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GENERIC BASES FOR CLUSTER ALGEBRAS AND THE CHAMBER ANSATZ

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Contents

1. Introduction and main results

1.1. In the recent literature on cluster algebras, calculations of Euler characteristics of certain varieties related to quiver representations play a prominent role. In [GLS2, GLS3, GLS5], cluster variables of coordinate rings of unipotent cells of algebraic groups and Kac-Moody groups were shown to be expressible in terms of Euler characteristics of varieties of flags of submodules of preprojective algebra representations. In another direction, starting with a formula of Caldero and Chapoton [CC], the coefficients of the Laurent polynomial expansions of cluster variables of some cluster algebras were described as Euler characteristics of Grassmannians of submodules of quiver representations. This was first achieved for acyclic cluster algebras [CK], later for cluster algebras admitting a 2-Calabi-Yau categorification [P, FK], and more recently for general antisymmetric cluster algebras of geometric type [DWZ2]. There is a posterior but essentially different proof in [Pl]; see also [N]. The first aim of this paper is to compare these two types of formulas for the large class of cluster algebras which can be realized as coordinate rings of unipotent cells of Kac-Moody groups.

To do this, we will return to the very source of cluster algebras, namely to the Chamber Ansatz of Berenstein, Fomin and Zelevinsky [BFZ, BZ], which describes parametrizations of Lusztig's totally positive parts of unipotent subgroups and

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Schubert varieties. The second aim of this paper is to provide a new understanding of the Chamber Ansatz formulas in terms of representations of preprojective algebras, together with a generalization to the Kac-Moody case. In particular the mysterious twist automorphisms of the unipotent cells needed in these formulas turn out to be just shadows of the Auslander-Reiten translations of the corresponding Frobenius categories of modules over the preprojective algebras. Our treatment of the Chamber Ansatz shows that the numerators of the twisted minors of [BFZ, BZ] form a cluster, and that the Laurent expansions with respect to these special clusters have coefficients equal to Euler characteristics of varieties of flags of submodules of preprojective algebra representations. This provides the desired link between the two types of Euler characteristics mentioned above, and it allows us to show that the cluster characters of Fu and Keller [FK] coincide after an appropriate change of variables with the φ-functions of [GLS2, GLS5].

Finally, our third aim is to exploit these results for studying natural bases of cluster algebras containing the cluster monomials. We consider the class of coefficient-free cluster algebras obtained by specializing to 1 the coefficients of the cluster algebra structures on unipotent cells. In [GLS5, Section 15.6] we have found such bases, consisting of appropriate subsets of Lusztig's dual semicanonical bases. Here, using the above connection with Fu-Keller cluster characters, we give a new description of the same bases in terms of module varieties of endomorphism algebras of cluster-tilting modules. In the special case when the cluster algebra is acyclic, this proves Dupont's generic basis conjecture [D]. In general, the elements of these bases are generating functions of Euler characteristics of quiver Grassmannians, at generic points of some particular irreducible components of the module varieties. These special irreducible components can be characterized in terms of the new E-invariant introduced by Derksen, Weyman and Zelevinsky [DWZ2] for representations of quivers with potential, and one may therefore conjecture that a similar description of a generic basis can be extended to any antisymmetric cluster algebra.

1.2. To state our results more precisely, we need to introduce some notation. Let Q be a finite quiver with vertex set $\{1, \dots, n\}$ and without oriented cycles. Denote by Λ the corresponding preprojective algebra. Let $\mathfrak{g}$ be the Kac-Moody Lie algebra with Cartan datum given by Q, and let W be the Weyl group of $\mathfrak{g}$. The graded dual $U(\mathfrak{n})^*_{\rm gr}$ of the universal enveloping algebra $U(\mathfrak{n})$ of the positive part $\mathfrak{n}$ of $\mathfrak{g}$ can be identified with the coordinate ring $\mathbb{C}[N]$ of an associated pro-unipotent pro-group N with Lie algebra $\mathfrak{n}$.

For $w \in W$, let $N^w := N \cap (B_-wB_-)$ be the corresponding unipotent cell in N, where B_- denotes the standard negative Borel subgroup of the Kac-Moody group G attached to $\mathfrak{g}$. Here we use the same notation as in [GLS5]. For details on Kac-Moody groups we refer to [Ku, Sections 6 and 7.4]. Let $x_i(t)$ denote the one-parameter subgroup of N associated to the simple root α_i. For each reduced expression $\mathbf{i} = (i_r, \dots, i_1)$ of w, the map

$$\underline{x}_{\mathbf{i}} : (t_r, \dots, t_2, t_1) \mapsto x_{i_r}(t_r) \cdots x_{i_2}(t_2) x_{i_1}(t_1)$$

gives a birational isomorphism from $\mathbb{C}^r$ to N^w. In [GLS5] we have described a cluster algebra structure on $\mathbb{C}[N^w]$ in terms of the representation theory of the preprojective algebra Λ.

For a nilpotent Λ-module X and $\mathbf{a} = (a_r, \dots, a_1) \in \mathbb{N}^r$, let $\mathcal{F}_{\mathbf{i},\mathbf{a},X}$ be the projective variety of flags

$$X_\bullet = (0 = X_r \subseteq \cdots \subseteq X_1 \subseteq X_0 = X)$$

of submodules of X such that $X_{k-1}/X_k \cong S_{i_k}^{a_k}$ for all $1 \le k \le r$, where S_j denotes the one-dimensional Λ-module supported on the vertex j of Q. The varieties $\mathcal{F}_{\mathbf{i},\mathbf{a},X}$ were first introduced by Lusztig [L1] for his Lagrangian construction of $U(\mathfrak{n})$. Dualizing Lusztig's construction, we can associate with X a regular function $\varphi_X \in \mathbb{C}[N]$ satisfying

$$\varphi_X(\underline{x}_{\mathbf{i}}(\mathbf{t})) = \sum_{\mathbf{a}\in\mathbb{N}^r} \chi(\mathcal{F}_{\mathbf{i},\mathbf{a},X})\mathbf{t}^{\mathbf{a}}.$$

Here $\mathbf{t} = (t_r, \dots, t_1) \in \mathbb{C}^r$, $\mathbf{t}^{\mathbf{a}} := t_r^{a_r} \cdots t_2^{a_2} t_1^{a_1}$, and χ denotes the topological Euler characteristic.

Buan, Iyama, Reiten, and Scott [BIRS] have attached to w a 2-Calabi-Yau Frobenius subcategory $\mathcal{C}_w$ of the category of finite-dimensional nilpotent Λ-modules. (The same categories were studied independently in [GLS4] for special elements w called adaptable.) In [GLS5] we showed that the $\mathbb{C}$-span of

$$\{\varphi_X \mid X \in \mathcal{C}_w\}$$

is a subalgebra of $\mathbb{C}[N]$, which becomes isomorphic to $\mathbb{C}[N^w]$ after localization at the multiplicative subset $\{\varphi_P \mid P \text{ is } \mathcal{C}_w\text{-projective-injective}\}$. Moreover, we showed that $\mathbb{C}[N^w]$ carries a cluster algebra structure, whose cluster variables are of the form φ_X for indecomposable modules X in $\mathcal{C}_w$ without self-extension. In Section 2 we explain this in more detail.

The category $\mathcal{C}_w$ comes with a remarkable module $V_{\mathbf{i}}$ for each reduced expression $\mathbf{i}$ of w (see [BIRS, Section III.2], [GLS5, Section 2.4]). The φ-functions of the indecomposable direct summands of $V_{\mathbf{i}}$ are some generalized minors on N which form a natural initial cluster of $\mathbb{C}[N^w]$. We introduce the new module

$$W_{\mathbf{i}} := I_w \oplus \Omega_w(V_{\mathbf{i}}),$$

where $\Omega_w = \tau_w^{-1}$ is the inverse Auslander-Reiten translation of $\mathcal{C}_w$, and I_w is the direct sum of the indecomposable $\mathcal{C}_w$-projective-injectives. For a Λ-module X, the set $\operatorname{Ext}^1_\Lambda(W_{\mathbf{i}}, X)$ is in a natural way a left module over the stable endomorphism algebra

$$\underline{\mathcal{E}} := \underline{\operatorname{End}}_{\mathcal{C}_w}(W_{\mathbf{i}})^{\mathrm{op}} \cong \underline{\operatorname{End}}_{\mathcal{C}_w}(V_{\mathbf{i}})^{\mathrm{op}}.$$

Denote by $\operatorname{Gr}_{\mathbf{d}}^{\underline{\mathcal{E}}}(\operatorname{Ext}^1_\Lambda(W_{\mathbf{i}}, X))$ the projective variety of $\underline{\mathcal{E}}$-submodules of $\operatorname{Ext}^1_\Lambda(W_{\mathbf{i}}, X)$ with dimension vector $\mathbf{d}$, a so-called quiver Grassmannian. Our first main result is

Theorem 1. *For $X \in \mathcal{C}_w$ and all $\mathbf{a} \in \mathbb{N}^r$, there is an isomorphism of algebraic varieties*

$$\mathcal{F}_{\mathbf{i},\mathbf{a},X} \cong \operatorname{Gr}_{d_{\mathbf{i},X}(\mathbf{a})}^{\underline{\mathcal{E}}}(\operatorname{Ext}^1_\Lambda(W_{\mathbf{i}}, X)),$$

where $d_{\mathbf{i},X}$ is an explicit bijection from $\{\mathbf{a} \mid \mathcal{F}_{\mathbf{i},\mathbf{a},X} \neq \varnothing\}$ to $\{\mathbf{d} \mid \operatorname{Gr}_{\mathbf{d}}^{\underline{\mathcal{E}}}(\operatorname{Ext}^1_\Lambda(W_{\mathbf{i}}, X)) \neq \varnothing\}$.

It follows easily that the set $\{\mathbf{a} \mid \chi(\mathcal{F}_{\mathbf{i},\mathbf{a},X}) \neq 0\}$ has a unique element if and only if $\operatorname{Ext}^1_\Lambda(W_{\mathbf{i}}, X) = 0$. Now by construction, $W_{\mathbf{i}}$ is a cluster-tilting module of $\mathcal{C}_w$; that is, $\operatorname{Ext}^1_\Lambda(W_{\mathbf{i}}, X) = 0$ if and only if X belongs to the additive hull $\operatorname{add}(W_{\mathbf{i}})$

of $W_{\mathbf{i}}$. Moreover, in this case, $\mathcal{F}_{\mathbf{i},\mathbf{a},X}$ is reduced to a point. Hence Theorem 1 has the following important consequence:

Theorem 2. *For $X \in \mathcal{C}_w$, the polynomial function $\mathbf{t} \mapsto \varphi_X(\underline{x}_{\mathbf{i}}(\mathbf{t}))$ is reduced to a single monomial $\mathbf{t}^{\mathbf{a}}$ if and only if $X \in \mathrm{add}(W_{\mathbf{i}})$.*

1.3. Let $W_{\mathbf{i},1}, \dots, W_{\mathbf{i},r}$ denote the indecomposable direct summands of $W_{\mathbf{i}}$. The r-tuple of regular functions $(\varphi_{W_{\mathbf{i},1}}, \dots, \varphi_{W_{\mathbf{i},r}})$ is a cluster of $\mathbb{C}[N^w]$, and it follows from Theorem 2 that the $\varphi_{W_{\mathbf{i},k}}(\underline{x}_{\mathbf{i}}(\mathbf{t}))$ are monomials in the variables $t_1, \dots, t_r$. Inverting this monomial transformation yields expressions of the t_k's as explicit rational functions on N^w, a result originally called the Chamber Ansatz by Berenstein, Fomin and Zelevinsky [BFZ] in type A_n, because of a convenient description of these formulas in terms of chambers in a wiring diagram. To present these formulas in the general Kac-Moody setting, we need more notation. By construction, the summands $V_{\mathbf{i},k}$ of $V_{\mathbf{i}}$ are related to the modules $W_{\mathbf{i},k}$ by short exact sequences

$$0 \to W_{\mathbf{i},k} \to P(V_{\mathbf{i},k}) \to V_{\mathbf{i},k} \to 0,$$

where for $X \in \mathcal{C}_w$, $P(X)$ denotes the projective cover in $\mathcal{C}_w$. We set

$$\varphi'_{V_{\mathbf{i},k}} := \frac{\varphi_{W_{\mathbf{i},k}}}{\varphi_{P(V_{\mathbf{i},k})}},$$

a Laurent monomial in the $\varphi_{W_{\mathbf{i},k}}$ (since $\mathrm{add}(W_{\mathbf{i}})$ contains all $\mathcal{C}_w$-projectives). As will be explained in Section 1.6 below, the regular functions $\varphi'_{V_{\mathbf{i},k}}$ on N^w are the twisted generalized minors of [BZ] corresponding to $\mathbf{i}$ (in the Dynkin case).

Denote by $q(i,j)$ the number of edges between two vertices i and j of the underlying unoriented graph of the quiver Q. For $1 \le k \le r$, put

$$C_{\mathbf{i},k} := \frac{1}{\varphi'_{V_{\mathbf{i},k}} \varphi'_{V_{\mathbf{i},k^-(i_k)}}} \cdot \prod_{j=1}^{n} \left(\varphi'_{V_{\mathbf{i},k^-(j)}} \right)^{q(i_k,j)}, \tag{1.1}$$

where $k^-(j) := \max\{0, 1 \le s \le k-1 \mid i_s = j\}$ and $V_{\mathbf{i},0}$ is by convention the zero module.

Theorem 3. *For $1 \le k \le r$ and $\mathbf{t} = (t_r, \dots, t_1)$ we have $C_{\mathbf{i},k}(\underline{x}_{\mathbf{i}}(\mathbf{t})) = t_k$. Therefore, for $X \in \mathcal{C}_w$ we get an equality in $\mathbb{C}[N^w]$:*

$$\varphi_X = \sum_{\mathbf{a} \in \mathbb{N}^r} \chi(\mathcal{F}_{\mathbf{i},\mathbf{a},X}) C_{\mathbf{i},r}^{a_r} \cdots C_{\mathbf{i},2}^{a_2} C_{\mathbf{i},1}^{a_1}. \tag{1.2}$$

1.4. Using Theorem 1, we now want to compare Equation (1.2) with similar formulas of Fu and Keller. To simplify our notation, we define

$$\begin{aligned} R &:= \{1, 2, \dots, r\}, \\ R_{\max} &:= \{k \in R \mid \text{there is no } k < s \le r \text{ with } i_s = i_k\}, \\ R_- &:= R \setminus R_{\max}. \end{aligned}$$

Let $T = T_1 \oplus \cdots \oplus T_r$ be a basic cluster-tilting module in $\mathcal{C}_w$, where the numbering is chosen so that T_k is $\mathcal{C}_w$-projective-injective for $k \in R_{\max}$. Assume that $(\varphi_{T_1}, \dots, \varphi_{T_r})$ is a cluster of $\mathbb{C}[N^w]$, i.e., that it can be obtained from $(\varphi_{V_{\mathbf{i},1}}, \dots, \varphi_{V_{\mathbf{i},r}})$ by a sequence of mutations. In this case, T is called $V_{\mathbf{i}}$-*reachable*. (One conjectures that this is always the case.) The endomorphism algebra $\mathcal{E}_T := \mathrm{End}_\Lambda(T)^{\mathrm{op}}$ has global dimension 3; see [GLS5, Proposition 2.19]. Thus we may consider

$$B^{(T)} := (B^{(T)}_{k,l})_{k,l \in R} := \left((\dim \mathrm{Hom}_\Lambda(T_k, T_l))_{k,l \in R} \right)^{-t},$$

the matrix of the Ringel bilinear form for $\mathcal{E}_T$. (For a matrix B, we denote the inverse of its transpose by B^{-t}.)

For a general 2-Calabi-Yau Frobenius category $\mathcal{C}$ with a cluster-tilting object, Fu and Keller [FK, Section 3] (extending previous work of Palu [P]) have attached to every object of $\mathcal{C}$ a Laurent polynomial called its cluster character. When applied to the category $\mathcal{C}_w$ and the cluster-tilting object T, the formula for this cluster character can be written as

$$\theta_X^T := \varphi_T^{(\underline{\dim}\operatorname{Hom}_\Lambda(T,X))\cdot B^{(T)}} \cdot \sum_{\mathbf{d}\in\mathbb{N}^{R_-}} \chi(\operatorname{Gr}_{\mathbf{d}}^{\mathcal{E}_T}(\operatorname{Ext}^1_\Lambda(T,X)))\,\hat{\varphi}_T^{\mathbf{d}} \qquad (X\in\mathcal{C}_w). \tag{1.3}$$

Here we use the abbreviations

$$\begin{aligned}
\varphi_T^{\mathbf{g}} &:= \textstyle\prod_{k\in R} \varphi_{T_k}^{g_k} && \text{for } \mathbf{g}=(g_1,g_2,\dots,g_r)\in\mathbb{Z}^r,\\
\hat{\varphi}_{T,k} &:= \textstyle\prod_{l\in R} \varphi_{T_l}^{B_{l,k}^{(T)}} && \text{for } k\in R_-,\\
\hat{\varphi}_T^{\mathbf{d}} &:= \textstyle\prod_{k\in R_-} \hat{\varphi}_{T,k}^{d_k} && \text{for } \mathbf{d}=(d_k)_{k\in R_-}\in\mathbb{N}^{R_-}.
\end{aligned}$$

By [FK, Theorem 4.3] and [GLS5, Theorem 3.3], the cluster variables of $\mathbb{C}[N^w]$ are of the form θ_X^T for indecomposable rigid modules X of $\mathcal{C}_w$, and (1.3) gives therefore a representation-theoretic description of their cluster expansions with respect to the cluster $(\varphi_{T_1},\dots,\varphi_{T_r})$. However, for an arbitrary $X\in\mathcal{C}_w$, not much is known about the function θ_X^T. For instance it is *a priori* only a rational function on N^w. Using Theorem 1 and the Chamber Ansatz Theorem 3, we prove our next main result:

Theorem 4. *For every $X\in\mathcal{C}_w$ we have*

$$\theta_X^T = \varphi_X.$$

In particular, θ_X^T is a regular function on N^w for every $X\in\mathcal{C}_w$; that is, the image of the cluster character $X\mapsto\theta_X^T$ is in the cluster algebra $\mathbb{C}[N^w]$.

1.5. In the last part of this paper, we deduce from Theorem 4 a new description of a generic basis for the coefficient-free cluster algebra obtained from $\mathbb{C}[N^w]$ by specializing to 1 the functions φ_P for all $\mathcal{C}_w$-projective-injectives P. (This algebra can be seen as the coordinate ring of the subvariety $N\cap(N_-wN_-)$ of N^w, but we will not use it.) In [GLS5, Section 15.6] we have already described such a basis in terms of generic modules over the preprojective algebra Λ. Here we want to express it in terms of generic modules over the stable endomorphism algebra $\underline{\mathcal{E}}_T$ of the cluster-tilting module T.

The quiver $\underline{\Gamma}_T$ of $\underline{\mathcal{E}}_T$ has the set R_- as vertices, with $k\in R_-$ corresponding to T_k, and it has $[B_{l,k}^{(T)}]_+$ arrows from k to l, where we write, for short, $[z]_+=\max(z,0)$. We consider the cluster algebra $\mathcal{A}(\underline{\Gamma}_T)\subset\mathbb{C}((x_k)_{k\in R_-})$ with initial seed $((x_k)_{k\in R_-},\underline{\Gamma}_T)$. We have a unique ring homomorphism $\Pi_T\colon\mathbb{C}[N^w]\to\mathbb{C}((x_k)_{k\in R_-})$ such that $\Pi_T(\varphi_{T_k})=x_k$ for $k\in R_-$, and $\Pi_T(\varphi_{T_k})=1$ for $k\in R_{\max}$. The homomorphism Π_T restricts to an epimorphism $\mathbb{C}[N^w]\to\mathcal{A}(\underline{\Gamma}_T)$, which we also denote by Π_T.

Following Palu [P], for an $\underline{\mathcal{E}}_T$-module Y we put

$$\psi_Y := x^{\mathbf{g}_Y}\cdot\sum_{\mathbf{d}\in\mathbb{N}^{R_-}} \chi(\operatorname{Gr}_{\mathbf{d}}^{\underline{\mathcal{E}}_T}(Y))\,\hat{x}_T^{\mathbf{d}}, \tag{1.4}$$

where

$$\mathbf{g}_Y := (g_k)_{k\in R_-} := \Big(\dim \operatorname{Ext}^1_{\underline{\mathcal{E}}_T}(S_k, Y) - \dim \operatorname{Hom}_{\underline{\mathcal{E}}_T}(S_k, Y)\Big)_{k\in R_-}$$

and

$$x^{\mathbf{g}_Y} := \prod_{k\in R_-} x_k^{g_k}, \qquad \hat{x}_{T,k} := \prod_{l\in R_-} x_l^{B^{(T)}_{l,k}}, \qquad \hat{x}_T^{\mathbf{d}} := \prod_{k\in R_-} \hat{x}_{T,k}^{d_k}.$$

(Here S_k, $k \in R_-$ are the simple $\underline{\mathcal{E}}_T$-modules.) In fact, if $Y = \operatorname{Ext}^1_\Lambda(T, X)$ for some $X \in \mathcal{C}_w$, in view of Theorem 4 we have $\psi_Y = \Pi_T(\varphi_X)$.

For $\mathbf{d} \in \mathbb{N}^{R_-}$ let $\operatorname{mod}(\underline{\mathcal{E}}_T, \mathbf{d})$ be the affine variety of representations of $\underline{\mathcal{E}}_T$ with dimension vector $\mathbf{d}$. It will be convenient to consider $\operatorname{mod}(\underline{\mathcal{E}}_T, \mathbf{d})$ with the *right* action of

$$\operatorname{GL}_{\mathbf{d}} := \prod_{k\in R_-} \operatorname{GL}_{\mathbf{d}(k)}(\mathbb{C})$$

by conjugation. For each irreducible component $\mathcal{Z}$ of $\operatorname{mod}(\underline{\mathcal{E}}_T, \mathbf{d})$ there is a dense open subset $\mathcal{U} \subseteq \mathcal{Z}$ such that for all $U, U' \in \mathcal{U}$ we have $\psi_U = \psi_{U'}$. Define $\psi_{\mathcal{Z}} := \psi_U$, where $U \in \mathcal{U}$. An irreducible component $\mathcal{Z}$ of $\operatorname{mod}(\underline{\mathcal{E}}_T, \mathbf{d})$ is called *strongly reduced* if there is a dense open subset $\mathcal{U} \subseteq \mathcal{Z}$ such that

$$\operatorname{codim}_{\mathcal{Z}}(U.\operatorname{GL}_{\mathbf{d}}) = \dim \operatorname{Hom}_{\underline{\mathcal{E}}_T}\left(\tau^{-1}_{\underline{\mathcal{E}}_T}(U), U\right)$$

for all $U \in \mathcal{U}$, where $\tau_{\underline{\mathcal{E}}_T}$ denotes the Auslander-Reiten translation of $\operatorname{mod}(\underline{\mathcal{E}}_T)$. It follows from Voigt's Lemma [G, Proposition 1.1] that strongly reduced components are (scheme-theoretically) generically reduced, hence the name. But contrary to what the terminology might suggest, being strongly reduced is not a property of the scheme $\mathcal{Z}$ equipped with its $\operatorname{GL}_{\mathbf{d}}$-action, since the definition uses additionally the representation theory of the algebra $\underline{\mathcal{E}}_T$. Note also that $\underline{\mathcal{E}}_T$ is given by a quiver with potential [BIRSm] and that

$$\dim \operatorname{Hom}_{\underline{\mathcal{E}}_T}(\tau^{-1}_{\underline{\mathcal{E}}_T}(U), U) = E^{\mathrm{inj}}(U)$$

is the E-invariant defined in [DWZ2].

Let $\operatorname{Irr}(\operatorname{mod}(\underline{\mathcal{E}}_T, \mathbf{d}))$ be the set of irreducible components of $\operatorname{mod}(\underline{\mathcal{E}}_T, \mathbf{d})$ and set

$$\operatorname{Irr}(\underline{\mathcal{E}}_T) := \bigcup_{\mathbf{d}\in\mathbb{N}^{R_-}} \operatorname{Irr}(\operatorname{mod}(\underline{\mathcal{E}}_T, \mathbf{d})).$$

Let $\operatorname{Irr}^{\mathrm{sr}}(\underline{\mathcal{E}}_T)$ denote the set of all strongly reduced irreducible components in $\operatorname{Irr}(\underline{\mathcal{E}}_T)$. For $\mathcal{Z} \in \operatorname{Irr}(\underline{\mathcal{E}}_T, \mathbf{d})$ define $\operatorname{Null}(\mathcal{Z}) := \{\mathbf{m} \in \mathbb{N}^{R_-} \mid \mathbf{m}(k) = 0 \text{ if } \mathbf{d}(k) \neq 0\}$.

Finally, let us denote by $\widetilde{\mathcal{S}}_w^*$ the dual semicanonical basis of $\mathbb{C}[N^w]$ constructed in [GLS5]. We can now state

Theorem 5. *The set*

$$\mathcal{G}_w^T := \{x^{\mathbf{m}} \cdot \psi_{\mathcal{Z}} \mid \mathcal{Z} \in \operatorname{Irr}^{\mathrm{sr}}(\underline{\mathcal{E}}_T),\ \mathbf{m} \in \operatorname{Null}(\mathcal{Z})\}$$

is a basis of the cluster algebra $\mathcal{A}(\underline{\Gamma}_T)$. It is equal to the image of the dual semicanonical basis $\widetilde{\mathcal{S}}_w^*$ under $\Pi_T\colon \mathbb{C}[N^w] \to \mathcal{A}(\underline{\Gamma}_T)$.

Each finite-dimensional path algebra is isomorphic to $\underline{\mathcal{E}}_T$ for some appropriate Λ, w and T; see [GLS5, Section 16]. In this case, $\operatorname{mod}(\underline{\mathcal{E}}_T, \mathbf{d})$ is an (irreducible) affine space for all $\mathbf{d}$, and it is easy to see that $\operatorname{mod}(\underline{\mathcal{E}}_T, \mathbf{d})$ is strongly reduced. Thus Theorem 5 implies Dupont's conjecture [D, Conjecture 6.1]. On the other hand, even if $\underline{\mathcal{E}}_T$ is not hereditary but mutation equivalent to an acyclic quiver, it is quite easy to find examples of irreducible components of varieties $\mathrm{mod}(\underline{\mathcal{E}}_T, \mathbf{d})$ which are not strongly reduced.

Since Theorem 5 gives a description of the generic basis $\mathcal{G}_w^T$ of $\mathcal{A}(\underline{\Gamma}_T)$ entirely in terms of the varieties of representations of the algebra $\underline{\mathcal{E}}_T$, it is natural in view of [DWZ2] to ask if the first statement of Theorem 5 generalizes to other classes of cluster algebras.

1.6. The paper closes with our categorical interpretation of the twist automorphisms of the unipotent cells, introduced by Berenstein, Fomin and Zelevinsky in connection with the Chamber Ansatz. For $x \in N^w$, the intersection $N \cap (B_- w x^T)$ consists of a unique element, which, following [BFZ, BZ, GLS5], we denote by $\eta_w(x)$. (The anti-automorphism $g \mapsto g^T$ of the Kac-Moody group is defined in [GLS5, Section 7.1]. For more details on η_w, we refer to [GLS5, Section 8].) The map η_w is in fact a regular automorphism of N^w, and we denote by $(\eta_w^*)^{-1}$ the $\mathbb{C}$-algebra automorphism of $\mathbb{C}[N^w]$, defined by

$$((\eta_w^*)^{-1} f)(x) = f(\eta_w^{-1}(x)) \qquad (f \in \mathbb{C}[N^w]).$$

Theorem 6. *For every $X \in \mathcal{C}_w$, we have*

$$(\eta_w^*)^{-1}(\varphi_X) = \frac{\varphi_{\Omega_w(X)}}{\varphi_{P(X)}}.$$

Moreover, η_w^* preserves the dual semicanonical basis $\widetilde{\mathcal{S}}_w^*$ of $\mathbb{C}[N^w]$ and permutes its elements.

Thus, the regular functions $\varphi'_{V_{\mathbf{i},k}}$ occurring in Theorem 3 are obtained by twisting the generalized minors $\varphi_{V_{\mathbf{i},k}}$ with η_w^{-1}, in agreement with [BFZ, BZ] in the Dynkin case. We believe that Theorem 6 provides a conceptual explanation of the existence of the automorphism η_w, and of its compatibility with total positivity [BZ, Proposition 5.3].

1.7. The article is organized as follows: In Section 2 we give a short reminder on cluster algebras and some previous results. In Section 3 we construct isomorphisms between flag varieties and quiver Grassmannians in a very general setup. The isomorphisms stated in Theorem 1 turn out to be special cases. Section 4 contains the proofs of Theorems 1 and 2 and of the Chamber Ansatz Theorem 3 together with some illustrating examples. The proof of the cluster character identities stated in Theorem 4 and a detailed example are in Sections 5 and 6. The proof of Theorem 5 is in Sections 7 and 8. Finally, Section 9 contains the proof of Theorem 6.

1.8. **Notation.** Throughout, we work over the field $\mathbb{C}$ of complex numbers. For a $\mathbb{C}$-algebra A let $\mathrm{mod}(A)$ be the category of finite-dimensional left A-modules. By an A-*module* we always mean a module in $\mathrm{mod}(A)$, unless stated otherwise. Often we do not distinguish between a module and its isomorphism class. Let $D := \mathrm{Hom}_{\mathbb{C}}(-, \mathbb{C})$ be the usual duality functor.

For a quiver Q, let $\mathrm{rep}(Q)$ be the category of finite-dimensional representations of Q over $\mathbb{C}$. It is well known that we can identify $\mathrm{rep}(Q)$ and $\mathrm{mod}(\mathbb{C}Q)$.

By a *subcategory* we always mean a full subcategory. For an A-module M let $\mathrm{add}(M)$ be the subcategory of all A-modules which are isomorphic to finite direct sums of direct summands of M. A subcategory $\mathcal{U}$ of $\mathrm{mod}(A)$ is an *additive subcategory* if any finite direct sum of modules in $\mathcal{U}$ is again in $\mathcal{U}$. By $\mathrm{Fac}(M)$ (resp.

$\mathrm{Sub}(M)$) we denote the subcategory of all A-modules X such that there exists some $t \geq 1$ and some epimorphism $M^t \to X$ (resp. monomorphism $X \to M^t$).

For an A-module M let $\Sigma(M)$ be the number of isomorphism classes of indecomposable direct summands of M. An A-module is called *basic* if it can be written as a direct sum of pairwise non-isomorphic indecomposable modules. An A-module M is called *rigid* if $\mathrm{Ext}^1_A(M,M) = 0$.

For an A-module M and a simple A-module S let $[M:S]$ be the Jordan-Hölder multiplicity of S in a composition series of M. Let $\underline{\dim}(M) := \underline{\dim}_A(M) := ([M:S])_S$ be the *dimension vector* of M, where S runs through all isomorphism classes of simple A-modules.

For a set U we denote its cardinality by $|U|$. If $f\colon X \to Y$ and $g\colon Y \to Z$ are maps, then the composition is denoted by $gf = g \circ f\colon X \to Z$.

If U is a subset of a $\mathbb{C}$-vector space V, then let $\mathrm{Span}_{\mathbb{C}}\langle U \rangle$ be the subspace of V generated by U.

Let $\mathbb{N} = \{0,1,2,\ldots\}$ be the natural numbers, including 0, and let $\mathbb{Z}$ be the ring of integers. For a domain R let $R(X_1,\ldots,X_r)$, $R[X_1,\ldots,X_r]$ and $R[X_1^{\pm 1},\ldots,X_r^{\pm 1}]$ be the field of rational functions, the polynomial ring, and the ring of Laurent polynomials in the variables $X_1,\ldots,X_r$ with coefficients in R, respectively.

2. Reminder on cluster algebras

2.1. Let $\mathcal{F} := \mathbb{Q}(X_1,\ldots,X_r)$ be the field of rational functions in r variables. We fix a subset $F \subseteq \{1,\ldots,r\}$.

A *seed* in $\mathcal{F}$ is a pair (x,Γ), where $\Gamma = (\Gamma_0,\Gamma_1,s,t)$ is a finite quiver without loops and without 2-cycles with set of vertices $\Gamma_0 = \{1,\ldots,r\}$, and $x = (x_1,\ldots,x_r)$ with $x_1,\ldots,x_r$ algebraically independent elements in $\mathcal{F}$. The vertices in $\{1,\ldots,r\} \setminus F$ are called *mutable*, and the ones in F are *frozen*.

Given a seed (x,Γ) in $\mathcal{F}$ and a mutable vertex k of Γ, we define the *mutation* of (x,Γ) at k as

$$\mu_k(x,\Gamma) := (x',\Gamma').$$

The quiver Γ' is obtained from Γ by applying the Fomin-Zelevinsky quiver mutation at k, which is defined as follows: For $1 \leq i,j \leq r$ let

$$\gamma_{ij} := |\text{number of arrows } j \to i \text{ in } \Gamma| - |\text{number of arrows } i \to j \text{ in } \Gamma|.$$

(Recall that there are no 2-cycles in Γ. So at least one of the numbers on the right-hand side is 0.) By definition also Γ' has no loops and no 2-cycles, and the corresponding numbers γ'_{ij} for Γ' are

$$\gamma'_{ij} := \begin{cases} -\gamma_{ij} & \text{if } i = k \text{ or } j = k, \\ \gamma_{ij} + \dfrac{|\gamma_{ik}|\gamma_{kj} + \gamma_{ik}|\gamma_{kj}|}{2} & \text{otherwise.} \end{cases}$$

Finally, $x' = (x'_1,\ldots,x'_r)$ is defined by

$$x'_s := \begin{cases} x_k^{-1}\left(\prod_{k \to i} x_i + \prod_{j \to k} x_j\right) & \text{if } s = k, \\ x_s & \text{otherwise,} \end{cases}$$

where the products are taken over all arrows of Γ which start, respectively end, in k. Set $\mu_{(x,\Gamma)}(x_k) := x'_k$. It is easy to check that (x',Γ') is again a seed in $\mathcal{F}$ and that $\mu_k^2(x,\Gamma) = (x,\Gamma)$.

Two seeds (x,Γ) and (y,Σ) are *mutation equivalent* if there is a sequence $(k_1,\dots,k_t)$ with $k_i \in \{1,\dots,r\} \setminus F$ for all i such that

$$\mu_{k_t} \dots \mu_{k_2}\mu_{k_1}(x,\Gamma) = (y,\Sigma).$$

In this case, we write $(y,\Sigma) \sim (x,\Gamma)$.

For a seed (x,Γ) in $\mathcal{F}$ let

$$\mathcal{X}_{(x,\Gamma)} := \bigcup_{(y,\Sigma)\sim(x,\Gamma)} \{y_1,\dots,y_r\},$$

where the union is over all seeds (y,Σ) with $(y,\Sigma) \sim (x,\Gamma)$. By definition, the *cluster algebra* $\mathcal{A}(x,\Gamma)$ associated to (x,Γ) is the subalgebra of $\mathcal{F}$ generated by $\mathcal{X}_{(x,\Gamma)}$.

We call (y,Σ) a *seed in* $\mathcal{A}(x,\Gamma)$ if $(y,\Sigma) \sim (x,\Gamma)$. In this case, y is a *cluster* in $\mathcal{A}(x,\Gamma)$, the elements $y_1,\dots,y_r$ are *cluster variables* and $y_1^{m_1}\cdots y_r^{m_r}$ with $m_i \geq 0$ for all i are *cluster monomials* in $\mathcal{A}(x,\Gamma)$.

For any seed of the form (y,Γ) in $\mathcal{F}$ we obtain an isomorphism $\mathcal{A}(x,\Gamma) \to \mathcal{A}(y,\Gamma)$ given by $x_i \mapsto y_i$ for all $1 \leq i \leq r$. So one sometimes writes just $\mathcal{A}(\Gamma)$ instead of $\mathcal{A}(x,\Gamma)$.

Note that for any cluster y in $\mathcal{A}(x,\Gamma)$ we have $y_i = x_i$ for all $i \in F$. These cluster variables are also called *coefficients* of $\mathcal{A}(x,\Gamma)$. Localizing $\mathcal{A}(x,\Gamma)$ at $\prod_{i\in F} x_i$ yields an algebra $\mathcal{A}(x,\Gamma,F^{\pm})$, which we also call a *cluster algebra*.

There are algebra epimorphisms

$$\mathcal{A}(x,\Gamma) \to \mathcal{A}(\underline{x},\underline{\Gamma}) \quad \text{and} \quad \mathcal{A}(x,\Gamma,F^{\pm}) \to \mathcal{A}(\underline{x},\underline{\Gamma})$$

defined by

$$x_i \to \begin{cases} 1 & \text{if } i \in F, \\ x_i & \text{otherwise,} \end{cases}$$

where $\mathcal{A}(\underline{x},\underline{\Gamma}) \subseteq \mathbb{Q}((x_i)_{i\in\{1,\dots,r\}\setminus F})$ is again a cluster algebra with $\underline{x} := (x_i)_{i\in\{1,\dots,r\}\setminus F}$, and the quiver $\underline{\Gamma}$ is obtained from Γ by deleting all vertices in F and all arrows starting or ending in one of the vertices in F. We say that the cluster algebra $\mathcal{A}(\underline{x},\underline{\Gamma})$ is obtained from $\mathcal{A}(x,\Gamma)$ by *specialization of coefficients* to 1, and the two epimorphisms defined above are called *specialization morphisms*. Clearly, the specialization morphisms induce a surjective map $\mathcal{X}_{(x,\Gamma)} \setminus \{x_i \mid i \in F\} \to \mathcal{X}_{(\underline{x},\underline{\Gamma})}$.

Using the identification $\mathbb{C}[N^w] \equiv \mathcal{A}(\Gamma_T)$, the epimorphism Π_T defined in Section 1.5, can be seen as a specialization morphism. Thus the cluster algebra $\mathcal{A}(\underline{\Gamma}_T)$ is obtained from $\mathbb{C}[N^w]$ by the specialization of coefficients to 1.

2.2. **Cluster algebra structures for coordinate rings of unipotent cells.** In a series of papers [GLS1, GLS2, GLS5] we constructed a map

$$\varphi\colon \operatorname{nil}(\Lambda) \to \mathbb{C}[N]$$

which maps a nilpotent Λ-module X to a function $\varphi_X \in \mathbb{C}[N]$. This map satisfies the following properties:

(i) For all $X, Y \in \operatorname{nil}(\Lambda)$ we have

$$\varphi_X\varphi_Y = \varphi_{X\oplus Y}.$$

(ii) Let $X, Y \in \operatorname{nil}(\Lambda)$ with $\dim \operatorname{Ext}^1_\Lambda(X,Y) = \dim \operatorname{Ext}^1_\Lambda(Y,X) = 1$, and let

$$0 \to X \to E' \to Y \to 0 \quad \text{and} \quad 0 \to Y \to E'' \to X \to 0$$

be non-split short exact sequences. Then we have

$$\varphi_X \varphi_Y = \varphi_{E'} + \varphi_{E''}.$$

(iii) Restriction yields a map

$$\varphi\colon \mathcal{C}_w \to \mathbb{C}[N^w].$$

(Again we identified $\mathbb{C}[N^w]$ with the localization of the $\mathbb{C}$-span of $\{\varphi_X \mid X \in \mathcal{C}_w\}$ at $\{\varphi_P \mid P \text{ is } \mathcal{C}_w\text{-projective-injective}\}$.)

(iv) Let $\mathbf{i} = (i_r, \dots, i_1)$ be a reduced expression of w, and let $\Gamma := \Gamma_{\mathbf{i}}$ and $F := R_{\max}$. (The definitions of $\Gamma_{\mathbf{i}}$ and $R_{\max}$ can be found in Sections 3.6 and 1.4, respectively.) Then there is an algebra isomorphism

$$\eta_{\mathbf{i}}\colon \mathcal{A}(x, \Gamma, F^{\pm}) \to \mathbb{C}[N^w]$$

with $\eta_{\mathbf{i}}(x_k) = \varphi_{V_{\mathbf{i},k}}$ for all $1 \le k \le r$.

Using the isomorphism $\eta_{\mathbf{i}}$ one can now speak of *cluster variables* and *cluster monomials* in $\mathbb{C}[N^w]$. For example, an r-tuple $(\varphi_{T_1}, \dots, \varphi_{T_r})$ is a cluster in $\mathbb{C}[N^w]$ if and only if there is a seed (y, Σ) in $\mathcal{A}(x, \Gamma, F^{\pm})$ with $\eta_{\mathbf{i}}(y_i) = \varphi_{T_i}$ for all i. In this case, let $T := T_1 \oplus \cdots \oplus T_r$. The vertices of the quiver Γ_T of the endomorphism algebra $\mathrm{End}_\Lambda(T)^{\mathrm{op}}$ are naturally parametrized by $1, \dots, r$ and the following hold:

(v) With the exception of arrows between coefficients $c, d \in F$, the quivers Σ and Γ_T coincide. The seed (y, Σ) in $\mathcal{A}(x, \Gamma)$ is already determined by y.

(vi) The module T is a basic cluster-tilting module in $\mathcal{C}_w$. For any mutable vertex k there is a unique indecomposable $T_k' \in \mathcal{C}_w$ with $T_k' \ncong T_k$ such that

$$\mu_k(T) := T_k' \oplus T/T_k$$

is a basic cluster-tilting module in $\mathcal{C}_w$. For $y_k' := \mu_{(y,\Sigma)}(y_k)$ we have

$$\eta_{\mathbf{i}}(y_k') = \varphi_{T_k'}.$$

We say that $(\varphi_{T_1}, \dots, \varphi_{T_k'}, \dots, \varphi_{T_r})$ is obtained from $(\varphi_{T_1}, \dots, \varphi_{T_k}, \dots, \varphi_{T_r})$ by *mutation in direction* k. We also say that $\mu_k(T)$ is obtained from T by *mutation in direction* k.

(vii) We have $\dim \mathrm{Ext}^1_\Lambda(T_k, T_k') = \dim \mathrm{Ext}^1_\Lambda(T_k', T_k) = 1$, and there are short exact sequences

$$0 \to T_k \to \bigoplus_{j \to k} T_j \to T_k' \to 0 \quad \text{and} \quad 0 \to T_k' \to \bigoplus_{k \to i} T_i \to T_k \to 0,$$

where we sum over all arrows in Γ_T ending and starting in k, respectively. Furthermore, the identity

$$y_k y_k' = \prod_{k \to i} y_i + \prod_{j \to k} y_j$$

in $\mathcal{A}(x, \Gamma)$ corresponds to the identity

$$\varphi_{T_k} \varphi_{T_k'} = \prod_{k \to i} \varphi_{T_i} + \prod_{j \to k} \varphi_{T_j}$$

in $\mathbb{C}[N^w]$. For $i, j \in R$, the number of arrows $k \to i$ in Γ_T equals $[B_{i,k}^{(T)}]_+$ and the number of arrows $j \to k$ is $[-B_{j,k}^{(T)}]_+$.

(viii) The cluster monomials in $\mathbb{C}[N^w]$ are

$$\varphi_{T_1}^{m_1} \cdots \varphi_{T_r}^{m_r},$$

where $m_i \geq 0$ for all i, and $T := T_1 \oplus \cdots \oplus T_r$ runs through the set of $V_{\mathbf{i}}$-reachable cluster-tilting modules in $\mathcal{C}_w$.

(ix) All cluster monomials in $\mathbb{C}[N^w]$ belong to the dual semicanonical basis of $\mathbb{C}[N^w]$.

3. Partial flag varieties and quiver Grassmannians

3.1. **Basic algebras and nilpotent modules.** Let $\Gamma = (\Gamma_0, \Gamma_1, s, t)$ be a finite quiver with set of vertices $\Gamma_0 = \{1, \ldots, n\}$, and set of arrows Γ_1. For an arrow $a\colon i \to j$ in Γ let $s(a) := i$ and $t(a) := j$ be its start vertex and terminal vertex, respectively.

A *path* of *length* m in Γ is an m-tuple $p = (a_1, \ldots, a_m)$ of arrows in Γ such that $s(a_i) = t(a_{i+1})$ for all $1 \leq i \leq m-1$. We define $s(p) := s(a_m)$ and $t(p) := t(a_1)$. Additionally, for each vertex $i \in \Gamma_0$ there is a path e_i of length 0 with $s(e_i) = t(e_i) = i$. An arrow a in Γ is a *loop* if $s(a) = t(a)$. A path $p = (a_1, a_2)$ is a 2-*cycle* if $s(p) = t(p)$.

The *path algebra* $\mathbb{C}\Gamma$ of Γ has the paths in Γ as a $\mathbb{C}$-basis, and the multiplication of two paths p and q is defined by

$$pq := \begin{cases} (a_1, \ldots, a_m, b_1, \ldots, b_l) & \text{if } s(p) = t(q),\ p = (a_1, \ldots, a_m) \text{ and } q = (b_1, \ldots, b_l), \\ p & \text{if } q = e_{s(p)}, \\ q & \text{if } p = e_{t(q)}, \\ 0 & \text{if } s(p) \neq t(q). \end{cases}$$

Extending this rule linearly turns $\mathbb{C}\Gamma$ into an associative $\mathbb{C}$-algebra with unit element.

For $m \geq 0$ let $\mathbb{C}\Gamma_{\geq m}$ be the ideal in $\mathbb{C}\Gamma$ generated by all paths of length m. An algebra A is called *basic* if $A = \mathbb{C}\Gamma/J$, where J is an ideal in $\mathbb{C}\Gamma$ with $J \subseteq \mathbb{C}\Gamma_{\geq 2}$. For the rest of this section, we assume that $A = \mathbb{C}\Gamma/J$ is a basic algebra.

Let $S_1, \ldots, S_n$ be the 1-dimensional A-modules associated to the vertices of Γ. (If A is finite-dimensional, then $S_1, \ldots, S_n$ are all simple A-modules up to isomorphism.) We focus on A-modules having only $S_1, \ldots, S_n$ as composition factors. These modules are called *nilpotent*. The category of all nilpotent A-modules is denoted by $\operatorname{nil}(A)$. (If A is finite-dimensional, then $\operatorname{nil}(\Lambda) = \operatorname{mod}(A)$.) Let $\widehat{I}_1, \ldots, \widehat{I}_n$ be the injective envelopes of $S_1, \ldots, S_n$, respectively. (The modules $\widehat{I}_j$ are in general infinite-dimensional A-modules.)

Let J_i be the maximal ideal of A spanned by all residue classes $\overline{p} := p + J$ of paths, where p runs through all paths except e_i. Thus A/J_i is 1-dimensional and (as an A-module) isomorphic to S_i. (In the following, we sometimes do not distinguish between a path p in $\mathbb{C}\Gamma$ and its residue class $\overline{p}$.)

Each (not necessarily finite-dimensional) A-module X can be interpreted as a representation $X = (X(i), X(a))_{i \in \Gamma_0, a \in \Gamma_1}$ of the quiver Γ, where the vector space $X(i)$ is defined by $e_i X$, and the linear map $X(a)\colon X(s(a)) \to X(t(a))$ is defined by $x \mapsto ax$. Recall that a *subrepresentation* of X is given by $U = (U(i))_{i \in \Gamma_0}$, where $U(i)$ is a subspace of $X(i)$ for all i, and for all $a \in \Gamma_1$ we have $X(a)(U(s(a))) \subseteq U(t(a))$. When passing from modules to representations, the submodules obviously correspond to the subrepresentations. The *dimension vector* of a representation $X = (X(i), X(a))_{i\in\Gamma_0, a\in\Gamma_1}$ is by definition $\underline{\dim}_\Gamma(X) := (\dim X(i))_{i\in Q_0}$.

Definition 3.1. For a dimension vector $\mathbf{d}$, let $\mathrm{Gr}_{\mathbf{d}}^A(X)$ be the projective variety of subrepresentations Y of X with $\underline{\dim}_\Gamma(Y) = \mathbf{d}$. Such a variety is called a *quiver Grassmannian.*

If X is nilpotent, then $\dim X(i) = [M : S_i]$ for all $i \in Q_0$. We study Grassmannians $\mathrm{Gr}_{\mathbf{d}}^A(X)$ only for nilpotent A-modules X, so there is no danger of confusing the two types of dimension vectors $\underline{\dim}_\Gamma(-)$ and $\underline{\dim}_A(-)$ associated to X and its submodules.

3.2. **Refined socle and top series.** For an arbitrary (not necessarily finite-dimensional) A-module X and a simple A-module S, let $\mathrm{soc}_S(X)$ be the sum of all submodules U of X with $U \cong S$. (If there is no such U, then $\mathrm{soc}_S(X) = 0$.) Similarly, let $\mathrm{top}_S(X) = X/V$, where V is the intersection of all submodules U of X such that $X/U \cong S$. (If there is no such U, then $V = X$ and $\mathrm{top}_S(X) = 0$.) Define $\mathrm{rad}_S(X) := V$.

Let us interpret X as a representation $X = (X(i), X(a))_{i\in\Gamma_0, a\in\Gamma_1}$ of Γ, and let $1 \le j \le n$. Then $\mathrm{soc}_{S_j}(X)$ can be seen as a subrepresentation $(X'(i))_{i\in\Gamma_0}$ of X, where

$$X'(i) = \begin{cases} 0 & \text{if } i \neq j, \\ \bigcap_{a\in\Gamma_1, s(a)=j} \mathrm{Ker}(X(a)) & \text{if } i = j. \end{cases}$$

Similarly, $\mathrm{rad}_{S_j}(X)$ can be seen as a subrepresentation $(X'(i))_{i\in\Gamma_0}$ of X, where

$$X'(i) = \begin{cases} X(i) & \text{if } i \neq j, \\ \sum_{a\in\Gamma_1, t(a)=j} \mathrm{Im}(X(a)) & \text{if } i = j. \end{cases}$$

It follows that $\mathrm{soc}_{S_j}(X)$ and $\mathrm{top}_{S_j}(X)$ are isomorphic to (possibly infinite) direct sums of copies of S_j.

Now fix some sequence $\mathbf{i} = (i_r, \dots, i_1)$ with $1 \le i_k \le n$ for all k. There exists a unique chain

$$(0 = X_r \subseteq \cdots \subseteq X_1 \subseteq X_0 \subseteq X)$$

of submodules X_k of X such that $X_{k-1}/X_k = \mathrm{soc}_{S_{i_k}}(X/X_k)$ for all $1 \le k \le r$. We define $\mathrm{soc}_{\mathbf{i}}(X) := X_0$,

$$X_k^+ := X_k^{\mathbf{i},+} := X_k$$

for all $0 \le k \le r$, and $X_\bullet^+ := X_\bullet^{\mathbf{i},+} := (X_r^+ \subseteq \cdots \subseteq X_1^+ \subseteq X_0^+)$. If $\mathrm{soc}_{\mathbf{i}}(X) = X$, then we call this chain the *refined socle series of type* $\mathbf{i}$ of X. Similarly, there exists a unique chain

$$(0 \subseteq X_r \subseteq \cdots \subseteq X_1 \subseteq X_0 = X)$$

of submodules X_k of X such that $X_{k-1}/X_k = \mathrm{top}_{S_{i_k}}(X_{k-1})$ for all $1 \le k \le r$. Set $\mathrm{top}_{\mathbf{i}}(X) := X/X_r$, $\mathrm{rad}_{\mathbf{i}}(X) := X_r$, and

$$X_k^- := X_k^{\mathbf{i},-} := X_k$$

for all $0 \le k \le r$. Define $X_\bullet^- := X_\bullet^{\mathbf{i},-} := (X_r^- \subseteq \cdots \subseteq X_1^- \subseteq X_0^-)$. If $\mathrm{rad}_{\mathbf{i}}(X) = 0$, then $X_\bullet^-$ is called the *refined top series of type* $\mathbf{i}$ of X.

The following lemma is straightforward:

Lemma 3.2. *For arbitrary (not necessarily finite-dimensional) A-modules X and Y and every A-module homomorphism $f\colon X \to Y$ the following hold:*

(i) $f(\mathrm{soc}_{\mathbf{i}}(X)) \subseteq \mathrm{soc}_{\mathbf{i}}(Y)$ *and* $f(\mathrm{rad}_{\mathbf{i}}(X)) \subseteq \mathrm{rad}_{\mathbf{i}}(Y)$.
(ii) *If f is a monomorphism (resp. epimorphism), then the induced maps*

$$X/\mathrm{soc}_{\mathbf{i}}(X) \to Y/\mathrm{soc}_{\mathbf{i}}(Y) \quad \textit{and} \quad \mathrm{rad}_{\mathbf{i}}(X) \to \mathrm{rad}_{\mathbf{i}}(Y)$$

are both monomorphisms (resp. epimorphisms).
(iii) *If* $\mathrm{soc}_{\mathbf{i}}(Y) = Y$, *then* $f(\mathrm{rad}_{\mathbf{i}}(X)) = 0$.

For $1 \leq k, s \leq r$ define

$$J_{k,s} := \begin{cases} J_{i_k} J_{i_{k-1}} \cdots J_{i_s} & \text{if } k \geq s, \\ A & \text{otherwise.} \end{cases}$$

Also the next lemma is easy to show:

Lemma 3.3. *For an arbitrary (not necessarily finite-dimensional) A-module X and $1 \leq k \leq r$ we have $J_{k,1}X = X_k^- = \mathrm{rad}_{(i_k,\dots,i_1)}(X)$.*

Corollary 3.4. *The algebra $A/J_{k,1}$ is finite-dimensional for all $1 \leq k \leq r$.*

Proof. Use Lemma 3.3 and the fact that the quiver Γ of A is finite. □

Let $\mathcal{D}_{\mathbf{i}}$ be the category of all A-modules X in $\mathrm{mod}(A)$ such that $\mathrm{soc}_{\mathbf{i}}(X) = X$.

Lemma 3.5. *For an A-module X the following are equivalent:*
(i) $X \in \mathcal{D}_{\mathbf{i}}$.
(ii) $\mathrm{soc}_{\mathbf{i}}(X) = X$.
(iii) $\mathrm{rad}_{\mathbf{i}}(X) = 0$.

Proof. By definition, (i) and (ii) are equivalent. The equivalence of (ii) and (iii) follows by an obvious induction on the length r of the sequence $\mathbf{i}$. □

Let $A_{\mathbf{i}} := A/J_{r,1}$. We identify the category $\mathrm{mod}(A_{\mathbf{i}})$ of finite-dimensional $A_{\mathbf{i}}$-modules with the category of all X in $\mathrm{nil}(A)$ such that $J_{r,1}X = 0$. Under this identification we obviously get the following:

Lemma 3.6. *We have $\mathcal{D}_{\mathbf{i}} = \mathrm{mod}(A_{\mathbf{i}})$.*

3.3. **Partial composition series.**

Definition 3.7. For $X \in \mathcal{D}_{\mathbf{i}}$ and $\mathbf{a} = (a_r, \dots, a_1)$ with $a_j \geq 0$, let $\mathcal{F}_{\mathbf{i},\mathbf{a},X}$ be the (possibly empty) set of chains $X_\bullet = (0 = X_r \subseteq \cdots \subseteq X_1 \subseteq X_0 = X)$ of submodules X_k of X such that $X_{k-1}/X_k \cong S_{i_k}^{a_k}$ for all $1 \leq k \leq r$. We call $(0 = X_r \subseteq \cdots \subseteq X_1 \subseteq X_0 = X)$ a *partial composition series* of *type* $\mathbf{i}$ of X.

Clearly, $\mathcal{F}_{\mathbf{i},\mathbf{a},X}$ is a projective variety. The *weight* of $X_\bullet \in \mathcal{F}_{\mathbf{i},\mathbf{a},X}$ is defined by

$$\mathrm{wt}(X_\bullet) := (a_r, \dots, a_2, a_1).$$

If $X_\bullet = X_\bullet^-$ (resp. $X = X_\bullet^+$), we define $\mathbf{a}^-(X) := \mathrm{wt}(X_\bullet^-)$ and $\mathbf{a}_k^-(X) := a_k$ (resp. $\mathbf{a}^+(X) := \mathrm{wt}(X_\bullet^+)$ and $\mathbf{a}_k^+(X) := a_k$) for all $1 \leq k \leq r$.

Lemma 3.8. *For $X \in \mathcal{D}_{\mathbf{i}}$ and $(X_r \subseteq \cdots \subseteq X_1 \subseteq X_0) \in \mathcal{F}_{\mathbf{i},\mathbf{a},X}$ we have*

$$X_k^- \subseteq X_k \subseteq X_k^+$$

for all $1 \leq k \leq r$.

Proof. For $1 \leq k \leq r$ we show that $X_k \subseteq X_k^+$ by decreasing induction on k. Clearly, we have $X_r \subseteq X_r^+$. (By definition, $X_r = X_r^+ = 0$.) Next, assume that $X_s \subseteq X_s^+$ for some $1 \leq s \leq r$. Thus, there is an epimorphism $\pi\colon X/X_s \to X/X_s^+$. We have $X_{s-1}/X_s \subseteq \mathrm{soc}_{S_{i_s}}(X/X_s)$, and by definition $X_{s-1}^+/X_s^+ = \mathrm{soc}_{S_{i_s}}(X/X_s^+)$. This implies that $\pi(X_{s-1}/X_s) \subseteq X_{s-1}^+/X_s^+$. In other words, $x + X_s^+ \in X_{s-1}^+/X_s^+$ for all $x \in X_{s-1}$. For each such x there exists some $y \in X_{s-1}^+$ with $x + X_s^+ = y + X_s^+$. This implies that $x - y$ is in X_s^+. Since $X_s^+ \subseteq X_{s-1}^+$ we get $x \in X_{s-1}^+$. Thus we have proved that $X_k \subseteq X_k^+$ for all $1 \leq k \leq r$. Similarly, one shows by induction on k that $X_k^- \subseteq X_k$ for all $1 \leq k \leq r$. □

The next lemma follows from the uniqueness of refined socle and top series.

Lemma 3.9. *Let $X \in \mathcal{D}_{\mathbf{i}}$. If $\mathbf{a}$ is equal to $\mathrm{wt}(X_\bullet^-)$ or $\mathrm{wt}(X_\bullet^+)$, then $\chi(\mathcal{F}_{\mathbf{i},\mathbf{a},X}) = 1$.*

Corollary 3.10. *For every $X \in \mathcal{D}_{\mathbf{i}}$ there exists some $\mathbf{a}$ such that $\chi(\mathcal{F}_{\mathbf{i},\mathbf{a},X}) \neq 0$.*

3.4. **The modules $V_{\mathbf{i}}$.** Let $A = \mathbb{C}\Gamma/J$ be a basic algebra, and let $\mathbf{i} = (i_r, \ldots, i_1)$ with $1 \leq i_k \leq n$ for all k. Without loss of generality we assume that for each $1 \leq j \leq n$ there exists some k with $i_k = j$. For $1 \leq k \leq r$ and $1 \leq j \leq n$, let

$$\begin{aligned} k^- &:= \max\{0, 1 \leq s \leq k-1 \mid i_s = i_k\}, \\ k^+ &:= \min\{k+1 \leq s \leq r, r+1 \mid i_s = i_k\}, \\ k_{\max} &:= \max\{1 \leq s \leq r \mid i_s = i_k\}, \\ k_{\min} &:= \min\{1 \leq s \leq r \mid i_s = i_k\}, \\ k_j &:= \max\{1 \leq s \leq r \mid i_s = j\}. \end{aligned}$$

For $1 \leq k \leq r$, define

$$V_k := V_{\mathbf{i},k} := \mathrm{soc}_{(i_k,\ldots,i_1)}(\widehat{I}_{i_k})$$

and $V_{\mathbf{i}} := V_1 \oplus \cdots \oplus V_r$. We also set $V_0 := 0$. For every $1 \leq j \leq n$ let $I_{\mathbf{i},j} := V_{k_j}$ and $I_{\mathbf{i}} := I_{\mathbf{i},1} \oplus \cdots \oplus I_{\mathbf{i},n}$. The modules in $\mathrm{add}(I_{\mathbf{i}})$ are called **i**-*injective*.

Lemma 3.11. *For $1 \leq k \leq r$ we have $V_k \cong D(e_{i_k}(A/J_{k,1}))$. In particular, V_k is an indecomposable injective $A/J_{k,1}$-module.*

Proof. Clearly, $J_{k,1}V_k = 0$. Thus V_k is an $A/J_{k,1}$-module. We have $\mathrm{soc}_{S_{i_k}}(V_k) \cong S_{i_k}$. Thus V_k can be embedded into the indecomposable injective $A/J_{k,1}$-module $D(e_{i_k}(A/J_{k,1}))$. We have $\mathrm{soc}_{S_{i_k}}(D(e_{i_k}(A/J_{k,1}))) \cong S_{i_k}$. Therefore $D(e_{i_k}(A/J_{k,1}))$ can be embedded into $\widehat{I}_{i_k}$. Thus we get two monomorphisms

$$V_k \xrightarrow{\iota_1} D(e_{i_k}(A/J_{k,1})) \xrightarrow{\iota_2} \widehat{I}_{i_k}.$$

Since $\mathrm{soc}_{(i_k,\ldots,i_1)}(D(e_{i_k}(A/J_{k,1}))) = D(e_{i_k}(A/J_{k,1}))$, we can apply Lemma 3.2(i) and get $\iota_2(D(e_{i_k}(A/J_{k,1}))) \subseteq \mathrm{soc}_{(i_k,\ldots,i_1)}(\widehat{I}_{i_k})$. Since $D(e_{i_k}(A/J_{k,1}))$ is finite-dimensional by Corollary 3.4, this implies that $V_k \cong D(e_{i_k}(A/J_{k,1}))$. □

Corollary 3.12. $V_{\mathbf{i}} \in \mathcal{D}_{\mathbf{i}}$.

Proof. We have $\mathrm{soc}_{\mathbf{i}}(V_{\mathbf{i}}) = V_{\mathbf{i}}$, and $V_{\mathbf{i}}$ is finite-dimensional by Corollary 3.4 and Lemma 3.11. □

Lemma 3.13. *An $A_{\mathbf{i}}$-module X is injective if and only if $X \in \mathrm{add}(I_{\mathbf{i}})$.*

Proof. One easily checks that $e_j J_{r,1} = e_j J_{k_j,1}$. This implies $D(e_j(A/J_{r,1})) = D(e_j(A/J_{k_j,1}))$. But $D(e_j(A/J_{k_j,1})) = I_{\mathbf{i},j}$ by Lemma 3.11. Thus the modules in $\mathrm{add}(I_{\mathbf{i}})$ are the injective $A_{\mathbf{i}}$-modules. □

Lemma 3.14. *For every $1 \leq k \leq r$ there is a monomorphism $V_{k^-} \to V_k$.*

Proof. We have $J_{k,1} \subseteq J_{k^-,1}$. Thus there is a short exact sequence

$$0 \to J_{k^-,1}/J_{k,1} \to A/J_{k,1} \to A/J_{k^-,1} \to 0.$$

Applying $e_{i_k}\cdot$ and then the duality D yields a short exact sequence

$$0 \to D(e_{i_k}(A/J_{k^-,1})) \to D(e_{i_k}(A/J_{k,1})) \to D(e_{i_k}(J_{k^-,1}/J_{k,1})) \to 0.$$

Now the result follows from Lemma 3.11. □

The following lemma is well known and easy to prove:

Lemma 3.15. *For any A-module X and any idempotent e in A the following hold:*

(i) *There is an isomorphism of $(eAe)^{\mathrm{op}}$-modules*

$$D(eX) \cong \mathrm{Hom}_A(X, D(eA))$$

defined by $\eta \mapsto f_\eta := [x \mapsto (ea \mapsto \eta(eax))]$.

(ii) *Assume that X is finite-dimensional. Then there is an isomorphism of eAe-modules*

$$eX \cong D\mathrm{Hom}_A(X, D(eA))$$

defined by $ex \mapsto [f \mapsto f(x)(e)]$.

The vector space $D\mathrm{Hom}_A(X, D(eA))$ is an $\mathrm{End}_A(D(eA))^{\mathrm{op}}$-module in an obvious way, and we have $eAe \cong \mathrm{End}_A(D(eA))^{\mathrm{op}}$. Under the isomorphisms $eX \cong D\mathrm{Hom}_A(X, D(eA))$ and $eAe \cong \mathrm{End}_A(D(eA))^{\mathrm{op}}$, the action of $\mathrm{End}_A(D(eA))^{\mathrm{op}}$ on $D\mathrm{Hom}_A(X, D(eA))$ turns into the action $eae \cdot ex := eaex$ of eAe on eX.

Lemma 3.16. *For any A-module X we have*

$$\mathrm{Hom}_A(X, V_k) = \mathrm{Hom}_A(X/X_k^-, V_k) = \mathrm{Hom}_{A/J_{k,1}}(X/X_k^-, V_k).$$

Proof. We have $\mathrm{soc}_{(i_k,\ldots,i_1)}(V_k) = V_k$, and $\mathrm{rad}_{(i_k,\ldots,i_1)}(X) = X_k^-$. By Lemma 3.2(iii) this implies that $f(X_k^-) = 0$ for every $f \in \mathrm{Hom}_A(X, V_k)$. This yields the identification $\mathrm{Hom}_A(X, V_k) = \mathrm{Hom}_A(X/X_k^-, V_k)$. Now X/X_k^- and V_k are annihilated by $J_{k,1}$. Thus X/X_k^- and V_k are $A/J_{k,1}$-modules. This implies that $\mathrm{Hom}_A(X/X_k^-, V_k) = \mathrm{Hom}_{A/J_{k,1}}(X/X_k^-, V_k)$. □

Corollary 3.17. *For any finite-dimensional A-module X we have*

$$D\mathrm{Hom}_A(X, V_k) \cong e_{i_k}(X/X_k^-).$$

Proof. The A-modules X/X_k^- and V_k can be regarded as an $A/J_{k,1}$-module, since both are annihilated by $J_{k,1}$, and V_k is injective as an $A/J_{k,1}$-module. Now we apply Lemma 3.15. □

3.5. **Balanced modules.** An A-module X is called $\mathbf{i}$-*balanced* if $X \in \mathcal{D}_{\mathbf{i}}$ and $X_\bullet^- = X_\bullet^+$. Thus, X is $\mathbf{i}$-balanced if and only if $X_k^- = X_k^+$ for all $0 \le k \le r$.

Proposition 3.18. *Let $X \in \mathcal{D}_{\mathbf{i}}$. Then the following are equivalent:*

(i) *X is $\mathbf{i}$-balanced.*
(ii) *There is a unique $\mathbf{b}$ such that $\mathcal{F}_{\mathbf{i},\mathbf{b},X} \neq \varnothing$.*
(iii) *There is a unique $\mathbf{b}$ such that $\chi(\mathcal{F}_{\mathbf{i},\mathbf{b},X}) \neq 0$.*

Proof. (i) $\implies$ (ii): Since $X \in \mathcal{D}_{\mathbf{i}}$, we know that $\mathrm{soc}_{\mathbf{i}}(X) = X$ and $\mathrm{rad}_{\mathbf{i}}(X) = 0$. This implies that $\mathcal{F}_{\mathbf{i},\mathrm{wt}(X_\bullet^-),X}$ and $\mathcal{F}_{\mathbf{i},\mathrm{wt}(X_\bullet^+),X}$ are both non-empty. Set $\mathbf{b} := \mathrm{wt}(X_\bullet^+)$. Since X is $\mathbf{i}$-balanced, we have $X_k^- = X_k^+$ for all k. In other words, $\mathbf{b} = \mathrm{wt}(X_\bullet^-) = \mathrm{wt}(X_\bullet^+)$. The uniqueness of $\mathbf{b}$ follows now from Lemma 3.8.
(ii) $\implies$ (iii): This follows directly from Lemma 3.9.
(iii) $\implies$ (i): Since $X \in \mathcal{D}_{\mathbf{i}}$, Lemma 3.9 implies $\chi(\mathcal{F}_{\mathbf{i},\mathrm{wt}(X_\bullet^-),X}) = \chi(\mathcal{F}_{\mathbf{i},\mathrm{wt}(X_\bullet^+),X}) = 1$. Since we assume $\mathbf{b}$ to be unique, we get $\mathrm{wt}(X_\bullet^-) = \mathrm{wt}(X_\bullet^+)$. Now (i) follows from Lemma 3.8. □

Lemma 3.19. *Let X and Y be A-modules. Then the following hold:*

(i) *If X and Y are $\mathbf{i}$-balanced, then $X \oplus Y$ is $\mathbf{i}$-balanced.*
(ii) *If X is $\mathbf{i}$-balanced, then each direct summand of X is $\mathbf{i}$-balanced.*

Proof. One easily checks that for every direct sum decomposition $M = M_1 \oplus M_2$ of an A-module M and every sequence $\mathbf{j} = (j_t, \dots, j_1)$ with $1 \le j_s \le n$ for all s, we have $\mathrm{soc}_{\mathbf{j}}(M) = \mathrm{soc}_{\mathbf{j}}(M_1) \oplus \mathrm{soc}_{\mathbf{j}}(M_2)$ and $\mathrm{rad}_{\mathbf{j}}(M) = \mathrm{rad}_{\mathbf{j}}(M_1) \oplus \mathrm{rad}_{\mathbf{j}}(M_2)$. This implies both (i) and (ii). □

We say that the pair $(A, \mathbf{i})$ is *balanced* if for each $1 \le k \le r$ the A-module $V_k = V_{\mathbf{i},k}$ is $(i_k, \dots, i_1)$-balanced. The following lemma follows directly from the definitions:

Lemma 3.20. *Assume that $(A, \mathbf{i})$ is balanced. For $1 \le k \le r$ and $0 \le s < k$ we have*

$$\mathrm{rad}_{(i_s,\dots,i_1)}(V_{\mathbf{i},k}) = (V_{\mathbf{i},k})_s^{\mathbf{i},-} = (V_{\mathbf{i},k})_s^{(i_k,\dots,i_1),-} = (V_{\mathbf{i},k})_s^{(i_k,\dots,i_1),+} = \mathrm{soc}_{(i_k,\dots,i_{s+1})}(V_{\mathbf{i},k}).$$

Lemma 3.21. *Assume that $(A, \mathbf{i})$ is balanced. Then the modules $V_{\mathbf{i},1}, \dots, V_{\mathbf{i},r}$ are pairwise non-isomorphic.*

Proof. Assume $V_{\mathbf{i},k} \cong V_{\mathbf{i},s}$ with $k > s$. By definition $V_{\mathbf{i},k} = \mathrm{soc}_{(i_k,\dots,i_s,\dots,i_1)}(\widehat{I}_{i_k})$ and $V_{\mathbf{i},s} = \mathrm{soc}_{(i_s,\dots,i_1)}(\widehat{I}_{i_s})$. Clearly, $\mathrm{rad}_{(i_s,\dots,i_1)}(V_{\mathbf{i},s}) = 0$. Since $V_{\mathbf{i},k} \cong V_{\mathbf{i},s}$ we also get $\mathrm{rad}_{(i_s,\dots,i_1)}(V_{\mathbf{i},k}) = 0$. But $V_{\mathbf{i},k}$ is $(i_k, \dots, i_1)$-balanced. By Lemma 3.20 this implies that $\mathrm{soc}_{(i_k,\dots,i_{s+1})}(V_{\mathbf{i},k}) = \mathrm{rad}_{(i_s,\dots,i_1)}(V_{\mathbf{i},k}) = 0$. But we have $\mathrm{soc}_{S_{i_k}}(V_{\mathbf{i},k}) \cong S_{i_k}$. This implies that $\mathrm{soc}_{(i_k,\dots,i_{s+1})}(V_{\mathbf{i},k}) \neq 0$, a contradiction. □

Proposition 3.22. *Assume that $(A, \mathbf{i})$ is balanced. For $1 \le k, s \le r$ we have*

$$\mathrm{Hom}_A(V_k, V_s) \cong e_{i_k}\,(J_{k,s+1}/J_{k,1})\,e_{i_s}.$$

Proof. Recall that $V_k = D(e_{i_k}(A/J_{k,1}))$ and $V_s = D(e_{i_s}(A/J_{s,1}))$. By Lemma 3.16 we have $\mathrm{Hom}_A(V_k, V_s) = \mathrm{Hom}_A(V_k/(V_k)_s^-, V_s)$. We have

$$(V_k)_s^- = J_{s,1}V_k = D(e_{i_k}(A/J_{k,s+1})).$$

For the second equality we used that V_k is $(i_k, \dots, i_1)$-balanced. Note that $(V_k)_s^- = 0$ if $k \le s$. We get

$$\begin{aligned}\operatorname{Hom}_A(V_k/(V_k)_s^-, V_s) &\cong D(e_{i_s}(V_k/(V_k)_s^-)) \\ &= D\left(e_{i_s}(D(e_{i_k}(A/J_{k,1}))/D(e_{i_k}(A/J_{k,s+1})))\right).\end{aligned}$$

For the first isomorphism we used Lemma 3.15. Now we first apply $e_{i_k}\cdot$ and then the duality D to the short exact sequence

$$0 \to J_{k,s+1}/J_{k,1} \to A/J_{k,1} \to A/J_{k,s+1} \to 0,$$

and we obtain

$$D(e_{i_k}(A/J_{k,1}))/D(e_{i_k}(A/J_{k,s+1})) \cong D(e_{i_k}(J_{k,s+1}/J_{k,1})).$$

Now $D(e_{i_s}D(e_{i_k}(J_{k,s+1}/J_{k,1}))) = D(D(e_{i_k}(J_{k,s+1}/J_{k,1})e_{i_s})) \cong e_{i_k}(J_{k,s+1}/J_{k,1})e_{i_s}$ implies that $\operatorname{Hom}_A(V_k, V_s) \cong e_{i_k}(J_{k,s+1}/J_{k,1})e_{i_s}$. $\square$

Using Lemma 3.15, the isomorphism $e_{i_k}J_{k,s+1}/J_{k,1}e_{i_s} \to \operatorname{Hom}_A(V_k, V_s)$ can be described more precisely: Let $e_{i_k}\overline{b}e_{i_s} \in e_{i_k}(J_{k,s+1}/J_{k,1})e_{i_s}$. Then $e_{i_k}\overline{b}e_{i_s}$ is mapped to the homomorphism $V_k \to V_s$, which maps a linear form $\eta\colon e_{i_k}(A/J_{k,1}) \to \mathbb{C}$ in $D(e_{i_k}(A/J_{k,1}))$ to the linear form $\psi \in D(e_{i_s}(A/J_{s,1}))$ defined by

$$\psi(e_{i_s}\overline{a}) := \eta(e_{i_k}\overline{b}e_{i_s}\overline{a}).$$

For A-modules X and Y let $\mathcal{I}_{\mathbf{i}}(X,Y)$ be the subspace of $\operatorname{Hom}_A(X,Y)$ consisting of the morphisms factoring through a module in $\operatorname{add}(I_{\mathbf{i}})$. Define

$$\overline{\operatorname{Hom}}_A(X,Y) := \operatorname{Hom}_A(X,Y)/\mathcal{I}_{\mathbf{i}}(X,Y).$$

Lemma 3.23. *Assume that $(A,\mathbf{i})$ is balanced. Then for each $X \in \mathcal{D}_{\mathbf{i}}$ and $1 \le k \le r$ we have*

$$\mathcal{I}_{\mathbf{i}}(X, V_k) = \operatorname{Hom}_A(X/X_k^+, V_k).$$

Proof. There is a short exact sequence

$$0 \to X_k^+ \to X \to X/X_k^+ \to 0.$$

Applying the functor $\operatorname{Hom}_A(-, V_k)$ we can identify $\operatorname{Hom}_A(X/X_k^+, V_k)$ with a subspace of $\operatorname{Hom}_A(X, V_k)$. Suppose that $f\colon X \to V_k$ is a homomorphism.

Assume first that $f = h \circ g$ with $g\colon X \to I$ and $I \in \operatorname{add}(I_{\mathbf{i}})$. It follows from Lemma 3.6 and Lemma 3.13 that we can assume without loss of generality that g is a monomorphism. By Lemma 3.2(i) we know that $g(X_k^+) \subseteq I_k^+$. By definition $I_k^- = \operatorname{rad}_{(i_k,\dots,i_1)}(I)$ and $\operatorname{soc}_{(i_k,\dots,i_1)}(V_k) = V_k$. Thus Lemma 3.2(iii) implies $h(I_k^-) = 0$. Since $(A,\mathbf{i})$ is balanced, we get $I_k^- = I_k^+$. This shows that $f(X_k^+) = 0$. In other words, $f \in \operatorname{Hom}_A(X/X_k^+, V_k)$. So we proved that $\mathcal{I}_{\mathbf{i}}(X, V_k) \subseteq \operatorname{Hom}_A(X/X_k^+, V_k)$.

To show the other inclusion, let $f\colon X \to V_k$ be a homomorphism with $f(X_k^+) = 0$. Thus there is a factorization $f = h_1 \circ g_1$, where $g_1\colon X \to X/X_k^+$ is the projection. Let $u_1\colon X \to I$ be a monomorphism with $I \in \operatorname{add}(I_{\mathbf{i}})$, and let $u_2\colon I \to I/I_k^+$ be the projection. By Lemma 3.2(ii) we get a monomorphism $g_2\colon X/X_k^+ \to I/I_k^+$ such that $u_2 \circ u_1 = g_2 \circ g_1$. Now X/X_k^+ and I/I_k^+ are $A/J_{k,1}$-modules, V_k is an injective $A/J_{k,1}$-module, and g_2 is a monomorphism. Thus there exists a homomorphism

$u_3\colon I/I_k^+ \to V_k$ such that $u_3 \circ g_2 = h_1$. The following commutative diagram illustrates the situation:

$$\begin{tikzcd}
& X \arrow[r, "f"] \arrow[dl, "u_1"'] \arrow[d, "g_1"] & V_k \\
I \arrow[d, "u_2"'] & X/X_k^+ \arrow[ur, "h_1"'] \arrow[dl, "g_2"] & \\
I/I_k^+ \arrow[uurr, bend right, "u_3"'] & &
\end{tikzcd}$$

It follows that

$$f = h_1 \circ g_1 = u_3 \circ g_2 \circ g_1 = u_3 \circ u_2 \circ u_1.$$

Thus we have proved that $\mathrm{Hom}_A(X/X_k^+, V_k) \subseteq \mathcal{I}_{\mathbf{i}}(X, V_k)$. Note that for the proof of this inclusion we did not use the assumption that $(A, \mathbf{i})$ is balanced. $\square$

Proposition 3.24. *Assume that $(A, \mathbf{i})$ is balanced, and let $X \in \mathcal{D}_{\mathbf{i}}$. For $1 \leq k \leq r$ we have*

$$D\overline{\mathrm{Hom}}_A(X, V_k) \cong e_{i_k}(X_k^+/X_k^-).$$

Proof. There is a short exact sequence

$$\eta\colon \quad 0 \to X_k^+/X_k^- \to X/X_k^- \to X/X_k^+ \to 0.$$

As noted in Lemma 3.16 we have $\mathrm{Hom}_A(X, V_k) = \mathrm{Hom}_A(X/X_k^-, V_k)$, and by Lemma 3.23 we know that $\mathcal{I}_{\mathbf{i}}(X, V_k) = \mathrm{Hom}_A(X/X_k^+, V_k)$. Note that X/X_k^+ and X_k^+/X_k^- are both annihilated by $J_{k,1}$. Thus they are $A/J_{k,1}$-modules, and $V_k = D(e_{i_k}(A/J_{k,1}))$ is an injective $A/J_{k,1}$-module. Now we apply $\mathrm{Hom}_A(-, V_k)$ to η and obtain $\overline{\mathrm{Hom}}_A(X, V_k) \cong \mathrm{Hom}_A(X_k^+/X_k^-, V_k)$. By Lemma 3.15 we get

$$\mathrm{Hom}_A(X_k^+/X_k^-, V_k) = \mathrm{Hom}_A(X_k^+/X_k^-, D(e_{i_k}(A/J_{k,1}))) \cong D(e_{i_k}(X_k^+/X_k^-)).$$

Thus we have proved that $D\overline{\mathrm{Hom}}_A(X, V_k) \cong e_{i_k}(X_k^+/X_k^-)$. $\square$

3.6. **The quiver of $\mathcal{E}_{\mathbf{i}}$.** Again, let $A = \mathbb{C}\Gamma/J$ be a basic algebra and let us fix some sequence $\mathbf{i} = (i_r, \dots, i_1)$. Define $\mathcal{E}_{\mathbf{i}} := \mathrm{End}_A(V_{\mathbf{i}})^{\mathrm{op}}$. Since we work over an algebraically closed field, Lemma 3.21 and a result by Gabriel (see for example [DK, Theorem 3.5.4 combined with Theorem 3.6.6]) imply that $\mathcal{E}_{\mathbf{i}}$ is a finite-dimensional basic algebra. We want to determine the quiver $\Gamma_{\mathcal{E}_{\mathbf{i}}}$ of $\mathcal{E}_{\mathbf{i}}$. The vertices of $\Gamma_{\mathcal{E}_{\mathbf{i}}}$ correspond to the indecomposable direct summands $V_1, \dots, V_r$ of $V_{\mathbf{i}}$.

Define a quiver $\Gamma_{\mathbf{i}}$ as follows: The set of vertices of $\Gamma_{\mathbf{i}}$ is just $\{1, 2, \dots, r\}$. For each pair (k, s) with $1 \leq s, k \leq r$ and $k^+ \geq s^+ \geq k > s$ and each arrow $a\colon i_s \to i_k$ in the quiver Γ of A, there is an arrow $\gamma_a^{k,s}\colon s \to k$ in $\Gamma_{\mathbf{i}}$. These are called the *ordinary arrows* of $\Gamma_{\mathbf{i}}$. Furthermore, for each $1 \leq k \leq r$ there is an arrow $\gamma_k\colon k \to k^-$ provided $k^- > 0$. These are the *horizontal arrows* of $\Gamma_{\mathbf{i}}$.

Proposition 3.25. *Assume that $(A, \mathbf{i})$ is balanced. Then there is a quiver isomorphism $\Gamma_{\mathbf{i}} \to \Gamma_{\mathcal{E}_{\mathbf{i}}}$ with $k \mapsto V_k$ for all $1 \leq k \leq r$.*

Proof. One can almost copy the proof of [BIRS, Theorem III.4.1]. One only has to replace the ideals I_j used in [BIRS] by our ideals J_j. (We have $I_j = J_j$ if and only if Γ has no loop at the vertex j.) Furthermore, everything has to be dualized. $\square$

In Proposition 3.25 we identify the vertex of $\Gamma_{\mathcal{E}_{\mathbf{i}}}$ corresponding to V_k with the vertex k of $\Gamma_{\mathbf{i}}$. Some examples can be found in Section 3.10.

3.7. **The $\mathcal{E}_{\mathbf{i}}$-module $D\overline{\mathrm{Hom}}_A(X, V_{\mathbf{i}})$.** Using Lemma 3.15 together with Propositions 3.22 and 3.24, we arrive at the following conclusion: Assume $(A, \mathbf{i})$ is balanced, and let $X \in \mathcal{D}_{\mathbf{i}}$. Using the identifications

$$\begin{aligned}
\mathrm{Hom}_A(V_k, V_s) &= e_{i_k}(J_{k,s+1}/J_{k,1})e_{i_s},\\
\mathrm{Hom}_A(V_s, V_k) &= e_{i_s}(A/J_{s,1})e_{i_k},\\
D\overline{\mathrm{Hom}}_A(X, V_k) &= e_{i_k}(X_k^+/X_k^-),
\end{aligned}$$

the algebra $\mathcal{E}_{\mathbf{i}}$ acts on $Y := D\overline{\mathrm{Hom}}_A(X, V_{\mathbf{i}})$ as follows: Assume $1 \le s \le k \le r$.

$$\begin{tikzcd}
e_{i_k}(X_k^+/X_k^-) \arrow[rr, bend right, "e_{i_s}(A/J_{s,1})e_{i_k}\cdot"'] & & e_{i_s}(X_s^+/X_s^-) \arrow[ll, bend right, "e_{i_k}(J_{k,s+1}/J_{k,1})e_{i_s}\cdot"']
\end{tikzcd}$$

For $e_{i_k}\overline{b}e_{i_s} \in e_{i_k}(J_{k,s+1}/J_{k,1})e_{i_s}$ and $\overline{x_s} \in e_{i_s}(X_s^+/X_s^-) = e_{i_s}X_s^+/e_{i_s}X_s^-$ we have

$$e_{i_k}\overline{b}e_{i_s} \cdot \overline{x_s} = e_{i_k}\overline{bx_s},$$

and for $e_{i_s}\overline{b}e_{i_k} \in e_{i_s}(A/J_{s,1})e_{i_k}$ and $\overline{x_k} \in e_{i_k}(X_k^+/X_k^-) = e_{i_k}X_k^+/e_{i_k}X_k^-$ we have

$$e_{i_s}\overline{b}e_{i_k} \cdot \overline{x_k} = e_{i_s}\overline{bx_k}.$$

We consider Y as a representation $Y = (Y(k), Y(\gamma))_{k,\gamma}$ of the quiver $\Gamma_{\mathbf{i}}$ of $\mathcal{E}_{\mathbf{i}}$. To describe Y, we just need to know how the maps $Y(\gamma)$ act on the vector spaces $Y(k) = e_{i_k}(X_k^+/X_k^-)$, where $1 \le k \le r$. Again using the description of $\Gamma_{\mathcal{E}_{\mathbf{i}}}$ based on [BIRS, Theorem III.4.1] we obtain the following result. First, assume $\gamma_k \colon k \to k^-$ is a horizontal arrow of $\Gamma_{\mathbf{i}}$. Then $Y(\gamma_k)$ acts as left multiplication with e_{i_k}:

$$e_{i_k}(X_k^+/X_k^-) \xrightarrow{e_{i_k}\cdot} e_{i_k}(X_{k^-}^+/X_{k^-}^-).$$

Next, let $\gamma_a^{k,s} \colon s \to k$ be an ordinary arrow of $\Gamma_{\mathbf{i}}$. Then $Y(\gamma_a^{k,s})$ acts as left multiplication with a:

$$e_{i_k}(X_k^+/X_k^-) \xleftarrow{a\cdot} e_{i_s}(X_s^+/X_s^-).$$

Remark 3.26. For $X \in \mathcal{D}_{\mathbf{i}}$ the following hold:

(i) $\mathcal{I}_{\mathbf{i}}(X, V_{\mathbf{i}})$ is a submodule of the $\mathrm{End}_A(V_{\mathbf{i}})$-module $\mathrm{Hom}_A(X, V_{\mathbf{i}})$. This implies that $D\overline{\mathrm{Hom}}_A(X, V_{\mathbf{i}})$ is a submodule of the $\mathcal{E}_{\mathbf{i}}$-module $D\mathrm{Hom}_A(X, V_{\mathbf{i}})$. Clearly, $D\overline{\mathrm{Hom}}_A(X, V_{\mathbf{i}})$ is also a module over the algebra $\overline{B}_{\mathbf{i}} := (\overline{\mathrm{End}}_A(V_{\mathbf{i}}))^{\mathrm{op}}$.

(ii) For $X \in \mathcal{D}_{\mathbf{i}}$ we have

$$\mathrm{Hom}_A(X, V_{\mathbf{i}}) = \mathrm{Hom}_{A_{\mathbf{i}}}(X, V_{\mathbf{i}}).$$

Since $\mathrm{add}(I_{\mathbf{i}})$ are the injective $A_{\mathbf{i}}$-modules, we can apply the Auslander-Reiten formula to obtain an isomorphism of $\overline{B}_{\mathbf{i}}$-modules

$$D\overline{\mathrm{Hom}}_{A_{\mathbf{i}}}(X, V_{\mathbf{i}}) \cong \mathrm{Ext}^1_{A_{\mathbf{i}}}(\tau_{A_{\mathbf{i}}}^{-1}(V_{\mathbf{i}}), X),$$

where $\tau_{A_{\mathbf{i}}}$ denotes the Auslander-Reiten translation of the finite-dimensional algebra $A_{\mathbf{i}}$.

3.8. **An isomorphism between partial flag varieties and quiver Grassmannians.** In this section we prove that the varieties $\mathcal{F}_{\mathbf{i},\mathbf{a},X}$ of partial composition series of modules $X \in \mathcal{D}_{\mathbf{i}}$ are isomorphic to certain quiver Grassmannians $\mathcal{G}_{\mathbf{i},\mathbf{a},X}$. In the proof we first construct a (rather trivial) isomorphism between partial flag varieties $\widetilde{\mathcal{F}}_{\mathbf{i},\mathbf{a},X}$ of graded vector spaces and the image $\widetilde{\mathcal{G}}_{\mathbf{i},\mathbf{a},X}$ of the usual embedding of $\widetilde{\mathcal{F}}_{\mathbf{i},\mathbf{a},X}$ into a product of classical subspace Grassmannians. Then we show that the restriction to the subvarieties $\mathcal{F}_{\mathbf{i},\mathbf{a},X} \subseteq \widetilde{\mathcal{F}}_{\mathbf{i},\mathbf{a},X}$ and $\mathcal{G}_{\mathbf{i},\mathbf{a},X} \subseteq \widetilde{\mathcal{G}}_{\mathbf{i},\mathbf{a},X}$ yields an isomorphism $\mathcal{F}_{\mathbf{i},\mathbf{a},X} \to \mathcal{G}_{\mathbf{i},\mathbf{a},X}$.

Let $X \in \mathcal{D}_{\mathbf{i}}$ for some $\mathbf{i} = (i_r, \dots, i_1)$. We define a map $d_{\mathbf{i},X} \colon \mathbb{N}^r \to \mathbb{Z}^r$ by $(a_r, \dots, a_1) \mapsto (f_1, \dots, f_r)$, where

$$f_k := (a_k^- - a_k) + (a_{k^-}^- - a_{k^-}) + \cdots + (a_{k_{\min}}^- - a_{k_{\min}})$$

for all $1 \le k \le r$, and $(a_r^-, \dots, a_1^-) := \mathbf{a}^-(X)$. In the following theorem, if $d_{\mathbf{i},X}(\mathbf{a}) \notin \mathbb{N}^r$, then $\mathrm{Gr}^{\mathcal{E}_{\mathbf{i}}}_{d_{\mathbf{i},X}(\mathbf{a})}(Y)$ is by definition the empty set.

Theorem 3.27. *Assume that $(A, \mathbf{i})$ is balanced, and let $X \in \mathcal{D}_{\mathbf{i}}$. Then for each $\mathbf{a} \in \mathbb{N}^r$ there exists an isomorphism of algebraic varieties*

$$\mathcal{F} \colon \mathcal{F}_{\mathbf{i},\mathbf{a},X} \to \mathrm{Gr}^{\mathcal{E}_{\mathbf{i}}}_{d_{\mathbf{i},X}(\mathbf{a})}(Y),$$

where Y is the $\mathcal{E}_{\mathbf{i}}$-module $D\overline{\mathrm{Hom}}_A(X, V_{\mathbf{i}})$. Furthermore, the map $\mathbf{a} \mapsto d_{\mathbf{i},X}(\mathbf{a})$ yields a bijection $\{\mathbf{a} \in \mathbb{N}^r \mid \mathcal{F}_{\mathbf{i},\mathbf{a},X} \neq \varnothing\} \to \left\{\mathbf{f} \in \mathbb{N}^r \mid \mathrm{Gr}^{\mathcal{E}_{\mathbf{i}}}_{\mathbf{f}}(Y) \neq \varnothing\right\}$.

Our proof of Theorem 3.27 will show that $\underline{\dim}_{\mathcal{E}_{\mathbf{i}}}(Y) = d_{\mathbf{i},X}(\mathbf{a}^+(X))$. Furthermore, if $\mathcal{F}_{\mathbf{i},\mathbf{a},X} \neq \varnothing$, and $X_\bullet = (X_r \subseteq \cdots \subseteq X_1 \subseteq X_0) \in \mathcal{F}_{\mathbf{i},\mathbf{a},X}$, then $f_k = \dim(e_{i_k}(X_k/X_k^-))$ for all $1 \le k \le r$. Note that $f_k = 0$ if $k^+ = r+1$.

3.9. **Proof of Theorem 3.27.**

3.9.1. Assume that $(A, \mathbf{i})$ is balanced. For the rest of this section, besides $\mathbf{i}$, we also fix some $\mathbf{a} = (a_r, \dots, a_1) \in \mathbb{N}^r$ and some $X \in \mathcal{D}_{\mathbf{i}}$. With the same notation as in Theorem 3.27, we define

$$\mathcal{G}_{\mathbf{i},\mathbf{a},X} := \mathrm{Gr}^{\mathcal{E}_{\mathbf{i}}}_{\mathbf{f}}(Y),$$

where $\mathbf{f} := d_{\mathbf{i},X}(\mathbf{a})$.

We consider X as a representation $X = (X(j), X(a))_{j \in \Gamma_0, a \in \Gamma_1}$ of the quiver Γ of A, and the $\mathcal{E}_{\mathbf{i}}$-module Y is considered as a representation $Y = (Y(k), Y(\gamma))_{k,\gamma}$ of the quiver $\Gamma_{\mathbf{i}}$ of $\mathcal{E}_{\mathbf{i}}$. Given $X_\bullet = (0 = X_r \subseteq \cdots \subseteq X_1 \subseteq X_0 = X)$ in $\mathcal{F}_{\mathbf{i},\mathbf{a},X}$ we consider each X_k as a subrepresentation of X. Thus we have $X_k = (X_k(j))_{j \in \Gamma_0}$ such that

$$X(a)(X_k(s(a))) \subseteq X_k(t(a))$$

for all arrows a of Γ.

Our aim is the construction of two mutually inverse isomorphisms of varieties

$$\mathcal{F}_{\mathbf{i},\mathbf{a},X} \underset{\mathcal{G}}{\overset{\mathcal{F}}{\rightleftarrows}} \mathcal{G}_{\mathbf{i},\mathbf{a},X}.$$

3.9.2. Recall that $\Gamma_0 = \{1,\dots,n\}$. We need to work with the category of Γ_0-graded vector spaces. Its objects are just tuples $W = (W(j))_{j\in\Gamma_0}$ of $\mathbb{C}$-vector spaces $W(j)$. Set $e_jW := W(j)$ for all $j \in \Gamma_0$. The morphisms are defined in the obvious way. The *degree* of W is $\underline{\dim}(W) := (\dim(W(j)))_{j\in\Gamma_0}$. Let $\mathbf{e}_1,\dots,\mathbf{e}_n$ denote the canonical coordinate vectors of $\mathbb{Z}^n$. (Thus the jth entry of $\mathbf{e}_j$ is 1, and all other entries are 0.) Each representation $X = (X(j), X(a))_{j\in\Gamma_0, a\in\Gamma_1}$ of Γ yields a Γ_0-graded vector space $\mathrm{gr}(X) := (X(j))_{j\in\Gamma_0}$.

Let $\widetilde{\mathcal{F}}_{\mathbf{i},\mathbf{a},X}$ be the projective variety of chains

$$X_\bullet = (0 = X_r \subseteq \cdots \subseteq X_1 \subseteq X_0 = \mathrm{gr}(X))$$

of Γ_0-graded subspaces of $\mathrm{gr}(X)$ such that $\mathrm{gr}(X_k^-) \subseteq X_k \subseteq \mathrm{gr}(X_k^+)$ and $\underline{\dim}(X_{k-1}/X_k) = a_k\mathbf{e}_{i_k}$ for all $1 \le k \le r$.

For a vector space L let $\mathrm{Gr}_d(L)$ be the projective variety of d-dimensional subspaces of L. Clearly, the variety of $(f_k + \dim(e_{i_k}X_k^-))$-dimensional subspaces U_k of $e_{i_k}X$ such that $e_{i_k}X_k^- \subseteq U_k \subseteq e_{i_k}X_k^+$ is isomorphic to $\mathrm{Gr}_{f_k}(e_{i_k}(X_k^+/X_k^-))$. The isomorphism is given by

$$U_k \mapsto \overline{U}_k := U_k/e_{i_k}X_k^-.$$

Let $\widetilde{\mathcal{G}}_{\mathbf{i},\mathbf{a},X}$ be the projective variety formed by the r-tuples $\overline{U} := (\overline{U}_k)_{1\le k\le r}$ in

$$\prod_{k=1}^{r} \mathrm{Gr}_{f_k}(e_{i_k}(X_k^+/X_k^-))$$

such that $U_k \subseteq U_{k^-}$ for all $1 \le k \le r$.

We construct two morphisms

$$\begin{tikzcd} \widetilde{\mathcal{F}}_{\mathbf{i},\mathbf{a},X} \arrow[r, shift left, "\widetilde{\mathcal{F}}"] & \widetilde{\mathcal{G}}_{\mathbf{i},\mathbf{a},X} \arrow[l, shift left, "\widetilde{\mathcal{G}}"] \end{tikzcd}$$

as follows: First, we define $\widetilde{\mathcal{F}}$. Let $X_\bullet = (0 = X_r \subseteq \cdots \subseteq X_1 \subseteq X_0 = \mathrm{gr}(X))$ be in $\widetilde{\mathcal{F}}_{\mathbf{i},\mathbf{a},X}$. To each X_k we assign the subspace

$$\overline{U}_k := e_{i_k}(X_k/X_k^-)$$

of $e_{i_k}(X_k^+/X_k^-)$. Set $\overline{U} := (\overline{U}_k)_{1\le k\le r}$. Then $\widetilde{\mathcal{F}}(X_\bullet) := \overline{U}$ defines a morphism of varieties

$$\widetilde{\mathcal{F}} \colon \widetilde{\mathcal{F}}_{\mathbf{i},\mathbf{a},X} \to \widetilde{\mathcal{G}}_{\mathbf{i},\mathbf{a},X}.$$

Second, we define the morphism $\widetilde{\mathcal{G}}$. Let $\overline{U} = (\overline{U}_k)_{1\le k\le r}$ be in $\widetilde{\mathcal{G}}_{\mathbf{i},\mathbf{a},X}$. We define a chain

$$X_\bullet := (0 = X_r \subseteq \cdots \subseteq X_1 \subseteq X_0 = \mathrm{gr}(X))$$

of Γ_0-graded vector spaces as follows: For $j \in \Gamma_0$ set $X_k := (X_k(j))_{j\in\Gamma_0}$, where

$$X_k(j) := U_p$$

and $p := \min\{k \le s \le r, r+1 \mid i_s = j\}$. (Here we set $U_{r+1} := 0$.) Then $\widetilde{\mathcal{G}}(\overline{U}) := X_\bullet$ defines a morphism of varieties

$$\widetilde{\mathcal{G}} \colon \widetilde{\mathcal{G}}_{\mathbf{i},\mathbf{a},X} \to \widetilde{\mathcal{F}}_{\mathbf{i},\mathbf{a},X}.$$

Just using the definitions of $\widetilde{\mathcal{F}}$ and $\widetilde{\mathcal{G}}$ we obtain the following:

Lemma 3.28. *The morphisms $\widetilde{\mathcal{F}}$ and $\widetilde{\mathcal{G}}$ are isomorphisms of algebraic varieties, and we have $\widetilde{\mathcal{G}} \circ \widetilde{\mathcal{F}} = \mathrm{id}_{\widetilde{\mathcal{F}}_{\mathbf{i},\mathbf{a},X}}$ and $\widetilde{\mathcal{F}} \circ \widetilde{\mathcal{G}} = \mathrm{id}_{\widetilde{\mathcal{G}}_{\mathbf{i},\mathbf{a},X}}$.*

3.9.3. The following lemma is needed in order to ensure that the quiver Grassmannian $\mathcal{G}_{\mathbf{i},\mathbf{a},X}$ is a subvariety of $\widetilde{\mathcal{G}}_{\mathbf{i},\mathbf{a},X}$:

Lemma 3.29. *Let $\overline{U} = (\overline{U}_k)_{1\le k\le r}$ be a submodule of the $\mathcal{E}_{\mathbf{i}}$-module Y. Then we have $U_k \subseteq U_{k^-}$ for all $1 \le k \le r$.*

Proof. Let $\gamma_k\colon k \to k^-$ be a horizontal arrow of $\Gamma_{\mathbf{i}}$. We know that $Y(\gamma_k)$ acts on Y as follows:

$$e_{i_k}(X_k^+/X_k^-) \xrightarrow{e_{i_k}\cdot} e_{i_k}(X_{k^-}^+/X_{k^-}^-).$$

In other words, $Y(\gamma_k)(x_k + e_{i_k}X_k^-) = x_k + e_{i_k}X_{k^-}^-$ for all $x_k \in e_{i_k}X_k^+$. Since $\overline{U}$ is a submodule of Y, we know that $u_k + e_{i_k}X_{k^-}^-$ is contained in $U_{k^-}/e_{i_k}X_{k^-}^-$ for all $u_k \in U_k$. This implies that $u_k \in U_{k^-}$ for all $u_k \in U_k$. Thus $U_k \subseteq U_{k^-}$. □

Lemma 3.30. *The following hold:*

(i) *$\mathcal{F}_{\mathbf{i},\mathbf{a},X}$ is a Zariski closed subset of $\widetilde{\mathcal{F}}_{\mathbf{i},\mathbf{a},X}$.*
(ii) *Under the identification*

$$Y = \bigoplus_{k=1}^{r} e_{i_k}(X_k^+/X_k^-)$$

the variety $\mathcal{G}_{\mathbf{i},\mathbf{a},X}$ is a Zariski closed subset of $\widetilde{\mathcal{G}}_{\mathbf{i},\mathbf{a},X}$.

Proof. For $X_\bullet \in \widetilde{\mathcal{F}}_{\mathbf{i},\mathbf{a},X}$, the condition that all X_k are submodules is closed. This implies (i). Similarly, for $\overline{U} \in \widetilde{\mathcal{G}}_{\mathbf{i},\mathbf{a},X}$, the condition that $\overline{U}$ is a subrepresentation is closed. Now (ii) follows directly from Lemma 3.29. □

We claim that $\widetilde{\mathcal{F}}$ and $\widetilde{\mathcal{G}}$ restrict to isomorphisms $\mathcal{F}\colon \mathcal{F}_{\mathbf{i},\mathbf{a},X} \to \mathcal{G}_{\mathbf{i},\mathbf{a},X}$ and $\mathcal{G}\colon \mathcal{G}_{\mathbf{i},\mathbf{a},X} \to \mathcal{F}_{\mathbf{i},\mathbf{a},X}$. Thus, we have to show the following:

(a) If $X_\bullet \in \mathcal{F}_{\mathbf{i},\mathbf{a},X}$, then $\widetilde{\mathcal{F}}(X_\bullet) \in \mathcal{G}_{\mathbf{i},\mathbf{a},X}$.
(b) If $\overline{U} \in \mathcal{G}_{\mathbf{i},\mathbf{a},X}$, then $\widetilde{\mathcal{G}}(\overline{U}) \in \mathcal{F}_{\mathbf{i},\mathbf{a},X}$.

Note that (a) and (b) imply Theorem 3.27.

3.9.4. *Proof of (a).* Let $X_\bullet = (0 = X_r \subseteq \cdots \subseteq X_1 \subseteq X_0 = X)$ be in $\mathcal{F}_{\mathbf{i},\mathbf{a},X}$. Define $\overline{U} := (\overline{U}_k)_{1\le k\le r}$, where $\overline{U}_k := e_{i_k}(X_k/X_k^-)$. Thus $\widetilde{\mathcal{F}}(X_\bullet) = \overline{U}$. We have to show that $\overline{U}$ is a subrepresentation of Y.

For each horizontal arrow $\gamma_k\colon k \to k^-$ of $\Gamma_{\mathbf{i}}$ we have $Y(\gamma_k)(\overline{U}_k) \subseteq \overline{U}_{k^-}$. This holds, since $X_k \subseteq X_{k^-}$ and therefore $e_{i_k}X_k \subseteq e_{i_k}X_{k^-}$. Next, let $\gamma_a^{k,s}\colon s \to k$ be an ordinary arrow of $\Gamma_{\mathbf{i}}$. It follows that $k > s$. We have

$$X_k \subseteq X_{k-1} \subseteq \cdots \subseteq X_{s+1} \subseteq X_s.$$

By definition of $\mathcal{F}_{\mathbf{i},\mathbf{a},X}$ we have $X_{t-1}/X_t \cong S_{i_t}^{a_t}$ for all $1 \le t \le r$. By the definition of $\Gamma_{\mathbf{i}}$ we know that $i_t \ne i_s$ for all $k \ge t \ge s+1$. This implies that $e_{i_s}X_s = e_{i_s}X_k$. It follows that

$$aX_s = ae_{i_s}X_s = ae_{i_s}X_k \subseteq e_{i_k}X_k.$$

(To get the inclusion $ae_{i_s}X_k \subseteq e_{i_k}X_k$ we used our assumption that X_k is an A-module.) This implies that $Y(\gamma_a^{k,s})(\overline{U}_s) \subseteq \overline{U}_k$. Thus we proved that $\overline{U} \in \mathcal{G}_{\mathbf{i},\mathbf{a},X}$.

3.9.5. *Proof of (b).* Let $\overline{U} = (\overline{U}_k)_{1 \le k \le r}$ be a subrepresentation of the $\mathcal{E}_{\mathbf{i}}$-module Y. Thus we have $\overline{U}_k = U_k/e_{i_k}X_k^-$. Let $X_\bullet := \widetilde{\mathcal{G}}(\overline{U})$. Recall that

$$X_\bullet = (0 = X_r \subseteq \cdots \subseteq X_1 \subseteq X_0 = \mathrm{gr}(X))$$

is defined as follows: For $1 \le k \le r$ and $j \in \Gamma_0$ we have $X_k(j) = U_p$, where $p = \min\{k \le s \le r, r+1 \mid i_s = j\}$. (We set $U_{r+1} := 0$.) Clearly, for all $0 \le s \le r-1$ we have $X_{s+1} \subseteq X_s$, since by Lemma 3.29 we know that $U_{s^+} \subseteq U_s$. It remains to show that X_s is a subrepresentation of X for all $1 \le s \le r$.

By induction, we can assume that $X_r, \dots, X_{s+1}$ are subrepresentations of X. So we only have to investigate how A acts on the subspace U_s of $e_{i_s}X_s$. Obviously, $e_jX_t = X_t(j)$ for all $1 \le t \le r$ and $j \in \Gamma_0$. Next, assume that $\gamma_a^{k,s}\colon s \to k$ is an ordinary arrow of $\Gamma_{\mathbf{i}}$. We know that $Y(\gamma_a^{k,s})$ acts on $\overline{U}_s$ as follows: For all $u_s \in U_s$ we have

$$Y(\gamma_a^{k,s})(u_s + e_{i_s}X_s^-) = (au_s) + e_{i_k}X_k^-.$$

Since by our assumption, $\overline{U}$ is a subrepresentation of Y, we get that $(au_s) + e_{i_k}X_k^-$ is contained in $\overline{U}_k = U_k/e_{i_k}X_k^-$ for all $u_s \in U_s$. Thus $au_s \in U_k$ for all $u_s \in U_s$. Now it follows from the definition of X_s and Lemma 3.29 that $U_k \subseteq e_{i_k}X_s = X_s(i_k)$. Thus X_s is a subrepresentation of X. It follows that $X_\bullet \in \mathcal{F}_{\mathbf{i},\mathbf{a},X}$. This finishes the proof of Theorem 3.27.

3.10. **Examples.**

3.10.1. Let Γ be the quiver with just one vertex 1 and arrows a and b. Set $A := \mathbb{C}\Gamma/J$, where J is generated by $\{ab, ba\}$. For $\mathbf{i} = (i_4, \dots, i_1) = (1,1,1,1)$ the modules $V_k = V_{\mathbf{i},k}$ look as follows:

$$V_1 = 1, \qquad V_2 = \begin{smallmatrix} 1 & & 1 \\ & 1 & \end{smallmatrix}, \qquad V_3 = \begin{smallmatrix} 1 & & & & 1 \\ & 1 & & 1 & \\ & & 1 & & \end{smallmatrix}, \qquad V_4 = \begin{smallmatrix} 1 & & & & & & 1 \\ & 1 & & & & 1 & \\ & & 1 & & 1 & & \\ & & & 1 & & & \end{smallmatrix}.$$

Obviously, V_k is $(i_k, \dots, i_1)$-balanced. The quiver $\Gamma_{\mathbf{i}}$ of $\mathcal{E}_{\mathbf{i}}$ looks as follows:

$$\begin{tikzcd}
4 \arrow[r, "\gamma_4"] & 3 \arrow[l, bend right, "\gamma_a^{4,3}"'] \arrow[l, bend left, "\gamma_b^{4,3}"] \arrow[r, "\gamma_3"] & 2 \arrow[l, bend right, "\gamma_a^{3,2}"'] \arrow[l, bend left, "\gamma_b^{3,2}"] \arrow[r, "\gamma_2"] & 1 \arrow[l, bend right, "\gamma_a^{2,1}"'] \arrow[l, bend left, "\gamma_b^{2,1}"]
\end{tikzcd}.$$

Let X be the A-module

$$\begin{tikzcd}
& b_2 \arrow[dl, "a"'] \arrow[dr, "b"] & & \\
b_1 & & b_3 \arrow[dr, "b"] & \\
& & & b_4.
\end{tikzcd}$$

(Here $\{b_1, \dots, b_4\}$ is a basis of X, and the arrows show how the generators a and b of A act on this basis.) As a representation of Γ, we have $X = (\mathbb{C}^4, X(a), X(b))$, where

$$X(a) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad X(b) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

In the following we just write $\langle \cdots \rangle$ instead of $\mathrm{Span}_{\mathbb{C}}\langle \cdots \rangle$. The chains $X_\bullet^+$ and $X_\bullet^-$ look as follows:

$$\begin{aligned} X_\bullet^+ &= (0 \subseteq \langle b_1, b_4 \rangle \subseteq \langle b_1, b_3, b_4 \rangle \subseteq \langle b_1, b_2, b_3, b_4 \rangle \subseteq \langle b_1, b_2, b_3, b_4 \rangle), \\ X_\bullet^- &= (0 \subseteq 0 \subseteq \langle b_4 \rangle \subseteq \langle b_1, b_3, b_4 \rangle \subseteq \langle b_1, b_2, b_3, b_4 \rangle). \end{aligned}$$

As a representation of $\Gamma_{\mathbf{i}}$, the $\mathcal{E}_{\mathbf{i}}$-module $Y = D\overline{\mathrm{Hom}}_A(X, V_{\mathbf{i}})$ looks as follows:

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{C}^2 \arrow[l, bend right=50] \arrow[l, bend left=50] \arrow[r, "{\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)}"] & \mathbb{C}^2 \arrow[l, bend right=50, "{\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)}"'] \arrow[l, bend left=50, "{\left(\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right)}"] \arrow[r, "{\left(\begin{smallmatrix}0&0\end{smallmatrix}\right)}"] & \mathbb{C} \arrow[l, bend right=50, "{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)}"'] \arrow[l, bend left=50, "{\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)}"] .
\end{tikzcd}$$

More precisely, the four vector spaces in the above quiver representation are (from left to right)

$$\begin{aligned} 0 &\equiv e_1(X_4^+/X_4^-), \\ \mathbb{C}^2 &\equiv e_1(X_3^+/X_3^-) = \langle b_1, b_4 \rangle \text{ with basis } (b_1, b_4), \\ \mathbb{C}^2 &\equiv e_1(X_2^+/X_2^-) = \langle b_1, b_3, b_4 \rangle / \langle b_4 \rangle \text{ with basis } (\overline{b_1}, \overline{b_3}), \\ \mathbb{C} &\equiv e_1(X_1^+/X_1^-) = \langle b_1, b_2, b_3, b_4 \rangle / \langle b_1, b_3, b_4 \rangle \text{ with basis } (\overline{b_2}). \end{aligned}$$

(Here $\overline{b_i}$ denotes the corresponding residue class of b_i.) One easily checks that the elements in $\mathcal{F}_{\mathbf{i},(1,1,1,1),X}$ are

$$\begin{aligned} f_\lambda &:= (0 \subset \langle b_4 \rangle \subset \langle b_1 + \lambda b_3, b_4 \rangle \subset \langle b_1, b_3, b_4 \rangle \subset X), \\ f_\infty &:= (0 \subset \langle b_4 \rangle \subset \langle b_3, b_4 \rangle \subset \langle b_1, b_3, b_4 \rangle \subset X), \\ g_\lambda &:= (0 \subset \langle b_1 + \lambda b_4 \rangle \subset \langle b_1, b_4 \rangle \subset \langle b_1, b_3, b_4 \rangle \subset X), \end{aligned}$$

where $\lambda \in \mathbb{C}$. It follows that the Euler characteristic of $\mathcal{F}_{\mathbf{i},(1,1,1,1),X}$ is 3. In this example, the isomorphism $\mathcal{F}_{\mathbf{i},(1,1,1,1),X} \to \mathrm{Gr}^{\mathcal{E}_{\mathbf{i}}}_{(0,1,1,0)}(Y)$ from Theorem 3.27 looks as follows:

$$\begin{aligned} f_\lambda &\mapsto (0, \langle b_4 \rangle, \langle \overline{b_1} + \lambda \overline{b_3} \rangle, 0), \\ f_\infty &\mapsto (0, \langle b_4 \rangle, \langle \overline{b_3} \rangle, 0), \\ g_\lambda &\mapsto (0, \langle b_1 + \lambda b_4 \rangle, \langle \overline{b_1} \rangle, 0). \end{aligned}$$

3.10.2. *Springer fibres.* Let Γ be the quiver with just one vertex 1 and one arrow a. Set $A := \mathbb{C}\Gamma/J$, where J is generated by a^m for some $m \geq 2$. Let $\mathbf{i} = (i_m, \ldots, i_2, i_1) = (1, \ldots, 1, 1)$. For $1 \leq k \leq m$ the module $V_k = V_{\mathbf{i},k}$ is uniserial of length k, and V_k is $(i_k, \ldots, i_1)$-balanced. We have $\mathrm{add}(V_{\mathbf{i}}) = \mathrm{nil}(A) = \mathrm{mod}(A)$. The quiver $\Gamma_{\mathbf{i}}$ of $\mathcal{E}_{\mathbf{i}}$ looks as follows:

$$\begin{tikzcd}
m \arrow[r, bend left, "\gamma_m"] & \cdots \arrow[l, bend left, "\gamma_a^{m,m-1}"] \arrow[r, bend left, "\gamma_4"] & 3 \arrow[l, bend left, "\gamma_a^{4,3}"] \arrow[r, bend left, "\gamma_3"] & 2 \arrow[l, bend left, "\gamma_a^{3,2}"] \arrow[r, bend left, "\gamma_2"] & 1 \arrow[l, bend left, "\gamma_a^{2,1}"] .
\end{tikzcd}$$

Let $\lambda = (\lambda_t, \ldots, \lambda_1)$ be a partition of m; i.e., the λ_j are integers such that $\lambda_t \geq \cdots \geq \lambda_1 \geq 1$ and $\lambda_t + \cdots + \lambda_1 = m$. Define $V_\lambda := V_{\lambda_t} \oplus \cdots \oplus V_{\lambda_1}$. This yields a bijection between the set of partitions of m and the set of isomorphism classes of

m-dimensional A-modules. For $\mathbf{a} := (a_m, \dots, a_2, a_1) := (1, \dots, 1, 1)$ the varieties $\mathcal{F}_\lambda := \mathcal{F}_{\mathbf{i},\mathbf{a},V_\lambda}$ are just the classical Springer fibres of Dynkin type A_{m-1}.

For example, let $m = 7$ and $\lambda = (3, 2, 2)$. Then $V_\lambda = V_3 \oplus V_2 \oplus V_2$. Set $Y := D\overline{\mathrm{Hom}}_A(V_\lambda, V_{\mathbf{i}})$. By Theorem 3.27 we get $\mathcal{F}_\lambda \cong \mathrm{Gr}_{\mathbf{f}}^{\mathcal{E}_{\mathbf{i}}}(Y)$, where $\mathbf{f} = (f_1, \dots, f_7) = (2, 4, 4, 3, 2, 1, 0)$ and $\underline{\dim}_{\mathcal{E}_{\mathbf{i}}}(Y) = (h_1, \dots, h_7) = (3, 6, 7, 7, 6, 3, 0)$. It is an easy exercise to write Y explicitly as a representation of $\Gamma_{\mathbf{i}}$.

3.10.3. *Balanced modules over preprojective algebras.* Let Λ be the preprojective algebra associated to a finite connected acyclic quiver Q. Recall that $\Lambda = \mathbb{C}\overline{Q}/(c)$, where $\overline{Q}$ is the *double quiver* obtained by adding to each arrow $a\colon i \to j$ in Q an arrow $a^*\colon j \to i$ pointing in the opposite direction, and (c) is the ideal generated by the element

$$c = \sum_{a \in Q_1} (a^*a - aa^*).$$

Let $\mathbf{i} = (i_r, \dots, i_1)$ be a reduced expression for some element w of the Weyl group W of Q. In this situation, the module $V_{\mathbf{i}}$ defined in Section 3.4 coincides with the cluster-tilting module $V_{\mathbf{i}}$ of $\mathcal{C}_w$ mentioned in Section 1.2 (see [GLS5]). The following result is then a direct consequence of [GLS5, Proposition 9.6].

Theorem 3.31. $(\Lambda, \mathbf{i})$ *is balanced.*

Let $(-,-)$ denote the usual W-invariant bilinear form on $\mathfrak{h}^*$, the dual of the Cartan subalgebra of the symmetric Kac-Moody Lie algebra $\mathfrak{g}$ associated to Q. For $i \in Q_0$ let α_i and ϖ_i be the corresponding simple root and fundamental weight, respectively. By [GLS5, Proposition 9.6], for $V_k = V_{\mathbf{i},k}$ we have

$$\mathbf{a}_l^-(V_k) = \begin{cases} -(s_{i_l} s_{i_{l+1}} \cdots s_{i_k}(\varpi_{i_k}), \alpha_{i_l}) & \text{if } 1 \le l \le k, \\ 0 & \text{otherwise.} \end{cases} \tag{3.1}$$

4. Categorification of the Chamber Ansatz

In this section we prove Theorems 1, 2 and 3. The proofs of Theorems 1 and 2 follow rather easily from Theorem 3.27. As a main ingredient for the proof of Theorem 3 we describe the twisted φ-functions $\varphi'_{V_{\mathbf{i},k}}$ in Proposition 4.4. These can be seen as module-theoretic versions of the twisted generalized minors introduced by Berenstein and Zelevinsky [BZ] (in the Dynkin case).

4.1. **Proof of Theorems 1 and 2.** We know from Theorem 3.31 that $(\Lambda, \mathbf{i})$ is balanced. Let $X \in \mathcal{C}_w$, and let

$$\mathcal{E} := \mathcal{E}_{\mathbf{i}} := \mathrm{End}_\Lambda(V_{\mathbf{i}})^{\mathrm{op}} \quad \text{and} \quad \underline{\mathcal{E}} := \underline{\mathcal{E}}_{\mathbf{i}} := \underline{\mathrm{End}}_{\mathcal{C}_w}(V_{\mathbf{i}})^{\mathrm{op}}.$$

As before, let $W_{\mathbf{i}} := I_w \oplus \Omega_w(V_{\mathbf{i}})$.

Recall that a cluster-tilting module $T \in \mathcal{C}_w$ is called $V_{\mathbf{i}}$*-reachable* if one can obtain T via a finite sequence of mutations starting with the initial cluster-tilting module $V_{\mathbf{i}}$. By [GLS5, Proposition 13.4], the module $W_{\mathbf{i}}$ is $V_{\mathbf{i}}$-reachable.

Since $\underline{\mathcal{C}}_w$ is a triangulated category with shift functor Ω_w^{-1}, we get an $\underline{\mathcal{E}}$-module isomorphism

$$D\overline{\mathrm{Hom}}_\Lambda(X, V_{\mathbf{i}}) \cong D\mathrm{Ext}^1_\Lambda(X, \Omega_w(V_{\mathbf{i}})).$$

The module I_w is $\mathcal{C}_w$-projective-injective; thus $\mathrm{Ext}^1_\Lambda(X, \Omega_w(V_{\mathbf{i}})) \cong \mathrm{Ext}^1_\Lambda(X, W_{\mathbf{i}})$. The category $\underline{\mathcal{C}}_w$ is a 2-Calabi-Yau category; see [BIRS, Proposition III.2.3] and also

[GLS4] for a special case. Thus there is an $\underline{\mathcal{E}}$-module isomorphism $D\mathrm{Ext}^1_\Lambda(X, W_{\mathbf{i}}) \cong \mathrm{Ext}^1_\Lambda(W_{\mathbf{i}}, X)$. Combining these isomorphisms, we have an $\underline{\mathcal{E}}$-module isomorphism

$$D\overline{\mathrm{Hom}}_\Lambda(X, V_{\mathbf{i}}) \cong \mathrm{Ext}^1_\Lambda(W_{\mathbf{i}}, X).$$

We can regard $D\overline{\mathrm{Hom}}_\Lambda(X, V_{\mathbf{i}})$ and $\mathrm{Ext}^1_\Lambda(W_{\mathbf{i}}, X)$ as modules over $\mathcal{E}$ and $\mathrm{End}_\Lambda(W_{\mathbf{i}})^{\mathrm{op}}$, which are annihilated by the ideals $\mathcal{I}_\Lambda(V_{\mathbf{i}}, V_{\mathbf{i}})$ and $\mathcal{I}_\Lambda(W_{\mathbf{i}}, W_{\mathbf{i}})$, respectively. Since $\Omega_w \colon \underline{\mathcal{C}}_w \to \underline{\mathcal{C}}_w$ is an equivalence, we have an isomorphism of stable endomorphism algebras

$$\underline{\mathcal{E}} \cong \underline{\mathrm{End}}_{\mathcal{C}_w}(W_{\mathbf{i}})^{\mathrm{op}}.$$

Recall that $W_{\mathbf{i}}$ is a cluster-tilting module in $\mathcal{C}_w$. In particular, we have $\mathrm{Ext}^1_\Lambda(W_{\mathbf{i}}, X) = 0$ for some $X \in \mathcal{C}_w$ if and only if $X \in \mathrm{add}(W_{\mathbf{i}})$.

By Theorem 3.27, the varieties $\mathcal{F}_{\mathbf{i},\mathbf{a},X}$ and $\mathrm{Gr}^{\mathcal{E}}_{d_{\mathbf{i},X}(\mathbf{a})}(Y)$ are isomorphic for all $\mathbf{a} \in \mathbb{N}^r$, where $Y := D\overline{\mathrm{Hom}}_\Lambda(X, V_{\mathbf{i}})$. Furthermore, the map $\mathbf{a} \mapsto d_{\mathbf{i},X}(\mathbf{a})$ yields a bijection

$$\{\mathbf{a} \in \mathbb{N}^r \mid \mathcal{F}_{\mathbf{i},\mathbf{a},X} \neq \varnothing\} \to \mathcal{U} := \left\{\mathbf{f} \in \mathbb{N}^r \mid \mathrm{Gr}^{\mathcal{E}}_{\mathbf{f}}(Y) \neq \varnothing\right\}.$$

(If $\mathrm{Gr}^{\mathcal{E}}_{\mathbf{f}}(Y) \neq \varnothing$, then $f_k = 0$ for all $k \in R_{\max}$, where $R_{\max}$ is defined as in Section 1.4. Thus we can identify $\mathrm{Gr}^{\mathcal{E}}_{\mathbf{f}}(Y)$ and $\mathrm{Gr}^{\mathcal{E}}_{\mathbf{d}}(Y)$, where $\mathbf{d} := (f_k)_{k \in R_-}$. Being a bit sloppy, we often just write $\mathrm{Gr}^{\mathcal{E}}_{\mathbf{f}}(Y)$ instead of $\mathrm{Gr}^{\mathcal{E}}_{\mathbf{d}}(Y)$.) Clearly, $\mathcal{U}$ contains always the elements $\mathbf{f} = \underline{\dim}(Y)$ and the 0-dimension vector $\mathbf{f} = (0, \dots, 0)$. (In both cases, $\mathrm{Gr}^{\mathcal{E}}_{\mathbf{f}}(Y)$ is a single point.) Thus there is a unique $\mathbf{a} \in \mathbb{N}^r$ with $\mathcal{F}_{\mathbf{i},\mathbf{a},X} \neq \varnothing$ if and only if $Y = 0$. This finishes the proof of Theorems 1 and 2.

4.2. **Example.** Let Q be a quiver with underlying graph $1 \text{ --- } 2 \text{ --- } 3 \text{ --- } 4$ and let $\mathbf{i} := (i_{10}, \dots, i_1) := (1, 3, 2, 4, 1, 3, 2, 4, 1, 3)$, which is a reduced expression of the longest element in the Weyl group W_Q. It follows that $\mathcal{C}_w = \mathrm{nil}(\Lambda) = \mathrm{mod}(\Lambda)$. Let $V_{\mathbf{i}} = V_1 \oplus \cdots \oplus V_{10}$ and $W_{\mathbf{i}} = I_w \oplus \Omega_w(V_{\mathbf{i}}) = W_1 \oplus \cdots \oplus W_{10}$. We first display the indecomposable direct summands which are not $\mathcal{C}_w$-projective-injective:

$$V_1 = 3, \quad V_2 = 1, \quad V_3 = \begin{smallmatrix} 3 & \\ & 4 \end{smallmatrix}, \quad V_4 = \begin{smallmatrix} 1 & & 3 \\ & 2 & \end{smallmatrix}, \quad V_5 = \begin{smallmatrix} 1 & & 3 & \\ & 2 & & 4 \\ & & 3 & \end{smallmatrix}, \quad V_6 = \begin{smallmatrix} & & 3 \\ & 2 & \\ 1 & & \end{smallmatrix},$$

$$W_1 = \begin{smallmatrix} & 2 & & 4 \\ 1 & & 3 & \\ & 2 & & \end{smallmatrix}, \quad W_2 = \begin{smallmatrix} 2 & & \\ & 3 & \\ & & 4 \end{smallmatrix}, \quad W_3 = \begin{smallmatrix} & 2 & \\ 1 & & 3 \\ & 2 & \end{smallmatrix}, \quad W_4 = \begin{smallmatrix} & 2 & & 4 \\ 1 & & 33 & \\ & 2 & & 4 \end{smallmatrix}, \quad W_5 = \begin{smallmatrix} & 2 & & \\ 1 & & 3 & \\ & 2 & & 4 \end{smallmatrix}, \quad W_6 = \begin{smallmatrix} & & 4 \\ & 3 & \\ 2 & & \end{smallmatrix}.$$

Finally, the indecomposable $\mathcal{C}_w$-projective-injectives look as follows:

$$V_7 = W_7 = \begin{smallmatrix} 1 & & & \\ & 2 & & \\ & & 3 & \\ & & & 4 \end{smallmatrix}, \quad V_8 = W_8 = \begin{smallmatrix} & & 3 & \\ & 2 & & 4 \\ 1 & & 3 & \\ & 2 & & \end{smallmatrix}, \quad V_9 = W_9 = \begin{smallmatrix} & 2 & & \\ 1 & & 3 & \\ & 2 & & 4 \\ & & 3 & \end{smallmatrix}, \quad V_{10} = W_{10} = \begin{smallmatrix} & & & 4 \\ & & 3 & \\ & 2 & & \\ 1 & & & \end{smallmatrix}.$$

Using our language of φ-functions, the functions Z_a and T_J appearing in [BFZ, Example 3.2.2] can be written as follows:

$$Z_1 = \varphi_{V_7}, \quad Z_2 = \varphi_{V_9}, \quad Z_3 = \varphi_{V_8}, \quad Z_4 = \varphi_{V_{10}},$$
$$T_2 = \varphi_{W_2}, \quad T_4 = \varphi_{W_6}, \quad T_{124} = \varphi_{W_1}, \quad T_{1245} = \varphi_{W_3}, \quad T_{245} = \varphi_{W_5}, \quad T_{24} = \varphi_{W_4},$$

and the equality $T_{24} = \Delta^{1345}(x)\Delta^{25}(x) - \Delta^{2345}(x)\Delta^{15}(x)$ translates to $\varphi_{W_4} = \varphi_{W_2}\varphi_X - \varphi_{W_7}\varphi_Y$, where

$$X := \begin{smallmatrix} & & & 4 \\ 1 & & 3 & \\ & 2 & & \end{smallmatrix} \quad \text{and} \quad Y := \begin{smallmatrix} & & 4 \\ & 3 & \\ 2 & & \end{smallmatrix}.$$

4.3. **Proof of Theorem 3.** As before, let $\mathbf{i} = (i_r, \dots, i_2, i_1)$ be a reduced expression of w. Define $\mathbf{j} := (i_r, \dots, i_2)$. (This is a reduced expression of $v := ws_{i_1}$.) By Theorem 3.31 we have

$$\mathrm{rad}_{S_{i_1}}(V_{\mathbf{i},k}) \cong V_{\mathbf{j},k-1}.$$

This yields the following result:

Lemma 4.1. *If P is $\mathcal{C}_w$-projective-injective, then $\mathrm{rad}_{S_{i_1}}(P)$ is $\mathcal{C}_v$-projective-injective.*

Corollary 4.2. *If $X \in \mathcal{C}_w$, then $\mathrm{rad}_{S_{i_1}}(X) \in \mathcal{C}_v$.*

Proof. There is an epimorphism $P \to X$, where P is $\mathcal{C}_w$-projective-injective. This yields an epimorphism $\mathrm{rad}_{S_{i_1}}(P) \to \mathrm{rad}_{S_{i_1}}(X)$; see Lemma 3.2(ii). Now apply Lemma 4.1. $\square$

For $1 \le k \le r$ with $k^+ \neq r+1$ we have a short exact sequence

$$\eta\colon 0 \to W_{\mathbf{i},k} \to P(V_{\mathbf{i},k}) \to V_{\mathbf{i},k} \to 0$$

in $\mathcal{C}_w$. The module S_{i_1} is a direct summand of the rigid module $V_{\mathbf{i}}$. Thus applying $\mathrm{Hom}_\Lambda(-, S_{i_1})$ to η yields

$$\mathrm{top}_{S_{i_1}}(W_{\mathbf{i},k}) \oplus \mathrm{top}_{S_{i_1}}(V_{\mathbf{i},k}) \cong \mathrm{top}_{S_{i_1}}(P(V_{\mathbf{i},k})).$$

Thus by restriction we obtain a short exact sequence

$$0 \to \mathrm{rad}_{S_{i_1}}(W_{\mathbf{i},k}) \to \mathrm{rad}_{S_{i_1}}(P(V_{\mathbf{i},k})) \to \mathrm{rad}_{S_{i_1}}(V_{\mathbf{i},k}) \to 0.$$

By Lemma 4.1, the module $\mathrm{rad}_{S_{i_1}}(P(V_{\mathbf{i},k}))$ is $\mathcal{C}_v$-projective-injective, and by Corollary 4.2 we know that $\mathrm{rad}_{S_{i_1}}(W_{\mathbf{i},k}) \in \mathcal{C}_v$. Since

$$\mathrm{rad}_{S_{i_1}}(V_{\mathbf{i},k}) \cong V_{\mathbf{j},k-1},$$

we get $\mathrm{rad}_{S_{i_1}}(W_{\mathbf{i},k}) \cong P \oplus W_{\mathbf{j},k-1}$ for some $\mathcal{C}_v$-projective-injective module P. (Here we just use the basic properties of the syzygy functor Ω_v; see [H].) Thus we have proved the following:

Lemma 4.3. *If we apply $\mathrm{rad}_{S_{i_1}}(-)$ to η, we get a short exact sequence*

$$0 \to P \oplus W_{\mathbf{j},k-1} \to P \oplus P(V_{\mathbf{j},k-1}) \to V_{\mathbf{j},k-1} \to 0,$$

where P is $\mathcal{C}_v$-projective-injective.

For $1 \le l \le k \le r$ define

$$b_{\mathbf{i}}(l,k) := -(s_{i_l} s_{i_{l+1}} \cdots s_{i_k}(\varpi_{i_k}), \alpha_{i_l}).$$

(For $l > k$ we define $b(l,k) := 0$.) Note that if $l > 1$, then $b_{\mathbf{i}}(l,k) = b_{\mathbf{j}}(l-1,k-1)$.

Proposition 4.4. *For $1 \le k \le r$ we have*

$$\varphi'_{V_{\mathbf{i},k}}(\underline{x}_{\mathbf{i}}(\mathbf{t})) = \prod_{l=1}^{k} t_l^{-b_{\mathbf{i}}(l,k)} = \mathbf{t}^{-\mathbf{a}^-(V_k)}.$$

Proof. Let $1 \le k \le r$. If $k^+ = r+1$, then the statement follows directly from [GLS5, Proposition 9.6]. Thus assume $k^+ \le r$. By induction we get

$$\varphi'_{V_{\mathbf{j},k-1}}(\underline{x}_{\mathbf{j}}(t_r, \dots, t_2)) = \prod_{l=2}^{k} t_l^{-b_{\mathbf{j}}(l-1,k-1)} = \prod_{l=2}^{k} t_l^{-b_{\mathbf{i}}(l,k)}.$$

Now Lemma 4.3 together with Theorem 2 and [GLS5, Proposition 9.6] yield the result. □

The following statement is a direct consequence by Proposition 4.4.

Corollary 4.5. *For $1 \leq k \leq r$ we have*

$$\mathbf{a}^+(W_{\mathbf{i},k}) - \mathbf{a}^+(P(V_{\mathbf{i},k})) = -\mathbf{a}^-(V_{\mathbf{i},k}) = -(0,\dots,0,b_{\mathbf{i}}(k,k),\dots,b_{\mathbf{i}}(2,k),b_{\mathbf{i}}(1,k)).$$

Now we can finish the proof of Theorem 3. For $1 \leq k \leq r$ we have to show that $t_k = C_{\mathbf{i},k}(\underline{x}_{\mathbf{i}}(\mathbf{t}))$, where

$$C_{\mathbf{i},k} := \frac{1}{\varphi'_{V_{\mathbf{i},k}}\varphi'_{V_{\mathbf{i},k^-(i_k)}}} \cdot \prod_{j=1}^{n} \left(\varphi'_{V_{\mathbf{i},k^-(j)}}\right)^{q(i_k,j)}$$

and $k^-(j) := \max\{0, 1 \leq s \leq k-1 \mid i_s = j\}$. We know from Proposition 4.4 that

$$\varphi'_{V_{\mathbf{i},k}}(\underline{x}_{\mathbf{i}}(\mathbf{t})) = \prod_{l=1}^{k} t_l^{-b_{\mathbf{i}}(l,k)}. \tag{4.1}$$

We insert (1.1) in the right-hand side of equation (4.1) and obtain

$$\prod_{l=1}^{k} C_{\mathbf{i},l}(\underline{x}_{\mathbf{i}}(\mathbf{t}))^{-b_{\mathbf{i}}(l,k)}.$$

To prove Theorem 3, we need to show that for all $1 \leq k \leq r$ we have

$$\varphi'_{V_{\mathbf{i},k}} = \prod_{l=1}^{k} C_{\mathbf{i},l}^{-b_{\mathbf{i}}(l,k)}. \tag{4.2}$$

This is done in exactly the same way as in [BZ, Section 4]. Namely, one first shows that the exponent of $\varphi'_{V_{\mathbf{i},k}}$ on the right-hand side of equation (4.2) is equal to $b(k,k) = 1$. Then one shows that for $1 \leq s < k$ the exponent of $\varphi'_{V_{\mathbf{i},s}}$ on the right-hand side of (4.2) is equal to

$$\zeta(s) := b(s,k) + b(s^+,k) - \sum_{s^+ > m > s} q(i_m, i_s) b(m,k).$$

A straightforward calculation shows that $\zeta(s) = 0$ for all $1 \leq s < k$. Finally, by [GLS5, Corollary 15.7] we know that a function $\varphi_X \in \mathbb{C}[N]$ with $X \in \mathcal{C}_w$ is already uniquely determined by its values on $\mathrm{Im}(\underline{x}_{\mathbf{i}})$. This finishes the proof of Theorem 3.

Remark 4.6. Recall that the module $W_{\mathbf{i}}$ is $V_{\mathbf{i}}$-reachable. This shows that $(\varphi_{W_{\mathbf{i},1}},\dots,\varphi_{W_{\mathbf{i},r}})$ is a cluster of the cluster structure on $\mathbb{C}[N^w]$ defined by the initial seed $((\varphi_{V_{\mathbf{i},1}},\dots,\varphi_{V_{\mathbf{i},r}}),\Gamma_{\mathbf{i}})$. By Theorem 3, the cluster $(\varphi_{W_{\mathbf{i},1}},\dots,\varphi_{W_{\mathbf{i},r}})$ gives a total positivity criterion for N^w in the sense of [BFZ, BZ]. Therefore, every $V_{\mathbf{i}}$-reachable cluster-tilting module of $\mathcal{C}_w$ also provides a total positivity criterion.

4.4. **Example.** Let Q be a quiver with underlying graph $1 \,\text{——}\, 2 \,\text{——}\, 3$ and let $\mathbf{i} := (i_6, \dots, i_1) := (1,2,1,3,2,1)$, which is a reduced expression of the longest element in the Weyl group W_Q. It follows that $\mathcal{C}_w = \mathrm{nil}(\Lambda) = \mathrm{mod}(\Lambda)$. The modules $V_{\mathbf{i}} = V_1 \oplus \cdots \oplus V_6$ and $W_{\mathbf{i}} = W_1 \oplus \cdots \oplus W_6$ look as follows:

$$V_1 = {\scriptstyle 1}, \quad V_2 = \begin{smallmatrix}1&\\&2\end{smallmatrix}, \quad V_3 = \begin{smallmatrix}1&&\\&2&\\&&3\end{smallmatrix}, \quad V_4 = \begin{smallmatrix}&2\\1&\end{smallmatrix}, \quad V_5 = \begin{smallmatrix}&2&\\1&&3\\&2&\end{smallmatrix}, \quad V_6 = \begin{smallmatrix}&&3\\&2&\\1&&\end{smallmatrix},$$

$$W_1 = \begin{smallmatrix}2&\\&3\end{smallmatrix}, \quad W_2 = {\scriptstyle 3}, \quad W_3 = V_3, \quad W_4 = \begin{smallmatrix}&3\\2&\end{smallmatrix}, \quad W_5 = V_5, \quad W_6 = V_6.$$

Besides the modules V_k and W_k there are only three other indecomposable Λ-modules:

$$L_1 = \begin{smallmatrix}1&&3\\&2&\end{smallmatrix}, \qquad L_2 = {\scriptstyle 2}, \qquad L_4 = \begin{smallmatrix}&2&\\1&&3\end{smallmatrix}.$$

(The reason for naming the third module L_4 and not L_3 will become clear in Section 8.6.) Here we used the same conventions for displaying Λ-modules as explained in [GLS5]. The φ-functions of the indecomposable Λ-modules are the following:

$$\begin{aligned}
\varphi_{V_1}(\underline{x}_{\mathbf{i}}(\mathbf{t})) &= t_6 + t_4 + t_1, & \varphi_{W_1}(\underline{x}_{\mathbf{i}}(\mathbf{t})) &= t_3 t_2,\\
\varphi_{V_2}(\underline{x}_{\mathbf{i}}(\mathbf{t})) &= t_5 t_4 + t_5 t_1 + t_2 t_1, & \varphi_{W_2}(\underline{x}_{\mathbf{i}}(\mathbf{t})) &= t_3,\\
\varphi_{V_3}(\underline{x}_{\mathbf{i}}(\mathbf{t})) &= t_3 t_2 t_1, & \varphi_{W_4}(\underline{x}_{\mathbf{i}}(\mathbf{t})) &= t_5 t_3,\\
\varphi_{V_4}(\underline{x}_{\mathbf{i}}(\mathbf{t})) &= t_6 t_5 + t_6 t_2 + t_4 t_2, & \varphi_{L_1}(\underline{x}_{\mathbf{i}}(\mathbf{t})) &= t_5 t_4 t_3 + t_5 t_3 t_1,\\
\varphi_{V_5}(\underline{x}_{\mathbf{i}}(\mathbf{t})) &= t_5 t_4 t_3 t_2, & \varphi_{L_2}(\underline{x}_{\mathbf{i}}(\mathbf{t})) &= t_5 + t_2,\\
\varphi_{V_6}(\underline{x}_{\mathbf{i}}(\mathbf{t})) &= t_6 t_5 t_3, & \varphi_{L_4}(\underline{x}_{\mathbf{i}}(\mathbf{t})) &= t_6 t_3 t_2 + t_4 t_3 t_2.
\end{aligned}$$

The modules $P(V_k)$ are the following:

$$P(V_1)=V_3, \quad P(V_2)=V_3, \quad P(V_3)=V_3, \quad P(V_4)=V_5, \quad P(V_5)=V_5, \quad P(V_6)=V_6.$$

Thus, we obtain the twisted minors $\varphi'_{V_k} = \varphi_{\Omega_w(V_k)} \varphi^{-1}_{P(V_k)}$:

$$\varphi'_{V_1}(\underline{x}_{\mathbf{i}}(\mathbf{t})) = \frac{t_3 t_2}{t_3 t_2 t_1}, \qquad \varphi'_{V_2}(\underline{x}_{\mathbf{i}}(\mathbf{t})) = \frac{t_3}{t_3 t_2 t_1}, \qquad \varphi'_{V_3}(\underline{x}_{\mathbf{i}}(\mathbf{t})) = \frac{1}{t_3 t_2 t_1},$$

$$\varphi'_{V_4}(\underline{x}_{\mathbf{i}}(\mathbf{t})) = \frac{t_5 t_3}{t_5 t_4 t_3 t_2}, \qquad \varphi'_{V_5}(\underline{x}_{\mathbf{i}}(\mathbf{t})) = \frac{1}{t_5 t_4 t_3 t_2}, \qquad \varphi'_{V_6}(\underline{x}_{\mathbf{i}}(\mathbf{t})) = \frac{1}{t_6 t_5 t_3}.$$

Finally, we compute the maps $C_{\mathbf{i},k}$:

$$C_{\mathbf{i},1} = \frac{1}{\varphi'_{V_1}}, \qquad C_{\mathbf{i},2} = \frac{1}{\varphi'_{V_2}} \cdot \varphi'_{V_1}, \qquad C_{\mathbf{i},3} = \frac{1}{\varphi'_{V_3}} \cdot \varphi'_{V_2},$$

$$C_{\mathbf{i},4} = \frac{1}{\varphi'_{V_4} \varphi'_{V_1}} \cdot \varphi'_{V_2}, \qquad C_{\mathbf{i},5} = \frac{1}{\varphi'_{V_5} \varphi'_{V_2}} \cdot \varphi'_{V_4} \varphi'_{V_3}, \qquad C_{\mathbf{i},6} = \frac{1}{\varphi'_{V_6} \varphi'_{V_4}} \cdot \varphi'_{V_5}.$$

5. Monomials of twisted minors

As defined before, let $V := V_{\mathbf{i}} = V_1 \oplus \cdots \oplus V_r$ and $W := W_{\mathbf{i}} = W_1 \oplus \cdots \oplus W_r$. For a cluster-tilting module T in $\mathcal{C}_w$ and any $X \in \mathcal{C}_w$, we consider $\mathrm{Hom}_\Lambda(T, X)$ as a module over $\mathcal{E}_T := \mathrm{End}_\Lambda(T)^{\mathrm{op}}$. The Ext-group $\mathrm{Ext}^1_\Lambda(T, X)$ is a module over $\mathcal{E}_T$ and over $\underline{\mathcal{E}}_T := \underline{\mathrm{End}}_{\mathcal{C}_w}(T)^{\mathrm{op}}$. By $\underline{\dim}\, \mathrm{Ext}^1_\Lambda(T, X)$ we mean the dimension vector of $\mathrm{Ext}^1_\Lambda(T, X)$ as an $\mathcal{E}_T$-module.

From now on, for the reduced expression $\mathbf{i} = (i_r, \dots, i_1)$ we assume without loss of generality that for each $1 \le j \le n$ there is some k with $i_k = j$. We can also assume that for at least one such j there are indices $k \neq s$ with $i_k = i_s = j$. (Otherwise all direct summands of T are $\mathcal{C}_w$-projective-injective, i.e., $R_- = \varnothing$.)

5.1. Apart from the definitions for Theorem 4 the following will be useful:

$$\begin{aligned}
\varphi'_k &:= \varphi'_{V_k} = \varphi_{\Omega_w(V_k)}\varphi^{-1}_{P(V_k)} && \text{for } k \in R, \text{ in particular,}\\
\varphi'_l &= \varphi_l^{-1} && \text{for } l \in R_{\max},\\
\hat{\varphi}'_k &:= \prod_{l\in R} (\varphi'_l)^{B^{(V)}_{l,k}} && \text{for } k \in R_-,\\
(\varphi'_\bullet)^{\mathbf{g}} &:= \prod_{k\in R} (\varphi'_k)^{g_k} && \text{for } \mathbf{g} = (g_1,\dots,g_r) \in \mathbb{Z}^r,\\
(\hat{\varphi}'_\bullet)^{\mathbf{d}} &:= \prod_{k\in R_-} (\hat{\varphi}'_k)^{d_k} && \text{for } \mathbf{d} = (d_l)_{l\in R_-} \in \mathbb{N}^{R_-}.
\end{aligned}$$

Recall from [GLS5, Proposition 9.1] that the functions φ_{V_k} can be seen as generalized minors. The functions φ'_{V_k} are the *twisted generalized minors*. The following proposition describes some special monomials in the functions φ'_{V_k}. These results play a crucial role in the proof of Theorem 4.

Proposition 5.1. *For* $\mathbf{t} = (t_r,\dots,t_1) \in (\mathbb{C}^*)^r$ *and* $k \in R_-$ *we have*

$$\hat{\varphi}_{W,k}(\underline{x}_{\mathbf{i}}(\mathbf{t})) = \hat{\varphi}'_k(\underline{x}_{\mathbf{i}}(\mathbf{t})) = t_{k^+}t_k^{-1}. \tag{5.1}$$

Moreover, for $X \in \mathcal{C}_w$ *we have*

$$\begin{aligned}
t^{\mathbf{a}^-(X)} &= \varphi_W^{(\underline{\dim}\,\mathrm{Hom}_\Lambda(W,X))\cdot B^{(W)}}(\underline{x}_{\mathbf{i}}(\mathbf{t})) && (5.2)\\
&= (\varphi'_\bullet)^{(\underline{\dim}\,\mathrm{Ext}^1_\Lambda(V,X)-\underline{\dim}\,\mathrm{Hom}_\Lambda(V,X))\cdot B^{(V)}}(\underline{x}_{\mathbf{i}}(\mathbf{t})). && (5.3)
\end{aligned}$$

Remark 5.2. It seems to be in general quite cumbersome to calculate the ingredients $\mathbf{a}^-(W_k) = \mathbf{a}^+(W_k)$ and $B^{(W)}$ of equation (5.2). In contrast, by Corollary 4.5 we have

$$\varphi'_k(\underline{x}_{\mathbf{i}}(\mathbf{t})) = \mathbf{t}^{-\mathbf{a}^-(V_k)}.$$

Similarly, $B^{(V)} = (\dim \mathrm{Hom}_\Lambda(V_k, V_l)_{1\le k,l\le r})^{-t}$ can be determined by our results in [GLS5]. Moreover, the dimension vector $\underline{\dim}\,\mathrm{Hom}_\Lambda(V,X)$ depends linearly on the multiplicities in the add$(M_{\mathbf{i}})$-filtration of X, so that equation (5.3) appears to be much more convenient for practical purposes; see [GLS5, Section 11].

The rest of this section is dedicated to the proof of Proposition 5.1.

5.2. **Proof of Equation** (5.1)**.** For $k \in R_-$ we have by definition (see equation (1.1))

$$\begin{aligned}
C_{\mathbf{i},k^+} &= \frac{1}{\varphi'_{k^+}\varphi'_k} \prod_{j\in Q_0} (\varphi'_{(k^+)^-(j)})^{q(i_k,j)} \\
&= \frac{1}{\varphi'_{k^+}\varphi'_k} \prod_{\substack{j\in Q_0 \\ k^+>(k^+)^-(j)>k}} (\varphi'_{(k^+)^-(j)})^{q(i_k,j)} \prod_{\substack{j\in Q_0 \\ k>(k^+)^-(j)}} (\varphi'_{(k^+)^-(j)})^{q(i_k,j)}, \\
C_{\mathbf{i},k} &= \frac{1}{\varphi'_k\varphi'_{k^-}} \prod_{j\in Q_0} (\varphi'_{k^-(j)})^{q(i_k,j)} \\
&= \frac{1}{\varphi'_k\varphi'_{k^-}} \prod_{\substack{j\in Q_0 \\ k^+>(k^-(j))^+>k}} (\varphi'_{k^-(j)})^{q(i_k,j)} \prod_{\substack{j\in Q_0 \\ (k^-(j))^+>k^+}} (\varphi'_{k^-(j)})^{q(i_k,j)}.
\end{aligned}$$

For $(k,j) \in R_- \times Q_0$ we have $k > (k^+)^-(j)$ if and only if $(k^-(j))^+ > k^+$, and in this case $(k^+)^-(j) = k^-(j)$. Thus

$$\prod_{\substack{j\in Q_0 \\ k>(k^+)^-(j)}} (\varphi'_{(k^+)^-(j)})^{q(i_k,j)} = \prod_{\substack{j\in Q_0 \\ (k^-(j))^+>k^+}} (\varphi'_{k^-(j)})^{q(i_k,j)}.$$

Using Theorem 3 we conclude that

$$\begin{aligned}
&t_{k^+}t_k^{-1} = (C_{\mathbf{i},k^+}C_{\mathbf{i},k}^{-1})(x_{\mathbf{i}}(\mathbf{t})) \\
&= \left(\varphi'_{k^-}(\varphi'_{k^+})^{-1} \prod_{\substack{j\in Q_0 \\ ((k^+)^-(j))^+>k^+>(k^+)^-(j)>k}} (\varphi'_{(k^+)^-(j)})^{q(i_k,j)} \prod_{\substack{j\in Q_0 \\ k^+>(k^-(j))^+>k>k^-(j)}} (\varphi'_{k^-(j)})^{-q(i_k,j)} \right) (x_{\mathbf{i}}(\mathbf{t})) \\
&= \hat{\varphi}'_k(x_{\mathbf{i}}(\mathbf{t}))
\end{aligned}$$

where the last equality follows from the description of the quiver of $\mathrm{End}_\Lambda(V)^{\mathrm{op}}$ in Section 3.6 and the definition of $\hat{\varphi}'_k$. This shows the second equality of equation (5.1).

For the first equality of (5.1) we compare mutations of V and W in direction k. Thus, for $k \in R_-$ we consider the short exact sequences

$$0 \to V_k \to \bigoplus_{l\in R} V_l^{[-B^{(V)}_{l,k}]_+} \to V'_k \to 0 \quad \text{and} \quad 0 \to V'_k \to \bigoplus_{l\in R} V_l^{[B^{(V)}_{l,k}]_+} \to V_k \to 0$$

as well as similar sequences for W. Since the stable endomorphism rings of V and W are isomorphic, we have $B_{k,l}^{(V)} = B_{k,l}^{(W)}$ for all $k, l \in R_-$. Thus, if we write

$$\begin{aligned}
\bar{V}_+^{(k)} &= \bigoplus_{l \in R_-} V_l^{[-B_{l,k}^{(V)}]_+}, & P_+^{(k)} &= \bigoplus_{l \in R_{\max}} V_l^{[-B_{l,k}^{(V)}]_+},\\
\bar{V}_-^{(k)} &= \bigoplus_{l \in R_-} V_l^{[B_{l,k}^{(V)}]_+}, & P_-^{(k)} &= \bigoplus_{l \in R_{\max}} V_l^{[B_{l,k}^{(V)}]_+},\\
\bar{W}_+^{(k)} &= \bigoplus_{l \in R_-} W_l^{[-B_{l,k}^{(W)}]_+}, & Q_+^{(k)} &= \bigoplus_{l \in R_{\max}} W_l^{[-B_{l,k}^{(W)}]_+},\\
\bar{W}_-^{(k)} &= \bigoplus_{l \in R_-} W_l^{[B_{l,k}^{(W)}]_+}, & Q_-^{(k)} &= \bigoplus_{l \in R_{\max}} W_l^{[B_{l,k}^{(W)}]_+},
\end{aligned}$$

we obtain by the Snake Lemma two commutative diagrams with exact rows and columns:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & W_k \arrow[r] \arrow[d] & \bar{W}_+^{(k)} \oplus Q_+^{(k)} \arrow[r] \arrow[d] & W_k' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P(V_k) \arrow[r] \arrow[d] & P(\bar{V}_+^{(k)}) \oplus P_+^{(k)} \oplus Q_+^{(k)} \arrow[r] \arrow[d] & P(V_k') \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & V_k \arrow[r] \arrow[d] & \bar{V}_+^{(k)} \oplus P_+^{(k)} \arrow[r] \arrow[d] & V_k' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

and

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & W_k' \arrow[r] \arrow[d] & \bar{W}_-^{(k)} \oplus Q_-^{(k)} \arrow[r] \arrow[d] & W_k \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P(V_k') \arrow[r] \arrow[d] & P(\bar{V}_-^{(k)}) \oplus P_-^{(k)} \oplus Q_-^{(k)} \arrow[r] \arrow[d] & P(V_k) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & V_k' \arrow[r] \arrow[d] & \bar{V}_-^{(k)} \oplus P_-^{(k)} \arrow[r] \arrow[d] & V_k \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0. &
\end{tikzcd}$$

Since in both diagrams the end term of the respective middle row is projective, both of them split, i.e.,

$$P(\bar{V}_+^{(k)}) \oplus P_+^{(k)} \oplus Q_+^{(k)} \cong P(V_k') \oplus P(V_k) \cong P(\bar{V}_-^{(k)}) \oplus P_-^{(k)} \oplus Q_-^{(k)}.$$

It follows that

$$\varphi_{Q_-^{(k)}} \varphi_{Q_+^{(k)}}^{-1} = \varphi_{P(\bar{V}_+^{(k)})} \varphi_{P_+^{(k)}} \left(\varphi_{P(\bar{V}_-^{(k)})} \varphi_{P_-^{(k)}} \right)^{-1}. \tag{5.4}$$

On the other hand, since $B_{k,l}^{(V)} = B_{k,l}^{(W)}$ for $k, l \in R_-$ we can write, with the above definitions,

$$\hat{\varphi}_k' = \frac{\varphi_{\bar{W}_-^k}}{\varphi_{P(\bar{V}_-^k)}} \cdot \frac{\varphi_{P(\bar{V}_+^k)}}{\varphi_{\bar{W}_+^k}} \cdot \frac{\varphi_{P_+^{(k)}}}{\varphi_{P_-^{(k)}}} \quad \text{and} \quad \hat{\varphi}_{W,k} = \varphi_{\bar{W}_-^k} \cdot \frac{1}{\varphi_{\bar{W}_+^k}} \cdot \frac{\varphi_{Q_-^{(k)}}}{\varphi_{Q_+^{(k)}}}.$$

Thus $\hat{\varphi}_k' = \hat{\varphi}_{W,k}$ by equation (5.4).

5.3. **Proof of Equation** (5.2)**.** Since W is a cluster-tilting module, for each $X \in \mathcal{C}_w$ we have a short exact sequence

$$0 \to \bigoplus_{k \in R} W_k^{g_k''} \to \bigoplus_{k \in R} W_k^{g_k'} \xrightarrow{p} X \to 0$$

for certain $\mathbf{g}' = (g_1', \dots, g_r') \in \mathbb{N}^r$ and $\mathbf{g}'' = (g_1'', \dots, g_r'') \in \mathbb{N}^r$. Note, that we can assume $g_k'' = 0$ for $k \in R_{\max}$ since W_k is $\mathcal{C}_w$-projective-injective in this case. Write W'' resp. W' for the first two terms of this sequence. Since W is rigid, the sequence remains exact under $\operatorname{Hom}_\Lambda(W, -)$. We conclude that

$$(\underline{\dim} \operatorname{Hom}_\Lambda(W, X)) \cdot B^{(W)} = \mathbf{g}' - \mathbf{g}''.$$

We know from Theorem 2 that there is a matrix $A \in \mathbb{N}^{r \times r}$ such that for $1 \le k \le r$ we have

$$\varphi_{W_k}(\underline{x}_{\mathbf{i}}(\mathbf{t})) = \prod_{l=1}^{r} t_l^{A_{k,l}}.$$

By Theorem 1, the modules W' resp. W'' have a unique partial composition series $W_\bullet' = (W')_\bullet^-$ resp. $W_\bullet'' = (W'')_\bullet^-$ of type $\mathbf{i}$ with

$$\mathbf{a}' := \operatorname{wt}(W_\bullet') = \mathbf{g}' \cdot A \qquad \text{resp.} \qquad \mathbf{a}'' := \operatorname{wt}(W_\bullet'') = \mathbf{g}'' \cdot A.$$

It follows that $W_k'' = W'' \cap W_k'$ for $1 \le k \le r$. With $X_k := p(W_k')$ we obtain for all $k \in R$ a commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & W_{k-1}'' \arrow[r] & W_{k-1}' \arrow[r] & X_{k-1} \arrow[r] & 0 \\ 0 \arrow[r] & W_k'' \arrow[r] \arrow[u] & W_k' \arrow[r] \arrow[u] & X_k \arrow[r] \arrow[u] & 0 \end{tikzcd} \tag{5.5}$$

and the vertical maps being the natural inclusions. By the Snake Lemma we obtain a short exact sequence

$$0 \to S_{i_k}^{a_k''} \to S_{i_k}^{a_k'} \to X_k / X_{k-1} \to 0$$

and conclude that $X_k / X_{k-1} \cong S_{i_k}^{a_k' - a_k''}$. Thus, $X_\bullet$ is a partial composition series of type $\mathbf{i}$ for X with $\operatorname{wt}(X_\bullet) = (\mathbf{g}' - \mathbf{g}'') \cdot A$. Applying $\operatorname{Hom}_\Lambda(-, S_{i_k})$ to the bottom row of the above diagram shows that $\operatorname{top}_{S_{i_k}}(X_k) = 0$ for all $k \in R$, since $W_\bullet' = (W')_\bullet^-$.

This shows that $X_\bullet$ is the refined top series of type $\mathbf{i}$ of X. All together we now have

$$\varphi_W^{(\underline{\dim}\operatorname{Hom}_\Lambda(W,X))\cdot B^{(W)}}(\underline{x}_{\mathbf{i}}(\mathbf{t})) = \varphi_W^{\mathbf{g}'-\mathbf{g}''}(\underline{x}_{\mathbf{i}}(\mathbf{t})) = \mathbf{t}^{(\mathbf{g}'-\mathbf{g}'')A} = \mathbf{t}^{\operatorname{wt}(X_\bullet)} = \mathbf{t}^{\mathbf{a}^-(X)},$$

which is our claim.

5.4. **Proof of Equation** (5.3)**.** For a $\mathcal{C}_w$-projective-injective module $X \in \mathcal{C}_w$ the claim is clear by the definition of φ'_k for $k \in R_{\max}$. So we can assume that X has no non-zero $\mathcal{C}_w$-projective-injective summands. Then we have a short exact sequence

$$0 \to X \to P(\Omega_w^{-1}(X)) \to \Omega_w^{-1}(X) \to 0$$

in $\mathcal{C}_w$ with $\Omega_w^{-1}(X)$ having no non-zero $\mathcal{C}_w$-projective-injective summands. Now we apply $\operatorname{Hom}_\Lambda(V,-)$ and obtain

$$\begin{aligned}&\underline{\dim}\operatorname{Hom}_\Lambda(V,\Omega_w^{-1}(X)) - \underline{\dim}\operatorname{Hom}_\Lambda(V,P(\Omega_w^{-1}(X)))\\ &\quad= \underline{\dim}\operatorname{Ext}^1_\Lambda(V,X) - \underline{\dim}\operatorname{Hom}_\Lambda(V,X).\end{aligned}$$

Thus, we have to show that

(5.6)
$$(\varphi'_\bullet)^{(\underline{\dim}\operatorname{Hom}_\Lambda(V,\Omega_w^{-1}(X))-\underline{\dim}\operatorname{Hom}_\Lambda(V,P(\Omega_w^{-1}(X))))\cdot B^{(V)}} = \varphi_W^{(\underline{\dim}\operatorname{Hom}_\Lambda(W,X))\cdot B^{(W)}}.$$

Using that Ω_w^{-1} is an autoequivalence of the stable category $\underline{\mathcal{C}}_w$ we obtain again by the Snake Lemma a commutative diagram with exact rows and columns:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & W'' \arrow[r] \arrow[d] & \bar{W}'\oplus P \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P(\Omega_w^{-1}(W'')) \arrow[r] \arrow[d] & P(\Omega_w^{-1}(\bar{W}'))\oplus P\oplus Q \arrow[r] \arrow[d] & P(\Omega_w^{-1}(X)) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Omega_w^{-1}(W'') \arrow[r] \arrow[d] & \Omega_w^{-1}(\bar{W}')\oplus Q \arrow[r] \arrow[d] & \Omega_w^{-1}(X) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0, &
\end{tikzcd}$$

where $\bar{W}'$ and W'' have no non-zero $\mathcal{C}_w$-projective-injective summands, P is $\mathcal{C}_w$-projective-injective, and $\bar{W}'\oplus P = W'_X \in \operatorname{add}(W)$. Similarly, we have $\Omega_w^{-1}(\bar{W}')$ without non-zero $\mathcal{C}_w$-projective-injective summands, Q is $\mathcal{C}_w$-projective-injective, and

$$\Omega_w^{-1}(\bar{W}')\oplus Q = V'_{\Omega_w^{-1}(X)} \in \operatorname{add}(V).$$

From this it is already clear that the components corresponding to R_- of the three vectors

$$\begin{aligned}&(\underline{\dim}\operatorname{Hom}_\Lambda(W,X))\cdot B^{(W)},\\ &(\underline{\dim}\operatorname{Hom}_\Lambda(V,\Omega_w^{-1}(X)))\cdot B^{(V)},\\ &(\underline{\dim}\operatorname{Hom}_\Lambda(V,\Omega_w^{-1}(X))-\underline{\dim}\operatorname{Hom}_\Lambda(V,P(\Omega_w^{-1}(X))))\cdot B^{(V)}\end{aligned}$$

coincide. Since $P(\Omega_w^{-1}(X))$ is $\mathcal{C}_w$-projective-injective, the middle row of our diagram splits. Thus we have

$$P(\Omega_w^{-1}(X)) \oplus P(\Omega_w^{-1}(W'')) \cong P(\Omega_w^{-1}(\bar{W}')) \oplus P \oplus Q.$$

Finally, from the above diagram we conclude that

$$\varphi_W^{(\underline{\dim}\,\mathrm{Hom}_\Lambda(W,X))\cdot B^{(W)}} = \varphi_{\bar{W}'}\varphi_{W''}^{-1}\varphi_P$$

and

$$(\varphi'_\bullet)^{(\underline{\dim}\,\mathrm{Hom}_\Lambda(V,\Omega_w^{-1}(X))-\underline{\dim}\,\mathrm{Hom}_\Lambda(V,P(\Omega_w^{-1}(X))))\cdot B^{(V)}} = \frac{\varphi_{\bar{W}'}}{\varphi_{P(\Omega_w^{-1}(\bar{W}'))}}\cdot\frac{\varphi_{P(\Omega_w^{-1}(W''))}}{\varphi_{W''}}\cdot\frac{\varphi_{P(\Omega_w^{-1}(X))}}{\varphi_Q}.$$

Thus (5.6) follows from the above isomorphism of $\mathcal{C}_w$-projective-injectives.

6. Cluster character identities

6.1. **Quivers with potential and mutations.** We review some material from [DWZ2, Section 4], which in turn is a review of [DWZ1]. Let $\mathcal{P}(\Gamma, W) := \mathbb{C}\langle\langle\Gamma\rangle\rangle/J(W)$ be the Jacobian algebra associated to a quiver $\Gamma = (\Gamma_0, \Gamma_1, s, t)$ and a potential $W \in \mathfrak{m}_{\mathrm{cyc}} \subset \mathbb{C}\langle\langle\Gamma\rangle\rangle$. For $k \in \Gamma_0$ we set

$$\begin{aligned}\Gamma^{(-,k)} &:= \{b \in \Gamma_1 \mid s(b) = k\},\\ \Gamma^{(k,+)} &:= \{a \in \Gamma_1 \mid t(a) = k\},\\ \Gamma^{(2,k)} &:= \Gamma^{(-,k)} \times \Gamma^{(k,+)}.\end{aligned}$$

For a reduced quiver potential (Γ, W) and Γ without 2-cycles at the vertex $k \in \Gamma_0$ let $\mu_k(\Gamma, W)$ be the mutation of (Γ, W) in direction k, as defined in [DWZ1]. For convenience, we briefly recall the construction. First, a possibly non-reduced quiver potential $\tilde{\mu}_k(\Gamma, W) := (\widetilde{\Gamma}, \widetilde{W})$ is defined as follows. The quiver $\widetilde{\Gamma}$ is obtained from Γ by inserting for each pair of arrows $(b, a) \in \Gamma^{(2,k)}$ a new arrow $[ba]$ from $s(a)$ to $t(b)$ and replacing each arrow c with $s(c) = k$ or $t(c) = k$ by a new arrow c^* in the opposite direction. Then $\widetilde{W} := [W] + \Delta$, where

$$\Delta := -\sum_{(b,a)\in\Gamma^{(2,k)}} [ba]a^*b^*$$

and $[W]$ is obtained by substituting each occurrence of a path ba with $(b, a) \in \Gamma^{(2,k)}$ by the arrow $[ba]$ (after some rotation, if necessary). Our definition of Δ deviates from the original one by a sign. However the resulting quiver potential is right equivalent to the original one and more convenient for our purpose. Finally, $\mu_k(\Gamma, W)$ is by definition the reduced part of $(\widetilde{\Gamma}, \widetilde{W})$.

For a representation M of $\mathcal{P}(\Gamma, W)$ and $k \in \Gamma_1$ we need the following notation:

$$\begin{aligned}
M^+(k) &:= \bigoplus_{a \in \Gamma^{(k,+)}} M(s(a)),\\
M^-(k) &:= \bigoplus_{b \in \Gamma^{(-,k)}} M(t(b)),\\
M(\alpha_k) &:= (M(a))_{a \in \Gamma^{(k,+)}} \colon M^+(k) \to M(k),\\
M(\beta_k) &:= (M(b))_{a \in \Gamma^{(-,k)}} \colon M(k) \to M^-(k),\\
M(\gamma_k) &:= (M(\partial_{b,a} W))_{(b,a) \in \Gamma^{(2,k)}} \colon M^-(k) \to M^+(k).
\end{aligned}$$

For an indecomposable representation M of $\mathcal{P}(\Gamma, W)$, which is not the simple representation S_k, the "premutation" $\tilde{\mu}_k(M) := \widetilde{M}$ is a representation of $\mathcal{P}(\widetilde{\Gamma}, \widetilde{W})$ which can be described as follows [DWZ1]:

- $\widetilde{M}(j) := M(j)$ for all $j \in \Gamma_0 \setminus \{k\}$ and $\widetilde{M}(a) := M(a)$ for all arrows $a \in \widetilde{\Gamma}_1 \cap \Gamma_1$.
- $\widetilde{M}([ba]) := M(b)M(a)$ for all pairs of arrows $(b,a) \in \Gamma^{(2,k)}$.
- It remains to define the maps
$$\begin{aligned}
\widetilde{M}(\alpha_k) &\colon \widetilde{M}^+(k) \to \widetilde{M}(k),\\
\widetilde{M}(\beta_k) &\colon \widetilde{M}(k) \to \widetilde{M}^-(k),
\end{aligned}$$
where $\widetilde{M}^+(k) := M^-(k)$ and $\widetilde{M}^-(k) := M^+(k)$.

Remark 6.1. It is an elementary exercise to verify that $\widetilde{M}$ is up to isomorphism uniquely determined by the following properties of those maps:

$$\operatorname{Ker}(\widetilde{M}(\alpha_k)) = \operatorname{Im}(M(\beta_k)), \tag{6.1}$$

$$\operatorname{Im}(\widetilde{M}(\beta_k)) = \operatorname{Ker}(M(\alpha_k)), \tag{6.2}$$

$$\operatorname{Ker}(\widetilde{M}(\beta_k)) \subseteq \operatorname{Im}(\widetilde{M}(\alpha_k)), \tag{6.3}$$

$$\widetilde{M}(\beta_k)\widetilde{M}(\alpha_k) = M(\gamma_k). \tag{6.4}$$

A concrete choice of a triple $(\widetilde{M}(k), \widetilde{M}(\alpha_k), \widetilde{M}(\beta_k))$ with properties (6.1)–(6.4) can be found in [DWZ2, p.765].

Next, we need to extract some material from [BIRSm]. Let $T = T_1 \oplus \cdots \oplus T_r$ be a $V_{\mathbf{i}}$-reachable cluster-tilting module in $\mathcal{C}_w$. We consider the quiver Γ_T of the endomorphism algebra $\mathcal{E}_T := \operatorname{End}_\Lambda(T)^{\rm op}$ and the quiver $\underline{\Gamma}_T$ of the corresponding stable endomorphism algebra $\underline{\mathcal{E}}_T := \underline{\operatorname{End}}_{\mathcal{C}_w}(T)^{\rm op}$.

We have $(\Gamma_T)_0 = R = \{1, 2, \dots, r\}$ with the vertex i corresponding to the direct summand T_i of T. We identify $\underline{\Gamma}_T$ with the full subquiver of Γ_T with vertices R_- corresponding to the non-$\mathcal{C}_w$-projective-injective indecomposable direct summands of T. Thus we get a surjective algebra homomorphism

$$\psi \colon \mathbb{C}\langle\langle \underline{\Gamma}_T \rangle\rangle \to \underline{\mathcal{E}}_T$$

such that $\psi(c) \in \underline{\operatorname{Hom}}_{\mathcal{C}_w}(T_{t(c)}, T_{s(c)})$ for all $c \in (\underline{\Gamma}_T)_1$.

We fix some $k \in R_-$. Then there are short exact sequences

$$0 \to T_k \xrightarrow{\alpha_k} \bigoplus_{a \in \Gamma_T^{(k,+)}} T_{s(a)} \xrightarrow{\beta'_k} T'_k \to 0 \tag{6.5}$$

and

$$0 \to T_k' \xrightarrow{\alpha_k'} \bigoplus_{b \in \Gamma_T^{(-,k)}} T_{t(b)} \xrightarrow{\beta_k} T_k \to 0 \tag{6.6}$$

such that $\mu_k(T) := T_k' \oplus T/T_k$ is also a basic cluster-tilting module in $\mathcal{C}_w$. It is convenient to label the components of the above maps as follows:

$$\alpha_k = (\alpha_{a,k})_{a \in \Gamma_T^{(k,+)}}, \qquad \beta_k' = (\beta_{k,a}')_{a \in \Gamma_T^{(k,+)}},$$
$$\alpha_k' = (\alpha_{b,k}')_{b \in \Gamma_T^{(-,k)}}, \qquad \beta_k = (\beta_{k,b})_{b \in \Gamma_T^{(-,k)}}.$$

Lemma 6.2. *With the above notation the following hold:*

(a) *One can choose* ψ *such that* $\mathrm{Ker}(\psi) = J(W_T)$ *for some potential* $W_T \in \mathfrak{m}_{\mathrm{cyc}} \subset \mathbb{C}\langle\langle \underline{\Gamma}_T \rangle\rangle$ *and*

$$\begin{aligned} \psi(a) &= \alpha_{a,k} \in \underline{\mathrm{Hom}}_{\mathcal{C}_w}(T_k, T_{s(a)}) && \text{for all } a \in \underline{\Gamma}_T^{(k,+)}, \\ \psi(b) &= \beta_{k,b} \in \underline{\mathrm{Hom}}_{\mathcal{C}_w}(T_{t(b)}, T_k) && \text{for all } b \in \underline{\Gamma}_T^{(-,k)}, \\ \psi(\partial_{b,a} W) &= \alpha_{b,k}' \beta_{k,a}' && \text{for all } (b,a) \in \underline{\Gamma}_T^{(2,k)}. \end{aligned}$$

(b) *We have a surjective algebra homomorphism*

$$\psi' \colon \mathbb{C}\langle\langle \widetilde{\underline{\Gamma}}_T \rangle\rangle \to \underline{\mathcal{E}}_{\mu_k(T)}$$

such that

$$\begin{aligned} \psi'(a^*) &= \beta_{k,a}' \in \underline{\mathrm{Hom}}_{\mathcal{C}_w}(T_{s(a)}, T_k') && \text{for all } a \in \underline{\Gamma}_T^{(k,+)}, \\ \psi'(b^*) &= \alpha_{b,k}' \in \underline{\mathrm{Hom}}_{\mathcal{C}_w}(T_k', T_{t(b)}) && \text{for all } b \in \underline{\Gamma}_T^{(-,k)}, \\ \psi'([ba]) &= \alpha_{a,k} \beta_{k,b} && \text{for all } (b,a) \in \underline{\Gamma}_T^{(2,k)}, \\ \psi'(c) &= \psi(c) && \text{for all } c \in (\widetilde{\underline{\Gamma}}_T)_1 \cap (\underline{\Gamma}_T)_1. \end{aligned}$$

Moreover, $\mathrm{Ker}(\psi') = J(\widetilde{W}_T)$.

Proof. By applying recursively [BIRSm, Theorem 5.3] it follows from [BIRSm, Theorem 6.6] and [BIRSm, Theorem 4.6] that we can find a *liftable* (in the sense of [BIRSm, 5.1]) isomorphism

$$\bar{\psi} \colon \mathcal{P}(\underline{\Gamma}_T, W_T) \to \underline{\mathcal{E}}_T.$$

Thus, by [BIRSm, Lemma 5.7] the conditions (O)-(IV) described in [BIRSm, Section 5.2] hold for our ψ. This shows (a). Part (b) follows from [BIRSm, Theorem 5.6] and the construction of ψ' (denoted by Φ' in [BIRSm]); see also [BIRSm, Theorem 4.5]. $\square$

We now consider the following special case of the above: For an indecomposable module $X \in \mathcal{C}_w \setminus \mathrm{add}(T \oplus T_k')$ we consider the indecomposable $\underline{\mathcal{E}}_T$-module $\mathrm{Ext}_\Lambda^1(T, X)$ via $\bar{\psi}$ as a representation M of $\mathcal{P}(\underline{\Gamma}_T, W_T)$. Similarly, we consider $\mathrm{Ext}_\Lambda^1(\mu_k(T), X)$ via $\bar{\psi}'$ as a representation M' of $\mathcal{P}(\widetilde{\underline{\Gamma}}_T, \widetilde{W}_T)$.

Proposition 6.3. *With the above notation, the representations* M' *and* $\tilde{\mu}_k(M)$ *of the Jacobian algebra* $\mathcal{P}(\widetilde{\underline{\Gamma}}_T, \widetilde{W}_T)$ *are isomorphic. Thus, we can consider the* $\underline{\mathcal{E}}_{\mu_k(T)}$*-module* $\mathrm{Ext}_\Lambda^1(\mu_k(T), X)$ *as the mutation of the* $\underline{\mathcal{E}}_T$*-module* $\mathrm{Ext}_\Lambda^1(T, X)$ *in direction* k.

Proof. Note that $M(\alpha_k) = \mathrm{Ext}^1_\Lambda(\alpha_k, X)$ and $M(\beta_k) = \mathrm{Ext}^1_\Lambda(\beta_k, X)$ by the first two equations in Lemma 6.2(a). Similarly, by Lemma 6.2(b) we have $M'(\alpha_k) = \mathrm{Ext}^1_\Lambda(\alpha'_k, X)$, $M'(\beta_k) = \mathrm{Ext}^1_\Lambda(\beta'_k, X)$, and $M'(k) = \mathrm{Ext}^1_\Lambda(T'_k, X)$. According to Remark 6.1 it is sufficient to verify (6.1)–(6.4) for this data. Indeed, (6.1) resp. (6.2) follows since $\mathrm{Ext}^1_\Lambda(-, X)$ is exact at the middle term of the short exact sequences (6.5) resp. (6.6). Next, (6.3) is clear, since $\mathrm{Ext}^1_\Lambda(\mu_k(T), X)$ is an indecomposable $\underline{\mathcal{E}}_{\mu_k(T)}$-module. Finally,

$$M'(\beta_k) \circ M'(\alpha_k) = \mathrm{Ext}^1_\Lambda(\alpha'_k\beta'_k, X) = \mathrm{Ext}^1_\Lambda(\psi(\partial_{b,a}W)_{(b,a)\in\underline{\Gamma}_T^{(2,k)}}, X) = M'(\gamma_k)$$

by the last equation in Lemma 6.2(a). Thus also (6.4) holds. □

6.2. **Transformation of g-vectors and F-polynomials.** We fix a $V_{\mathbf{i}}$-reachable cluster-tilting module T in $\mathcal{C}_w$, and define for any $X \in \mathcal{C}_w$ the (extended) *index* of X with respect to T as

$$\mathbf{g}_X^T := (\underline{\dim}\,\mathrm{Hom}_\Lambda(T, X)) \cdot B^{(T)} \in \mathbb{Z}^r, \tag{6.7}$$

and the *F-polynomial* of X with respect to T as

$$F_X^T((y_l)_{l\in R_-}) := \sum_{\mathbf{d}\in\mathbb{N}^{R_-}} \chi(\mathrm{Gr}_{\mathbf{d}}^{\underline{\mathcal{E}}_T}(\mathrm{Ext}^1_\Lambda(T, X)))\, y^{\mathbf{d}}.$$

Moreover, we will need the $\mathbf{h}$-*vector* of X with respect to T:

$$\mathbf{h}_X^T := (h_k)_{k\in R_-} \text{ where } h_k := -\dim \mathrm{Hom}_{\underline{\mathcal{E}}_T}(S_k, \mathrm{Ext}^1_\Lambda(T, X)).$$

Fix $k \in R_-$ and write $T' := \mu_k(T) = T'_k \oplus T/T_k$ for the cluster-tilting module obtained from T by mutation in direction k. Thus we have short exact sequences:
(6.8)

$$0 \to T_k \to \bigoplus_{l\in R} T_l^{[-B^{(T)}_{l,k}]_+} \to T'_k \to 0 \quad \text{and} \quad 0 \to T'_k \to \bigoplus_{l\in R} T_l^{[-B^{(T)}_{k,l}]_+} \to T_k \to 0.$$

Let us recall the following observation from [FK, Section 3]. We have a short exact sequence

$$0 \to \bigoplus_{k\in R_-} T_k^{-\tilde{h}_k} \to \bigoplus_{k\in R} T_k^{-\tilde{h}'_k} \xrightarrow{\pi_X} X \to 0$$

such that π_X is a minimal right $\mathrm{add}(T)$-approximation for certain non-positive integers $\tilde{h}_k$ and $\tilde{h}'_k$. It will be convenient to define $\tilde{h}_l := 0$ for $l \in R_{\max}$. From this we obtain the short exact sequence

$$0 \to \mathrm{Hom}_\Lambda\left(T, \bigoplus_{k\in R_-} T_k^{-\tilde{h}_k}\right) \to \mathrm{Hom}_\Lambda\left(T, \bigoplus_{k\in R} T_k^{-\tilde{h}'_k}\right) \to \mathrm{Hom}_\Lambda(T, X) \to 0, \tag{6.9}$$

which can be viewed as a projective resolution of $\mathrm{Hom}_\Lambda(T, X)$ over $\mathrm{End}_\Lambda(T)^{\mathrm{op}}$, and
(6.10)

$$0 \to \mathrm{Ext}^1_\Lambda(T, X) \to \mathrm{Ext}^1_\Lambda\left(T, \bigoplus_{k\in R_-} \Omega_w^{-1}(T_k)^{-\tilde{h}_k}\right) \to \mathrm{Ext}^1_\Lambda\left(T, \bigoplus_{k\in R_-} \Omega_w^{-1}(T_k)^{-\tilde{h}'_k}\right),$$

which is a minimal injective copresentation of $M := \mathrm{Ext}^1_\Lambda(T, X)$ over the stable endomorphism ring $\underline{\mathcal{E}}_T$. Thus, it follows from (6.9) that

$$\mathbf{g}_X^T = (g_k)_{1\le k\le r} = (\tilde{h}_k - \tilde{h}'_k)_{1\le k\le r}.$$

In particular, $g_k \geq 0$ for all $k \in R_{\max}$. On the other hand, we conclude from (6.10) that

$$\tilde{h}_k = h_k := -\dim \operatorname{Hom}_{\underline{\mathcal{E}}_T}(S_k, M) \quad \text{and} \quad \tilde{h}'_k = -\dim \operatorname{Ext}^1_{\underline{\mathcal{E}}_T}(S_k, M)$$

for $k \in R_-$.

We have the following easy application of deep results in [BIRSm] and [DWZ2]:

Lemma 6.4. *Let*

$$\begin{aligned} \mathbf{g}_X^T &= (g_k)_{k\in R}, & \mathbf{h}_X^T &= (h_l)_{l\in R_-}, & F &= F_X^T,\\ \mathbf{g}_X^{T'} &= (g'_k)_{k\in R}, & \mathbf{h}_X^{T'} &= (h'_l)_{l\in R_-}, & F' &= F_X^{T'}. \end{aligned}$$

Moreover, let $B = (B^{(T)}_{l,m})_{l,m\in R_-}$, and let $(B', (\hat{y}'_l)_{l\in R_-})$ be obtained from $(B, (\hat{y}_l)_{l\in R_-})$ by Y-seed mutation in direction k in $\mathbb{Q}_{sf}((y_l)_{l\in R_-})$. Then for $M := \operatorname{Ext}^1_\Lambda(T, X)$ and $k \in R_-$ we get

$$h'_k = -\dim \operatorname{Ext}^1_{\underline{\mathcal{E}}}(S_k, M) \quad \textit{and therefore} \quad g_k = h_k - h'_k. \tag{6.11}$$

Moreover, for $k \in R_-$ we have

$$(\hat{y}_k + 1)^{h_k} F((\hat{y}_l)_{l\in R_-}) = (\hat{y}'_k + 1)^{h'_k} F'((\hat{y}'_l)_{l\in R_-}) \tag{6.12}$$

and for $l \in R$ we have

$$g'_l = \begin{cases} g_l - h_k B^{(T)}_{l,k} + g_k [B^{(T)}_{l,k}]_+ & \textit{if } l \neq k,\\ -g_k & \textit{if } l = k. \end{cases} \tag{6.13}$$

Remark 6.5. Observe that (6.13) is just [DWZ2, (2.11)] (proved in [DWZ2, Lemma 5.2]) extended to our situation with coefficients. Our independent proof for this situation is quite different.

Proof. We know from [BIRSm, Theorems 5.3 and 6.4] that the stable endomorphism algebra $\underline{\mathcal{E}}_T$ is given by a quiver with potential, and $\underline{\mathcal{E}}_{T'}$ is obtained from $\underline{\mathcal{E}}_T$ by a mutation of quiver potentials in direction k. Then by Proposition 6.3 the $\underline{\mathcal{E}}_{T'}$-module $M' := \operatorname{Ext}^1_\Lambda(T', X)$ is obtained from the $\underline{\mathcal{E}}_T$-module $M = \operatorname{Ext}^1_\Lambda(T, X)$ by mutation in direction k in the sense of [DWZ2, (4.16) and (4.17)].

Now, equation (6.11) follows from the description of the minimal injective presentation of M in [DWZ2, Remark 10.8]. In fact, with the notation used there, we have obviously $\dim U_k^* = \dim \operatorname{Ext}^1_{\underline{\mathcal{E}}_T}(S_k, M) = \tilde{h}'_k$. On the other hand, by the definition of the mutation procedure for M in direction k and the construction of U_k^* we have $\dim U_k^* = \dim \operatorname{Hom}_{\underline{\mathcal{E}}_{T'}}(S_k, M') = h'_k$.

Similarly, equation (6.12) follows now from the "Key-Lemma" [DWZ2, Lemma 5.2].

Next, let Z_k be the $r \times r$ matrix defined by

$$Z_k := \begin{pmatrix} 1 & & 0 & 0 & 0 & & 0\\ & \ddots & & & & \ddots & \\ 0 & & 1 & 0 & 0 & & 0\\ [-B^{(T)}_{k,1}]_+ & \cdots & [-B^{(T)}_{k,k-1}]_+ & -1 & [-B^{(T)}_{k,k+1}]_+ & \cdots & [-B^{(T)}_{k,r}]_+\\ 0 & & 0 & 0 & 1 & & 0\\ & \ddots & & & & \ddots & \\ 0 & & 0 & 0 & 0 & & 1 \end{pmatrix}.$$

To prove equation (6.13), note that $B^{(T')} = Z_k^t B^{(T)} Z_k$; see [GLS2, Proposition 7.5]. Moreover, if we apply $\mathrm{Hom}_\Lambda(-, X)$ to the second short exact sequence in (6.8) we obtain with $M = \mathrm{Ext}^1_\Lambda(T, X)$ the following exact sequence:

$$0 \to \mathrm{Hom}_\Lambda(T_k, X) \to \mathrm{Hom}_\Lambda(\bigoplus_{l\in R} T_l^{[B_{l,k}^{(T)}]_+}, X) \to \mathrm{Hom}_\Lambda(T'_k, X)$$
$$\to M(k) \xrightarrow{M(\beta_k)} \bigoplus_{l\in R_-} M(l)^{[-B_{k,l}^{(T)}]_+},$$

where $\dim \mathrm{Ker}(M(\beta_k)) = -h_k$. Thus

$$\underline{\dim}\, \mathrm{Hom}_\Lambda(T', X) = (\underline{\dim}\, \mathrm{Hom}_\Lambda(T, X)) \cdot Z_k^t - h_k \mathbf{e}_k,$$

where $\mathbf{e}_k$ is the kth standard coordinate vector of $\mathbb{Z}^r$. Now, using $Z_k = Z_k^{-1}$ and $B_{k,k}^{(T)} = 0$, we find $\mathbf{g}' = \mathbf{g} Z_k + h_k \mathbf{e}_k B^{(T)}$. This implies our claim since $-B_{k,l}^{(T)} = B_{l,k}^{(T)}$ for all $(k, l) \in R_- \times R$. □

6.3. **Proof of Theorem 4.** Again, let $W := W_{\mathbf{i}}$. We show in a first step that

$$\varphi_X = \theta_X^W$$

for all $X \in \mathcal{C}_w$. It is well known that the morphism $\underline{x}_{\mathbf{i}} \colon (\mathbb{C}^*)^r \to N^w$ is dominant. In fact, by [L2, Proposition 2.7] it is injective, and N^w is irreducible and of dimension r. Since $\varphi_X \in \mathbb{C}[N^w]$ and θ_X^W is a rational function, it is sufficient to verify this equality on $\underline{x}_{\mathbf{i}}(\mathbf{t})$; see also [GLS5, Corollary 15.7]. Now, we have:

$$\begin{aligned}
\varphi_X(\underline{x}_{\mathbf{i}}(\mathbf{t})) &= \sum_{\mathbf{a}\in\mathbb{N}^r} \chi(\mathcal{F}_{\mathbf{i},\mathbf{a},X})\, \mathbf{t}^{\mathbf{a}} \\
&= \mathbf{t}^{\mathbf{a}^-(X)} \sum_{\mathbf{a}\in\mathbb{N}^r} \chi(\mathrm{Gr}^{\mathcal{E}}_{d_{\mathbf{i},X}(\mathbf{a})}(\mathrm{Ext}^1_\Lambda(W, X)))\, \mathbf{t}^{\mathbf{a}-\mathbf{a}^-(X)} \\
&= \varphi_W^{(\underline{\dim}\, \mathrm{Hom}_\Lambda(W,X))\cdot B^{(W)}}(\underline{x}_{\mathbf{i}}(\mathbf{t})) \sum_{\mathbf{d}\in\mathbb{N}^{R_-}} \chi(\mathrm{Gr}^{\mathcal{E}_W}_{\mathbf{d}}(\mathrm{Ext}^1_\Lambda(W, X)))\, \hat{\varphi}_W^{\mathbf{d}}(\underline{x}_{\mathbf{i}}(\mathbf{t})) \\
&= \theta_X^W(\underline{x}_{\mathbf{i}}(\mathbf{t})).
\end{aligned}$$

The first equality is just the description of φ_X given in [GLS5, Proposition 6.1], and the second equality follows directly from Theorem 1. To prove the third equality, we have to show that $\mathbf{t}^{\mathbf{a}-\mathbf{a}^-(X)} = \hat{\varphi}_W^{\mathbf{d}}(\underline{x}_{\mathbf{i}}(\mathbf{t}))$ for all $\mathbf{a} \in \mathbb{N}^r$ and $\mathbf{d} = (d_k)_{k\in R_-} := d_{\mathbf{i},X}(\mathbf{a})$. Proposition 5.1 yields

$$\hat{\varphi}_W^{\mathbf{d}}(\underline{x}_{\mathbf{i}}(\mathbf{t})) = \prod_{k\in R_-} (t_{k^+} t_k^{-1})^{d_k},$$

and by Theorem 3.27 we have

$$d_k = (a_k^- - a_k) + (a_{k^-}^- - a_{k^-}) + \cdots + (a_{k_{\min}}^- - a_{k_{\min}})$$

for all $k \in R_-$, where $\mathbf{a}^-(X) = (a_r^-, \dots, a_1^-)$. Observe that for $k \in R_{\max}$ we have $d_{k^-} = a_k - a_k^-$. Now an easy calculation yields the result. The last equality is just the definition of θ_X^W. Since the image of $\underline{x}_{\mathbf{i}}$ is dense in N^w and φ_X is regular (and thus continuous), we get $\varphi_X = \theta_X^W$.

Since W is $V_{\mathbf{i}}$-reachable, it remains to show the following: If a new cluster-tilting module T' is obtained from a cluster-tilting module T by mutation in direction k, then $\theta_X^{T'} = \theta_X^T$ for all $X \in \mathcal{C}_w$. This follows from Lemma 6.4 with the notation used there and a calculation inspired from the proof of [FZ2, Proposition 6.8]:

$$\begin{aligned}
\theta_X^T &= \varphi_T^{\mathbf{g}} F(\hat{\varphi}_T) \\
&= \varphi_T^{\mathbf{g}} (\hat{\varphi}_{T,k}+1)^{-h_k} F'(\hat{\varphi}_{T'}) (\hat{\varphi}_{T',k}+1)^{h'_k} \\
&= \varphi_T^{\mathbf{g}} (\hat{\varphi}_{T,k}+1)^{-h_k} \left(\frac{\hat{\varphi}_{T,k}+1}{\hat{\varphi}_{T,k}}\right)^{h'_k} F'(\hat{\varphi}_{T'}) \\
&= \varphi_T^{\mathbf{g}} (\hat{\varphi}_{T,k}+1)^{-g_k} \hat{\varphi}_{T,k}^{-h'_k} F'(\hat{\varphi}_{T'}) \\
&= \varphi_T^{\mathbf{g}} \left(\varphi_{T_k}\varphi_{T'_k} \prod_{l\in R} \varphi_{T_l}^{-[-B^{(T)}_{l,k}]_+}\right)^{-g_k} \left(\prod_{l\in R} \varphi_{T_l}^{B^{(T)}_{l,k}}\right)^{-h'_k} F'(\hat{\varphi}_{T'}) \\
&= \left(\prod_{l\in R} \varphi_{T_l}^{g_l+[-B^{(T)}_{l,k}]_+ g_k - B^{(T)}_{l,k} h'_k}\right) \left(\varphi_{T_k}\varphi_{T'_k}\right)^{-g_k} F'(\hat{\varphi}_{T'}) \\
&= \left(\prod_{l\in R} \varphi_{T_l}^{g_l+[B^{(T)}_{l,k}]_+ g_k - B^{(T)}_{l,k} h_k}\right) \left(\varphi_{T_k}\varphi_{T'_k}\right)^{-g_k} F'(\hat{\varphi}_{T'}) \quad (\text{since } g_k = h_k - h'_k) \\
&= \varphi_{T'}^{\mathbf{g}'} F'(\hat{\varphi}_{T'}) \qquad (\text{by } (6.13)) \\
&= \theta_X^{T'}.
\end{aligned}$$

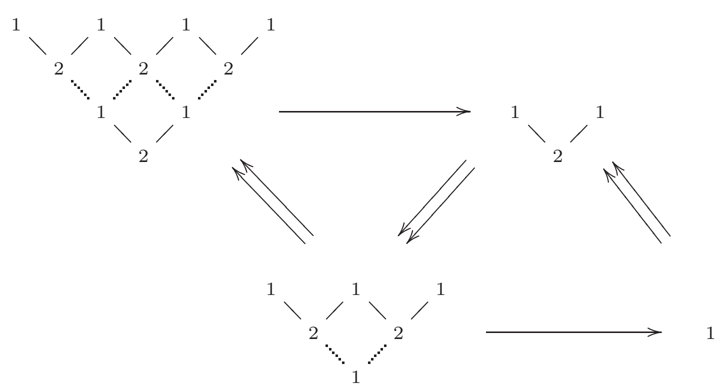
(Here we set $\mathbf{g} := \mathbf{g}_X^T$ and $\mathbf{g}' := \mathbf{g}_X^{T'}$.) At the beginning, we used that the Y-seed $((B^{(T')}_{k,l})_{k,l\in R_-}, \hat{\varphi}_{T'})$ is obtained from $((B^{(T)}_{k,l})_{k,l\in R_-}, \hat{\varphi}_T)$ by Y-seed mutation in direction k by [FZ2, Proposition 3.9].

6.4. **An example.** Let Q be the Kronecker quiver

$$1 \overset{a}{\underset{b}{\rightrightarrows}} 2$$

and let $\Lambda = \mathbb{C}\overline{Q}/(c)$ be the corresponding preprojective algebra. Then $\mathbf{i} = (2,1,2,1)$ is a reduced expression for the Weyl group element $w = s_2s_1s_2s_1$. (The stable category $\underline{\mathcal{C}}_w$ is equivalent to the cluster category of $\mathbb{C}Q$; see [GLS5, Section 16].)

The following picture describes the module $V := V_{\mathbf{i}} = V_1 \oplus \cdots \oplus V_4$ and the quiver Γ_V of $\mathcal{E}_V := \mathrm{End}_\Lambda(V)^{\mathrm{op}}$. (The numbers 1 and 2 in the picture are basis vectors of the modules V_k. The solid edges show how the arrows a and b of $\overline{Q}$ act on these vectors, and the dotted edges illustrate the actions of a^* and b^*.)



From the well-known description of $\mathcal{E}_V$ by a quiver with relations, we obtain $B^{(V)}$, the matrix of the Ringel form of $\mathcal{E}_V$ and its inverse:

$$B^{(V)} = \begin{pmatrix} 0 & -2 & 1 & 0 \\ 2 & 0 & -2 & 1 \\ -1 & 2 & 1 & -2 \\ 0 & -1 & 0 & 1 \end{pmatrix} \quad \text{and} \quad (B^{(V)})^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 1 & 2 & 4 & 6 \\ 0 & 1 & 2 & 4 \end{pmatrix}.$$

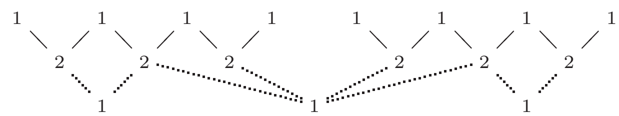
Note that the entries of $(B^{(V)})^{-1}$ describe the dimensions of $\mathrm{Hom}_\Lambda(V_j, V_i)$. Observe also that $\underline{\mathcal{E}}_V := \underline{\mathrm{End}}_{\mathcal{C}_w}(V)^{\mathrm{op}}$ is isomorphic to $\mathbb{C}Q$. Next, we describe

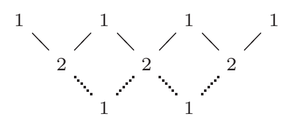
$$W := W_{\mathbf{i}} = I_w \oplus \Omega_w(V_{\mathbf{i}}) = W_1 \oplus \cdots \oplus W_4.$$

Note that $I_w = V_3 \oplus V_4$, $W_3 = V_3$ and $W_4 = V_4$. We have short exact sequences

$$0 \to W_1 \to V_3^3 \to V_1 \to 0 \qquad \text{and} \qquad 0 \to W_2 \to V_3^2 \to V_2 \to 0.$$

Thus

$W_1 =$ 

and $W_2 =$  .

In our situation, we have $W = (\mu_1 \circ \mu_2)(V)$. Using [GLS2, Section 7] we get

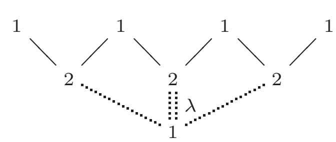
$$B^{(W)} = \begin{pmatrix} 0 & -2 & 3 & 0 \\ 2 & 0 & -4 & -1 \\ -3 & 4 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{pmatrix} \quad \text{and} \quad (B^{(W)})^{-1} = \begin{pmatrix} 25 & 14 & 9 & 14 \\ 16 & 9 & 6 & 9 \\ 11 & 6 & 4 & 6 \\ 6 & 3 & 2 & 4 \end{pmatrix}.$$

The quiver Γ_W of $\mathcal{E}_W := \mathrm{End}_\Lambda(W)^{\mathrm{op}}$ looks as follows:

$$\begin{tikzcd} V_4 \arrow[dr, "2"'] & & W_2 \arrow[ll] \arrow[dl, "4"] & \\ & V_3 \arrow[rr, "3"'] & & W_1. \arrow[ul, "2"'] \end{tikzcd}$$

(Here we write $i \xrightarrow{m} j$ in case there are m arrows from i to j.)

For $\lambda \in \mathbb{C}$ we consider the following Λ-module in $\mathcal{C}_w$:

$X_\lambda :=$ 

Note that $X_\lambda \cong \Omega_w(X_\lambda)$. A direct calculation shows that $\mathrm{Ext}^1_\Lambda(V, X_\lambda)$ is an (indecomposable) regular $\mathbb{C}Q$-module with dimension vector $(1,1)$ and that $\underline{\dim}\,\mathrm{Hom}_\Lambda(V, X_\lambda) = (1,2,3,5)$. This implies that

$$(\underline{\dim}\,\mathrm{Hom}_\Lambda(V, X_\lambda)) \cdot B^{(V)} = (1,-1,0,1).$$

Using the exact sequences

$$0 \to \mathrm{Hom}_\Lambda(V_k, X_\lambda) \to \mathrm{Hom}_\Lambda(P(V_k), X_\lambda) \to \mathrm{Hom}_\Lambda(W_k, X_\lambda) \to \mathrm{Ext}^1_\Lambda(V_k, X_\lambda) \to 0$$

for $k = 1, 2$, we obtain $\underline{\dim}\,\mathrm{Hom}_\Lambda(W, X_\lambda) = (9, 5, 3, 5)$, and therefore

$$(\underline{\dim}\,\mathrm{Hom}_\Lambda(W, X_\lambda)) \cdot B^{(W)} = (1, -1, 0, 0).$$

Finally, since $\Omega_w^{-1}(X_\lambda) \cong X_\lambda$, we get that also $\mathrm{Ext}^1_\Lambda(W, X_\lambda)$ is an indecomposable regular $\mathbb{C}Q$-module with dimension vector $(1, 1)$.

It is straightforward to calculate (using the Euler characteristics of the corresponding varieties of partial composition series) the following:

$$\begin{aligned}
\varphi_{X_\lambda}(\underline{x}_{\mathbf{i}}(\mathbf{t})) &= t_3t_2^3t_1^4 + t_4t_3t_2^2t_1^4 + t_4t_3^2t_2^2t_1^3,\\
\varphi_{V_1}(\underline{x}_{\mathbf{i}}(\mathbf{t})) &= t_3 + t_1,\\
\varphi_{V_2}(\underline{x}_{\mathbf{i}}(\mathbf{t})) &= t_4(t_3^2 + 2t_3t_1 + t_1^2) + t_2t_1^2,\\
\varphi_{V_3}(\underline{x}_{\mathbf{i}}(\mathbf{t})) &= t_3t_2^2t_1^3,\\
\varphi_{V_4}(\underline{x}_{\mathbf{i}}(\mathbf{t})) &= t_4t_3^2t_2^3t_1^4,\\
\varphi_{W_1}(\underline{x}_{\mathbf{i}}(\mathbf{t})) &= t_3^3t_2^6t_1^8,\\
\varphi_{W_2}(\underline{x}_{\mathbf{i}}(\mathbf{t})) &= t_3^2t_2^3t_1^4.
\end{aligned}$$

Note that the φ-functions of the direct summands of W evaluate on $\underline{x}_{\mathbf{i}}(\mathbf{t})$ to monomials.

The F-polynomial of each indecomposable representation of

$$\underline{\mathrm{End}}_{\mathcal{C}_w}(V)^{\mathrm{op}} \cong \underline{\mathrm{End}}_{\mathcal{C}_w}(W)^{\mathrm{op}} \cong \mathbb{C}Q$$

with dimension vector $(1, 1)$ is $1 + y_2 + y_1y_2$. Thus, from the definitions we get

$$\begin{aligned}
\theta^W_{X_\lambda} &= \varphi_{W_1}\varphi_{W_2}^{-1}(1 + \hat{\varphi}_{W,2} + \hat{\varphi}_{W,1}\hat{\varphi}_{W,2}),\\
\theta^V_{X_\lambda} &= \varphi_{V_1}\varphi_{V_2}^{-1}\varphi_{V_4}\,(1 + \hat{\varphi}_{V,2} + \hat{\varphi}_{V,1}\hat{\varphi}_{V,2}).
\end{aligned}$$

Thus, by Theorem 4 we should have

$$\varphi_{X_\lambda} = \theta^V_{X_\lambda} = \theta^W_{X_\lambda}. \tag{6.14}$$

From the matrices $B^{(V)}$ resp. $B^{(W)}$ we get

$$\begin{aligned}
\hat{\varphi}_{V,1} &= \varphi_{V_2}^2\varphi_{V_3}^{-1}, & \hat{\varphi}_{W,1} &= \varphi_{W_2}^2\varphi_{V_3}^{-3},\\
\hat{\varphi}_{V,2} &= \varphi_{V_1}^{-2}\varphi_{V_3}^2\varphi_{V_4}^{-1}, & \hat{\varphi}_{W,2} &= \varphi_{W_1}^{-2}\varphi_{V_3}^4\varphi_{V_4}.
\end{aligned}$$

We verify equation (6.14) by evaluation on $\underline{x}_{\mathbf{i}}(\mathbf{t})$. First, we observe that

$$\hat{\varphi}_{W,1}(\underline{x}_{\mathbf{i}}(\mathbf{t})) = t_3t_1^{-1} \quad \text{and} \quad \hat{\varphi}_{W,2}(\underline{x}_{\mathbf{i}}(\mathbf{t})) = t_4t_2^{-1}.$$

This implies that

$$(\varphi_{W_1}\varphi_{W_2}^{-1}(1+\hat{\varphi}_{W,2}+\hat{\varphi}_{W,1}\hat{\varphi}_{W,2}))(\underline{x}_{\mathbf{i}}(\mathbf{t})) = t_3t_2^3t_1^4(1+t_4t_2^{-1}+t_4t_3t_2^{-1}t_1^{-1}) = \varphi_{X_\lambda}(\underline{x}_{\mathbf{i}}(\mathbf{t})).$$

On the other hand, from the definitions we get

$$\varphi_{V_1}\varphi_{V_2}^{-1}\varphi_{V_4}\,(1 + \hat{\varphi}_{V,2} + \hat{\varphi}_{V,1}\hat{\varphi}_{V,2}) = \varphi_{V_1}^{-1}\varphi_{V_2}^{-1}(\varphi_{V_1}^2\varphi_{V_4} + \varphi_{V_3}^2 + \varphi_{V_2}^2\varphi_{V_3}).$$

Evaluation at $\underline{x}_{\mathbf{i}}(\mathbf{t})$ yields

$$\begin{aligned}
\frac{t_3t_2^2t_1^3(t_4(t_3+t_1)^2+t_2t_1^2)^2+t_4t_3^2t_2^3t_1^4(t_3+t_1)^2+t_3^2t_2^4t_1^6}{(t_3+t_1)(t_4(t_3+t_1)^2+t_2t_1^2)} &= t_4(t_3^2t_2^2t_1^3 + t_3t_2^2t_1^4) + t_3t_2^3t_1^4\\
&= \varphi_{X_\lambda}(\underline{x}_{\mathbf{i}}(\mathbf{t})).
\end{aligned}$$

7. E-invariant and Ext

For a cluster-tilting module T in $\mathcal{C}_w$, and $\underline{\mathcal{E}} := \underline{\mathcal{E}}_T$, let $E\colon \mathcal{C}_w \to \mathrm{mod}(\underline{\mathcal{E}})$ be the functor defined by $X \mapsto \mathrm{Ext}^1_\Lambda(T, X)$. This functor is known to be dense (i.e. up to isomorphism, all objects in $\mathrm{mod}(\underline{\mathcal{E}})$ are in the image of E); see for example [KR, Proposition 2.1(c)].

The following proposition shows that, for an $\underline{\mathcal{E}}$-module of the form EX, the E-invariant defined in [DWZ2] has a nice geometric description as the codimension of the orbit of X in the nilpotent variety. This result plays a crucial role in the proof of Proposition 8.6 and of Theorem 5.

Proposition 7.1. *Let $T \in \mathcal{C}_w$ be a $V_{\mathbf{i}}$-reachable cluster-tilting module. Then for any $X, Y \in \mathcal{C}_w$ there is a short exact sequence*

$$0 \to D\mathrm{Hom}_{\underline{\mathcal{E}}}(Y, \tau_{\underline{\mathcal{E}}}(X)) \to \mathrm{Ext}^1_\Lambda(X, Y) \to \mathrm{Hom}_{\underline{\mathcal{E}}}(X, \tau_{\underline{\mathcal{E}}}(Y)) \to 0.$$

In particular, if $\underline{\dim}_\Lambda(X) = \mathbf{d}$ we have

$$\mathrm{codim}_{\Lambda^w_{\mathbf{d}}}(\mathrm{GL}_{\mathbf{d}}\,.X) = \dim \mathrm{Hom}_{\underline{\mathcal{E}}}(EX, \tau_{\underline{\mathcal{E}}}(EX)) = \dim \mathrm{Hom}_{\underline{\mathcal{E}}}(\tau^{-1}_{\underline{\mathcal{E}}}(EX), EX).$$

Proof. The stable endomorphism algebra $\underline{\mathcal{E}}$ of T is the Jacobian algebra of a quiver with potential; see [BIRSm]. Denote by G_T the corresponding Ginzburg dg-algebra and by $\mathcal{D}_{\mathrm{perf}}(G_T)$ resp. $\mathcal{D}_{\mathrm{f.d.}}(G_T)$ the corresponding subcategory of perfect complexes resp. of complexes with total finite-dimensional cohomology of the derived category. The shift in $\mathcal{D}_{\mathrm{perf}}(G_T)$ is denoted by Σ. Following Amiot [A] we have the generalized cluster category as the triangulated quotient

$$\mathcal{C}_T := \mathcal{D}_{\mathrm{perf}}(G_T)/\mathcal{D}_{\mathrm{f.d.}}(G_T).$$

It follows from [ART] that $\mathcal{C}_w \cong \mathcal{C}_V$ as triangulated categories, and then from [BIRSm] and [KY] that $\mathcal{C}_V \cong \mathcal{C}_T$. Next, denote by $\mathcal{F} \subseteq \mathcal{D}_{\mathrm{perf}}(G_T)$ the subcategory which consists of the cones of maps in $\mathrm{add}(G_T)$. Then the canonical projection $\mathcal{D}_{\mathrm{perf}}(G_T) \to \mathcal{C}_T$ induces an equivalence of additive categories $\mathcal{F} \to \mathcal{C}_T$ [A, Proposition 2.9, Lemma 2.10]; see also [KY, Remark 4.1]. By [A, Proposition 2.12] we have for $X, Y \in \mathcal{F}$ a short exact sequence

$$0 \to \mathrm{Ext}^1_{\mathcal{D}_{\mathrm{perf}}(G_T)}(X, Y) \to \mathrm{Ext}^1_{\mathcal{C}_T}(X, Y) \to D\mathrm{Ext}^1_{\mathcal{D}_{\mathrm{perf}}(G_T)}(Y, X) \to 0.$$

We have to show that $D\mathrm{Ext}^1_{\mathcal{D}_{\mathrm{perf}}(G_T)}(X, Y) \cong \mathrm{Hom}_{\underline{\mathcal{E}}}(H^0(Y), \tau_{\underline{\mathcal{E}}}(H^0(X)))$, since $H^0(X) \cong \mathrm{Hom}_{\mathcal{D}_{\mathrm{perf}}(G_T)}(G_T, X) = \mathrm{Hom}_{\mathcal{C}_T}(T, X)$ for $X \in \mathcal{F}$.

To this end we choose a minimal presentation

$$P_1 \xrightarrow{p} P_0 \to X \to \Sigma P_1 \text{ with } P_0, P_1 \in \mathrm{add}(G_T). \tag{7.1}$$

Now, for $P \in \mathrm{add}(G_T)$ and $Y \in \mathcal{D}_{\mathrm{perf}}(G_T)$ we have

$$\mathrm{Hom}_{\mathcal{D}_{\mathrm{perf}}(G_T)}(P, Y) = \mathrm{Hom}_{\underline{\mathcal{E}}}(H^0(P), H^0(Y))$$

and $\mathrm{Hom}_{\mathcal{D}_{\mathrm{perf}}}(P, \Sigma Y) = 0$ if $Y \in \mathcal{F}$. Thus,

$$\mathrm{Ext}^1_{\mathcal{D}_{\mathrm{perf}}(G_T)}(X, Y) = \mathrm{Hom}_{\mathcal{D}_{\mathrm{perf}}(G_T)}(X, \Sigma Y) \cong \mathrm{Coker}(\mathrm{Hom}_{\underline{\mathcal{E}}}(H^0(p), H^0(Y))).$$

Next, the sequence

$$H^0(P_1) \xrightarrow{H^0(p)} H^0(P_0) \to H^0(X) \to 0$$

is a minimal projective presentation in $\mathrm{mod}(\underline{\mathcal{E}})$ because of the minimality of (7.1). Since we have $\mathrm{Hom}_{\underline{\mathcal{E}}}(H^0(P), L) \cong D\mathrm{Hom}_{\underline{\mathcal{E}}}(L, \nu_{\underline{\mathcal{E}}}(H^0(P)))$ for $L \in \mathrm{mod}(\underline{\mathcal{E}})$ and $P \in \mathrm{add}(G_T)$ we conclude that

$$\begin{aligned} D\mathrm{Hom}_{\mathcal{D}_{\mathrm{perf}}(G_T)}(X, \Sigma Y) &\cong \mathrm{Ker}(\mathrm{Hom}_{\underline{\mathcal{E}}}(H^0(Y), \nu_{\underline{\mathcal{E}}}(H^0(p)))) \\ &= \mathrm{Hom}_{\underline{\mathcal{E}}}(H^0(Y), \tau_{\underline{\mathcal{E}}}(H^0(X))). \end{aligned}$$

Here $\nu_{\underline{\mathcal{E}}}$ is the usual Nakayama functor. Finally, by Lemma 8.1 below, we have $\dim \mathrm{Ext}^1_\Lambda(X, X) = 2\,\mathrm{codim}_{\Lambda^w_{\mathbf{d}}}(\mathrm{GL}_{\mathbf{d}}\,.X)$. This yields the result. $\square$

8. Generic bases for cluster algebras

8.1. **Generically reduced components.** For a $V_{\mathbf{i}}$-reachable cluster-tilting module T in $\mathcal{C}_w$, the algebra $\underline{\mathcal{E}} := \underline{\mathcal{E}}_T$ is given by a quiver with potential [BIRSm]. Then by [DWZ2, Corollary 10.8],

$$\dim \mathrm{Hom}_{\underline{\mathcal{E}}}(\tau_{\underline{\mathcal{E}}}^{-1}(Y), Y) = E^{\mathrm{inj}}(Y)$$

is the E-*invariant* defined in [DWZ2]. Since this translates into a simple rank condition [DWZ2, Equation (1.17)], for each irreducible component $\mathcal{Z} \in \mathrm{Irr}(\underline{\mathcal{E}})$ the following hold:

(i) There is a dense open subset $\mathcal{U}' \subseteq \mathcal{Z}$ and a unique $h(\mathcal{Z}) \in \mathbb{N}$ such that

$$\dim \mathrm{Hom}_{\underline{\mathcal{E}}}(\tau_{\underline{\mathcal{E}}}^{-1}(U), U) = h(\mathcal{Z})$$

for all $U \in \mathcal{U}'$.

(ii) There is a dense open subset $\mathcal{U}'' \subseteq \mathcal{Z}$ and a unique $e(\mathcal{Z}) \in \mathbb{N}$ such that

$$\dim \mathrm{Ext}^1_{\underline{\mathcal{E}}}(U, U) = e(\mathcal{Z})$$

for all $U \in \mathcal{U}''$.

(iii) There is a dense open subset $\mathcal{U}''' \subseteq \mathcal{Z}$ and a unique $c(\mathcal{Z}) \in \mathbb{N}$ such that

$$\mathrm{codim}_{\mathcal{Z}}(U.\,\mathrm{GL}_{\mathbf{d}}) = c(\mathcal{Z})$$

for all $U \in \mathcal{U}'''$.

It is well known that

$$c(\mathcal{Z}) \le e(\mathcal{Z}) \le h(\mathcal{Z}).$$

(For the second inequality, one uses the Auslander-Reiten formula

$$\mathrm{Ext}^1_{\underline{\mathcal{E}}}(U, U) \cong D\underline{\mathrm{Hom}}_{\underline{\mathcal{E}}}(\tau_{\underline{\mathcal{E}}}^{-1}(Y), Y).)$$

For an algebra A and A-modules M and N, $\underline{\mathrm{Hom}}_A(M, N)$ denotes the homomorphism space $\mathrm{Hom}_A(M, N)$ modulo the subspace of homomorphisms factoring through projectives.) It follows from Voigt's Lemma [G, Proposition 1.1] that $\mathcal{Z}$ is (scheme-theoretically) generically reduced if and only if $c(\mathcal{Z}) = e(\mathcal{Z})$. Recall that $\mathcal{Z}$ is *strongly reduced* if $c(\mathcal{Z}) = h(\mathcal{Z})$. So, strongly reduced components are in particular generically reduced.

8.2. **Open subsets of nilpotent varieties.** For $\mathbf{d} \in \mathbb{N}^n$ let $\Lambda_{\mathbf{d}}$ be the affine variety of nilpotent representations with dimension vector $\mathbf{d}$ of the preprojective algebra Λ. Following Lusztig [L1, Section 12], $\Lambda_{\mathbf{d}}$ is equidimensional with

$$\dim(\Lambda_{\mathbf{d}}) = \sum_{a \in Q_1} \mathbf{d}(s(a))\mathbf{d}(t(a)). \tag{8.1}$$

On $\Lambda_{\mathbf{d}}$ acts the group $\mathrm{GL}_{\mathbf{d}} = \prod_{i \in Q_0} \mathrm{GL}_{\mathbf{d}(i)}(\mathbb{C})$ from the left by conjugation. We have the following surprising result, which we borrow from [GLS4, Lemma 4.3].

Lemma 8.1. *Let M be a nilpotent Λ-module with $\underline{\dim}_\Lambda(M) = \mathbf{d}$. Then*

$$2\operatorname{codim}_{\Lambda_{\mathbf{d}}}(\mathrm{GL}_{\mathbf{d}}.M) = \dim \operatorname{Ext}^1_\Lambda(M, M).$$

Proof. We have

$$2\operatorname{codim}_{\Lambda_{\mathbf{d}}}(\mathrm{GL}_{\mathbf{d}}.M) = 2(\dim(\Lambda_{\mathbf{d}}) - \dim(\mathrm{GL}_{\mathbf{d}}) + \dim \operatorname{End}_\Lambda(M)) = \dim \operatorname{Ext}^1_\Lambda(M, M),$$

where the last equality holds by (8.1) and [CB, Lemma 1]. □

Using the notation from Section 3, let $J_w := J_{r,1}$, where $\Lambda = A$ and $\mathbf{i} = (i_r, \dots, i_1)$ is a reduced expression of w. This definition does not depend on the choice of $\mathbf{i}$; see [BIRS, Proposition III.1.8].

Lemma 8.2. *We have*

$$\mathcal{C}_w = \{X \in \operatorname{mod}(\Lambda) \mid \operatorname{Ext}^1_\Lambda(D(\Lambda/J_w), X) = 0 = \operatorname{Hom}_\Lambda(X, D(J_w))\}.$$

Thus, the subset $\Lambda^w_{\mathbf{d}} := \{X \in \Lambda_{\mathbf{d}} \mid X \in \mathcal{C}_w\}$ is open in $\Lambda_{\mathbf{d}}$.

Proof. Since Λ/J_w is Gorenstein [BIRS, III Proposition 2.2 and Corollary 3.6] we have

$$\mathcal{C}_w := \operatorname{Fac}(\Lambda/J_w) = \{X \in \operatorname{mod}(\Lambda/J_w) \mid \operatorname{Ext}^1_{\Lambda/J_w}(D(\Lambda/J_w), X) = 0\}.$$

We consider the short exact sequence

$$0 \to D(\Lambda/J_w) \xrightarrow{i} D(\Lambda) \xrightarrow{p} D(J_w) \to 0.$$

Since $\operatorname{Hom}_\Lambda(X, D(\Lambda)) \cong D(X)$ and $\operatorname{Hom}_\Lambda(X, D(\Lambda/J_w)) \cong D(X/J_wX)$ naturally, we conclude that $X \in \operatorname{mod}(\Lambda/J_w)$ if and only if $\operatorname{Hom}_\Lambda(X, p)$ is an isomorphism.

For $X \in \operatorname{Fac}(\Lambda/J_w)$ we have by [BIRS, III.2.3],

$$0 = \operatorname{Ext}^1_\Lambda(D(\Lambda/J_w), X) \cong D\operatorname{Ext}^1_\Lambda(X, D(\Lambda/J_w)).$$

It follows that

$$0 \to \operatorname{Hom}_\Lambda(X, D(\Lambda/J_w)) \xrightarrow{\operatorname{Hom}_\Lambda(X,i)} \operatorname{Hom}_\Lambda(X, D(\Lambda)) \to \operatorname{Hom}_\Lambda(X, D(J_w)) \to 0$$

is exact. Now, $\operatorname{Hom}_\Lambda(X, i)$ is an isomorphism since $X \in \operatorname{mod}(\Lambda/J_w)$. This implies that $\operatorname{Hom}_\Lambda(J_w, X) = 0$.

Conversely, if $X \in \operatorname{mod}(\Lambda)$ fulfills the conditions of the lemma we conclude from $\operatorname{Hom}_\Lambda(X, D(J_w)) = 0$ that $\operatorname{Hom}_\Lambda(X, i)$ is an isomorphism. Thus $X \in \operatorname{mod}(\Lambda/J_w)$, and we infer $\operatorname{Ext}^1_{\Lambda/J_w}(D(\Lambda/J_w), X) = 0$ from $\operatorname{Ext}^1_\Lambda(D(\Lambda/J_w), X) = 0$.

If Λ is of Dynkin type, i.e. finite-dimensional, the conditions of the lemma define obviously an open subset in $\Lambda_{\mathbf{d}}$. Otherwise, J_w is as a tilting module a finitely presented Λ-module [BIRS, Section III.1]. Thus we get an injective resolution

$$0 \to D(\Lambda/J_w) \to \bigoplus_{i \in Q_0} D(e_i\Lambda)^{m_i} \to \bigoplus_{i \in Q_0} D(e_i\Lambda)^{n_i} \to 0$$

so that $\operatorname{Hom}_\Lambda(X, D(J_w)) = 0$ represents also in this case an open condition. □

8.3. **The syzygy functor preserves irreducible components.** The results of this subsection will only be used in Section 9.1, where we prove the second part of Theorem 6. We consider for $\mathbf{e} \in \mathbb{N}^R$ the subset

$$\Lambda^w_{\mathbf{d},\mathbf{e}} := \{X \in \Lambda^w_{\mathbf{d}} \mid \underline{\dim}\, \mathrm{Hom}_\Lambda(V, X) = \mathbf{e}\}.$$

By the upper semicontinuity of $\dim \mathrm{Hom}_\Lambda(V_k, -)$ and Lemma 8.2, $\Lambda^w_{\mathbf{d},\mathbf{e}}$ is a locally closed (possibly empty) subset of $\Lambda_{\mathbf{d}}$. Note that for a given $\mathbf{e}$ there is at most one $\mathbf{d} = \mathbf{d}(\mathbf{e})$ such that $\Lambda^w_{\mathbf{d},\mathbf{e}} \neq \varnothing$. We showed in [GLS5, Section 14] that in case $\Lambda^w_{\mathbf{d},\mathbf{e}} \neq \varnothing$, it is irreducible and of the same dimension as $\Lambda_{\mathbf{d}}$. In particular, if $\Lambda^w_{\mathbf{d},\mathbf{e}} \neq \varnothing$ its Zariski closure is an irreducible component of $\Lambda^w_{\mathbf{d}}$ and each irreducible component of $\Lambda^w_{\mathbf{d}}$ is of this form.

Proposition 8.3. *For $X \in \Lambda^w_{\mathbf{d},\mathbf{e}}$ let $h\colon P \to X$ be a $\mathcal{C}_w$-admissible epimorphism (i.e. $P, X \in \mathcal{C}_w$ and h is an epimorphism with $\mathrm{Ker}(h) \in \mathcal{C}_w$), where P is $\mathcal{C}_w$-projective-injective. Then for $\mathbf{d}' := \underline{\dim}_\Lambda(P) - \mathbf{d}$ and a unique $\mathbf{e}' \in \mathbb{N}^R$ there exists an irreducible variety $\mathcal{E}_{\mathbf{e}',\mathbf{e}}$ together with open morphisms*

$$\begin{tikzcd} & \mathcal{E}_{\mathbf{e}',\mathbf{e}} \arrow[dl, "\pi'"'] \arrow[dr, "\pi"] & \\ \Lambda^w_{\mathbf{d}',\mathbf{e}'} & & \Lambda^w_{\mathbf{d},\mathbf{e}} \end{tikzcd}$$

such that for each $E \in \mathcal{E}_{\mathbf{e}',\mathbf{e}}$ there exists a short exact sequence

$$0 \to \pi'(E) \xrightarrow{i} P \xrightarrow{p} \pi(E) \to 0.$$

Proof. The proof consists of four steps.

(i) Let $\tilde{\mathbf{e}}$ be the (componentwise) minimum value of the map $\Lambda^w_{\mathbf{d},\mathbf{e}} \to \mathbb{N}^{R_-}$ defined by $X \mapsto \underline{\dim}(\underline{\mathrm{Hom}}_{\mathcal{C}_w}(V, X)) = \underline{\dim}(D\mathrm{Ext}^2_\Lambda(X, V))$. Let $\Lambda^{w,-}_{\mathbf{d},\mathbf{e}}$ be the open subset of $\Lambda^w_{\mathbf{d},\mathbf{e}}$ defined by

$$\Lambda^{w,-}_{\mathbf{d},\mathbf{e}} := \{X \in \Lambda^w_{\mathbf{d},\mathbf{e}} \mid \underline{\dim}(\underline{\mathrm{Hom}}_{\mathcal{C}_w}(V, X)) = \tilde{\mathbf{e}}\}.$$

Suppose that $P \in \Lambda^w_{\mathbf{d}'',\mathbf{e}''}$ and set $\mathbf{e}' := \mathbf{e}'' - \mathbf{e} + \tilde{\mathbf{e}}$. It is easy to see that for a short exact sequence

$$0 \to X' \to P \to X \to 0$$

in $\mathcal{C}_w$ with $X \in \Lambda^w_{\mathbf{d},\mathbf{e}}$, we have $X' \in \Lambda^w_{\mathbf{d}',\mathbf{e}'}$ if and only if $X \in \Lambda^{w,-}_{\mathbf{d},\mathbf{e}}$, since $\mathrm{Ext}^1_\Lambda(V, X') \cong \underline{\mathrm{Hom}}_{\mathcal{C}_w}(V, X)$.

(ii) We claim that

$$\begin{aligned}\mathcal{E}_{\mathbf{e}',\mathbf{e}} := \{(X', i, p, X) \in \Lambda^w_{\mathbf{d}',\mathbf{e}'} \times \mathrm{Hom}_{Q_0}(\mathbb{C}^{\mathbf{d}'}, \mathbb{C}^{\mathbf{d}''}) \times \mathrm{Hom}_{Q_0}(\mathbb{C}^{\mathbf{d}''}, \mathbb{C}^{\mathbf{d}}) \times \Lambda^w_{\mathbf{d},\mathbf{e}} \mid \\ i \in \mathrm{Hom}_\Lambda(X', P) \text{ injective}, p \in \mathrm{Hom}_\Lambda(P, X) \text{ surjective }, p \circ i = 0\},\end{aligned}$$

together with the obvious projections has the required properties. By construction, we only have to show that π and π' are open.

(iii) As for the openness of π, consider the vector bundle

$$\mathcal{V}_{\mathbf{e}} := \{(p, X) \in \mathrm{Hom}_{Q_0}(\mathbb{C}^{\mathbf{d}''}, \mathbb{C}^{\mathbf{d}}) \times \Lambda^w_{\mathbf{d},\mathbf{e}} \mid p \in \mathrm{Hom}_\Lambda(P, X)\}.$$

This is a subbundle of the trivial vector bundle $\mathrm{Hom}_{Q_0}(\mathbb{C}^{\mathbf{d}''}, \mathbb{C}^{\mathbf{d}}) \times \Lambda^w_{\mathbf{d},\mathbf{e}}$. In particular, the projection $\pi_2\colon \mathcal{V}_{\mathbf{e}} \to \Lambda^w_{\mathbf{d},\mathbf{e}}$, $(p, X) \mapsto X$ is an open morphism. The set

$$\mathcal{V}_{\mathbf{e}}^{\mathrm{sur}} := \{(p,X) \in \mathcal{V}_{\mathbf{e}} \mid p \text{ surjective}\}$$

is a dense open subset of $\mathcal{V}_{\mathbf{e}}$.

It is a standard argument to check that

$$\begin{aligned}\mathcal{E}_{\mathbf{e}} := \{(X',i,p,X) \in \Lambda_{\mathbf{d}'} \times \mathrm{Hom}_{Q_0}(\mathbb{C}^{\mathbf{d}'},\mathbb{C}^{\mathbf{d}''}) \times \mathrm{Hom}_{Q_0}(\mathbb{C}^{\mathbf{d}''},\mathbb{C}^{\mathbf{d}}) \times \Lambda_{\mathbf{d},\mathbf{e}}^{w} \mid \\ i \in \mathrm{Hom}_{\Lambda}(X',P) \text{ injective}, p \in \mathrm{Hom}_{\Lambda}(P,X) \text{ surjective }, p \circ i = 0\}\end{aligned}$$

together with the obvious projection

$$\pi_{34} \colon \mathcal{E}_{\mathbf{e}} \to \mathcal{V}_{\mathbf{e}}^{\mathrm{sur}},\ (X',i,p,X) \mapsto (p,X)$$

is a $\mathrm{GL}_{\mathbf{d}'}$-principal bundle. In particular, π_{34} is open, and $\mathcal{E}_{\mathbf{e}}$ is an irreducible variety.

Now, by Lemma 8.2, and step (i) of the proof, $\mathcal{E}_{\mathbf{e}',\mathbf{e}}$ is a dense open subset of $\mathcal{E}_{\mathbf{e}}$. Thus π, as a composition of the open morphisms $\pi_2 \circ \pi_{34} \colon \mathcal{E}_{\mathbf{e}} \to \Lambda_{\mathbf{d},\mathbf{e}}^{w}$ and the inclusion $\mathcal{E}_{\mathbf{e}',\mathbf{e}} \hookrightarrow \mathcal{E}_{\mathbf{e}}$, is open.

(iv) Let

$$\Lambda_{\mathbf{d}',\mathbf{e}'}^{w,+} := \{X' \in \Lambda_{\mathbf{d}',\mathbf{e}'}^{w} \mid \underline{\dim}\,\mathrm{Ext}_{\Lambda}^{1}(V,X') = \tilde{\mathbf{e}}\}.$$

This is a locally closed subset of $\Lambda_{\mathbf{d}',\mathbf{e}'}^{w}$. A similar argument as in step (iii) shows that the restriction

$$\mathcal{E}_{\mathbf{e}',\mathbf{e}} \to \Lambda_{\mathbf{d}',\mathbf{e}'}^{w,+}$$

of π' is open. It remains to show that $\Lambda_{\mathbf{d}',\mathbf{e}'}^{w,+}$ is dense in $\Lambda_{\mathbf{d}',\mathbf{e}'}^{w}$. To this end we note that there exist constants $f, f' \in \mathbb{N}$ such that

$$\dim(\pi^{-1}(\pi(E))) = f \quad \text{and} \quad \dim((\pi')^{-1}(\pi'(E))) = f'$$

for all $E \in \mathcal{E}_{\mathbf{e}',\mathbf{e}}$; see step (iii) of the proof. Moreover, by Schanuel's Lemma, for $X \in \mathrm{Im}(\pi)$ we have $\pi'(\pi^{-1}(\mathrm{GL}_{\mathbf{d}}.X)) = \mathrm{GL}_{\mathbf{d}'}.X'$ for some $X' \in \mathrm{Im}(\pi')$. Thus, we have in this situation

$$\begin{aligned}\text{(8.2)} \quad \mathrm{codim}_{\Lambda_{\mathbf{d}}}(\mathrm{GL}_{\mathbf{d}}.X) &= \mathrm{codim}_{\Lambda_{\mathbf{d},\mathbf{e}}^{w}}(\mathrm{GL}_{\mathbf{d}}.X) \\ &= \mathrm{codim}_{\Lambda_{\mathbf{d}',\mathbf{e}'}^{w,+}}(\mathrm{GL}_{\mathbf{d}'}.X') \le \mathrm{codim}_{\Lambda_{\mathbf{d}'}}(\mathrm{GL}_{\mathbf{d}'}.X').\end{aligned}$$

On the other hand, since $\mathrm{Ext}_{\Lambda}^{1}(X,X) \cong \mathrm{Ext}_{\Lambda}^{1}(X',X')$, we have by Lemma 8.1 that $\mathrm{codim}_{\Lambda_{\mathbf{d}}}(\mathrm{GL}_{\mathbf{d}}.X) = \mathrm{codim}_{\Lambda_{\mathbf{d}'}}(\mathrm{GL}_{\mathbf{d}'}.X')$. This implies by (8.2) that

$$\dim(\Lambda_{\mathbf{d}',\mathbf{e}'}^{w,+}) = \dim(\Lambda_{\mathbf{d}',\mathbf{e}'}^{w}) = \dim(\Lambda_{\mathbf{d}'}).$$

□

8.4. **Bongartz's bundle construction.** By Lemma 8.2, we know that $\Lambda_{\mathbf{d}}^{w}$ is an open subset of $\Lambda_{\mathbf{d}}$ for all $\mathbf{d}$. The varieties $\Lambda_{\mathbf{d}}$ and $\Lambda_{\mathbf{d}}^{w}$ are equidimensional, and we know that their irreducible components for all possible dimension vectors $\mathbf{d}$ parametrize the dual semicanonical bases $\mathcal{S}^*$ of $\mathbb{C}[N]$, and $\mathcal{S}_w^*$ of $\mathbb{C}[N]^{N'(w)}$, respectively.

Given $\mathbf{e} \in \mathbb{N}^{R_-}$ and $c \in \mathbb{N}$ we consider the locally closed subset

$$\Lambda_{\mathbf{d}}^{(\mathbf{e},c)} := \left\{X \in \Lambda_{\mathbf{d}}^{w} \mid \dim \mathrm{Ext}_{\Lambda}^{1}(T_k,X) = \mathbf{e}(k) \text{ for } k \in R_- \text{ and } \dim \mathrm{End}_{\Lambda}(X) = c\right\}.$$

We say that a subset $\mathcal{Y} \subseteq \Lambda_{\mathbf{d}}^{w}$ is a *T-sheet of type* $\mathbf{e}$ if it is an irreducible component of some $\Lambda_{\mathbf{d}}^{(\mathbf{e},c)}$. In this case, we say that $\mathcal{Y}$ is *dense* if the Zariski closure $\overline{\mathcal{Y}}$ in $\Lambda_{\mathbf{d}}^{w}$ is an irreducible component of $\Lambda_{\mathbf{d}}^{w}$.

Clearly, $\Lambda_{\mathbf{d}}^w$ is a finite union of T-sheets, and each irreducible component contains a unique dense T-sheet.

Remark 8.4. (1) Let us recall some definitions and results from [CBS]. We write $\mathcal{Z} = \mathcal{Z}' \oplus \mathcal{Z}''$ for irreducible components, say $\mathcal{Z}' \subseteq \Lambda_{\mathbf{d}'}^w$ and $\mathcal{Z}'' \subseteq \Lambda_{\mathbf{d}''}^w$, if and only if $\mathcal{Z} \subseteq \Lambda_{\mathbf{d}'+\mathbf{d}''}^w$ is an irreducible component which contains a dense open subset $\mathcal{U}$ such that for all $X \in \mathcal{U}$ we have $X \cong X' \oplus X''$ for some $X' \in \mathcal{Z}'$ and $X'' \in \mathcal{Z}''$. This is possible if and only if $\mathrm{Ext}_\Lambda^1(\mathcal{Z}', \mathcal{Z}'') = 0$, i.e. if there are dense open subsets $\mathcal{U}' \subseteq \mathcal{Z}'$ and $\mathcal{U}'' \subseteq \mathcal{Z}''$ such that $\mathrm{Ext}_\Lambda^1(X', X'') = 0$ for all $X' \in \mathcal{U}'$ and $X'' \in \mathcal{U}''$. (Note, since $\underline{\mathcal{C}}_w$ is 2-Calabi-Yau, the conditions $\mathrm{Ext}_\Lambda^1(\mathcal{Z}', \mathcal{Z}'') = 0$ and $\mathrm{Ext}_\Lambda^1(\mathcal{Z}'', \mathcal{Z}') = 0$ are equivalent.) On the other hand, $\mathcal{Z} \subseteq \Lambda_{\mathbf{d}}^w$ is by definition *indecomposable* if it contains a dense open subset $\mathcal{U}$ such that all $X \in \mathcal{U}$ are indecomposable. For example, since $\mathrm{Ext}_\Lambda^1(T_k, T_k) = 0$, one can apply Voigt's Lemma to show that

$$\mathcal{T}_k := \overline{\mathrm{GL}_{\mathbf{d}} .T_k}$$

is an indecomposable irreducible component. (Here we set $\mathbf{d} := \underline{\dim}_\Lambda(T_k)$.)

With these definitions, each irreducible component $\mathcal{Z}$ admits an essentially unique decomposition into indecomposable irreducible components; see [CBS].

(2) We say that an irreducible component $\mathcal{Z} \subseteq \Lambda_{\mathbf{d}}^w$ is *generically T-free* if in the decomposition into indecomposable components, there is no summand of the form $\mathcal{T}_k$ for any $1 \leq k \leq r$. As a consequence of (1), each irreducible component $\mathcal{Z} \subseteq \Lambda_{\mathbf{d}}^w$ can be written uniquely as

$$\mathcal{Z} = \mathcal{Z}' \oplus \bigoplus_{k \in R} \mathcal{T}_k^{m_k}$$

with $\mathcal{Z}'$ generically T-free. If $\mathbf{e} \in \mathbb{N}^{R_-}$ is the type of the unique dense T-sheet $\mathcal{Y}' \subseteq \mathcal{Z}'$, then in the above decomposition $m_k = 0$ in case $\mathbf{e}(k) \neq 0$ by the definition of type. Moreover, the unique dense T-sheet $\mathcal{Y} \subseteq \mathcal{Z}$ has the same type as $\mathcal{Y}'$.

We have the following variant of a construction by Bongartz [Bo, Section 4.3]:

Lemma 8.5. *Let $\mathcal{Y} \subseteq \Lambda_{\mathbf{d}}^w$ be a T-sheet of type $\mathbf{e}$. Then there exists a $\mathrm{GL}_{\mathbf{d}}$-$\mathrm{GL}_{\mathbf{e}}$-variety $\mathcal{B}_{\mathcal{Y}}$ together with morphisms*

$$\begin{tikzcd} & \mathcal{B}_{\mathcal{Y}} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \mathcal{Y} & & \mathrm{mod}(\underline{\mathcal{E}}, \mathbf{e}) \end{tikzcd}$$

such that

- π_1 *is a* $\mathrm{GL}_{\mathbf{d}}$*-equivariant* $\mathrm{GL}_{\mathbf{e}}$*-principal bundle,*
- π_2 *is* $\mathrm{GL}_{\mathbf{e}}$*-equivariant and* $\mathrm{GL}_{\mathbf{d}}$*-invariant,*
- *if $Y \in \mathcal{Y}$, then $\pi_2(B) \cong \mathrm{Ext}_\Lambda^1(T, Y)$ for all $B \in \pi_1^{-1}(Y)$.*

Proof. It is easy to derive from [Bo, Section 2.4] that for any $k \in R_-$, the set

$$\mathcal{E}_{\mathcal{Y},k} := \left\{(Y, e) \mid Y \in \mathcal{Y} \text{ and } e \in \mathrm{Ext}_\Lambda^1(T_k, Y)\right\}$$

can be given the structure of an algebraic vector bundle of rank $\mathbf{e}(k)$ over $\mathcal{Y}$. It follows that

$$\mathcal{B}_{\mathcal{Y}} := \{(Y, (v_{k,l})_{k \in R_-, 1 \leq l \leq \mathbf{e}(k)}) \mid Y \in \mathcal{Y} \text{ and } (v_{k,l})_{1 \leq l \leq \mathbf{e}(k)}$$
$$\text{is a basis of } \mathrm{Ext}_\Lambda^1(T_k, Y) \text{ for all } k \in R_-\}$$

is together with the obvious projection π_1 a $\mathrm{GL}_{\mathbf{d}}$-equivariant $\mathrm{GL}_{\mathbf{e}}$-principal bundle over $\mathcal{Y}$. Then one proceeds as in [GLS5, Section 14]. $\square$

Proposition 8.6. *For $\mathcal{Z} \in \mathrm{Irr}(\mathrm{mod}(\underline{\mathcal{E}}, \mathbf{e}))$ the following are equivalent:*

(i) *$\mathcal{Z}$ is strongly reduced.*
(ii) *$\mathcal{Z} = \overline{\pi_2(\mathcal{B}_{\mathcal{Y}})}$ for some dense, generically T-free T-sheet of type $\mathbf{e}$.*

In this case, $\mathcal{Y}$ is uniquely determined, and $\mathcal{Z} = \overline{\pi_2(\mathcal{B}_{\mathcal{Y}'})}$ precisely for the dense T-sheets

$$\mathcal{Y}' \subseteq \overline{\mathcal{Y}} \oplus \bigoplus_{k \in \mathrm{Null}(\mathcal{Z})} \mathcal{T}_k^{m_k}$$

with $m_k \in \mathbb{N}$.

Proof. Since each $M \in \mathrm{mod}(\underline{\mathcal{E}})$ is of the form $M \cong \mathrm{Ext}^1_\Lambda(T, X)$ for some $X \in \mathcal{C}_w$ we conclude that $\mathcal{Z}$ is a countable union of constructible sets of the form $\pi_2(\mathcal{B}_{\mathcal{Y}})$ for certain T-sheets $\mathcal{Y}$ of type $\mathbf{e}$. Since $\mathbb{C}$ is not countable, the Baire category theorem implies that $\mathcal{Z} = \overline{\pi_2(\mathcal{B}_{\mathcal{Y}})}$ for one of these T-sheets, say $\mathcal{Y} \subseteq \Lambda^w_{\mathbf{d}}$.

There is some $c \in \mathbb{N}$ such that $\mathrm{codim}_{\mathcal{Y}}(\mathrm{GL}_{\mathbf{d}} .Y) = c$ for all $Y \in \mathcal{Y}$ by the definition of the T-sheets. We claim that then $\pi_2(\mathcal{B}_{\mathcal{Y}})$ contains a dense open subset $\mathcal{M}$ such that $\mathrm{codim}_{\mathcal{Z}}(M.\mathrm{GL}_{\mathbf{e}}) = c$ for all $M \in \mathcal{M}$. Indeed, for any $M \in \pi_2(\mathcal{B}_{\mathcal{Y}})$ we have $\mathrm{codim}_{\mathcal{B}_{\mathcal{Y}}}(\pi_2^{-1}(M.\mathrm{GL}_{\mathbf{e}})) = c$ since there are only finitely many orbits, say $\mathrm{GL}_{\mathbf{d}} .Y_s$ for $1 \le s \le t$ in $\mathcal{Y}$, such that $\mathrm{Ext}^1_\Lambda(T, Y_s) \cong M$, so that

$$\pi_2^{-1}(M.\mathrm{GL}_{\mathbf{e}}) = \bigcup_{s=1}^{t} \pi_1^{-1}(\mathrm{GL}_{\mathbf{d}} .Y_s).$$

Now, $\mathrm{codim}_{\mathcal{B}_{\mathcal{Y}}}(\pi_1^{-1}(\mathrm{GL}_{\mathbf{d}} .Y)) = \mathrm{codim}_{\mathcal{Y}}(\mathrm{GL}_{\mathbf{d}} .Y) = c$ for all $Y \in \mathcal{Y}$ since π_1 is a principal bundle. So our claim follows from Chevalley's theorem.

Finally, let $h := \mathrm{codim}_{\Lambda^w_{\mathbf{d}}}(\mathcal{Y})$. Then $\mathrm{codim}_{\Lambda^w_{\mathbf{d}}}(\mathrm{GL}_{\mathbf{d}} .Y) = c + h$ for all $Y \in \mathcal{Y}$. Thus, since each $M \in \pi_2(\mathcal{B}_{\mathcal{Y}})$ is of the form $M \cong \mathrm{Ext}^1_\Lambda(T, Y)$ for some $Y \in \mathcal{Y}$, we have by Proposition 7.1,

$$c + h = \dim \mathrm{Hom}_{\underline{\mathcal{E}}}(M, \tau_{\underline{\mathcal{E}}}(M)).$$

Thus, $\mathcal{Z}$ is strongly reduced if and only if $h = 0$. The rest is clear by Remark 8.4(2). $\square$

8.5. **Proof of Theorem 5.** Consider the epimorphism

$$\Pi_T : \mathbb{C}[N^w] \to \mathcal{A}(\underline{\Gamma}_T)$$

defined in Section 1.5. Then, by Theorem 4 and our results from Section 6.2, for $X \in \mathcal{C}_w$ we get

$$\Pi_T(\varphi_X) = \Pi_T(\theta^T_X) = x^{\mathbf{m}} \cdot \psi_{\mathrm{Ext}^1_\Lambda(T, X')},$$

where we write

$$X = X' \oplus \bigoplus_{k=1}^{r} T_k^{m_k}$$

in such a way that X' has no direct summand from $\mathrm{add}(T)$, and we set $\mathbf{m} := (m_k)_{k \in R_-}$. Thus, by Proposition 8.6, the image of the dual semicanonical basis of $\mathbb{C}[N^w]$ under Π_T is just the generic basis $\mathcal{G}^T_w$. (This is indeed a basis by [GLS5, Section 15].) This finishes the proof.

8.6. **An example.** We continue with the example from Section 4.4. Let $W := W_{\mathbf{i}} = W_1 \oplus \cdots \oplus W_6$. The quiver Γ_W of $\mathcal{E}_W$ looks as follows:

$$\begin{tikzcd}
 & & W_3 \arrow[dr] & & \\
 & W_5 \arrow[ur] \arrow[dr] & & W_1 \arrow[ll] \arrow[dr] & \\
W_6 \arrow[ur] & & W_4 \arrow[ll] \arrow[ur] & & W_2. \arrow[ll]
\end{tikzcd}$$

An easy computation yields

$$B^{(W)} = \begin{pmatrix} 0 & -1 & 1 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & -1 \\ 1 & 0 & -1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 \end{pmatrix} \quad \text{and} \quad (B^{(W)})^{-1} = \begin{pmatrix} 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 2 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}.$$

Note that $\underline{\mathcal{E}} := \underline{\mathcal{E}}_W$ is the path algebra of the quiver $\underline{\Gamma}_W$

$$\begin{tikzcd}
 & W_1 \arrow[dr, "a"] & \\
W_4 \arrow[ur, "c"] & & W_2 \arrow[ll, "b"]
\end{tikzcd}$$

modulo the ideal generated by $\{ba, cb, ac\}$. We have

$$\underline{B}^{(W)} = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}.$$

Let S_1, S_2, S_4 be the simple $\underline{\mathcal{E}}$-modules corresponding to the vertices of $\underline{\Gamma}_W$, and let P_1, P_2, P_4 and I_1, I_2, I_4 be their projective covers and injective envelopes, respectively. One easily checks that $I_1 = P_4$, $I_2 = P_1$ and $I_4 = P_2$, and that $S_1, S_2, S_4, I_1, I_2, I_4$ are the only indecomposable $\underline{\mathcal{E}}$-modules up to isomorphism. Let us write these modules as representations of $\underline{\Gamma}_W$:

S_1
$$\begin{tikzcd}
 & \mathbb{C} \arrow[dr] & \\
0 \arrow[ur] & & 0, \arrow[ll]
\end{tikzcd}$$

S_2
$$\begin{tikzcd}
 & 0 \arrow[dr] & \\
0 \arrow[ur] & & \mathbb{C}, \arrow[ll]
\end{tikzcd}$$

S_4
$$\begin{tikzcd}
 & 0 \arrow[dr] & \\
\mathbb{C} \arrow[ur] & & 0, \arrow[ll]
\end{tikzcd}$$

I_1
$$\begin{tikzcd}
 & \mathbb{C} \arrow[dr] & \\
\mathbb{C} \arrow[ur, "1"] & & 0, \arrow[ll]
\end{tikzcd}$$

I_2
$$\begin{tikzcd}
 & \mathbb{C} \arrow[dr, "1"] & \\
0 \arrow[ur] & & \mathbb{C}, \arrow[ll]
\end{tikzcd}$$

I_4
$$\begin{tikzcd}
 & 0 \arrow[dr] & \\
\mathbb{C} \arrow[ur] & & \mathbb{C}. \arrow[ll, "1"]
\end{tikzcd}$$

Next, we determine the $\underline{\mathcal{E}}$-modules $\mathrm{Ext}^1_\Lambda(W, X)$, where X runs through all 12 indecomposable Λ-modules.

$$\begin{aligned}
&\mathrm{Ext}^1_\Lambda(W, V_1) = I_1, \qquad &&\mathrm{Ext}^1_\Lambda(W, V_2) = I_2, \qquad &&\mathrm{Ext}^1_\Lambda(W, V_4) = I_3, \\
&\mathrm{Ext}^1_\Lambda(W, L_1) = S_1, \qquad &&\mathrm{Ext}^1_\Lambda(W, L_2) = S_2, \qquad &&\mathrm{Ext}^1_\Lambda(W, L_4) = S_4,
\end{aligned}$$

and we have $\mathrm{Ext}^1_\Lambda(W, W_k) = 0$ for all $1 \leq k \leq 6$. Since $\underline{\mathcal{E}}$ is a representation-finite algebra (it has only 6 indecomposable modules), each irreducible component in $\mathrm{Irr}(\underline{\mathcal{E}}, \mathbf{e})$ is the closure of some $\mathrm{GL}_\mathbf{e}$-orbit. For an $\underline{\mathcal{E}}$-module Y in $\mathrm{mod}(\underline{\mathcal{E}}, \mathbf{e})$ let $\mathcal{O}_Y := Y.\mathrm{GL}_\mathbf{e}$ be its $\mathrm{GL}_\mathbf{e}$-orbit, and let $\overline{\mathcal{O}_Y}$ be the Zariski closure of $\mathcal{O}_Y$. We get

$$\mathrm{Irr}(\underline{\mathcal{E}}) = \left\{\overline{\mathcal{O}_Y} \mid Y = I_1^{a_1} \oplus I_2^{a_2} \oplus I_4^{a_4} \oplus S_k^{s_k} \mid a_1, a_2, a_4, s_k \geq 0,\ k = 1, 2, 4\right\}.$$

An easy calculation shows that

$$\begin{aligned}\mathrm{Irr}^{\mathrm{sr}}(\underline{\mathcal{E}}) = \{\overline{\mathcal{O}_Y} \mid Y = I_1^{a_1} \oplus I_2^{a_2} \oplus I_4^{a_4} \oplus S_k^{s_k} \mid a_1, a_2, a_4, s_k \geq 0,\ k = 1, 2, 4,\\ s_1 a_4 = s_2 a_1 = s_4 a_2 = 0\}.\end{aligned}$$

Next, we compute the functions ψ_Y, where Y runs through the 6 indecomposable $\underline{\mathcal{E}}$-modules. Observe that $\hat{x}_{W,1} = x_2 x_4^{-1}$, $\hat{x}_{W,2} = x_1^{-1} x_4$ and $\hat{x}_{W,4} = x_1 x_2^{-1}$. We obtain

$$\begin{aligned}
\psi_{S_1} &= x_1^{-1}x_4(1 \cdot \hat{x}_W^{(0,0,0)}) + 1 \cdot \hat{x}_W^{(1,0,0)}) = x_1^{-1}x_4(1 + x_2x_4^{-1}),\\
\psi_{S_2} &= x_1x_2^{-1}(1 \cdot \hat{x}_W^{(0,0,0)}) + 1 \cdot \hat{x}_W^{(0,1,0)}) = x_1x_2^{-1}(1 + x_1^{-1}x_4),\\
\psi_{S_4} &= x_2x_4^{-1}(1 \cdot \hat{x}_W^{(0,0,0)}) + 1 \cdot \hat{x}_W^{(0,0,1)}) = x_2x_4^{-1}(1 + x_1x_2^{-1}),\\
\psi_{I_1} &= x_1^{-1}(1 \cdot \hat{x}_W^{(0,0,0)}) + 1 \cdot \hat{x}_W^{(1,0,0)} + 1 \cdot \hat{x}_W^{(1,0,1)}) = x_1^{-1}(1 + x_2x_4^{-1} + x_2x_4^{-1}x_1x_2^{-1}),\\
\psi_{I_2} &= x_2^{-1}(1 \cdot \hat{x}_W^{(0,0,0)}) + 1 \cdot \hat{x}_W^{(0,1,0)} + 1 \cdot \hat{x}_W^{(1,1,0)}) = x_1^{-1}(1 + x_1^{-1}x_4 + x_2x_4^{-1}x_1^{-1}x_4),\\
\psi_{I_4} &= x_4^{-1}(1 \cdot \hat{x}_W^{(0,0,0)}) + 1 \cdot \hat{x}_W^{(0,0,1)} + 1 \cdot \hat{x}_W^{(0,1,1)}) = x_1^{-1}(1 + x_1x_2^{-1} + x_1^{-1}x_4x_1x_2^{-1}).
\end{aligned}$$

The basis $\mathcal{G}_w^W$ consists then of the following 14 sets of monomials:

$$\begin{array}{cccc}
x_1^a x_2^b x_3^c, & \psi_{I_1}^a \psi_{I_2}^b \psi_{I_4}^c, & & \\[2ex]
\psi_{S_1}^a \psi_{I_1}^b \psi_{I_2}^c, & x_2^a \psi_{S_1}^b \psi_{I_1}^c, & x_4^a \psi_{S_1}^b \psi_{I_2}^c, & x_2^a x_4^b \psi_{S_1}^c,\\[2ex]
\psi_{S_2}^a \psi_{I_2}^b \psi_{I_4}^c, & x_4^a \psi_{S_2}^b \psi_{I_2}^c, & x_1^a \psi_{S_2}^b \psi_{I_4}^c, & x_1^a x_4^b \psi_{S_2}^c,\\[2ex]
\psi_{S_4}^a \psi_{I_4}^b \psi_{I_1}^c, & x_1^a \psi_{S_4}^b \psi_{I_4}^c, & x_2^a \psi_{S_4}^b \psi_{I_1}^c, & x_1^a x_2^b \psi_{S_4}^c,
\end{array}$$

where $a, b, c \geq 0$.

For example, from our calculations in Section 4.4 we get

$$\varphi_{V_1} = \varphi_{W_6}\varphi_{W_4}^{-1} + \varphi_{W_4}^{-1}\varphi_{W_5}\varphi_{W_1}^{-1}\varphi_{W_2} + \varphi_{W_3}\varphi_{W_1}^{-1}.$$

As predicted by Theorem 5 we get

$$(\Pi_T \circ \Phi_T)(\varphi_{V_1}) = x_4^{-1} + x_4^{-1}x_1^{-1}x_2 + x_1^{-1} = \psi_{I_1}.$$

9. Categorification of the twist automorphism

9.1. We define an isomorphism

$$\kappa_\mathbf{i} \colon \mathbb{C}[\varphi_{V_{\mathbf{i},1}}^{\pm 1}, \dots, \varphi_{V_{\mathbf{i},r}}^{\pm 1}] \to \mathbb{C}[\varphi_{W_{\mathbf{i},1}}^{\pm 1}, \dots, \varphi_{W_{\mathbf{i},r}}^{\pm 1}]$$

of Laurent polynomial rings by

$$\varphi_{V_{\mathbf{i},k}} \mapsto \varphi'_{V_{\mathbf{i},k}}$$

for $1 \leq k \leq r$. By the *Laurent phenomenon* [FZ1], each cluster variable of a cluster algebra is a Laurent polynomial in the cluster variables of any given cluster. It follows that $\mathbb{C}[N^w]$ is a subalgebra of both $\mathbb{C}[\varphi_{V_{\mathbf{i},1}}^{\pm 1}, \dots, \varphi_{V_{\mathbf{i},r}}^{\pm 1}]$ and $\mathbb{C}[\varphi_{W_{\mathbf{i},1}}^{\pm 1}, \dots, \varphi_{W_{\mathbf{i},r}}^{\pm 1}]$. Here we use that $W_\mathbf{i}$ is $V_\mathbf{i}$-reachable.

By Theorem 4 and the definition of $\kappa_\mathbf{i}$ we have for $X \in \mathcal{C}_w$,

$$\kappa_\mathbf{i}(\varphi_X) = \kappa_\mathbf{i}(\theta_X^{V_\mathbf{i}}) = (\varphi'_\bullet)^{(\underline{\dim}\,\mathrm{Hom}_\Lambda(V_\mathbf{i},X))\cdot B^{(V_\mathbf{i})}} F_{\Omega_w(X)}^{W_\mathbf{i}}(\hat{\varphi}'_\bullet).$$

Here we also used that $\mathrm{Ext}_\Lambda^1(V_\mathbf{i}, X)$ and $\mathrm{Ext}_\Lambda^1(W_\mathbf{i}, \Omega_w(X))$ are isomorphic as modules over $\underline{\mathrm{End}}_{\mathcal{C}_w}(V_\mathbf{i})^{\mathrm{op}} \cong \underline{\mathrm{End}}_{\mathcal{C}_w}(W_\mathbf{i})^{\mathrm{op}}$. Now, using the short exact sequence

$$0 \to \Omega_w(X) \to P(X) \to X \to 0,$$

where $P(X)$ is $\mathcal{C}_w$-projective-injective, we get

$$(\varphi'_\bullet)^{(\underline{\dim}\,\mathrm{Hom}_\Lambda(V_\mathbf{i},X))\cdot B^{(V_\mathbf{i})}} = (\varphi'_\bullet)^{(\underline{\dim}\,\mathrm{Ext}_\Lambda^1(V_\mathbf{i},\Omega_w(X)) - \underline{\dim}\,\mathrm{Hom}_\Lambda(V_\mathbf{i},\Omega_w(X)))\cdot B^{(V_\mathbf{i})}} \varphi_{P(X)}^{-1}.$$

Here we used that

$$(\varphi'_\bullet)^{(\underline{\dim}\,\mathrm{Hom}_\Lambda(V_\mathbf{i},P(X)))\cdot B^{(V_\mathbf{i})}} = \varphi_{P(X)}^{-1}.$$

Thus, we can continue with the help of Proposition 5.1 and Theorem 4:

$$\begin{aligned}(\varphi'_\bullet)^{(\underline{\dim}\,\mathrm{Hom}_\Lambda(V_\mathbf{i},X))\cdot B^{(V_\mathbf{i})}} F_{\Omega_w(X)}^{W_\mathbf{i}}(\hat{\varphi}'_\bullet) &= \varphi_{W_\mathbf{i}}^{(\underline{\dim}\,\mathrm{Hom}_\Lambda(W_\mathbf{i},\Omega_w(X)))\cdot B^{(W_\mathbf{i})}} F_X^{W_\mathbf{i}}(\hat{\varphi}_{W_\mathbf{i}})\varphi_{P(X)}^{-1} \\ &= \varphi_X \varphi_{P(X)}^{-1}.\end{aligned}$$

This proves that $\kappa_\mathbf{i}$ does not depend on the choice of the reduced word $\mathbf{i}$. Thus we can denote by κ_w the restriction of $\kappa_\mathbf{i}$ to $\mathbb{C}[N^w] \subset \mathbb{C}[\varphi_{V_{\mathbf{i},1}}^{\pm 1}, \dots, \varphi_{V_{\mathbf{i},r}}^{\pm 1}]$. Moreover, the fact that κ_w permutes the elements of the dual semicanonical basis follows directly from Proposition 8.3 and our results in [GLS5, Section 15]. It now remains to prove that $\kappa_w = (\eta_w^*)^{-1}$.

9.2. **Definitions and known results.** Before we proceed, we recall some definitions and known results. For more details we refer to [GLS5].

Let $u \in W$. We denote by $D_{\varpi_i, u(\varpi_i)}$ the restriction to N of the generalized minor $\Delta_{\varpi_i, u(\varpi_i)}$. Let O_w be the open subset of N defined by

$$O_w := \left\{ x \in N \mid D_{\varpi_j, w^{-1}(\varpi_j)}(x) \neq 0 \text{ for all } 1 \leq j \leq n \right\}.$$

Following [BZ] and [GLS5], we introduce a map $\widetilde{\eta}_w : N \cap O_w \to N^w$ by

$$\widetilde{\eta}_w(x) = [wz^T]_+.$$

Here, for g in the Kac-Moody group of $\mathfrak{g}$ admitting a Birkhoff decomposition, $[g]_+$ stands for the factor of this decomposition belonging to N. Let $N(w) = N \cap (w^{-1}N_-w)$ and $N'(w) = N \cap (w^{-1}Nw)$ be the unipotent groups associated to w [GLS5, Sections 5.2 and 8.2]. Multiplication in N induces a bijective map $N(w) \times N'(w) \to N$. The ring of $N'(w)$-invariant functions on N, denoted by $\mathbb{C}[N]^{N'(w)}$, is thus isomorphic to $\mathbb{C}[N(w)]$.

The restriction of $\widetilde{\eta}_w$ to $N(w) \cap O_w$ is an isomorphism from $N(w) \cap O_w$ to N^w. Also, $N^w \subset O_w$, and the restriction of $\widetilde{\eta}_w$ to N^w is precisely the automorphism η_w of N^w mentioned in Section 1.6.

Fix $x \in N^w$, and set $z = \eta_w^{-1}(x) \in N^w$. Also let y be the unique element of $N(w) \cap O_w$ such that $\widetilde{\eta}_w(y) = x$. It is known that $z^{-1}y \in N'(w)$. Hence, for every $\varphi \in \mathbb{C}[N]$ invariant by right translation by $N'(w)$, we have $\varphi(z) = \varphi(y)$.

Finally, let $\mathbf{i} = (i_r, \dots, i_1)$ be a reduced word for w. We know that when $\mathbf{t}$ varies over $(\mathbb{C}^*)^r$, $\underline{x}_{\mathbf{i}}(\mathbf{t})$ goes over a dense subset of N^w. For $1 \le l \le k \le r$, we set $w_k := s_{i_k} \cdots s_{i_1}$ and

$$\beta_{\mathbf{i}}(k) := w_{k-1}^{-1}(\alpha_{i_k}) = s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k}).$$

As before, let $b_{\mathbf{i}}(l,k) := -(s_{i_l} \cdots s_{i_k}(\varpi_{i_k}), \alpha_{i_l})$. Note that $b_{\mathbf{i}}(l,k) \ge 0$.

9.3. **End of the proof of Theorem 6.** To prove that $\kappa_w = (\eta_w^*)^{-1}$, we have to show that

$$(\eta_w^{-1})^*(\varphi_{V_k}) = \frac{\varphi_{\Omega_w(V_k)}}{\varphi_{P(V_k)}} = \varphi'_{V_k}$$

for all $1 \le k \le r$.

We know that $\varphi_{V_k} = D_{\varpi_{i_k}, w_k^{-1}(\varpi_{i_k})}$. Let $x := \underline{x}_{\mathbf{i}}(\mathbf{t})$ where $\mathbf{t} \in (\mathbb{C}^*)^r$, and attach to x the elements y and z as above. By Proposition 4.4 we have

$$\varphi'_{V_k}(x) = \prod_{1 \le l \le k} t_l^{-b_{\mathbf{i}}(l,k)}.$$

Hence, it is enough to show that

$$D_{\varpi_{i_k}, w_k^{-1}(\varpi_{i_k})}(z) = D_{\varpi_{i_k}, w_k^{-1}(\varpi_{i_k})}(y) = \prod_{1 \le l \le k} t_l^{-b_{\mathbf{i}}(l,k)}, \tag{9.1}$$

where the first equality follows from the fact that $D_{\varpi_{i_k}, w_k^{-1}(\varpi_{i_k})}$ is $N'(w)$-invariant for $1 \le k \le r$.

The proof is very similar to that of [BZ], so we just recall the main ideas, referring to appropriate places in [BZ] for some simple calculations. There are two steps.

(a) We first show (9.1) in the particular case when $k = k_j := \max\{s \in R \mid i_s = j\}$ for a given $1 \le j \le n$. In this case, (9.1) can be written as

$$D_{\varpi_j, w^{-1}(\varpi_j)}(y) = \prod_{1 \le l \le k} t_l^{-b_{\mathbf{i}}(l,k)}. \tag{9.2}$$

Since $\varphi'_{I_{\mathbf{i},j}} = 1/\varphi_{I_{\mathbf{i},j}}$, we already know by Proposition 4.4 that

$$D_{\varpi_j, w^{-1}(\varpi_j)}(x) = \varphi_{I_{\mathbf{i},j}}(x) = \prod_{1 \le l \le k} t_l^{b_{\mathbf{i}}(l,k)}.$$

Hence it is enough to check that

$$D_{\varpi_j, w^{-1}(\varpi_j)}(y) D_{\varpi_j, w^{-1}(\varpi_j)}(x) = 1. \tag{9.3}$$

This is proved in exactly the same way as in [BZ, Lemma 6.4 (a)], using some basic properties of generalized minors.

(b) We then show (9.1) for any $1 \le k \le r$. We write $x = x''x'$, where

$$x' := x_{i_k}(t_k) \cdots x_{i_1}(t_1) \qquad \text{and} \qquad x'' := x_{i_r}(t_r) \cdots x_{i_{k+1}}(t_{k+1}).$$

Let $N(\beta_{\mathbf{i}}(k)) = \exp(\mathfrak{n}_{\beta_{\mathbf{i}}(k)})$ denote the root subgroup of $N(w)$ associated to the root $\beta_{\mathbf{i}}(k)$. The product map gives an isomorphism of affine varieties

$$N(\beta_{\mathbf{i}}(1)) \times \cdots \times N(\beta_{\mathbf{i}}(r)) \to N(w).$$

Therefore we can write $y = y^{(1)} \cdots y^{(r)}$ with $y^{(k)} \in N(\beta_{\mathbf{i}}(k))$. Arguing as in [BZ, Propositions 5.3 and 5.4], one shows that

$$\widetilde{\eta}_{w_k}(y^{(1)} \cdots y^{(k)}) = x_{i_k}(t_k) \cdots x_{i_1}(t_1) = x'.$$

Moreover, $y' := y^{(1)} \cdots y^{(k)} \in N(w_k)$ and $y^{(k+1)}, \dots, y^{(r)} \in N'(w_k)$. Therefore, since $D_{\varpi_{i_k}, w_k^{-1}(\varpi_{i_k})}$ is $N'(w_k)$-invariant, we have

$$D_{\varpi_{i_k}, w_k^{-1}(\varpi_{i_k})}(y) = D_{\varpi_{i_k}, w_k^{-1}(\varpi_{i_k})}(y').$$

Now, using (a) with w replaced by w_k, we obtain that

$$D_{\varpi_{i_k}, w_k^{-1}(\varpi_{i_k})}(y') = \prod_{1 \le l \le k} t_l^{-b_{\mathbf{i}}(l,k)}.$$

This finishes the proof.

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