1969

Generators and Relations of Abelian Semigroups and Semigroup Rings.

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HERZOG, Juergen Reinhard Gerhard, 1941-
GENERATORS AND RELATIONS OF ABELIAN SEMIGROUPS AND SEMIGROUP RINGS.

The Louisiana State University and Agricultural and Mechanical College, Ph.D., 1969
Mathematics

University Microfilms, Inc., Ann Arbor, Michigan

GENERATORS AND RELATIONS
OF ABELIAN SEMIGROUPS AND SEMIGROUP RINGS

A Dissertation

Submitted to the Graduate Faculty of the
Louisiana State University and
Agricultural and Mechanical College
in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy

in

The Department of Mathematics

by
Juergen Reinhard Gerhard Herzog
Vordiplom, Universitaet Heidelberg, April, 1966
M.S., Purdue University, June, 1968
August, 1969

ACKNOWLEDGMENTS

The author wishes to express his gratitude to Professor Ernst Kunz for his guidance and assistance in the preparation of this dissertation, and especially for suggesting the questions studied here. Thanks are also due to Professor Otto F. G. Schilling, who was responsible for financial support for the author in the summer of 1968 at Purdue University where part of this dissertation was written. Last not least, the author expresses his appreciation to Mr. Klaus Nagel, whose knowledge in programming enabled the checking with a computer of some conjectures on numerical semigroups.

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ABSTRACT

The object of this paper is to find necessary and sufficient conditions for an affine space-curve C with parametric equations $x = t^{n_1}$, $y = t^{n_2}$, $z = t^{n_3}$ (n_1, n_2, n_3 positive integers with g.c.d.$(n_1, n_2, n_3) = 1$) to be an ideal theoretic complete intersection.

In Chapter I we study algebraic varieties, whose coordinate ring is the semigroup ring k[S], where k is an algebraically closed field, and S designates a geometric semigroup. A semigroup S is said to be a geometric semigroup if S is a finitely generated subsemigroup with 0 element of a finitely generated free abelian group. The curve $C = \{(t^{n_1}, t^{n_2}, t^{n_3}) / t \in k\}$ is a special case of these varieties. We prove the theorem that if R is an integral domain, S a semigroup of integers, and S' any finitely generated semigroup, then $R[S] \cong R[S']$ implies $S \cong S'$.

In Chapter II we give a new proof of the fact that each finitely generated abelian semigroup is finitely presented. Moreover, we show that if S is a geometric semigroup with no invertible elements, then the number of relations defining S is greater than or equal to the least number of generators of S minus the rank of the associated group of S. If equality holds, we say that S is a complete intersection. We close this chapter by proving that S is a complete intersection if and only if the variety belonging to k[S] is an ideal theoretic complete intersection.

Chapter III is devoted to the study of Sylvester-semigroups and their semigroup rings. We call a subsemigroup of the natural

numbers a Sylvester-semigroup if there exists an integer m such that $z \in S$ if and only $m-z \notin S$, for all integers z. We show that a semigroup S is a Sylvester-semigroup if and only if the localization O_S of the semigroup ring $k[S]$ at the origin is a Gorenstein ring. It turns out that the ideal theory of Sylvester-semigroups is similar to the ideal theory of Gorenstein rings. Using a result of Serre we conclude that a semigroup of natural numbers generated by 3 elements is a complete intersection if and only if S is a Sylvester-semigroup.

In Chapter IV a direct proof of this theorem is given. We also give two other equivalent conditions for a semigroup of natural numbers generated by 3 elements to be a complete intersection.

INTRODUCTION

This study will present necessary and sufficient conditions for a certain class of rational varieties to be ideal-theoretic complete intersections. The beginning of the investigation will be a classical example of a curve in the affine 3-space, which is not an ideal-theoretic complete intersection: this curve is given by the parametric representation

$$C = \{(t^3,t^4,t^5) \in k^3/\ t \in k\}$$

where k is an algebraically closed field. The coordinate ring of C is $k[T^3,T^4,T^5]$ where T is an indeterminate. The ring $k[T^3,T^4,T^5]$ may be considered as a semigroup ring k[S] where S is the semigroup generated by $\{3,4,5\}$. This point of view suggests the following generalization: Let S be a finitely generated subsemigroup (with 0 element) of a finitely generated free abelian group. Such a semigroup will be called a *geometric semigroup*. Let I_S be the kernel of the canonical epimorphism

$k[X_1,\ldots,X_n] \to k[S]$, the X_i's being mapped onto the generators of S. Let V_S be the affine variety in the affine n-space $A_n(k)$ defined by I_S. In other words:

$$V_S = \{x \in A_n(k)\ /\ F(x) = 0 \text{ for all } F \in I_S\}$$

We call V_S the monomial variety defined by S. These are actually the varieties we are interested in; the curve C is a special case of them.

In the first section of Chapter I we summarize some geometric properties of monomial varieties and turn, in the following section, to the question: If S and S' are geometric semigroups and V_S, $V_{S'}$ their affine varieties over k respectively, can V_S and $V_{S'}$ be bi-regularly equivalent with S and S' non-isomorphic?

Concerning this question, we prove that if R is an integral domain, S a subsemigroup of the natural numbers, and S' any finitely generated semigroup, then

$$R[S] \cong R[S'] \text{ implies that } S \cong S'$$

The proof of this theorem was indicated to the author by E. D. Davis.

In Chapter II we describe the ideal I_S in detail. It turns out that the generators of I_S are essentially given by the generators of the semigroup S. This fact will enable us to give a simple proof of the theorem that each finitely generated commutative semigroup is finitely presented. (cf L. Rédei, [4]). In fact, by our method, this is a special case of Hilbert's basis theorem.

At this point, the question of how many relations are necessary to define a given semigroup arises. We prove that if S is a geometric semigroup, then the least number of elements of a set of defining relations for S is greater or equal to the least number of generators of S minus the rank of the associated group $\hat{S}$ of S. If equality holds, we say that S is a complete intersection. We conclude this section by proving that if S is a geometric semigroup with no invertible elements, then V_S is an ideal-theoretic complete intersection if and only if S is a complete intersection. This theorem reduces the question of complete intersections for monomial varieties to the similar question for the defining semigroups. Moreover, we obtain as a corollary that V_S is an ideal-theoretic complete intersection if and only if V_S is locally an ideal-theoretic complete intersection.

These results are still somewhat unsatisfactory, because, in general, it is a hard problem to decide whether a given geometric semigroup is a complete intersection or not. Even if we are dealing only with subsemigroups of the natural numbers (i.e. numerical semigroups), this problem is so far unsolved. However, it is easier to determine those numerical semigroups whose semigroup ring k[S] is contained in a larger class of rings which are locally Gorensteinrings.

In Chapter III we investigate the special case where S is a numerical semigroup. In the first section we compile some known facts about numerical semigroups and introduce the notion of Sylvester-semigroups. We say that a cancellative semigroup S is a Sylvester-semigroup if there exists an element $x \in \hat{S}$ ($\hat{S}$ is the associated group of S) such that for all elements $s \in \hat{S}$ the following holds:

$$(*) \quad s \in S \text{ if and only if } x-s \notin S$$

(S is understood to be embedded in $\hat{S}$)

This notion was introduced by R. Apèry in his paper [1].

He calls a semigroup with the above peoperty "symmetric". According to L. Rédei, Sylvester proved that each numerical semigroup which is generated by two relatively prime elements has the property (*). Since the term "symmetric" is already used in semigroup theory in other contexts, we prefer to call such a semigroup a Sylvester-semigroup. In the following section of Chapter III, we show that the localization at the origin of the semigroup ring k[S], where k is a field and S a numerical semigroup, is a Gorensteinring if and only if S is a Sylvester-semigroup. This theorem suggests that the ideal theory of a Sylvester-semigroup should be similar to the ideal theory of a Gorensteinring. We show that this is the case. With the aid of this theorem and a result of Serre [5], we conclude: A numerical semigroup of rank 3 is a complete intersection if and only if it is a Sylvester-semigroup.

In Chapter IV we prove this theorem directly and give some equivalent conditions for numerical semigroups of rank 3 to be complete intersections. The main tool of the proof will be the notion of a minimal relation. Finally, we show by examples that many of our results do not hold for semigroups generated by more than 3 elements.

NOTATIONAL CONVENTIONS

The positive integers will be denoted by N; the positive integers together with 0 by N_0; and the integers by Z. If E is a set, E^n will always denote $\prod_{i=1}^{n} E_i$ where $E_i = E$ for $i = 1,\ldots,n$. If R[S] is a semigroup ring then the elements of the natural basis of R[S] over R will be denoted by x_s, $s \in S$.

CHAPTER I

Monomial Varieties

1.1. Basic definitions and some geometric properties of monomial varieties.

As indicated in the introduction we will study a certain class of rational varieties. To describe them we need some definitions.

Definition 1.1.1.: S is a geometric semigroup if S is a finitely generated subsemigroup with 0 element of a finitely generated free abelian group.

The reason why we call such a semigroup a geometric semigroup, is that the reader may visualize it as a subsemigroup of the groups of lattice points of R^n, n suitably chosen.

Let k be an algebraically closed field.

Definition 1.1.2.: An algebraic variety $V \subseteq A_m(k)$ is called a monomial variety, if its coordinate ring $k[V] = k[X_1, \ldots, X_m]/I(V)$, $I(V) = \{f \in k[X_1,\ldots,X_m]/f(x) = 0 \text{ for all } x \in V\}$ is isomorphic to the semigroup ring $k[S]$ over k of a geometric semigroup S.

For a geometric semigroup S there is a canonical construction of a monomial variety V_S : The associated group $\hat{S}$ of S is a finitely generated, free abelian group and, since S is cancellative, we have an embedding $S \to \hat{S}$. We will identify S with its image in $\hat{S}$. Also, since $\hat{S}$ is a finitely generated free abelian group, we may identify $\hat{S}$ with Z^n for some positive integer n. Thus we may assume that

$S \subseteq Z^n$ and $\hat{S} = Z^n$. Let $\{v_1,\ldots,v_m\} \subseteq Z^n$ be a minimal set of generators of S. Say $v_i = (z_{ij})$, $i = 1,\ldots,m, j = 1,\ldots,n$, $z_{ij} \epsilon Z$. Let k be an algebraically closed field. Let $T_1,\ldots,T_n$ be indeterminates and $T^{v_i} = \prod_{j=1}^{n} T_n^{z_{ij}}$, $i = 1,\ldots,m$. It is clear that the semigroup ring $k[S]$ is isomorphic to $k[T^{v_1}, \ldots, T^{v_m}]$. We have an epimorphism $\mathcal{E}$ from the polynomial ring $k[X_1,\ldots,X_m]$ in m-variables onto the ring $k[T^{v_1}\ldots T^{v_m}]$ defined by : $\mathcal{E}(X_i) = T^{v_i}$, $i=1,\ldots,m$. Let I_S be the kernel of $\mathcal{E}$. We define:

$$V_S = V(I_S)$$

$$(V(I_S) = \{x \epsilon A_m(k)/f(x) = 0 \text{ for all } f \epsilon I_S\})$$

We should keep in mind that the definition of V_S depends on the identifications we just made.

From the definition of V_S follows:

a) $V_S \subseteq A_m(k)$

b) The coordinate ring of V_S is $k[T^{v_1},\ldots,T^{v_m}]$

c) The function field of V_S is $k(T_1,\ldots,T_n)$

d) V_S is an irreducible rational variety

e) $\dim V_S = n = \text{rank}_Z \hat{S}$

Let H_i, $i = 1,\ldots,m$ be the hyperplanes in $A_m(k)$ defined by the equations: $\{X_i = 0\}_{i=1,\ldots,m}$

The following theorem summarizes some properties of monomial varieties:

Theorem 1.1.3.: With the above assumptions we have:

1) If $x \in V_S \cap C(H_1 \cup \ldots \cup H_m)$ then x is of the form $x = (t^{v_1}, \ldots, t^{v_m})$ where $t^{v_i} = \prod_{j=1}^{n} t_j^{z_{ij}}$ for $i = 1, \ldots, m$ and with $t_j \in k$ for $j = 1, \ldots, n$.

2) The singular locus of V_S is contained in

$$V_S \cap (H_1 \cup \ldots \cup H_m)$$

3) The origin is a point of V_S if and only if S has no invertible elements.

4) If $m > n$ and if the origin is a point of V_S, then it is a singular point.

Proof: 1.) Let $x \in V_S \cap C(H_1 \cup \ldots \cup H_m)$. To x corresponds a k-algebra homomorphism

$$\varphi_x : k[T^{v_1}, \ldots, T^{v_m}] \to k.$$

Let $R = k[T^{v_1}, \ldots, T^{v_m}]_{\{T^v\}_{v \in S}}$ be the quotient ring of $k[T^{v_1}, \ldots, T^{v_m}]$ with respect to the multiplicatively closed set $\{T^v\}_{v \in S}$ then

$$R = k[T_1, \ldots, T_n]_{\{T^v\}_{v \in Z^n}}$$

This is a special case of a general fact:

Lemma 1.1.4.: If R is an unitary ring, S a commutative semigroup, $\pi : S \to \hat{S}$ the natural morphism of S into its associated group $\hat{S}$ and $R[S] \xrightarrow{R[\pi]} R[\hat{S}]$ the ring homomorphism induced by π, then $R[\hat{S}]$ is isomorphic to $R[S]_{\{S\}}$, the quotient ring of $R[S]$ with respect to $\{S\}$ and $R[\pi]$ is the natural mapping of $R[S]$ into its quotient ring

$R[S]_{\{S\}}$.

Since $x \notin H_1 \cup \ldots \cup H_m$ we can lift φ_x to

$\tilde{\varphi}_x : R \to k$ such that the diagram

$$\begin{tikzcd} R \arrow[dr, "\tilde{\varphi}_x"] & \\ k[T^{v_1},\ldots,T^{v_m}] \arrow[u] \arrow[r] & k \end{tikzcd}$$

is commutative.

Say $\tilde{\varphi}_x(T_i) = t_i$, $t_i \in k$ then $\varphi_x(T^{v_j}) = t^{v_j}$ for $j = 1,\ldots,m$, since $\varphi_x = \tilde{\varphi}_{x/k[T^{v_1},\ldots,T^{v_m}]}$. In other words: $x = (t^{v_1},\ldots,t^{v_m})$.

2.) Let the kernel of φ_x be $\mathfrak{m}_x$ and the kernel of $\tilde{\varphi}_x$ be $\tilde{\mathfrak{m}}_x$. It is obvious that $k[T^{v_1},\ldots,T^{v_m}]_{\mathfrak{m}_x} \cong R_{\tilde{\mathfrak{m}}_x}$. But $R_{\tilde{\mathfrak{m}}_x}$ is a quotient ring of a polynomial ring and hence regular. This proves assertion 2.)

3.) If S has invertible elements, then there exists an equation $\prod_{i=1}^{m} (T^{v_i})^{\nu_i} = 1$, $\nu_i \geq 0$, ν_i integers. So the origin cannot be a point of V_S. Conversely, if S has no invertible elements, then $(T^{v_1},\ldots T^{v_m})$ is a maximal ideal in $k[R^{v_1},\ldots T^{v_m}]$ corresponding to the origin.

4.) If $m > n$ then I_S is generated by polynomials of the form $\prod_{i=1}^{m} X_i^{\nu_i} - \prod_{i=1}^{m} X_i^{\mu_i}$ with $\sum_{i=1}^{m} v_i > 1$ and $\sum_{i=1}^{m} \mu_i > 1$. (We will prove this in the next chapter). If we take this for granted, then it is clear that the Jacobian of V_S vanishes at the origin, so that in fact the origin is a singular point.

With the same assumptions we get:

Corollary 1.1.5.: V_S has outside of $H_1 \cup \ldots \cup H_m$ the parametric representation

$$\{(t^{v_1},\ldots,t^{v_m}) \in A_m(k)/t_i \in k,\ t_i \neq 0,\ i = 1,\ldots,n\}.$$

Let S be a numerical semigroup, (i.e. S is a subsemigroup of N_0). Say S is generated by $n_1,\ldots,n_\ell \in N$ and $\{n_1,\ldots,n_\ell\}$ is a minimal system of generators of S. Then we obtain as a special case:

Corollary 1.1.6.: V_S is a curve in $A_\ell(k)$ passing through the origin and having there its only singular point. Moreover, we have for V_S the parametric representation

$$V_S = \{(t^{n_1},\ldots,t^{n_\ell}) \in A_\ell(k)\ /\ t \in k\}$$

Remark: The parametric representation doesn't hold in general globally.

Example: Let S be generated by $v_1 = (1,1)$, $V_2 = (1,2)$, $v_3 = (1,3)$ then

$V_S = \{(x_1,x_2,x_3) \in A_3(k)\ /\ x_1x_3-x_2^2 = 0\}.$

Therefore $(1,0,0) \in V_S$, but

$(1,0,0) \notin \{(t_1\cdot t_2,\ t_1\cdot t_2^2,\ t_1\cdot t_2^3)\ /\ t_1,t_2 \in k\}$

1.2. An isomorphism theorem

Before we turn to the intersection problem we are going to prove a theorem, which is of a certain interest in this context.

Let S, S' be geometric semigroups and V_S, $V_S{}'$ their affine varieties over k. Can V_S and $V_{S'}$ be biregularly equivalent for non-isomorphic S, S'? More generally the question can be stated: If R is an arbitrary commutative, unitary ring, and S,S' arbitrary commutative semigroups with 0 element, does an R-isomorphism of the semigroup rings $R[S]$, $R[S']$ imply that S and S' are isomorphic?

Before proving the main theorem, we show:

Proposition 1.2.1.: For an abelian semigroup S the following conditions are equivalent:

a.) S is a geometric semigroup

b.) 1.) S is finitely generated and has a 0 element

2.) S is cancellative

3.) Whenever $n s_1 = n s_2$ for $n \in N$ and $s_1, s_2 \in S$ then $s_1 = s_2$

c.) 1.) S is finitely generated and has a 0 element

2.) If R is an integral domain then $R[S]$ is an integral domain.

Proof: a.) implies b.) is trivial and a.) implies c.) is clear, since $R[S]$ can be considered as subring of a polynomial ring $R[X_1,\dots,X_n]_T$, localized with respect to $T = \{X^\nu\}_{\nu \in N_0^n}$ (Lemma 1.1.4.).

b.) implies a.): Denote the associated group of S by $\hat{S}$. There is a canonical homomorphism $\pi : S \to \hat{S}$. By assumption S is cancellative, therefore π is an embedding. Since S is finitely generated, $\hat{S}$ is also finitely generated. It remains to show that $\hat{S}$ is free or equivalently; $\hat{S}$ is torsion free.

Consider the equation $n\hat{s} = 0$ with $n \in N$ and $\hat{s} \in \hat{S}$. We have to show that $\hat{s} = 0$. $\hat{s}$ can be written: $\hat{s} = \pi(s_1) - \pi(s_2)$ where $s_1, s_2 \in S$. Therefore $0 = \pi(ns_1) - \pi(ns_2)$, thus $\pi(ns_1) = \pi(ns_2)$.

Since π is injective this implies that $ns_1 = ns_2$ and by condition 3.) $s_1 = s_2$, therefore $\hat{s} = 0$.

c.) implies b.) : We only have to check 2.) and 3.).

2.) Assume S is not cancellative, then there exist $s, s_1, s_2 \in S$ $s_1 \neq s_2$ such that $s_1 + s = s_2 + s$. Then $(x_{s_1} - x_{s_2})x_s = x_{s_1+s} - x_{s_2+s} = 0$ but $x_{s_1} - x_{s_2} \neq 0$ and $x_s \neq 0$, a contradiction.

3.) Assume there exist $s_1, s_2 \in S$ such that $s_1 \neq s_2$ and $ns_1 = ns_2$ for some $n \in N$. Then $(x_{s_1} - x_{s_2})(\sum_{v=1}^{n-1} x_{(n-v)s_1+vs_2}) = x_{ns_1} - x_{ns_2} = 0$

but $x_{s_1} - x_{s_2} \neq 0$ and $\sum_{v=1}^{n-1} x_{(n-v)s_1+vs_2} \neq 0$, a contradiction.

<u>Theorem 1.2.2.</u>: Let R be an integral domain, S a numerical semigroup, and S' any finitely generated semigroup with 0 element. Then $R[S] \underset{R}{\cong} R[S']$ implies that $S \cong S'$.

<u>Proof</u>: Let k be the quotient field of R then also $k[S] \cong k[S']$. Since k[S] is an integral domain, so is k[S'] and by proposition 1.2.1. S' must be a geometric semigroup. Since $\dim k[S] = 1$ also $\dim k[S'] = 1$. In section 1.1., we showed that $\dim k[S'] = \operatorname{rank}_Z \hat{S}'$; hence, $\hat{S}' = Z$ and therefore $S' \subsetneq Z$. Assume S' has invertible elements, then $S' \cong Z$ and the elements $x_s \in k[S']$, $s \in S'$, are units in $k[S']$; whereas the units of the ring k[S] are just the elements of k, different from zero, a contradiction. Thus S' is isomorphic to some numerical semigroup.

Theorem 1.2.2 is therefore reduced to the special case (E. D. Davis):

If k is a field and S, S' are numerical semigroups such that $k[S] \cong k[S']$, then $S \cong S'$.

Proof: 1.) If $S = N$ then $k[S]$ is isomorphic to the polynomial ring in one variable $k[T]$ and therefore $k[S] \cong k[T]$.

Assume S' is a proper subsemigroup of N. Say S' is generated by $\{n_1', n_2', \ldots, n_\ell'\}$, $n_i' \in N$. Then $k[S'] \cong k[T^{n_1'}, \ldots, T^{n_\ell'}]$ and we know that $k[T^{n_1'}, \ldots, T^{n_\ell'}]_{(T^{n_1}, \ldots, T^{n_\ell})}$ is a non-regular local ring. However all localizations of the polynomial ring with respect to prime ideals are regular local rings, therefore $k[S]$ cannot be isomorphic to S', unless S' is also isomorphic to N.

2.) We may suppose that both S and S' are proper subsemigroups of N_0. Say S is generated by $\{n_1, \ldots, n_\ell\}$, $n_i \in N$. We may assume that the greatest common divisor of $n_1, n_2, \ldots, n_\ell$ is 1, because if $v = \gcd(n_1, n_2, \ldots, n_\ell)$ then $\tilde{S}$, generated by $\{\frac{n_1}{v}, \frac{n_2}{v}, \ldots, \frac{n_\ell}{v}\}$, is isomorphic to S. Let S' be generated by $\{n_1', \ldots, n_k'\}$, $n_i \in N$. We also assume that $\gcd(n_1', \ldots, n_k') = 1$. We get

$$k[T^{n_1}, \ldots, T^{n_\ell}] \cong k[S] \cong k[S'] \cong k[T^{n_1'}, \ldots, T^{n_k'}].$$

Let $\varphi : k[T^{n_1}, \ldots, T^{n_\ell}] \to k[T^{n_1'}, \ldots, T^{n_k'}]$ be this isomorphism. Let $\mathcal{Y} = (T^{n_1}, \ldots, T^{n_\ell})$ and $\mathcal{Y}' = (T^{n_1'}, \ldots, T^{n_\ell'})$ be the maximal ideals in $k[T^{n_1}, \ldots, T^{n_\ell}]$ and $k[T^{n_1'}, \ldots, T^{n_k'}]$ belonging to the origin, so $\varphi(\mathcal{Y}) = \mathcal{Y}'$, since the ideals $\mathcal{Y}$ and $\mathcal{Y}'$ define the only singular points of the corresponding varieties. The integral closure of $k[T^{n_1},\dots,T^{n_\ell}]$ as well as of $k[T^{n'_1},\dots,T^{n'_\ell}]$ is $k[T]$. Therefore φ induces an automorphism $\tilde{\varphi} : k[T] \to k[T]$ such that the diagram

$$\begin{tikzcd} k[T] \arrow[r, "\tilde{\varphi}"] & k[T] \\ k[T^{n_1},\dots,T^{n_\ell}] \arrow[u] \arrow[r, "\varphi"'] & k[T^{n'_1},\dots,T^{n'_\ell}] \arrow[u] \end{tikzcd}$$

is commutative.

By Lüroth's theorem we get $\tilde{\varphi}(T) = \alpha + \beta T$ with $\alpha, \beta \in k$ and $\beta \neq 0$. Since $\varphi(\mathfrak{g}) = \mathfrak{g}'$ we obtain $\tilde{\varphi}(T) = \tilde{\varphi}(\mathfrak{g}k[T]) = \mathfrak{g}'k[T] = (T)$, so that in our case α must be zero. Thus $\tilde{\varphi}(T) = \beta T$ with $\beta \in k$, $\beta \neq 0$. Since φ is the restriction of $\tilde{\varphi}$ to $k[T^{n_1},\dots,T^{n_\ell}]$, we have $\varphi(T^s) = \beta^s T^s$ for all $s \in S$, thereby proving that $S \cong S'$.

<u>Corollary 1.2.3.</u>: If k is a field and S a numerical semigroup then $\mathrm{Aut}_k(k[S], k[S]) \cong k^*$.

<u>Remarks</u>: The proof of Theorem 1.2.2. cannot be generalized to arbitrary geometric semigroups, because the integral closure of $k[S]$ need not to be a polynomial ring. Even if this is so, things are much more complicated, since the automorphism group is in general not as trivial as in the case of corollary 1.2.3.

<u>Example</u>: We define an automorphism

$\varphi : k[T_1^2, T_1T_2, T_2^2] \to k[T_1^2, T_1T_2, T_2^2]$ by

$$\varphi(T_1^2) = T_1^2 + 4T_1T_2 + 4T_2^2$$

$$\varphi(T_1T_2) = -T_1^2 - T_1T_2 + 2T_2^2$$

$$\varphi(T_2^2) = T_1^2 - 2T_1T_2 + T_2^2 .$$

CHAPTER II

Connections between semigroups and their associated semigroup rings

2.1. The defining relations of a semigroup and its semigroup ring.

First of all we introduce some notations and definitions and recall some well-known lemmas.

Definition 2.1.1.: S is a free semigroup of rank n if there exist $s_1,\ldots,s_n \epsilon S$ such that any equation $\sum_{i=1}^{n} v_i s_i = \sum_{i=1}^{n} v_i' s_i$ with $v_i', v_i \epsilon N$, $i = 1,\ldots,n$ yields $v_i = v_i'$ for all $i = 1,\ldots,n$.

Free semigroups have the following universal mapping property: Let F be a free semigroup of rank n and S a commutative semigroup. Assume F is generated by $\{e_1,\ldots,e_n\}$. Pick any elements $s_1,\ldots,s_n \epsilon S$. Then there exists exactly one morphism $\varphi : F \to S$ such that $\varphi(e_i) = s_i$ for $i = 1,\ldots,n$.

As corollaries we get:

1.) Any two free semigroups of the same rank are isomorphic.

2.) Any finitely generated commutative semigroup is an epimorphic image of a free semigroup.

Now let S be a finitely generated commutative semigroup with 0 element. Then there exists an epimorphism $\rho^*: F \to S$, where F is a free commutative semigroup. The morphism ρ^* defines a (binary) relation on F:

$$\rho = \{(v,w) \epsilon F\times F \ / \ \rho^*(v) = \rho^*(w)\}$$

Obviously ρ is an equivalence relation and moreover has the following compatibility property C : If $(v,v') \in \rho$ and $w \in F$ then $(v+w, v'+w) \in \rho$. According to Rédei we define:

Definition 2.1.2.: A subset $\rho \subseteq F \times F$ is called a congruence on F if 1.) ρ is an equivalence relation 2.) ρ has the property C. If ρ is a congruence on F, then the set of equivalence classes F/ρ can be given a semigroup structure in an obvious way and there is a natural epimorphism $\rho^* : F \to F/\rho$.

Lemma: If F and S are finitely generated commutative semigroups, F is free, $\rho^* : F \to S$ an epimorphism and $\rho \subseteq F \times F$ the congruence $\rho = \{(v,w) \in F \times F \,/\, \rho^*(v) = \rho^*(w)\}$ then F/ρ is isomorphic to S.

Hence any finitely generated commutative semigroup can be written as a free semigroup modulo a congruence.

The trivial congruence on F is $i = \{(v,v) \in F \times F / v \in F\}$ and clearly $F/i \cong F$. For any relation ρ on F we put $\rho^{-1} = \{(v,w) \in F \times F / (w,v) \in \rho\}$. It is rather obvious that for an arbitrary relation ρ on F there is a smallest congruence $\bar{\rho}$ containing ρ, namely the intersection of all congruences containing ρ. There is an explicit construction of $\bar{\rho}$:

1. Step: Put $\rho_o = \rho \cup \rho^{-1} v\ i$, then ρ_o is reflexive and symmetric and $\rho \subseteq \rho_o$.

2. Step: Put $\rho_1 = \{v+w, v'+w) \in F \times F / (v,v') \in \rho_o\ w \in F\}$, then ρ_1 is reflexive, symmetric and satisfies condition C and $\rho \subseteq \rho_o \subseteq \rho_1$.

3. Step: (Transitive closure) Define $(v,w)\epsilon F\times F$ to be an element of $\bar{\rho}$ if there exist $v_0, v_1, \ldots, v_\ell \epsilon F$, $v_0 = v$, $v_\ell = w$ with $(v_i, v_{i+1})\epsilon\rho$, for all $i = 0,1,\ldots,\ell-1$.

A simple verification shows that $\bar{\rho}$ is in fact the desired congruence.

Definition 2 1.3.: A congruence ρ on F is said to be finitely generated if there exists a finite subset $\sigma \subseteq \rho$ such that $\bar{\sigma} = \rho$.

For a finitely generated commutative semigroup S we define:

Definition 2.1.4.: S is finitely presented if $S \cong F/\rho$, where F is a free semigroup and ρ a finitely generated congruence.

We now prove a representation theorem for semigroup rings and introduce for this purpose some notations: Any free semigroup F of rank n is isomorphic to

$$N_0{}^n = \{(v_1,\ldots,v_n) \;/\; v_i \epsilon N_0,\; i=1,\ldots,n\}$$

with addition componentwise, and any commutative semigroup S of rank n is an epimorphic image $\rho^* : N_0{}^n \to S$ and $S \cong N_0{}^n/\rho$, where ρ is the congruence determined by ρ^*.

Let R be a commutative, unitary ring and $R[X_1,\ldots,X_n]$ be the polynomial ring in n variables over R. If $v\epsilon N_0{}^n$, $v = (v_1,\ldots,v_n)$, we define $X^v = \prod_{i=1}^{n} X_i^{v_i}$. $R[X_1,\ldots,X_n]$ is isomorphic to $R[N_0{}^n]$ by $X^v \to x_v$. The epimorphism $\rho^*: N_0{}^n \to S$ induces an epimorphism $R[X_1,\ldots,X_n] \xrightarrow{R[\rho^*]} R[S]$, where $R[\rho^*](X^v) = x_{\rho^*(v)}$.

We may consider $R[X_1,\ldots X_n]$ as an S-graded ring in the following sense: A polynomial $F\epsilon R[X_1,\ldots,X_n]$ is said to be homogeneous of degree s, $s\epsilon S$ if $F = \sum_{v\epsilon N_0^n} r_v X^v$ with $\rho^*(v) = s$ for all s such that $r_v \neq 0$. Also R[S] is in a trivial way an S-graded ring. From the definition of $R[\rho^*]$ follows, that $R[\rho^*]$ is a homogeneous homomorphism of degree 0. We denote the kernel of $R[\rho^*]$ by I_S and for $A\epsilon\rho$, $A = (v,w)$, $v,w\epsilon N_0^n$ we put

$$F_A = X^v - X^w$$

<u>Theorem 2.1.5</u>: $\qquad I_S = (\{F_A\}_{A\epsilon\rho})$.

In other words: $R[S] \cong R[N_0^n/\rho] \cong R[N_0^n]/(\{F_A\}_{A\epsilon\rho})$.

Proof: Let $J = (\{F_A\}_{A\epsilon\rho})$.

1.) $J \subseteq I_S$, clearly by the definition of $R[\rho^*]$

2.) Observe that I_S is a homogeneous ideal in the S-graded ring $R[X_1,\ldots,X_n]$. Therefore it suffices to prove: If $F\epsilon I_S$ and F is homogeneous, then $F\epsilon J$. Choose $F\epsilon I_S$, F homogeneous of degree s.

$F = \sum_{i=1}^{m} r_{v_i} X^{v_i}$ with $\rho^*(v_i) = s$ for $i=1,\ldots,m$ and $\sum_{i=1}^{m} r_{v_i} = 0$.

$F = \sum_{i=1}^{m-1} r_{v_i}(X^{v_i}-X^{v_m}) = \sum_{i=1}^{m-1} r_{v_i} F_{A_i}$, where $A_i = (v_i,v_m)$ for $i=1,\ldots,m-1$. Since $\rho^*(v_i) = \rho^*(v_m)$ for all $i=1,\ldots,m-1$, we get that $A_i\epsilon\rho$ for $i=1,\ldots,m-1$, thus $F\epsilon J$.

<u>Remark</u>: This theorem can easily be generalized to commutative semigroups S, which are not necessarily finitely generated.

With the same assumptions as for Theorem 2.1.5 we have the following:

Proposition 2.1.6.: Let $\sigma = \{A_1,\dots,A_m\}$, where $A_i \epsilon \rho$ for $i=1,\dots,m$. Then the following conditions are equivalent:

1.) $\bar{\sigma} = \rho$

2.) $I_S = (F_{A_1},\dots,F_{A_m})$

Proof: 1.) implies 2.):

a.) Let $\rho_0 = \sigma \cup \sigma^{-1} \cup i$, then $(\{F_A\}_{A\epsilon\rho_0}) = (F_{A_1},\dots,F_{A_m})$. This is trivial, because if $A\epsilon\sigma^{-1}$, then $A = A_i^{-1}$ for some $i=1,\dots,m$ and $F_A = F_{A_i^{-1}} = -F_{A_i}$ and if $A\epsilon i$ then $F_A = 0$.

b.) Let $\rho_1 = \{(v+w, v'+w)/(v,v')\epsilon\rho_0, w\epsilon N_0^n\}$ then $(\{F_A\}_{A\epsilon\rho_1}) = (\{F_A\}(\{F_A\}_{A\epsilon\rho_0})$, because if $A\epsilon\rho_1$ then there exists $w\epsilon N_0^n$ and $A'\epsilon\rho_0$ such that $F_A = X^w F_{A'}$.

c.) Let $A\epsilon\rho$, $A = (v,w)$. By assumption $\bar{\sigma} = \rho$, and we know that $\bar{\sigma}$ is the transitive closure of ρ_1. Therefore there exist $B_0,\dots,B_k \epsilon\rho_1$, $B_i = (v_i,w_i)$ such that $v = v_0$, $w = w_k$ and $v_{i+1} = w_i$ for $i = 0,\dots,k-1$. Hence $F_A = \sum_{i=0}^{k} F_{B_i}$. a.), b.) and c.) together establish our assertion.

2.) implies 1.):

We know already that $(F_{A_1},\dots,F_{A_m}) = (\{F_A\}_{A\epsilon\bar{\sigma}})$. Let $S' = N_0^n/\bar{\sigma}$ and $\sigma^* : N_0^n \to S'$ be the canonical epimorphism then, since $I_{S'} = (\{F_A\}_{A\epsilon\bar{\sigma}})$

$0 \to (F_{A_1},\dots,F_{A_m}) \to R[X_1,\dots,X_n] \xrightarrow{R[\sigma^*]} R[S] \to 0$ is exact. By assumption $I_S = (F_{A_1},\dots,F_{A_m}) = I_{S'}$. Therefore, if $A\epsilon\rho$, $A = (v,w)$, then $R[\sigma^*](F_A) = x_{\sigma^*(v)} - x_{\sigma^*(w)} = 0$. Hence $\sigma^*(v) = \sigma^*(w)$, or equivalently $A\epsilon\bar{\sigma}$.

Corollary 2.1.7.: A finitely generated commutative semigroup is finitely presented.

Proof: An easy consequence of 2.1.5., 2.1.6. and Hilbert's Basis Theorem.

Remarks: Rédei proved this fact in his book [4] and the proof also appears in [3].

2.2. Complete intersections.

In this section we prove a theorem on the number of relations for geometric semigroups and introduce the notion of "complete intersections" for semigroups.

Let S be a geometric semigroup.

Definition 2.2.1.: $\dim S =: \operatorname{rank}_Z \hat{S}$.

The following fact gives a justification of this definition: If k is a field then

$$(\text{Krull}) \dim k[S] = \dim S \quad (\text{See Chapter I}).$$

The next two definitions refer to arbitrary finitely generated semigroups S.

Definition 2.2.2.: The rank of S is the number of elements of a minimal system of generators of S.

If $\rho \subseteq F$ is a congruence we define:

Definition 2.2.3.: The rank of ρ is the number of elements of a minimal system of generators of ρ.

Theorem 2.2.4.: Let S be a geometric semigroup and represent S as : $S = F/\rho$, where F is free and ρ is a congruence. Then

$$\operatorname{rank} \rho \geq \operatorname{rank} S - \dim S.$$

Proof: We make use of the following theorem: If k is a field, $k[X_1,\ldots,X_n]$ the polynomial ring in n-variables over k, $\mathfrak{A} = (\{F_1,\ldots,F_r\})$ an ideal in $k[X_1,\ldots,X_n]$ and $A = k[X_1,\ldots,X_n]/\mathfrak{A}$ then

$$r \geq n - \dim A.$$

In our case we have:

$\dim S = \dim k[S] = \dim \left(k[F]/(\{F_A\}_{A\in\rho})\right)$. If $\operatorname{rank} \rho = r$ then there exist $A_1,\ldots,A_r \in \rho$ such that $(\{F_A\}_{A\in\rho}) = (\{F_{A_1},\ldots,F_{A_r}\})$, see 2.1.6.

We get therefore:

$$\begin{aligned}\operatorname{rank} \rho = r &= \dim k[F] - \dim k[S]\\ &= \dim F - \dim S\\ &\geq \operatorname{rank} S - \dim S,\end{aligned}$$

since $\operatorname{rank} S \leq \dim F$. This proves our theorem.

Let S be a geometric semigroup.

Definition 2.2.5.: S is a complete intersection if S can be represented as $S \cong F/\rho$, where F is free and ρ is a congruence, such that $\operatorname{rank} \rho = \operatorname{rank} S - \dim S$.

As a straightforward application of the next theorem we will find that a geometric semigroup with no invertible elements is a complete intersection if and only if the monomial variety V_S defined by S is an ideal theoretic complete intersection.

Let $R = \bigoplus_{s\in S} R_s$ be an S-graded ring, where S is a geometric semigroup. From now on we will assume that S has no invertible elements then

$P = \bigoplus_{\substack{s \in S \\ s \neq 0}} R_s$ becomes an ideal in R. If $\mathfrak{A} \subseteq R$ is an arbitrary finitely generated ideal in R, we define the rank of $\mathfrak{A}$ to be the least number of generators of $\mathfrak{A}$.

<u>Theorem 2.2.6.</u>: Assume R_0 is an integral domain. Let $\mathfrak{A}$ be a finitely generated homogeneous ideal in R and assume that $\mathfrak{A}$ is generated by the homogeneous elements $\{a_1,\ldots,a_n\}$ and rank $\mathfrak{A} = m$. Then there exist $a_{i_1},\ldots,a_{i_m} \in \{a_1,\ldots,a_n\}$ such that

$$\mathfrak{A} = (a_{i_1},\ldots,a_{i_m}) \text{ .}$$

In other words: We can find a minimal system of generators of $\mathfrak{A}$ in a set of homogeneous generators of $\mathfrak{A}$.

<u>Proof</u>: The ideal $P = \bigoplus_{\substack{s \in S \\ s \neq 0}} R_s$ is a prime ideal in R, since R_0 is an integral domain. By assumption rank $\mathfrak{A} = m$, therefore we can choose $a_{i_1},\ldots,a_{i_m} \in \{a_1,\ldots,a_n\}$ such that $\mathfrak{A}R_P = (\bar{a}_{i_1},\ldots,\bar{a}_{i_m})$, where the $\bar{a}_{i_j}$'s are the images of the a_{i_j}'s in R_P. This can be done, since in a local ring one can select a minimal system of generators of an ideal from any system of generators. We show now that $\{a_{i_1},\ldots,a_{i_m}\}$ is a global system of generators of $\mathfrak{A}$.

Since $\mathfrak{A}$ is homogeneneous we only have to prove: If $a \in \mathfrak{A}$ and a is homogeneous then $a \in (a_{i_1},\ldots,a_{i_n})$. We can certainly find $r_i \in R$, $i = 0, 1,\ldots,m$; $r_0 \notin P$ such that $r_0 a = \sum_{j=1}^{m} r_j a_{i_j}$. We may assume that $r_0 = 1-r_0'$, $r_0' \in P$. Therefore, $a = r_0'a + \sum_{j=1}^{m} r_j a_{i_j}$, For an element $r \in R$ we denote the homogeneous component of degree s of r by $h_s(r)$. The element a is assumed to be homogeneous. Say, $\deg a = s$. Then $a = h_s(r_0'a) + h_s(\sum_{j=1}^{m} r_j a_{i_j})$, but $h_s(r_0'a) = 0$, since $r_0' \in P$. Therefore $a = h_s(\sum_{j=1}^{m} r_j a_{i_j})$.

However, the ideal $(a_{i_1},\ldots,a_{i_m})$ is homogeneous so that the last equation yields $a \in (a_{i_1},\ldots,a_{i_m})$.

Let S be a geometric semigroup with no invertible elements. Represent S as F/ρ and let I_S be the kernel of $R[\rho^*]:R[X_1,\ldots,X_n] \to R[S]$.

As a corollary of 2.2.6 we get:

<u>Corollary 2.2.7.</u>: rank ρ = rank I_S .

Proof: Obviously rank $\rho \geq$ rank I_S, (see 2.1.6). For the converse remember that $R[X_1,\ldots,X_n]$ is an S-graded ring in the sense of section 2.1. and $I_S = (\{F_A\}_{A\in\rho})$ is a homogeneous ideal in $R[X_1,\ldots,X_n]$ generated by the homogeneous elements $\{F_A\}_{A\in\rho}$. The inequality rank $\rho \leq$ rank I_S follows now from 2.2.6 and 2.1.6.

<u>Corollary 2.2.8.</u>: Let S be a geometric semigroup with no invertible elements. The following conditions are equivalent:

a.) S is a complete intersection.

b.) V_S is an ideal theoretic complete intersection.

c.) V_S is locally an ideal theoretic complete intersection.

d.) V_S is locally at the origin an ideal theoretic complete intersection.

Proof: It is trivial that a.) implies b.), b.) implies c.) and c.) implies d.). We only have to show that d.) implies a.), but this is a simple consequence of 2.2.7 and 2.1.6.

Let $I_S^* = \{\deg F_A \,/\, A \in \rho\}$. Obviously I_S^* is an ideal in S. If S is a geometric semigroup with no invertible elements, whose rank is m and dimension is n, then we get the following necessary condition for V_S to be an ideal theoretic complete intersection.

Corollary 2.2.9.: If V_S is an ideal theoretic complete intersection, then rank $I_S^* \leq m - n$.

Proof: If V_S is an ideal theoretic complete intersection, then rank $\rho \,\#= m-n$ for a suitable representation F/ρ of S. Obviously rank $I_S^* \leq$ rank I_S. But rank I_S = rank ρ (2.2.7), therefore rank $I_S^* \leq m-n$.

Example: $V_S = \{(t^3,t^4,t^5)/t \in k\}$ is not an ideal theoretic complete intersection. (S is the semigroup generated by $\{3,4,5\}$). In fact, I_S^* is generated by $\{8,9,10\}$.

We will show later, that the condition given in 2.2.9. is also sufficient for numerical semigroups of rank 3.

CHAPTER III

Sylvester-semigroups

3.1. Properties of Sylvester-semigroups.

For numerical semigroups we do not have to require that they are finitely generated, because this property is common to all of them. In this section we will assume that the generators of a numerical semigroup S have greatest common divisor 1. Because if S is generated by $\{n_1, n_2, \ldots, n_\ell\}$ and gcd $(n_1, \ldots, n_\ell) = d$, then we obtain a numerical semigroup S' which is isomorphic to S and whose greatest common divisor of its generators is 1. We let S' be the semigroup generated by $\{\frac{n_1}{d}, \ldots, \frac{n_\ell}{d}\}$.

It is obvious that if S is such a numerical semigroup then there exists $s \in S$ such that $s+v \in S$ for all $v=0,1,2,\ldots$. Hence for a numerical semigroup S there exists a greatest integer m not belonging to S. In special cases we can compute m, according to the following theorem.

<u>Theorem 3.1.1</u>: Let $c_1, \ldots, c_\ell \in N$ such that $\gcd(c_i, c_j) = 1$ for all $i,j = 1,\ldots,\ell$, $i \neq j$. Let $n_i = \prod_{\substack{j=1 \\ j \neq i}}^{\ell} c_j$ for $i = 1,2,\ldots,\ell$. Then the semigroup S generated by $\{n_1, \ldots, n_\ell\}$ has the following property: There exists an element $m \in Z$ such that for all $z \in Z$ we have: $z \in S$ if and only $m-z \notin S$.

Moreover: $m = (\ell-1) \prod_{j=1}^{\ell} c_j - \sum_{i=1}^{\ell} \prod_{\substack{j=1 \\ j \neq i}}^{\ell} c_j$.

Remark: From the definition of m it follows that m is uniquely determined. Namely m is the greatest integer not belonging to S.

Proof of the Theorem:

a.) We first show that if we have an equation $\sum_{i=1}^{\ell} r_i n_i = 0$, $r_i \in Z$ then r_i is a multiple of c_i for $i = 1,\ldots,\ell$.

Proof: $\sum_{i=1}^{\ell} r_i \prod_{\substack{j=1\\ j\neq i}}^{\ell} c_j = \sum_{i=1}^{\ell} r_i n_i = 0$. Therefore, $r_i \prod_{\substack{j=1\\ j\neq i}}^{\ell} c_j \equiv 0 \bmod c_i$ for all $i = 1,\ldots,\ell$. But $\prod_{\substack{j=1\\ j\neq i}}^{\ell} c_j \not\equiv 0 \bmod c_i$, since the c_i's are assumed to be mutually prime. Hence $r_i \equiv 0 \bmod c_i$ for all i $i = 1,\ldots,\ell$. This is what we claimed.

b.) Let $m = (\ell-1) \prod_{j=1}^{\ell} c_j - \sum_{i=1}^{\ell} \prod_{\substack{j=1\\ j\neq i}}^{\ell} c_j$.

We show that m has the required properties stated in the theorem.

1.) Assume $m \in S$ then there exist $r_1,\ldots,r_\ell \in N_0$ such that $m = \sum_{i=1}^{\ell} r_i n_i$ (*).

Now $m = (\ell-1) \prod_{j=1}^{\ell} c_j - \sum_{i=1}^{\ell} n_i = \sum_{i=1}^{\ell-1} c_i n_i - \sum_{i=1}^{\ell} n_i$ and therefore (*) yields $\sum_{i=1}^{\ell-1} (r_i+1-c_i) n_i + (r_\ell+1) n_\ell = 0$. Since $r_\ell+1 > 0$, there exists $i \in \{1,\ldots,\ell-1\}$ such that $r_i+1-c_i < 0$. By a.) r_i+1-c_i must be divisible by c_i. Therefore r_i+1 is divisible by c_i. But $0 < r_i+1 < c_i$, a contradiction. Therefore $m \notin S$.

2.) We now prove the converse: If $s \notin S$ then $m-s \in S$.

Let $s \in Z$, $s \notin S$, $s = \sum_{i=1}^{\ell} r_i n_i$, $r_i \in Z$.

For $i,j = 1,\ldots,\ell$ we have the relations $c_i n_i = c_j n_j$. Say $r_1 = a_1 c_1 + r_1'$, $a_1, r_1' \in Z$ and $0 \le r_1' < c_1$.

Then $s = \sum_{i=1}^{\ell} r_i n_i - a_1 c_1 n_1 + a_1 c_2 n_2$

$$= r_1' n_1 + (r_2 + a_1 c_2) n_2 + \sum_{i=3}^{\ell} r_i n_i \ .$$

By the relation $c_2 n_2 = c_3 n_3$ we make, in a similar way, the coefficient of n_2 less than c_2 and greater or equal to 0. We can proceed in this way until $i = \ell-1$. Hence we may assume from the beginning that $s = \sum_{i=1}^{\ell} r_i n_i$ and $0 \le r_i < c_i$ for $i=1,\ldots,\ell-1$. Since $s \notin S$, r_ℓ must then be less than zero. Therefore

$m-s = \sum_{i=1}^{\ell-1} (c_i - 1 - r_i) n_i + (1 - r_\ell) n_\ell$ is an element of S, since $c_i - 1 - r_i \ge 0$ and $1 - r_\ell \ge 0$.

This theorem gives rise to a general definition. Let S be a cancellative semigroup with 0 element.

<u>Definition 2.1.2.</u>: S is a Sylvester-semigroup if there exists $m \in \hat{S}$ such that for all $s \in \hat{S}$, $s \in S$ if and only if $m-s \notin S$.

<u>Remarks:</u> We call cancellative semigroups with the above property Sylvester-semigroups, because Sylvester proved Theorem 3.1.1. in the case $\ell=2$. Some authors call these semigroups symmetric semigroups; however, the term "symmetric" is also used more commonly to describe semigroups in a different sense.

Of course, Theorem 2.1.1 does not determine all possible numerical semigroups, which are Sylvester-semigroups. As an example, consider the semigroup S generated by $\{4,5,6\}$. S is a Sylvester-semigroup, but is not of the type as described in 3.1.1. If S is a numerical semigroup then we let M be the maximal ideal $M = S - \{0\}$ and M^- be the set $M^- = \{z \in Z \ / \ z+M \subseteq S\}$.

<u>Theorem 3.1.3.</u>: Let S be a numerical semigroup then the following conditions are equivalent:

a.) S is a Sylvester-semigroup.

b.) $M^- = \{m\} \cup S$, where m is the greatest integer not belonging to S.

c.) Each proper principal ideal is irreducible, (i.e.,: If $s \in S$, $s \neq 0$, then the ideal (s) cannot be written as the intersection of two ideals in S, which both properly contain (s)).

d.) There exists a proper principal ideal, which is irreducible.

<u>Proof</u>: a.) implies b.) : Let $z \in M^-$. If $z \in S$, then there is nothing to show. Assume $z \notin S$. If $z \neq m$, then $m-z \in M$, since S is a Sylvester-semigroup and therefore $z + M \nsubseteq S$, since $m \notin S$. Thus z must be equal to m. Obviously, $m \in M^-$.

b.) implies a.) : Assume S is not symmetric. Then there exists a greatest integer $m_1 \notin S$ such that $m-m_1 \notin S$.

We clain: $M^- \supsetneq \{m_1, m\} \cup S$.

<u>Proof</u>: We only have to show, that $m_1 \in M^-$. Assume $m_1 \notin M^-$, then there exists $s \in M$ such that $m_1+s \notin S$. By the definition of m_1, it follows that $m-m_1-s \in S$. Therefore $m-m_1 = (m-m_1-s)+s$ is an element of S, a contradiction.

a.) implies c.): Let (s) be a principal ideal and $s_1 \epsilon S$ such that $s_1 \notin (s)$, then $s+m \epsilon (s_1)$.

Proof: Since $s_1 \notin (s)$, it follows that $s_1 - s \notin S$. Therefore $m-(s_1-s) \epsilon S$, since S is a Sylvester-semigroup. In other words: $(m+s)-s_1 \epsilon S$ or equivalently $m+s \epsilon (s_1)$.

c.) follows now easily. Assume (s) is reducible then there exists $s_1, s_2 \epsilon S$, $s_1, s_2 \notin (s)$ such that $(s_1) \cap (s_2) \subseteq (s)$. However, from the considerations above, it follows that $s+m \epsilon (s_1) \cap (s_2)$, whereas $s+m \notin (s)$, a contradiction.

It is trivial that c.) implies d.)

d.) implies a.) : We show, if S is not a Sylvester-semigroup, then each principal ideal is reducible.

Claim: If (s) is a proper principal ideal then $(s) = (s,s+m) \cap (s,s+m_1)$, where m is the greatest integer not belonging to S and m_1 the greatest integer not belonging to S such that $m-m_1 \notin S$.

Remark: By the definition of m and m_1, it follows that $s+m \epsilon S$ and $s+m_1 \epsilon S$. Also $(s,s+m)$ and $(s,s+m_1)$ both contain (s) properly. Thus, once we have proved the claim our assertion follows.

Proof of the claim: It is sufficient to show that $(s+m) \cap (s+m_1) \subseteq (s)$. Let $s' \epsilon (s+m) \cap (s+m_1)$ then $s' = s+m+s_1 = s+m_1+s_2$, $s_1, s_2 \epsilon S$. Assume $s_1 = 0$, then $m-m_1 \epsilon S$, a contradiction. Therefore $s_1 > 0$ and this implies that $m+s_1 \epsilon S$. Hence $s' \epsilon (s)$.

3.2. The semigroup ring of a Sylvester-semigroup.

Let S be a numerical semigroup generated by $\{n_1, \ldots, n_\ell\}$ and let k be a field. We know that $k[S] \cong k[T^{n_1}, T^{n_2}, \ldots, T^{n_\ell}]$, T an indeterminate. Let $\mathfrak{m}$ be the maximal ideal $(T^{n_1},\dots,T^{n_\ell})$ in $k[S]$, K the quotient field of $k[S]$ and O_S the local ring $k[S]_{\mathfrak{m}}$ with maximal ideal $\mathfrak{M} = \mathfrak{m}k[S]_{\mathfrak{m}}$. We are interested in the length of the module $\mathfrak{M}^{-1}/O_S$ and denote this length by $\ell(\mathfrak{M}^{-1}/O_S)$. Remember, if R is an integral domain with quotient field K and $\mathfrak{a} \subseteq R$ is an ideal, one defines:

$$\mathfrak{a}^{-1} = \{x \in K / x\mathfrak{a} \subseteq R\}$$

Of course, $\mathfrak{a}^{-1}$ is an R-module and contains R. Hence one can form the R-module $\mathfrak{a}^{-1}/R$.

The following theorem is of the type as 3.1.3., however on the ring level.

<u>Theroem 3.2.1</u>.: The following conditions are equivalent:

a.) S is a Sylvester-semigroup.

b) $\ell(\mathfrak{M}^{-1}/O_S) = 1$

c.) Each proper principal ideal in O_S is irreducible.

d) There exists a proper principal ideal in O_S, which is irreducible

<u>Proof</u>: a.) implies b.)

1.) We first show that $\ell(\mathfrak{m}^{-1}/_{k[S]}) = 1$ and this follows at once as soon as we proved, that

$$\mathfrak{m}^{-1} = k\cdot T^{\mu} + k[S], \text{ where } \mu \text{ is the greatest integer}$$

not belonging to S .

Let $x \in \mathfrak{m}^{-1}$, then $x = \dfrac{f(T)}{g(T)}$, $f(T),\ g(T) \in k[T]$.

We may assume that gcd $(f(T), g(T)) = 1$. We have $\frac{f(T)}{g(T)} T^{n_1} \in k[S]$, since $x \in \mathfrak{m}^{-1}$. Therefore $g(T) = \varkappa T^{\nu}$, $\varkappa \in k$, $0 \leq \nu \leq n_1$, because $\gcd(g(T), f(T)) = 1$. Assume that $\nu > 0$, then $f(T)$ must be of the form $f(T) = \varkappa' + \ldots$, where $\varkappa' \in k$, $\varkappa' \neq 0$. Hence we get $\frac{f(T)T^{n_1-\nu}}{\varkappa} = \frac{\varkappa'}{\varkappa} T^{n_1-\nu} + \ldots$ is an element of $k[S]$. But this can only happen, if $\nu = n_1$. If S is generated by just one element, i.e. $n_1 = 1$, then our assertion follows at once. Therefore we may assume, that S is generated by more than one element. In this case, there exists $s \in S$, with $s - n_1 \notin S$. Hence $x\, T^{s} = \frac{\varkappa'}{\varkappa} T^{s-n_1} + \ldots$ is not an element of $k[S]$. Thus ν must be zero and $g(T)$ is a constant. Therefore $\mathfrak{m}^{-1} \in k[T]$. Now let $f(T) \in \mathfrak{m}^{-1}$ and assume that $f(T) = \varkappa T^{\nu} + \ldots$, $\varkappa \in k$ and ν a positive integer not belonging to S. We show that ν must be equal to μ, thereby proving 1.). Assume $\nu \neq \mu$, then $\mu - \nu > 0$ and $\mu - \nu \in S$, since S is a Sylvester-semigroup. Therefore $T^{\mu-\nu} \in \mathfrak{m}$, but $f(T)T^{\mu-\nu} = \varkappa T^{\mu} + \ldots$ is not an element of $k[S]$, a contradiction. On the other hand, if $f(T) = \varkappa T^{\mu} + h(T)$, $h(T) \in k[S]$, then $f(T)$ is certainly an element of $\mathfrak{m}^{-1}$.

2.) We want to show that $\ell(\mathfrak{M}^{-1}/0_S) = 1$. This follows immediately from the fact that $\mathfrak{M}^{-1} = \mathfrak{m}^{-1} 0_S$ and 1.), because

$$\mathfrak{M}^{-1}/_{0_S} = \mathfrak{m}^{-1} 0_S/_{0_S} = \mathfrak{m}^{-1}/_{k[S]} \underset{k[S]}{\otimes} 0_S .$$

Let us prove that $\mathfrak{M}^{-1} = \mathfrak{m}^{-1} 0_S$. It is trivial that $\mathfrak{M}^{-1} \subseteq \mathfrak{m}^{-1} 0_S$. Conversely, if $x \in \mathfrak{M}^{-1}$ then $x \mathfrak{M} \subseteq 0_S$. Therefore $x\, \mathfrak{m} 0_S \subseteq 0_S$.

Hence $xT^{n_i} \in O_S$ for $i = 1,\ldots,\ell$. Therefore there exist $t, r_1, r_2, \ldots, r_\ell \in k[S]$ such that $xT^{n_i} = \frac{r_i}{t}$ for $i = 1,\ldots,\ell$. This means, however, that $tx \in \mathfrak{m}^{-1}$ and thus $x \in \mathfrak{m}^{-1} O_S$.

b) implies c.): This is a general fact: If $(R, \mathfrak{m})$ is a one dimensional local domain, then the conditions b.) and c.) are equivalent. A proof of this theorem is in [2].

It is trivial that c.) implies d.)

d.) implies a.): Assume S is not a Sylvester-semigroup. Let μ_1 be the greatest positive integer such that $\mu_1 \notin S$ and $\mu - \mu_1 \notin S$. Let (c) be any proper principal ideal in O_S. We may assume that $c \in k[S]$. By the definition of μ and μ_1, it follows that $cT^{\mu} \notin (c)$ and $cT^{\mu_1} \notin (c)$. Moreover cT^{μ} and cT^{μ_1} are elements in $k[S]$. We claim that $(c) = (c, cT^{\mu_1}) \cap (c, cT^{\mu})$. This shows that (c) is reducible.

Proof of the claim: Let $a \in (c, cT^{\mu_1}) \cap (c, cT^{\mu})$.

There exist $a_1, a_2, b_1, b_2 \in O_S$ such that

$$a = a_1 c + a_2 cT^{\mu_1} = b_1 c + b_2 cT^{\mu}.$$

Hence $a_2 cT^{\mu_1} - b_2 cT^{\mu} = b_1 c - a_1 c$. Multiplying this equation by the product of the denominators of a_1, a_2, b_1, b_2, we may assume that a_1, a_2, b_1, b_2 are already elements in $k[S]$.

Let $a_2 = \varkappa + a_2'$, $\varkappa \in k$, $a_2' \in (T^{n_1}, \ldots, T^{n_\ell})$ and $b_2 = \varkappa' + b_2'$, $\varkappa' \in k$, $b_2' \in (T^{n_1}, \ldots, T^{n_\ell})$. By the definition of μ and μ_1, it follows that $a_2' T^{\mu_1} \in k[S]$ and $b_2' T^{\mu} \in k[S]$. Therefore the above equation yields $\varkappa cT^{\mu_1} - \varkappa' cT^{\mu} = dc$, $d \in k[S]$. In fact, $d = b_1 - a_1 - a_2' T^{\mu_1} - b_2' T^{\mu}$. Hence we obtain in $k[T]$ that $\varkappa T^{\mu_1} - \varkappa' T^{\mu} = d$. Since $d \in k[S]$, it follows that $\varkappa = \varkappa' = 0$. In other words: $a_2 T^{\mu_1}$ and $b_2 T^{\mu}$ are elements in $k[S]$. Therefore $\varepsilon \cdot a = (b_1 + b_2 T^{\mu})c$, ε a unit in O_S and $b_1 + b_2 T^{\mu} \in k[S]$. Hence $a \in (c)$.

Remarks: If S is a Sylvester-semigroup, then it does not follow in general that each principal ideal in $k[S]$ is irreducible.

Example: Let S be generated by $\{2,3\}$. The ideal (T^2+T^3) is reducible in $k[T^2,T^3]$. In fact, $(T^2+T^3)=(T^2+T^3,T^4) \quad (T^2+T^3,T^3+T^4)$.

Theorem 3.2.1 is related to a general concept: A noetherian local ring $(R,\mathfrak{m})$ is said to be a Gorenstein ring if

1.) $(R,\mathfrak{m})$ is a Cohen Macauley ring.
2.) Each ideal in $(R,\mathfrak{m})$ generated by a parameter system is irreducible.

As a special case it follows easily that: A one dimensional noetherian local domain is a Gorenstein ring if and only if each proper principal ideal is irreducible.

Corollary 3.2.2.: O_S is a Gorenstein ring if and only if S is a Sylvester-semigroup.

It is known, that a noetherian local ring $(R,\mathfrak{m})$ which is a complete intersection is a Gorenstein ring.

Corollary 3.2.3: If a numerical semigroup S is a complete intersection then S is a Sylvester-semigroup.

The converse is not true, as we show by an example in Chapter IV. However, there is a stronger result: Serre proved in [5] the following: Let k be an algebraically closed field, C a curve in the affine 3-space $A_3(k)$, $P \in C$ a point of C and $k[C]_P$ the local ring of P on C. Then the following conditions are equivalent:

a.) $k[C]_P$ is a complete intersection

b.) $k[C]_P$ is a Gorenstein ring.

<u>Corollary 3.2.4.</u>: If S is a numerical semigroup of rank 3, then S is a complete intersection if and only if S is a Sylvester-semigroup.

In Chapter IV we will prove 3.2.4 directly and give some other equivalent conditions for a numerical semigroup of rank 3 to be a complete intersection.

CHAPTER IV

Numerical Semigroups of Rank 3

4.1. The minimal relations of numerical semigroups of rank 3.

Preliminary remarks: Let S be a finitely generated cancellative semigroup, say $S = N_0^n/\rho$. To a relation $A\epsilon\rho$, $A = (v,v')$ we assign a vector w_A in Z^n in the following way:

$$A \to w_A = v-v' .$$

It is easily verified that the set

$$M_\rho(S) = \{w_A \epsilon Z^n / A\epsilon\rho\}$$

is a subgroup of Z^n.

We call $M_\rho(S)$ the relation module of S (with respect to ρ). If we are given $M_\rho(S)$ we can recover ρ. To show this, we introduce some notions: Let $v, v'\epsilon Z^n$, $v = (z_i)$, $v' = (z_i')$. We define

$$v \vee v' = (\max(z_i, z_i')), \quad v \wedge v' = (\min(z_i, z_i'))$$

and $v^+ = v \vee 0$, $v^- = (-v) \vee 0$. For any $v\epsilon Z^n$ we have $v = v^+ - v^-$.

We define the set of reduced relations of ρ to be

$\rho_r = \{A\epsilon\rho \ / \ A=(v,w), \ v\wedge w = 0\}$. It is clear that $\rho = \bar{\rho}_r$. In fact

$$\rho = \{(v+w, \ v'+w) \ / \ w\epsilon Z^n, \ (v,v')\epsilon\rho_r\} .$$

Therefore ρ is completely determined by ρ_r. There is a bijection between the elements of ρ_r and the elements of $M_\rho(S)$, which is established in the following way:

$$M_\rho(S) \rightleftarrows \rho_r$$

$$w \longmapsto (w^+, w^-) = A_w$$

$$w_A = v - v' \longleftarrow (v, v') = A.$$

The above considerations show how to obtain ρ from $M_\rho(S)$. Therefore S is completely determined by $M_\rho(S)$.

We say that $v_1, \ldots, v_k \in M_\rho(S)$ generate ρ if the relations $A_{v_i} = (v_i^+, v_i^-)$, $i = 1, \ldots, k$ generate ρ. If $\{A_i = (v_i, v_i')/i=1, \ldots, k\}$ is a minimal system of generators of ρ, then the relations A_i are reduced, since S is cancellative. Therefore to find a minimal system of generators of ρ is equivalent to finding $v_1, \ldots, v_k \in M_\rho(S)$, k minimal, such that $v_1, \ldots, v_k$ generate ρ.

<u>The minimal relations</u>:

Let us return to the case where S is a numerical semigroup of rank 3. Say, S is generated by $\{n_1, n_2, n_3\}$, $n_i \in N$.

Let $e_1 = (1,0,0)$, $e_2 = (0,1,0)$, $e_3 = (0,0,1)$. We have an epimorphism $\rho^* : N_0^3 \to S$, where $\rho^*(e_i) = n_i$ for $i = 1,2,3$. Hence $S = {N_0^3}/\rho$,

$\rho = \{(v, v')/\rho^*(v) = \rho^*(v')\}$ and

$$M_\rho(S) = \{(z_1, z_2, z_3) \in Z^3 / \sum_{i=1}^{3} z_i n_i = 0\}.$$

If $v \in M_\rho(S)$, $v \neq 0$, $v = (z_1, z_2, z_3)$, then there is one component of v, say the i - component, such that either:

1.) $z_i > 0$ and $z_j \leq 0$ for $j \neq i$

or 2.) $z_i < 0$ and $z_j \geq 0$ for $j \neq i$.

We say in this case, that v is of type i and define:

Definition 4.1.1: $v \in M_\rho(S)$ is a mimimal relation of type i if for all relations $v' \in M_\rho(S)$ of type i, $v' = (z_j')$, we have

$$|z_i'| \geq |z_i| .$$

A relation is said to be minimal if it is minimal of any type. We denote the set of minimal relations of $M_\rho(S)$ by $\mathfrak{M}$.

Remark: The minimal relations of type 1 are obtained in the following way: Take the smallest multiple c_1 of n_1 such there exist r_2, $r_3 \in N_0$ with $c_1 n_1 = r_2 n_2 + r_3 n_3$. Then $v = (-c_1, r_2, r_3)$ is a minimal relation of type 1. Similarly one finds the minimal relations of type 2 and type 3. From this description follows immediately that $\mathfrak{M}$ is finite.

Let $\{v_1, v_2, \ldots, v_k\} \subseteq M_\rho(S)$ be any minimal set of generators of ρ. We will prove later that $\{v_1, v_2, \ldots, v_k\} \subseteq \mathfrak{M}$. This shows that it is worthwhile to study the minimal relations of $M_\rho(S)$.

Let us choose $v_1, v_2, v_3 \in \mathfrak{M}$,

$$v_1 = (-c_1, r_{12}, r_{13})$$

$$v_2 = (r_{21}, -c_2, r_{23}) \quad , c_i > 0 \text{ for } i = 1,2,3$$

$$v_3 = (r_{31}, r_{32}, -c_3) \quad \text{and } r_{ij} \geq 0 \text{ for } i, j = 1,2,3 .$$

We will distinguish two cases:

Case I: $r_{ij} \neq 0$ for all $i,j = 1,2,3$.

Case II: There exists $i,j \in \{1,2,3\}$ such that $r_{ij} = 0$.

First we consider Case I:

Proposition 4.1.3.: In case I we have,

$$v = v_1 + v_2 + v_3 = 0 \ .$$

Proof: Assume $v \neq 0$ and say v is of type 1. If the first component of v is less than zero, then $-c_1 + r_{21} + r_{31} < 0$. However, since c_1 is minimal for relations of type 1, we have that $-c_1 + r_{21} + r_{31} \leq -c_1$, a contradiction, since $r_{21} > 0$ and $r_{31} > 0$. If the first component of v is positive, then $-c_1 + r_{21} + r_{31} > 0$. And again, since c_1 is minimal we have $-c_1 + r_{21} + r_{31} \geq c_1$. But then either $r_{21} \geq c_1$ or $r_{31} \geq c_1$. Say $r_{21} \geq c_1$, then $v_1 + v_2 = (r_{21} - c_1, r_{12} - c_2, r_{13} + r_{23})$, where $r_{21} - c_1 \geq 0$ and $r_{13} + r_{23} > 0$. Hence $r_{12} - c_2 < 0$ and therefore $v_1 + v_2$ is a relation of type 2. But c_2 is minimal for relations of type 2, therefore $r_{12} - c_2 \leq -c_2$, or equivalently $r_{12} \leq 0$. This is again a contradiction, since $r_{12} > 0$ by assumption.
The proof works similarly, if we assume that v is of type 2 or type 3.

Proposition 4.1.3.: In case I we have

$$\mathfrak{M} = \{\pm v_1, \pm v_2, \pm v_3\} \ .$$

Proof: Let $v_1' = (-c_1, r_{12}', r_{13}')$ be an element of $\mathfrak{M}$. If $r_{13}' = 0$, then $v_1' - v_1 = (0, r_{12}' - r_{12}, -r_{13})$ is a relation of type 3. Therefore $r_{13} \geq c_3$. We know, however, by 4.1.2 that $r_{13} = c_3 - r_{23}$ and hence $r_{13} < c_3$, since $r_{23} > 0$, a contradiction. Similarly it can be shown that $r_{12}' \neq 0$. Thus we may restrict our attention to the case where $v_1' = (-c_1, r_{12}', r_{13}')$ with $r_{12}' > 0$ and $r_{13}' > 0$. We apply again 4.1.2. and obtain that $v_1' + v_2 + v_3 = 0$. Therefore $v_1' = -(v_2+v_3) = v_1$. Similarly it is shown that v_2 and v_3 are uniquely determined. This proves 4.1.3.

Next we consider case II:

<u>Proposition 4.1.4</u>: In case II either

$\alpha.)$ $(0, -c_2, c_3) \in \mathfrak{M}$ or

$\beta.)$ $(c_1, 0, -c_3) \in \mathfrak{M}$ or

$\gamma.)$ $(-c_1, c_2, 0) \in \mathfrak{M}$

<u>Proof</u>: We may assume without loss of generality that $r_{21} = 0$. Therefore, $v_2 = (0, -c_2, r_{23})$. If $r_{23} = c_3$, then our assertion follows. Thus we may assume that $r_{23} > c_3$, since c_3 is minimal. Consider $v_2 + v_3 = (r_{31}, r_{32} - c_2, r_{23} - c_3)$. Since $r_{31} \geq 0$ and $r_{23} - c_3 > 0$, we get : $r_{32} - c_3 < 0$. This implies, that $r_{32} - c_2 \leq -c_2$, hence $r_{32} \leq 0$. However, $r_{32} \geq 0$, therefore $r_{32} = 0$ and $r_{31} \geq c_1$. If $r_{31} = c_1$, then our assertion follows again. Thus, let us assume that $r_{31} > c_1$. Now $v_1 + v_3 = (r_{31} - c_1, r_{12}, r_{13} - c_3)$ is a relation of type 3, since $r_{31} - c_1 > 0$ and $r_{12} \geq 0$. Therefore $r_{13} - c_3 < 0$. By the minimality of c_3 this means that $r_{13} - c_3 \leq -c_3$, hence $r_{13} = 0$. Thus $v_1 = (-c_1, r_{12}, 0)$ and $r_{12} \geq c_2$. This is, however, a contradiction, since

$v_1 + v_2 + v_3 = (r_{31} - c_1, r_{12} - c_2, r_{23} - c_3)$ is a relation with $r_{31} - c_1 > 0$, $r_{12} - c_2 \geq 0$ and $r_{23} - c_3 > 0$. Therefore $r_{31} = c_1$ and $v_3 = (c_1, 0, -c_3)$ which proves our proposition.

In the following proposition we describe explicitely the set of minimal relations in case II. If x is a real number, we define $[x]$ to be the greatest integer less than or equal to x.

Proposition 4.1.5.: In case II the set of minimal relations is either:

α.) $\mathfrak{M} = \{\pm(0,-c_2,c_3),\ \pm(d(0,-c_2,c_3) + (-c_1,r_{12},r_{13}))\}$,

where $d \in Z$ and $-[\frac{r_{13}}{c_3}] \leq d \leq [\frac{r_{12}}{c_2}]$, $r_{12}, r_{13} \geq 0$.

or β.) $\mathfrak{M} = \{\pm(c_1,0,-c_3),\ \pm(d(c_1,0,-c_3) + (r_{21},-c_2,r_{23}))\}$,

where $d \in Z$ and $-[\frac{r_{21}}{c_1}] \leq d \leq [\frac{r_{23}}{c_3}]$, $r_{21}, r_{23} \geq 0$.

or γ.) $\mathfrak{M} = \{\pm(-c_1,c_2,0),\ \pm(d(-c_1,c_2,0) + (r_{31},r_{32},-c_3))\}$,

where $d \in Z$ and $-[\frac{r_{32}}{c_2}] \leq d \leq [\frac{r_{31}}{c_1}]$, $r_{31}, r_{32} \geq 0$.

Proof: Referring to 4.1.4 we may assume that $(0,-c_2,c_3) \in \mathfrak{M}$. Let $v_1 = (c_1,r_{12},r_{13})$, $v_2 = (0,-c_2,c_3)$ and $\mathfrak{N} = \{\pm v_1,\ \pm(dv_2+v_1)\}$, $-[\frac{r_{13}}{c_3}] \leq d \leq [\frac{r_{12}}{c_2}]$. We show that $\mathfrak{M} = \mathfrak{N}$.

It is clear that $\mathfrak{M} \supseteq \mathfrak{N}$. Conversely, assume we are given a minimal relation $v_2' = (r_{21}',-c_2,r_{23}')$ of type 2. If $r_{23}' = c_3$, then $v_2'-v_2 = (r_{21}',0,0)$. therefore $r_{21}' = 0$. In other words $v_2' = v_2$ and thus $v_2' \in \mathfrak{N}$. If $r_{23}' \neq c_3$, then $v_2'-v_2 = (r_{21}',0,r_{23}'-c_3)$ is a minimal relation of type 3. Since $r_{21}' \geq 0$ we have $r_{23}'-c_3 < 0$.

Then $r_{23}'-c_3 \leq -c_3$, because of the minimality of c_3. So that in this case $r_{23}' = 0$. Hence $v_2' = (r_{21}', -c_2, 0)$. Now $v_1+v_2' = (r_{21}'-c_1, r_{12}-c_2, r_{13})$ is a relation of type 2, since $r_{21}'-c_1 \geq 0$ and $r_{13} \geq 0$. Therefore $r_{12}-c_2 \leq 0$. If $r_{12}-c_2 = 0$, then $r_{21}' = c_1$, and $r_{13} = 0$. Hence $v_2' = v_1$ and thus $v_2' \in \mathfrak{M}$. If $r_{12}-c_2 < 0$, then $r_{12}-c_2 \leq -c_2$. But this implies that $r_{12} = 0$, so that $v_1 = (-c_1, 0, r_{13})$. Now $v_1+v_2'-v_2 = (r_{21}'-c_1, 0, r_{13}-c_3)$ is a relation with $r_{21}'-c_1 \geq 0$ and $r_{13}-c_3 \geq 0$. Hence $r_{21}' = c_1$ and $r_{13} = c_3$ and $v_2' = v_2-v_1$.

In any case we get that $v_2' \in \mathfrak{M}$. Similarly it is shown that any minimal relation v_3' of type 3 is an element of $\mathfrak{M}$.

If $v_1' = (-c_1, r_{12}', r_{13}')$ is any minimal relation of type 1, then $v_1'-v_1 = (0, r_{12}'-r_{12}, r_{13}'-r_{13}) = dv_2$, where $d \in Z$. The side conditions for d are obvious. Hence $v_1' = v_1+dv_2$ and our proposition follows.

Our next aim is, to prove that the minimal relations generate ρ. Let R be a commutative ring with 1. If $v \in M_\rho(S)$, we define $F_v \in R[X_1, X_2, X_3]$ to be $F_v = X^{v+} - X^{v-}$. As an immediate consequence of 2.1.5. we obtain,

$$R[S] \cong R[X_1, X_2, X_3] / (\{F_v\}_{v \in M_{\rho}(S)}) .$$

In order to show that the minimal relations generate ρ we only have to prove, according to 2.1.6., that $(\{F_v\}_{v \in \mathfrak{M}}) = (\{F_v\}_{v \in M_\rho(S)})$ (*) .

We define on S an ordering : If $s_1, s_2 \in S$, then $s_1 \geq s_2$ if $s_1-s_2 \in S$ and we say that $s_1 > s_2$ if $s_1 \geq s_2$ and $s_1 \neq s_2$.

It is clear that:

1.) If $s_1, s_2 \in S$, then $s_1 \geq s_2$, $s_2 \geq s_1$, if and only if $s_1 = s_2$.

2.) If $\{s_i\}_{i=1,2,\ldots} \subseteq S$ such that $s_{i+1} \geq s_i$ for all $i = 1,2,\ldots,$ then there exists i_0 such that for all $i \geq i_0$, $s_{i+1} = s_i$.

In Chapter II we introduced on $R[X_1,X_2,X_3]$ an S-graduation. If $F \in R[X_1,X_2,X_3]$ is a homogeneous polynomial, then we denote its homogeneous degree by deg (F) and for two homogeneous polynomials F, G, we mean by deg (F) < deg(G), that deg (F) is less than deg (G) with respect to the ordering on S. Now (*) will be a simple consequence of the following proposition.

<u>Proposition 4.1.6</u>. Let $v \in M_\rho(S)$. Then F_v can be written as $F_v = F' + QF_w$, where $F' \in (\{F_v\}_{v \in \mathfrak{M}})$, $w \in M_\rho(S)$, $Q \in R[X_1,X_2,X_3]$ and $\deg(F_w) < \deg(F_v)$.

<u>Proof</u>: Case I: $\mathfrak{M} = \{\pm v_1, \pm v_2, \pm v_3\}$, where $v_1 = (-c_1, r_{12}, r_{13})$, $v_2 = (r_{21}, -c_2, r_{23})$, $v_3 = (r_{31}, r_{32}, -c_3)$ and for all $i,j = 1,2,3$, $r_{ij} > 0$.

Let $v \in M_\rho(S)$. If $v \in \mathfrak{M}$, then the assertion is trivial. Otherwise we may assume without loss of generality that $v = (-a_1,a_2,a_3)$, $a_1 > c_1$, $a_2 \geq 0$, $a_3 \geq 0$. We get $F_v - X_1^{a_1-c_1} F_{v_1} =$

$$X_2^{\min(a_2,r_{12})} \cdot X_3^{\min(a_3,r_{13})} \cdot F_w, \text{ where}$$

$$F_w = X_2^{a_2-\min(a_2,r_{12})} X_3^{a_3-\min(a_3,r_{13})} - X_1^{a_1-c_1} X_2^{r_{12}-\min(a_2,r_{12})} X_3^{r_{13}-\min(a_3,r_{13})}.$$

Of course $w \in M_\rho(S)$ and

$$\deg(F_w) = \deg(F_v - X_1^{a_1-c_1}\cdot F_{v_1}) - \deg(X_2^{\min(a_2,r_{12})}\cdot X_3^{\min(a_3,r_{13})})$$

$$= \deg(F_v) - \deg(X_2^{\min(a_2,r_{12})}\cdot X_3^{\min(a_3,r_{13})}).$$

It cannot be that $(a_2,a_3) = (0,0)$. Therefore

$\deg(X_2^{\min(a_2,r_{12})}\cdot X_3^{\min(a_3,r_{13})}) > 0$. Hence $\deg(F_w) < \deg(F_v)$.

This proves our assertion in case I.

Case II: We may restrict our attention to considering only II α.), $\mathfrak{M} = \{\pm v_2, \pm(d\,v_2+v_1)\}$, $v_1 = (-c_1,r_{12},r_{13})$, $v_2 = (0,-c_2,c_3)$ and $-[\frac{r_{13}}{c_3}] \leq d \leq [\frac{r_{12}}{c_2}]$.

1.) Let $v = (-a_1,a_2,a_3)$ be a relation of type 1. Hence $a_1 \geq c_1$, and $a_2 \geq 0$, $a_3 \geq 0$. If $a_1 = c_1$ then our assertion is trivial. Therefore we may assume that $a_1 > c_1$. We get similarly as in Case I,

$F_v - X_1^{a_1-c_1}\, F_{v_1} = X_2^{\min(a_2,r_{12})}\cdot X_3^{\min(a_3,r_{13})}\cdot F_w$ with $w \in M_\rho(S)$.

If $a_2 > 0$ and $a_3 > 0$ then $\deg F_w < \deg F_v$ and the proposition follows. Now assume that $a_2 = 0$. If $r_{13} > 0$, then again $\deg F_w < \deg F_v$, since $a_3 > 0$. If $r_{13} = 0$, then $r_{12} \geq c_2$ and $v_1' = v_1+v_2 = (-c_1,r_{12}-c_2,c_3)$ is a minimal of type I, whose third component is greater than zero. Therefore $F_v - X_1^{a_1-c_1}\, F_{v_1'} = X_3^{\min(a_3,c_3)}\cdot F_{w'}$, $w' \in M_\rho(S)$ and $\deg F_{w'} < \deg F_v$, since $\min(a_3,c_3) > 0$. The proof works similarly if we assume that $a_3 = 0$.

2.) Let $v = (a_1, -a_2, a_3)$ be a relation of type 2. We may assume that $a_2 > c_2$, otherwise there is nothing to prove. If $a_3 = 0$, then v is also a relation of type 1 and this case is already treated in 1. If $a_3 > 0$, then $F_v - X_2^{a_2 - c_2} \cdot F_{v_2} = X_3^{\min(a_3, c_3)} \cdot F_w$, where $w \in M_\rho(S)$ and $\deg F_w < \deg F_v$, since $\min(a_3, c_3) > 0$. The case that $v \in M_\rho(S)$ is a relation of type 3 is similarly proved as the case above. This completes the proof.

As a corollary of 4.1.6 we get :

<u>Theorem 4.1.7</u>: The minimal relations of $M_\rho(S)$ generate ρ.

<u>Theorem 4.1.8.</u>:

In case I : S is not a complete intersection.

In case II: S is a complete intersection.

<u>Proof</u>: Case I : $I_S = (F_{v_1}, F_{v_2}, F_{v_3})$, where

$$F_{v_1} = X_2^{r_{12}} X_3^{r_{13}} - X_1^{c_1}$$

$$F_{v_2} = X_1^{r_{21}} X_3^{r_{23}} - X_2^{c_2}$$

$$F_{v_3} = X_1^{r_{31}} X_2^{r_{32}} - X_3^{c_3} \quad \text{and } r_{ij} > 0 \text{ for } i,j = 1,2,3.$$

Assume we have an equation

$$Q_1 F_{v_1} = Q_2 F_{v_2} + Q_3 F_{v_3} \quad , Q_i \in R[X_1, X_2, X_3], \ i = 1,2,3.$$

This equation yields

$$-\bar{Q}_1 X_1^{c_1} = \bar{Q}_2 X_1^{r_{21}} X_3^{r_{23}} - \bar{Q}_3 X_3^{c_3} \quad \text{mod } X_2, \text{ where}$$

$$\bar{Q}_i = Q_i \text{ mod } X_2, \ i = 1,2,3.$$

Since $r_{23} > 0$ we have $\bar{Q}_1 = X_3 \cdot P$ with $P \in R[X_1,X_2]$. On the other hand: $\bar{Q}_1 = Q_1 - X_2 \cdot T$ with $T \in R[X_1,X_2,X_3]$. Hence $Q_1 = X_2T + X_3P$. Therefore $Q_1 \in (X_1,X_2,X_3)$. The ideal I_S is contained in (X_1,X_2,X_3). Let $\mathfrak{P}$ be a prime ideal in $R[X_1,X_2,X_3]$ containing (X_1,X_2,X_3), then Q_1 is a non-unit in the local ring $R[X_1,X_2,X_3]_{\mathfrak{P}}$. Similarly, starting with equations $Q_2F_{v_2} = Q_1F_{v_1} + Q_3F_{v_3}$, $Q_3F_{v_3} = Q_1F_{v_1} + Q_2F_{v_3}$ leads to the conclusion that Q_2, Q_3 are non-units in $R[X_1,X_2,X_3]$. Therefore rank $(I_S\, R[X_1,X_2,X_3]_{\mathfrak{P}}) = 3$ and S cannot be a complete intersection.

Case II: We may assume that

$$\mathfrak{M} = \{\pm(0, -c_2,c_3), \pm(d(0,-c_2,c_3) + (-c_1,r_{12},r_{13}))\} \text{ with}$$

$d \in Z$ and $-[\frac{r_{13}}{c_3}] \le d \le [\frac{r_{12}}{c_2}]$. Let $v_2 = (0, -c_2,c_3)$ and

$$v_1 = (-c_1, r_{12} + [\frac{r_{13}}{c_3}]c_2, r_{13} - [\frac{r_{13}}{c_3}]c_3) = (-c_1,\bar{r}_{12},\bar{r}_{13}).$$

Then for any $v \in \mathfrak{M}$ we have either $v = \pm v_2$ or $v = \pm(dv_2+v_1)$, $d = 0,1,2,\ldots,[\frac{r_{13}}{c_3}] + [\frac{r_{12}}{c_2}]$. We claim: $I_S = (F_{v_1},F_{v_2})$.

<u>Proof</u>: Let $v \in \mathfrak{M}$, $v = dv_2+v_1$, $d > 0$, then $v = (-c_1,\bar{r}_{12}-dc_2,\bar{r}_{13}+dc_3)$ and $F_v = X_2^{\bar{r}_{12}-dc_2} X_3^{\bar{r}_{13}+dc_3} - X_1^{c_1}$. Obviously there exists a polynomial $Q \in R[X_1,X_2,X_3]$ such that $F_{dv_2} = QF_{v_2}$.

$$\text{Then } X_2^{\bar{r}_{12}-dc_2} X_3^{\bar{r}_{13}} QF_{v_2}+F_{v_1} = X_2^{\bar{r}_{12}-dc_2} X_3^{\bar{r}_{13}}F_{dv_2} + F_{v_1}$$

$$= X_2^{\bar{r}_{12}-dc_2} X_3^{\bar{r}_{13}}(X_3^{dc_3} - X_2^{dc_2}) + (X_2^{\bar{r}_{12}} X_3^{\bar{r}_{13}} - X_1^{c_1})$$

$$= X_2^{\bar{r}_{12}-dc_2} X_3^{\bar{r}_{23}+dc_3} - X_2^{\bar{r}_{12}} X_3^{\bar{r}_{13}} + X_2^{\bar{r}_{12}} X_3^{\bar{r}_{13}} - X_1^{c_1}$$

$$= X_2^{\bar{r}_{12}-dc_2} X_3^{\bar{r}_{23}+dc_3} - X_1^{c_1} = F_v.$$ This proves the claim and completes the proof.

Corollary 4.1.9.: In any case rank $\rho \leq 3$.

In the following section we give some equivalent conditions for a numerical semigroup to be a complete intersection.

4.2. Equivalent conditions for complete intersections.

Let S be a numerical semigroup of rank 3 generated by $\{n_1,n_2,n_3\}$, $n_i \in N$ and let $\rho^* : N_0^3 \to S$ be the canonical epimorphism, defined as in 4.1. We define $I_S^* = \{s \in S /$ There exist $\mathcal{W}_1, \mathcal{W}_2 \in N_0^3$, $\mathcal{W}_1 \neq \mathcal{W}_2$ such that $\rho(\mathcal{W}_1) = \rho(\mathcal{W}_2) = s\}$

I_S^* is an ideal in S. In fact $I_S^* = (\{\deg F_v\}_{v \in M_\rho(S)})$. For $v \in M_\rho(S)$ we let $s(v) = \rho^*(v^+) = \rho^*(v-)$ and put

$$\mu = \min_{\substack{v_1, v_2 \in M_\rho(s) \\ v_1 \neq z v_2, z \in Z}} \{s(v_1)+s(v_2)\} - n_1 - n_2 - n_3.$$

Theorem 4.2.1.: The following conditions are equivalent:

a.) S is a complete intersection.

b.) I_S^* is generated by 2 elements.

c.) $\mu \notin S$.

d.) S is a Sylvester-semigroup.

<u>Proof</u>: We have proved already that a.) implies b.), see 2.2.9.

b.) implies a.): If S is not a complete intersection then we know that $\mathfrak{M} = \{\pm v_1, \pm v_2, \pm v_3\}$ with

$$v_1 = (-c_1, r_{12}, r_{13})$$

$$v_2 = (r_{21}, -c_2, r_{23})$$

$$v_3 = (r_{31}, r_{32}, -c_3) \text{ and } r_{ij} > 0 \text{ for } i,j = 1,2,3.$$

I_S^* is certainly generated by $\{s(v_1), s(v_2), s(v_3)\}$. We only have to show that for $i,j = 1,2,3$, $i \neq j$, $s(v_i)-s(v_j) \notin S$. We may assume that $i = 2$ and $j = 1$. The other cases are proved similar.

Assume $s(v_2) - s(v_1) \in S$. Then there exist $a_1, a_2, a_3 \in N_0$ such that $c_2n_2 - c_1n_1 = a_1n_1 + a_2n_2 + a_3n_3$. Hence $(c_2-a_2)n_2 + (-c_1-a_1)n_1 - a_3n_3 = 0$. Since $-a_3 \leq 0$ and $-c_1-a_1 < 0$, we have $c_2-a_2 > 0$. This implies that $c_2-a_2 \geq c_2$. Therefore $a_2 = 0$. Thus we obtain the relation $v = (-c_1-a_1, c_2, a_3)$, which is minimal of type 2. Therefore $v = -v_2$. Hence $r_{21} = a_1 + c_1 \geq c_1$. However $r_{21} = c_1 - r_{31} < c_1$, since $r_{31} \neq 0$, a contradiction.

a.) implies c.) : We may assume that we have the minimal relations $v_1 = (-c_1, r_{12}, r_{13})$ and $v_2 = (0, -c_2, c_3)$.

Obviously, $\mu = c_1n_1 + c_2n_2 - n_1 - n_2 - n_3$

$$= c_1n_1 + c_3n_3 - n_1 - n_2 - n_3 .$$

We have to show that $\mu \notin S$. Assume there exist $a_1, a_2, a_3 \in N_0$, such that $\mu = a_1n_1 + a_2n_2 + a_3n_3$. Then $(a_1+1-c_1)n_1 + (a_2+1)n_2 + (a_3+1-c_3)n_3 = 0$.

Using relation v_2 we may assume that $0 \leq a_3 < c_3$. Hence $a_2+1 > 0$ and $a_3+1-c_3 \geq 0$ and therefore $a_1+1-c_1 < 0$. This implies that $a_1+1-c_1 \leq -c_1$. Therefore $a_1 < 0$, a contradiction.

c.) implies a.) : If S is not a complete intersection then

$\mathfrak{M} = \{\pm v_1, \pm v_2, \pm v_3\}$ with

$v_1 = (-c_1, r_{12}, r_{13})$

$v_2 = (r_{21}, -c_2, r_{23})$

$v_3 = (r_{31}, r_{32}, -c_3)$ and $r_{ij} > 0$ for $i,j = 1,2,3$. We want to show that $\mu \in S$. Without loss of generality we may assume that $s(v_1) < s(v_2) < s(v_3)$. Then

$$\mu = s(v_1) + s(v_2) - n_1 - n_2 - n_3 = c_1 n_1 + c_2 n_2 - n_1 - n_2 - n_3$$

$$= r_{12}n_2 + r_{13}n_3 + r_{21}n_1 + r_{23}n_3 - n_1 - n_2 - n_3$$

$$= (r_{21}-1)n_1 + (r_{12}-1)n_2 + (c_3-1)n_3.$$ The last equality follows from $v_1 + v_2 + v_3 = 0$. Since $r_{21} > 0$, $r_{12} > 0$ and $c_3 > 0$, $\mu \in S$.

a.) implies d.) : We may assume that we have the minimal relations $v_1 = (-c_1, r_{12}, r_{13})$ and $v_2 = (0, -c_2, c_3)$. In order to show that S is a Sylvester-semigroup we prove: If $z \in Z$, then $z \in S$ if and only $\mu - z \notin S$.

One direction is clear, since we know already that if S is a complete intersection then $\mu \notin S$.

To prove the converse, assume $z \in Z$ and $\mu - z \notin S$. We want to show that $z \in S$.

Assume $z \notin S$, $z = a_1 n_1 + a_2 n_2 + a_3 n_3$, $a_i \in Z$. Using the relations v_1, v_2

we may assume that $0 \leq a_1 < c_1$ and $0 \leq a_2 < c_2$ and therefore $a_3 < 0$, since $z \notin S$. Therefore $\mu - z = (-a_3-1)n_3 + (c_1-1-a_1)n_1 + (c_1-1-a_2)n_2$ and $-a_3 - 1 \geq 0$, $c_1-1-a_1 \geq 0$ and $c_2-1-a_2 \geq 0$, hence $\mu - z \in S$, a contradiction.

d.) implies a.) : Assume S is not a complete intersection.

Then $\mathfrak{M} = \{\pm v_1, \pm v_2, \pm v_3\}$ with

$v_1 = (-c_1, r_{12}, r_{13})$

$v_2 = (r_{21}, -c_2, r_{23})$

$v_3 = (r_{31}, r_{32}, -c_3)$ and $r_{ij} > 0$ for $i,j = 1,2,3$. We clain that S is not a Sylvester-semigroup. Let us first prove a simple lemma.

If S is a Sylvester-semigroup (numerical) and $z \in Z$, $z \notin S$, then there exists $s \in S$, $s \neq 0$, such that $z+s \notin S$, or z is the greatest integer not belonging to S.

Proof of the Lemma: Let m be the greatest integer not belonging to S. We may assume that $z \neq m$, therefore $z < m$, since $z \notin S$. Since S is a Sylvester-semigroup, we have $m-z \in S$. Therefore $z + (m-z) = m$ is not an element of S and also $m-z \neq 0$.

Proof of the claim:

Let $\mu_1 = c_1n_1 + c_2n_2 - n_1-n_2-n_3-r_{12}n_2$ and $\mu_2 = c_1n_1 + c_2n_2-n_1-n_2-n_3-r_{21}n_1$.

<u>Claim 1</u>: $\mu_i \notin S$, $i = 1,2$.

<u>Proof</u>: Assume $\mu_1 \in S$. Then there exist $a_1,a_2,a_3 \in N_0$ such that

$\mu_1 = a_1n_1 + a_2n_2 + a_3n_3$. Now $\mu_1 = c_1n_1 + c_2n_2-n_1-n_2-n_3-r_{12}n_2 =$

$(c_1-1)n_1 + (r_{32}-1)n_2-n_3$. For the last equality we used $v_1+v_2+v_3 = 0$.

We may assume that $0 \leq a_3 < c_3$ and it follows that

$(c_1-1-a_1)n_1 + (r_{32}-1-a_2)n_2 + (-1-a_3)n_3 = 0.$

Assume $c_1-1-a_1 \le 0$, then $r_{32}-1-a_2 > 0$. This implies that $r_{32}-1-a_2 \ge c_2$. Since $r_{12} + r_{32} = c_2$ we get $-1-a_2 \ge r_{12} > 0$, hence $a_2 < 0$, a contradiction.

Assume $r_{32}-1-a_2 \le 0$, then $c_1-1-a_1 > 0$ and therefore $c_1-1-a_1 \ge c_1$, hence $a_1 < 0$, a contradiction.

We have, therefore, that $c_1-1-a_1 > 0$ and $r_{32}-1-a_2 > 0$ hence $-1-a_3 \le -c_3$. It cannot be that $1-a_3 < -c_3$, since by assumption $0 \le a_3 < c_3$. On the other hand, assuming that $-1-a_3 = c_3$, then $(c_1-1-a_1, r_{32}-1-a_2, -1-a_3) = v_3$. Therefore $r_{32}-1-a_2 = r_{32}$. Hence $a_3 = -1$, a contradiction. In the same way it is shown that $\mu_2 \notin S$.

<u>Claim 2</u>: $\mu_i + s \in S$ for all $s \in S$, $s \ne 0$, $i=1,2$.

<u>Proof</u>: We show this only for μ_1. Similar arguments work for μ_2. It is enough to prove that $\mu_1 + n_i \in S$ for $i = 1,2,3$.

$$\begin{aligned}\mu_1 + n_1 &= c_1n_1 + c_2n_2-n_2-n_3-r_{12}n_2 \\ &= (c_2-1)n_2 + (r_{13}-1)n_3\end{aligned}$$

Hence $\mu_1 + n_1 \in S$, since $c_2-1 \ge 0$ and $r_{12}-1 \ge 0$,

$$\begin{aligned}\mu_1 + n_2 &= c_1n_1 + c_2n_2-n_1-n_3-r_{12}n_2 \\ &= (r_{21}-1)n_1 + (c_3-1)n_3\end{aligned}$$

Hence $\mu_1 + n_2 \in S$, since $r_{21}-1 \ge 0$ and $c_3-1 \ge 0$.

$$\begin{aligned}\mu_1 + n_3 &= c_1n_1 + c_2n_2-n_1-n_2-r_{12}n_2 \\ &= (c_1-1)n_1 + (r_{32}-1)n_2 .\end{aligned}$$

Hence $\mu_1 + n_3 \in S$ since $c_1-1 \ge 0$ and $r_{32}-1 \ge 0$.

Now assume that S is a Sylvester-semigroup and m the greatest integer not belonging to S. We apply our lemma and it follows that $\mu_1 = m$ and $\mu_2 = m$. Therefore, $\mu_1 = \mu_2$ and, hence, $r_{21}n_1 = r_{12}n_2$ which is a contradiction, since $0 \le r_{12} < c_2$.

Corollary 4.2.2.: If S is a numerical semigroup generated by $\{n_1, n_2, n_3\}$ and S is a Sylvester-semigroup then

$$\mu = \min_{\substack{v_1, v_2 \in M_\rho(S) \\ v_1 \neq z v_2,\ z \in Z}} \{s(v_1) + s(v_2)\} - n_1 - n_2 - n_3 \text{ is the greatest}$$

integer not belonging to S.

Conjecture: Referring to the proof of 4.2.1. the author conjectures that $\max(\mu_1, \mu_2)$ is the greatest integer not belonging to S.

For the sake of completeness we show by examples that most of the equivalent statements of Theorem 4.2.1 are no longer equivalent if the rank of the semigroup is greater than 3.

Example A: Let S be the semigroup generated by $\{5,6,7,8\}$. S is a Sylvester-semigroup but is not a complete intersection, because I_S^* is generated by $\{12,13,14,15\}$ and hence rank $I_S^* = 4$. However, rank I_S^* s hould be 3. A reasonable definition of the number μ for numerical semigroups S of rank 4, generated by $\{n_1, n_2, n_3, n_4\}$, would be

$$\mu = \min_{\substack{v_1, v_2, v_3 \in M_\rho(S) \\ v_1, v_2, v_3 \text{ linearly independent.}}} \{s(v_1) + s(v_2) + s(v_3)\} - n_1 - n_2 - n_3 - n_4 .$$

If we take this as definition of μ, then in our case $\mu = 13$, hence $\mu \in S$. Therefore, if S is a Sylvester-semigroup then it does not follow in general that $\mu \notin S$.

Example B: Let S be the numerical semigroup generated by $\{6,8,9,10\}$. In this case rank $I_S^* = 3$. In fact I_S^* is generated by $\{16,18,20\}$. We show, however, that S is not a complete intersection. We have relations

$$v_1 = (1, -2, 1, 0) \quad , \quad s(v_1) = 16$$
$$v_2 = (0, 1, -2, 1) \quad , \quad s(v_2) = 18$$
$$v_3 = (-3, 1, 0, 1) \quad , \quad s(v_3) = 18$$
$$v_4 = (-3, 0, 2, 0) \quad , \quad s(v_4) = 18$$
$$v_5 = (2, 1, 0, -2) \quad , \quad s(v_5) = 20$$

and so on. If $v \in M_\rho(\mathfrak{s})$ and $v \neq v_i$ for $i = 1,2,3,4,5$ then $s(v) > 20$. We see that $F_{v_4} = F_{v_3} - F_{v_1}$.

Assume $F_{v_3} = Q_1 F_{v_1} + Q_2 F_{v_3}$, $Q_i \in R[X_1,X_2,X_3,X_4]$ then $-X_1^3 \equiv 0 \mod(X_2,X_3,X_4)$, a contradiction. Assume $F_{v_2} = Q_1 F_{v_2} + Q_3 F_{v_3}$, $Q_i \in R[X_1,X_2,X_3,X_4]$ then $-X_3^2 \equiv 0 \mod (X_1,X_2,X_3)$, a contradiction.

Therefore, if S were a complete intersection then I_S would have to be generated by $\{F_{v_1}, F_{v_2}, F_{v_3}\}$. But $F_{v_5} \notin (F_{v_1}, F_{v_2}, F_{v_3})$, since $s(v_5) - s(v_i) \notin S$ for $i = 1,2,3$.

__Question:__ Let S be a numerical semigroup generated by $\{n_1, n_2, n_3, \ldots, n_\ell\}$ and define

$$\mu = \min_{\substack{v_1,\ldots,v_{\ell-1} \in M_\rho(s) \\ v_1,\ldots,v_{\ell-1} \text{ linearly independent.}}} \{s(v_1) + \ldots s(v_{\ell-1})\} - n_1 - n_2 - \ldots - n_\ell.$$

Are the following conditions equivalent?

1.) S is a complete intersection.

2.) $\mu \notin S$.

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BIOGRAPHY

The author was born on December 21, 1941, in Heidelberg, Germany. In 1963 he began graduate study at the Universitaet Kiel and in May 1964 he transferred to the Universitaet Heidelberg where he received his Vordiplom. The following year he entered Purdue University and received a Master of Science in Mathematics in June 1968. In September 1968 he continued his graduate work at Louisiana State University where he is presently a candidate for the degree of Doctor of Philosophy in Mathematics.

EXAMINATION AND THESIS REPORT

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Major Field: Mathematics

Title of Thesis: GENERATORS AND RELATIONS OF ABELIAN SEMIGROUPS AND SEMIGROUP RINGS

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Date of Examination:

June 12, 1969