1}^N f_j \overline{(f_j \partial f_i - f_i \partial f_j)}.$$

If we put

$$\beta_{ij} = \|\bar{f}\|^{-4} u^2 \overline{(f_j \partial f_i - f_i \partial f_j)},$$

then we get $\beta = (\beta_{ij}) \in L_1^2$ by Lemma 7. Clearly, $\bar{\partial} \alpha = P_f \beta$, and there exists $\gamma \in L_0^2$ with $\bar{\partial} \gamma = \beta$ by Lemma 5. Then, if we put

$$g = \alpha - P_f \gamma \in L_0^1,$$

then $\bar{\partial} g = 0$, hence $g_j \in F^+$, $j = 1, \dots, N$, by Lemma 3, and

$$P_f g = u^2, \quad \text{which shows that } u^2 \in I(f_1, \dots, f_N). \quad \text{q.e.d.}$$

4. Proofs of Theorem 1*, 2* and 3.

PROOF OF THEOREM 1*. Let $f, g \in F^+$. If we take in (2.2) $N = 2$, $f_1 = f^2, f_2 = g^2$ and $u = fg$, then (2.2) holds. If it were true that (2.2) would imply $u \in I(f_1, \dots, f_N)$, we would have $fg \in I(f^2, g^2)$ for any $f, g \in F^+$. We will show that this is not the case for some f and g .

Suppose $fg \in I(f^2, g^2)$, i.e., $fg = Af^2 + Bg^2$ with $A, B \in F^+$. Then

$$(4.1) \quad Af^2/g = f - Bg \in F^+, \quad Bg^2/f = g - Af \in F^+.$$

We put

$$f(z) = \prod_{k=1}^{\infty} ((z - z_k)/(1 - \bar{z}_k z)),$$

where

$$z_k = 1 - b^k \quad \text{with a constant } b, 0 < b < 1/3,$$

and

$$g(z) = \exp \left[-c \frac{1+z}{1-z} \right] \quad \text{with a constant } c > 0.$$

Then by (4.1), B/f is holomorphic.

Then, as we shall see shortly later, if $H(z)$ is holomorphic in D ,

$$(4.2) \quad f \times H \in F^+ \quad \text{implies} \quad H \in F^+.$$

Thus

$$\begin{aligned} A/g &= p \in F^+ & \text{since } (A/g)f^2 \in F^+, \\ B/f &= q \in F^+ & \text{since } B \in F^+. \end{aligned}$$

Hence

$$(4.3) \quad 1 = p \times f + q \times g .$$

This is impossible, since $p(z_k)f(z_k) = 0$ and $q(z_k)g(z_k) \rightarrow 0$ as seen from (1.4) and the definition of $g(z)$.

Now we will show (4.2). First we note that

$$|f(z)| \geq \prod \|z| - |z_k|\| / (1 - |z_k||z|) .$$

Put

$$r^{(n)} = 1 - b^n(1 + b)/2 = (z_n + z_{n+1})/2 .$$

Then, if $|z| = r^{(n)}$,

$$\begin{aligned} \|z| - |z_k|\| &= |b^k - b^n(1 + b)/2| , \\ 1 - |z_k||z| &\leq b^k + b^n(1 + b)/2 . \end{aligned}$$

Thus, if $|z| = r^{(n)}$,

$$\begin{aligned} (4.4) \quad |f(z)| &= \prod_{k \leq n} \prod_{k > n} \geq \prod_{k \leq n} \frac{1 - b^{n-k}(1 + b)/2}{1 + b^{n-k}(1 + b)/2} \prod_{k > n} \frac{1 - b^{k-n} \times 2/(1 + b)}{1 + b^{k-n} \times 2/(1 + b)} \\ &\geq \prod_{m \geq 0} \frac{1 - b^m(1 + b)/2}{1 + b^m(1 + b)/2} \prod_{m \geq 0} \frac{1 - b^m \times 2b/(1 + b)}{1 + b^m \times 2b/(1 + b)} = K > 0 . \end{aligned}$$

Let

$$h(z) = f(z)H(z) \in F^+ .$$

Then, for any constant $a > 0$,

$$M(r, h) \exp \left[\frac{-a}{1 - r} \right] \rightarrow 0 \quad \text{as } r \rightarrow 1 .$$

By (4.4), we have

$$M(r^{(n)}, H) \leq M(r^{(n)}, h)/K .$$

Thus, for $r^{(n-1)} \leq r \leq r^{(n)}$

$$M(r, H) \leq M(r^{(n)}, h)/K .$$

Hence, for $r^{(n-1)} \leq r \leq r^{(n)}$,

$$M(r, H) \exp \left[\frac{-a}{1 - r} \right] \leq K^{-1} M(r^{(n)}, h) \exp [-a/(1 - r^{(n-1)})]$$

$$\leq K^{-1}M(r^{(n)}, h) \exp[-ab/(1 - r^{(n)})] \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

hence

$$M(r, H) \exp\left[\frac{-a}{1-r}\right] \rightarrow 0 \quad \text{as } r \rightarrow 1$$

for any $a > 0$, which shows that $H \in F^+$.

q.e.d.

PROOF OF THEOREM 2*. Let $\nu(t)$, $0 \leq t \leq 2\pi$, be a continuous and monotone increasing function such that $\nu(0) = 0$, $\nu(2\pi) = 1$, and $\nu'(t) = 0$ almost everywhere on $[0, 2\pi]$. We put, for $2n\pi \leq t \leq (2n+2)\pi$,

$$\mu(t) = n + \nu(t - 2n\pi), \quad n = 0, \pm 1, \pm 2, \dots$$

and

$$f(z) = \exp\left[-\int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} d\mu(t)\right].$$

Then $f(z) \in H^\infty \subset N^+$ and

$$1/f(z) \in F^+,$$

as shown in [12, Proof of Theorem 1]. Therefore, $f(z)$ satisfies (2.4) but does not generate N^+ , since $f(z)$ is not invertible in N^+ while $f(z)$ is invertible in F^+ .

PROOF OF THEOREM 3. Put

$$E = \left\{ \exp\left[-c \frac{1+z}{1-z}\right]; c > 0 \right\}.$$

Then $E \subset N^+ \subset F^+$. If we write $I = \bigcup_{f \in F^+} fE$, I is a proper ideal. It is easy to see that

$$\exp\left[-c \frac{1+z}{1-z}\right] \rightarrow 1 \quad \text{as } c \rightarrow 0.$$

Hence the maximal ideal containing I is not closed.

5. Closed maximal ideals in F^+ . Let A be a topological algebra with identity 1, locally convex and commutative, over the complex number field \mathbb{C} . Topology of A is defined by a countable family of semi-norms $\{\|\cdot\|_\alpha\}_{\alpha \in I}$, which are supposed to satisfy that $\|1\|_\alpha = 1$ and for $a, b \in A$

$$(5.1) \quad \|ab\|_\alpha \leq \|a\|_\alpha \|b\|_\alpha \quad \text{for every } \alpha \in I.$$

For an $\alpha \in I$, let $E_\alpha = \{a \in A; \|a\|_\alpha = 0\}$. E_α is obviously an ideal in A . For $a \in A$, we write $\hat{a} = a + E_\alpha \in A/E_\alpha$. Then A/E_α is a normed space with $\|\hat{a}\|_\alpha = \|a\|_\alpha$. We have, by (5.1), for $a, b \in A$

$$(5.1') \quad \|a^\wedge b^\wedge\|_\alpha \leq \|a^\wedge\|_\alpha \|b^\wedge\|_\alpha.$$

The completion of A/E_α with res. to the norm $\|\cdot\|_\alpha$ is denoted as A_α^* .

LEMMA 8. *Let $g \in A$. Suppose $|1 - \mu g|$ is invertible for $|\mu| < \delta$. Then, for any $\alpha \in I$ there is a $\delta(\alpha) > 0$ such that $\|(1 - \mu g)^{-1}\|_\alpha$ is bounded for $|\mu| \leq \delta(\alpha)$.*

PROOF. Put $\delta(\alpha) = \min(\delta, 1/2 \|g\|_\alpha)$ and

$$h_n = 1 + \mu g + \cdots + \mu^n g^n \in A, \quad h_n^\wedge \in A/E_\alpha.$$

$\{h_n^\wedge\}$ is a Cauchy sequence in A_α^* if $|\mu| \leq \delta(\alpha)$. Then

$$h^\wedge = \lim_{n \rightarrow \infty} h_n^\wedge \in A_\alpha^*.$$

For a fixed μ , we define a linear operator T on A/E_α by

$$T a^\wedge = (1 - \mu g)^\wedge a^\wedge \in A/E_\alpha \text{ for } a \in A.$$

T is continuous on A/E_α by (5.1'), and continuously extended on A_α^* . Then

$$T h^\wedge = \lim_{n \rightarrow \infty} T h_n^\wedge = \lim_{n \rightarrow \infty} ((1 - \mu g) h_n)^\wedge = \lim_{n \rightarrow \infty} (1 - \mu^{n+1} g^{n+1})^\wedge = 1^\wedge.$$

Thus $h^\wedge = ((1 - \mu g)^{-1})^\wedge \in A/E_\alpha$. Then we have, for $|\mu| \leq \delta(\alpha)$

$$\|(1 - \mu g)^{-1}\|_\alpha = \lim_{n \rightarrow \infty} \|h_n^\wedge\|_\alpha = \lim_{n \rightarrow \infty} \|h_n\|_\alpha \leq 1 + \sum_{n=1}^{\infty} \|\mu g\|_\alpha^n \leq 2.$$

LEMMA 9. *Let $f \in A$ and $\lambda_0 \in \mathcal{C}$. Suppose $\lambda - f$ is invertible for $|\lambda - \lambda_0| < \delta, \delta > 0$. Then $(\lambda - f)^{-1}$ is continuous with respect to λ .*

PROOF. Put $(\lambda_0 - f)^{-1} = g$ and $\mu = \lambda_0 - \lambda$. Then

$$\begin{aligned} & (\lambda - f)^{-1} - (\lambda_0 - f)^{-1} \\ &= (\lambda_0 - f)^{-1} [(1 - \mu g)^{-1} - 1] \\ &= \mu g (\lambda_0 - f)^{-1} (1 - \mu g)^{-1} = \mu g^2 (1 - \mu g)^{-1}. \end{aligned}$$

Then for any $\alpha \in I$, if $|\mu| \leq \delta(\alpha)$,

$$\begin{aligned} & \|(\lambda - f)^{-1} - (\lambda_0 - f)^{-1}\|_\alpha \\ & \leq |\mu| \|g\|_\alpha^2 \|(1 - \mu g)^{-1}\|_\alpha \\ & \leq 2 |\mu| \|g\|_\alpha^2 \rightarrow 0 \text{ as } \lambda \rightarrow \lambda_0, \mu \rightarrow 0, \end{aligned}$$

hence $(\lambda - f)^{-1}$ is continuous.

LEMMA 10. *For any $f \in A$, there is a number λ_f such that $\lambda_f - f$ is not invertible.*

PROOF. Suppose $\lambda - f$ were invertible for any $\lambda \in \mathcal{C}$. Then $(\lambda - f)^{-1}$

is continuous with respect to λ . Let L be a continuous linear functional on A . Then

$$G(\lambda) = L((\lambda - f)^{-1})$$

is an entire function. For,

$$G(\lambda) - G(\lambda_0) = -(\lambda - \lambda_0)L((\lambda - f)^{-1}(\lambda_0 - f)^{-1}),$$

hence, by Lemma 9, we obtain

$$G'(\lambda_0) = -L((\lambda_0 - f)^{-1}(\lambda_0 - f)^{-1}).$$

Further, by the continuity of L ,

$$|G(\lambda)| = |L((\lambda - f)^{-1})| \leq K \|(\lambda - f)^{-1}\|_\alpha$$

with an $\alpha \in I$ and a constant K . Thus, by Lemma 8, if $|\lambda| > 2\|f\|_\alpha$,

$$|G(\lambda)| \leq K|\lambda|^{-1} \|(1 - f/\lambda)^{-1}\|_\alpha \leq 2K/|\lambda| \rightarrow 0$$

as $|\lambda| \rightarrow \infty$. Therefore $G(\lambda) \equiv 0$, i.e.,

$$L((\lambda - f)^{-1}) \equiv 0 \quad \text{for } \lambda \in \mathbb{C}$$

for any continuous linear functional L on A , which is absurd.

As a characterization of closed maximal ideals we have, in analogy with the well known theorem of Igusa [4], the following

THEOREM 4. *Let M be a maximal ideal in F^+ . The following conditions for M are equivalent:*

(i) M is closed in the topology of uniform convergence on every disk $|z| \leq r$, $r < 1$.

(ii) $F^+/M \cong \mathbb{C}$.

(iii) M corresponds to a point $z_0 \in D$, i.e., M consists of all functions of F^+ which vanish at z_0 .

PROOF. (i) \rightarrow (ii): Obviously, $F^+/M \supset \mathbb{C}$. For $f \in F^+$, we denote

$$[f] = f + M \in F^+/M.$$

We introduce the family of semi-norms in F^+/M as follows:

$$\|[f]\|_r = \inf_{h \in M} (\max_{|z|=r} |f(z) + h(z)|), \quad 0 \leq r < 1.$$

Then clearly

$$\|[fg]\|_r \leq \|[f]\|_r \|[g]\|_r, \quad 0 \leq r < 1.$$

By Lemma 10, to each $[f] \in F^+/M$, there corresponds a number $\lambda \in \mathbb{C}$ such that $\lambda - [f]$ is not invertible. But, since F^+/M is a field by the maximality of M , $\lambda - f$ must belong to M , i.e., $\lambda \in [f]$. Thus we obtain

$$F^+/M \cong C .$$

(ii) \rightarrow (iii): Let z_0 be the coset $[z] \in F^+/M$. Then $z - z_0 \in M$, hence $z_0 \in D$.

For each $f(z) \in F^+$, we have

$$(5.2) \quad f(z) - f(z_0) = A(z)(z - z_0) .$$

As easily seen, $A(z) \in F^+$, thus $f(z) - f(z_0) \in M$. If $f(z) \in M$, then $f(z_0) \in M$, whence $f(z_0) = 0$. Thus M corresponds to the point $z_0 \in D$.

(iii) \rightarrow (i): This is evident from the theorem of Hurwitz.

6. Maximal ideals in F^+ . Now we will study some structures of maximal ideal space of the algebra F^+ .

The complex w -sphere is denoted by W . Let Q be the set of all continuous functions $\omega(r)$, $0 \leq r < 1$, satisfying (3.1).

Taking a function $f(z) \in F^+$, we define a topology $\tau_Q(f)$ in W .

For a number $\varepsilon > 0$ and a function $\omega(r) \in Q$, we define neighborhood $U(a)$ of $a \in W$ as follows:

(A) $a \neq \infty$.

A(i) Suppose there is a point $z_0 \in D$ such that $f(z_0) = a$. Then we put for a number $\eta > 0$,

$$U(a) = U(a; \varepsilon, \omega, z_0, \eta) = \left\{ w; w = f(z), \text{ where } |z - z_0| < \eta \text{ and } \exp \left[\frac{\omega(|z|)}{1 - |z|} \right] |f(z) - a| < \varepsilon \right\}$$

A(ii) Suppose there is a point ζ_0 , $|\zeta_0| = 1$, such that

$$\lim_{z \rightarrow \zeta_0} \exp \left[\frac{\omega(|z|)}{1 - |z|} \right] |f(z) - a| = 0 .$$

Then we put for a number $\eta > 0$,

$$U(a) = U(a; \varepsilon, \omega, \zeta_0, \eta) = \left\{ w; w = a + \rho e^{i\theta}, 0 \leq \theta \leq 2\pi, \text{ and } \rho < \varepsilon \exp \left[\frac{-\omega(|z|)}{1 - |z|} \right] \text{ for a point } z \in D \text{ such that } |z - \zeta_0| < \eta, \exp \left[\frac{\omega(|z|)}{1 - |z|} \right] |f(z) - a| < \varepsilon \right\} \cup \{a\} .$$

A(iii) Suppose there is neither z_0 in A(i) nor ζ_0 in A(ii). We put

$$U(a) = U(a; \varepsilon, \omega) = \{a\} .$$

(B) $a = \infty$.

B(i) Suppose there is a point ζ_0 , $|\zeta_0| = 1$, such that

$$\overline{\lim}_{z \rightarrow \zeta_0} \exp \left[\frac{-\omega(|z|)}{1 - |z|} \right] |f(z)| = \infty .$$

Then we put for a number $\eta > 0$,

$$U(\infty) = U(\infty; \varepsilon, \omega, \zeta_0, \eta) = \{w; w = \rho e^{i\theta}, 0 \leq \theta \leq 2\pi, \text{ and}$$

$$\rho > (1/\varepsilon) \exp \left[\frac{\omega(|z|)}{1 - |z|} \right] \text{ for a point } z \in D \text{ such that}$$

$$|z - \zeta_0| < \eta, \exp \left[\frac{-\omega(|z|)}{1 - |z|} \right] |f(z)| > 1/\varepsilon\} \cup \{\infty\} .$$

B(ii) Suppose there is no point ζ_0 in B(i). Then

$$U(\infty) = U(\infty; \varepsilon, \omega) = \{\infty\} .$$

By this system of neighborhoods, W becomes a Hausdorff space. We note that the topology depends on the function $f(z)$.

Let $f(D) \subset W$ be the range of $f(z)$ in D , and $(f(D))^a$ be the closure of $f(D)$ with respect to the topology determined by f . Since $f(z) \in F^+$, $(f(D))^a$ does not contain ∞ . We compactify $(f(D))^a$ as follows:

Let P_f be an (abstract) element. Neighborhoods of P_f are defined to be open sets (in the sense of the usual Riemann sphere topology) containing $W - (f(D))^a$.

Then, $A_f = (f(D))^a \cup \{P_f\}$ is obviously compact. We note that $A_f - \{P_f\}$ satisfies the Hausdorff separation axiom, although A_f does not. A_f might be considered, in a sense, as an Alexandroff compactification of $(f(D))^a$.

Further, let C_0 be the set of all continuous complex valued functions with compact supports in D .

Put

$$(6.1) \quad T = \prod_{f \in F^+} A_f \cdot \prod_{\phi \in C_0} W_\phi \quad (W_\phi = W \text{ with the usual Riemann sphere topology})$$

T is compact with the Tychonoff topology. We denote by π_f or π_ϕ the projection of T on A_f or on W_ϕ , respectively. We write, for $z \in D$,

$$\psi(z) = \{f(z), \phi(z)\}_{f \in F^+, \phi \in C_0}$$

ψ is a continuous and one-to-one mapping from D into T . We write the closure of $\psi(D)$ in T as D^* . Then D^* is compact and $\psi(D)$ is dense in D^* .

ψ is an open mapping. To see this, for $z_0 \in D$, let U be a relatively compact neighborhood of z_0 , and ϕ be a function of C_0 with support

contained in U and $\phi(z_0) \neq 0$. Put

$$V = \{p \in D^*; \pi_\psi(p) \neq 0\}.$$

V is a neighborhood of $\psi(z_0)$ on D^* . $V - \psi(\bar{U})$ is an open set, as we shall see shortly later. But we have

$$(V - \psi(\bar{U})) \cap \psi(D) = \text{void},$$

hence $V - \psi(\bar{U})$ is void, for $\psi(D)$ is dense in D^* . Therefore we obtain

$$V \subset \psi(\bar{U}) \subset \psi(D), \quad V \subset \psi(U),$$

and ψ is an open mapping.

Now we will show that $V - \psi(\bar{U})$ is open, i.e., $\psi(\bar{U})$ is closed. Let $q \notin \psi(\bar{U})$. If there is an $f_0 \in F^+$ such that $\pi_{f_0}(q) \neq P_{f_0}$, then there is a neighborhood $U(\pi_{f_0}(q))$ such that $U(\pi_{f_0}(q)) \cap \pi_{f_0}(\psi(\bar{U})) = \text{void}$. If $\pi_f(q) = P_f$ for any $f \in F^+$, then there is, for an $f_0 \in F^+$, a neighborhood $U(P_{f_0})$ such that $U(P_{f_0}) \cap \pi_{f_0}(\psi(\bar{U})) = \text{void}$, since $\pi_{f_0}(\psi(\bar{U})) = f_0(\bar{U})$ is compact in $f_0(D)$. Thus, if $U(q)$ is a neighborhood of q such that $\pi_{f_0}(U(q)) = U(P_{f_0})$, then $U(q) \cap \psi(\bar{U}) = \text{void}$, and $(\psi(\bar{U}))^c$ is open, hence $\psi(\bar{U})$ is closed.

Thus ψ is homeomorphic, and D and $\psi(D)$ may be identified.

π_f is the continuous extension of f onto D^* . For $a, b \in D^*$, $a \neq b$, there is an $f \in F^+$ with

$$\pi_f(a) \neq \pi_f(b),$$

since for any point $p \in D^* - \psi(D)$ we have $\phi(p) = 0$ for each $\phi \in C_0$.

We put

$$P = \prod_{f \in F^+} \{P_f\} \cdot \prod_{\phi \in C_0} W_\phi$$

and

$$D^{**} = D^* - P.$$

Let \mathfrak{M} be the set of all maximal ideals in F^+ . Then

THEOREM 5. *Elements of \mathfrak{M} and points of the space D^{**} correspond in a one-to-one way.*

PROOF. Let z_0 be a point of D . It is easy to see that the set of all functions $f(z) \in F^+$ with $f(z_0) = 0$ forms a maximal ideal in F^+ .

Let J be a maximal ideal in F^+ . We suppose that there are no common zeros in D for functions of J .

Let $f_{\alpha_1}, \dots, f_{\alpha_N}$ be functions of J . Thus, by Theorem 2, we have for any $\omega(r) \in Q$,

$$(6.3) \quad \inf_{r_0 \leq r < 1} \exp[\omega(r)/(1-r)](|f_{\alpha_1}(z)| + \dots + |f_{\alpha_N}(z)|) = 0$$

($r = |z|$) for any $r_0 < 1$. Thus, there is a sequence $\{z_n\} \subset D$, $r_n = |z_n| \rightarrow 1$, such that

$$\liminf_{\substack{n \rightarrow \infty \\ k \geq n}} \exp [\omega(r_k)/(1 - r_k)] (|f_{\alpha_1}(z_k)| + \cdots + |f_{\alpha_N}(z_k)|) = 0.$$

We denote by $E(\alpha_1, \dots, \alpha_N; \omega)$ the set of points of D^* such that

$$\zeta^* \in E(\alpha_1, \dots, \alpha_N; \omega) \text{ if for any neighborhood } U(\zeta^*), \\ \inf_{z \in U(\zeta^*) \cap D} \exp [\omega(|z|)/(1 - |z|)] (|f_{\alpha_1}(z)| + \cdots + |f_{\alpha_N}(z)|) = 0.$$

$E(\alpha_1, \dots, \alpha_N; \omega)$ is a closed non-void subset of the compact space D^* , for every $\omega(r) \in Q$, by Theorem 2, since J is a proper ideal. If $\omega_1(r), \dots, \omega_M(r) \in Q$, we have

$$E(\alpha_1, \dots, \alpha_N; \omega_1) \cap \cdots \cap E(\alpha_1, \dots, \alpha_N; \omega_M) \\ \supset E(\alpha_1, \dots, \alpha_N; \omega_1 + \cdots + \omega_M) \neq \text{void}.$$

Hence

$$E(\alpha_1, \dots, \alpha_N) = \bigcap_{\omega \in Q} E(\alpha_1, \dots, \alpha_N; \omega)$$

is non-void. Since

$$E(\alpha_1, \dots, \alpha_N) \cap E(\alpha'_1, \dots, \alpha'_K) \supset E(\alpha_1, \dots, \alpha_N, \alpha'_1, \dots, \alpha'_K) \neq \text{void},$$

we have

$$E = \bigcap_{(\alpha_1, \dots, \alpha_N)} E(\alpha_1, \dots, \alpha_N)$$

is non-void.

Let $\zeta^* \in E$ and $M(\zeta^*)$ be the set of all functions $f \in F^+$ such that

$$(6.4) \quad \exp \left[\frac{\omega(|z|)}{1 - |z|} \right] |f(z)| \rightarrow 0 \text{ for each } \omega(r) \in Q,$$

as $z \rightarrow \zeta^*$ in D^* , $z \in D$.

$M(\zeta^*)$ is obviously a proper ideal. Take a function $f \in J$. For any $\varepsilon > 0$ and $\omega(r) \in Q$, we choose a neighborhood $U(\zeta^*)$ as

$$\pi_f(U(\zeta^*)) = U(\pi_f(\zeta^*); \varepsilon, \omega, \zeta_0, \eta)$$

as defined in A(ii) with suitable ζ_0 , $|\zeta_0| = 1$, and $\eta > 0$. Thus, if $z \in U(\zeta^*)$,

$$\exp [\omega(|z|)/(1 - |z|)] |f(z) - \pi_f(\zeta^*)| < \varepsilon.$$

But, by the definition of the set E , we have $\pi_f(\zeta^*) = 0$, hence f satisfies (6.4) and $J \subset M(\zeta^*)$, hence $J = M(\zeta^*)$.

We have that $E \subset D^{**}$. E contains only one point, since the extensions of functions of F^+ separate points of D^* . Thus we obtain the proof of our theorem.

REFERENCES

- [1] N. DUNFORD AND J. T. SCHWARTZ, Linear operators, part I, Interscience Publishers Inc., New York, 1964.
- [2] P. L. DUREN, Theory of H^p spaces, Academic Press, New York and London, 1970.
- [3] L. HÖRMANDER, Generators for some rings of analytic functions, Bull. Amer. Math. Soc., 73 (1967), 943-949.
- [4] J. IGUSA, On a property of the domain of regularity, Mem. Coll. Sci. University of Kyoto, Ser. A, 27 (1952), 95-97.
- [5] J. J. KELLEHER AND B. A. TAYLOR, Finitely generated ideals in rings of analytic functions, Math. Ann., 193 (1971), 225-237.
- [6] J. J. KELLEHER AND B. A. TAYLOR, Closed ideals in locally convex algebras of analytic functions, J. Reine Angew. Math., 255 (1972), 190-209.
- [7] G. KÖTHE, Topologische Lineare Räume I, Zweite Auflage, Springer-Verlag, Berlin-Heidelberg-New York, 1966.
- [8] K. V. RAJESWARA RAO, On a generalized corona problem, J. Analyse Math., 18 (1967), 277-278.
- [9] N. YANAGIHARA, Multipliers and linear functionals for the class N^+ , Trans. Amer. Math. Soc., 180 (1973), 449-461.
- [10] N. YANAGIHARA, The containing Fréchet space for the class N^+ , Duke Math. J., 40 (1973), 93-103.
- [11] N. YANAGIHARA, The second dual space for the space N^+ , Proc. Japan Acad., 49 (1973), 33-36.
- [12] N. YANAGIHARA, The class N^+ of holomorphic functions and its containing Fréchet space F^+ , Boll. Un. Mat. Ital. (Bologna), 8 (1973), 230-245.

DEPARTMENT OF MATHEMATICS, CHIBA UNIVERSITY,
1-33 YAYOI-CHO, CHIBA CITY, CHIBA-KEN, JAPAN.

