

GENERIC MODAL CUT ELIMINATION APPLIED TO CONDITIONAL LOGICS

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ABSTRACT. We develop a general criterion for cut elimination in sequent calculi for propositional modal logics, which rests on absorption of cut, contraction, weakening and inversion by the purely modal part of the rule system. Our criterion applies also to a wide variety of logics outside the realm of normal modal logic. We give extensive example instantiations of our framework to various conditional logics. For these, we obtain fully internalised calculi which are substantially simpler than those known in the literature, along with leaner proofs of cut elimination and complexity. In one case, conditional logic with modus ponens and conditional excluded middle, cut elimination and complexity were explicitly stated as open in the literature.

1. INTRODUCTION

Cut elimination, originally conceived by Gentzen [7], is one of the core concepts of proof theory and plays a major role in particular for algorithmic aspects of logic, including the complexity of automated reasoning and, via interpolation, modularity issues. The large number of logical calculi that are currently in use, in particular in various areas of computer science, motivates efforts to define families of sequent calculi that cover a variety of logics and admit uniform proofs of cut elimination, enabled by suitable sufficient conditions. Here, we present such a method for modal sequent calculi that applies to possibly non-normal modal logics, which appear, e.g. in concurrency and knowledge representation. We use a separation of the modal calculi into a fixed underlying propositional part and a modal part. The core of our criterion, that we call the absorption of cut, stipulates that an application of the cut rule to conclusions of modal rules can be replaced by a single rule application. This concept generalises the notion of resolution closed rule set [14, 18], dropping the assumption that the logic at hand is rank-1, i.e. axiomatised by formulas in which the nesting depth of modal operators is uniformly equal to 1 (such as K).

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Our method is reasonably simple and intuitive, and nevertheless applies to a wide range of modal logics. While we use normal modal logics such as K and T as running examples to illustrate our concepts at the time of introduction, our main applications are conditional logics, which have a binary modal operator \Rightarrow read as a non-monotonic implication (unlike default logics, conditional logics allow nested non-monotonic implications). In particular, we prove cut-elimination (hence, since the generic systems under consideration are analytic, the subformula property) for the conditional logic CK and all its extensions by any of the axioms conditional modus ponens (MP) $(A \Rightarrow B) \rightarrow A \rightarrow B$, where \rightarrow denotes material implication, conditional excluded middle (CEM) $(A \Rightarrow B) \vee (A \Rightarrow \neg B)$, and conditional identity (ID) $A \Rightarrow A$ using our generic procedure. An easy analysis of proof search in the arising cut-free calculi moreover establishes that the satisfiability problem of each of these logics is in PSPACE. This is a tight bound for logics not containing CEM, whereas the provability problem in CKCEM and CKIDCEM can be solved in coNP, as we show by a slightly adapted algorithmic treatment of our calculus using a dynamic programming approach in the spirit of [20]. We point out that while (different) cut-free labelled sequent calculi for CK, CKMP, CKCEM, and some further conditional logics, as well as the ensuing upper complexity bounds, have previously been presented by Olivetti et al., the corresponding issues for CKMPCEM have explicitly been left as open problems [13]; moreover, our coNP upper bounds for CKCEM and CKIDCEM improve previous upper PSPACE bounds.

Related work. A set of sufficient conditions for a sequent calculus to admit cut elimination and a subsequent analysis of the complexity of *cut elimination* (not of proof search) is presented in [16]. The range of application of this method is very wide and encompasses, e.g. first-order logic, the modal logic $S4$, linear logic, and intuitionistic propositional logic. This generality is reflected in the fact that the method as a whole is substantially more involved than ours. A simpler method for a different and comparatively restrictive class of calculi, so-called canonical calculi, is considered in [1]; this method does not apply to typical modal systems, as it considers only so-called *canonical* rules, i.e., left and right introduction rules for connectives which permit adding a common context simultaneously in the premise and the conclusion. (In fact, it might be regarded as the essence of modal logic that its rules fail to be canonical, e.g. the necessitation rule $A/\Box A$ does not generalise to $\Gamma, A/\Gamma, \Box A$ for a sequent Γ .) Moreover, the format of the rules in *op.cit.* does not allow for the introduction of more than one occurrence of a logical connective, which is necessary even for the most basic modal logics. The same applies to [5]. In [4], logical rules are treated on an individual basis, which precludes the treatment of cuts between two rule conclusions. Overall, our notion of absorption is substantially more general when compared to similar notions in the papers discussed above, which stipulate that cuts between left and right rules for the same connective are absorbed by structural rules. In our own earlier work [14], we have considered a special case of the method presented here in the restricted context of *rank-1* logics; in particular, these results did not cover logics such as $K4$, CKMP, or CKMPCEM.

This work is an extended and revised version of [15].

2. PRELIMINARIES AND NOTATION

A *modal similarity type* (or modal signature) is a set Λ of modal operators with associated arities that we keep fixed throughout the paper. Given a set V of propositional

$ \begin{array}{llll} \text{(N)} \frac{A}{\Box A} & \text{(D)} \Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B) & \text{(4)} \Box A \rightarrow \Box \Box A & \text{(R)} \Box A \rightarrow A \end{array} $
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Figure 1: Axioms and Rules of modal Hilbert Systems

variables, the set $\mathcal{F}(\Lambda)$ of Λ -formulas is given by the grammar

$$\mathcal{F}(\Lambda) \ni A, B ::= \perp \mid p \mid \neg A \mid A \wedge B \mid \heartsuit(A_1, \dots, A_n)$$

where $p \in V$ and $\heartsuit \in \Lambda$ is n -ary. We use standard abbreviations of the other propositional connectives \top , \vee and \rightarrow . A Λ -*sequent* is a finite multiset of Λ -formulas, and the set of Λ -sequents is denoted by $\mathcal{S}(\Lambda)$. We write the multiset union of Γ and Δ as Γ, Δ and identify a formula $A \in \mathcal{F}(\Lambda)$ with the singleton sequent containing only A . If $S \subseteq \mathcal{F}(\Lambda)$ is a set of formulas, then an S -*substitution* is a mapping $\sigma : V \rightarrow S$. We denote the result of uniformly substituting $\sigma(p)$ for p in a formula A by $A\sigma$. This extends pointwise to Λ -sequents so that $\Gamma\sigma = A_1\sigma, \dots, A_n\sigma$ if $\Gamma = A_1, \dots, A_n$. If $S \subseteq \mathcal{F}(\Lambda)$ is a set of Λ -formulas and $A \in \mathcal{F}(\Lambda)$, we say that A is a *propositional consequence* of S if there exist $A_1, \dots, A_n \in S$ such that $A_1 \wedge \dots \wedge A_n \rightarrow A$ is a substitution instance of a propositional tautology. We write $S \vdash_{\text{PL}} A$ if A is a propositional consequence of S and $A \vdash_{\text{PL}} B$ for $\{A\} \vdash_{\text{PL}} B$ for the case of single formulas.

3. MODAL DEDUCTION SYSTEMS

To facilitate the task of comparing the notion of provability in both Hilbert and Gentzen type proof systems, we introduce the following notion of a proof rule that can be used, without any modifications, in both systems.

Definition 3.1. A Λ -*rule* is of the form $\frac{\Gamma_1 \dots \Gamma_n}{\Gamma_0}$ where $n \geq 0$ and $\Gamma_0, \dots, \Gamma_n$ are Λ -sequents. The sequents $\Gamma_1, \dots, \Gamma_n$ are the *premises* of the rule and Γ_0 its *conclusion*. A rule $\frac{}{\Gamma_0}$ without premises is called a Λ -*axiom*, which we denote by just its conclusion, Γ_0 . A *rule set* is just a set of Λ -rules, and we say that a rule set R is *substitution closed*, if $\Gamma_1\sigma \dots \Gamma_n\sigma / \Gamma_0\sigma \in R$ whenever $\Gamma_1 \dots \Gamma_n / \Gamma_0 \in R$ and $\sigma : V \rightarrow \mathcal{F}(\Lambda)$ is a substitution.

In view of the sequent calculi that we introduce later, we read sequents disjunctively. Consequently, a rule $\Gamma_1 \dots, \Gamma_n / \Gamma_0$ can be used to prove the disjunction of Γ_0 , provided that $\bigvee \Gamma_i$ is provable, for all $1 \leq i \leq n$. We emphasise that a rule is an expression of the object language, i.e. it does not contain meta-linguistic variables. As such, it represents a specific deduction step rather than a family of possible deductions, which helps to economise on syntactic categories. In our examples, concrete rule sets are presented as instances of rule schemas.

Example 3.2. For the modal logics K , $K4$ and T , we fix the modal signature $\Lambda = \{\Box\}$ consisting of a single modal operator \Box with arity one. The language of conditional logic is given by the similarity type $\Lambda = \{\Rightarrow\}$ where the conditional arrow \Rightarrow has arity 2. We use infix notation and write $A \Rightarrow B$ instead of $\Rightarrow(A, B)$ for $A, B \in \mathcal{F}(\Lambda)$. Formulas $A \Rightarrow B$ are interpreted as various forms of conditionals, e.g. default implication ‘if A then normally B ’, relevant implication and others, depending on the choice of semantics and imposed logical principles. Deduction over modal and conditional logics are governed by the following rule sets:

$$\begin{array}{c}
\text{(RCEA)} \frac{A \leftrightarrow A'}{(A \Rightarrow B) \leftrightarrow (A' \Rightarrow B)} \quad \text{(RCK)} \frac{B_1 \wedge \dots \wedge B_n \rightarrow B}{(A \Rightarrow B_1) \wedge \dots \wedge (A \Rightarrow B_n) \rightarrow (A \Rightarrow B)} \\
\text{(ID)} A \Rightarrow A \quad \text{(MP)} (A \Rightarrow B) \rightarrow (A \rightarrow B) \quad \text{(CEM)} (A \Rightarrow B) \vee (A \Rightarrow \neg B)
\end{array}$$

Figure 2: Axioms and Rules of conditional Hilbert Systems

1. The rule set \mathbf{K} associated to the modal logic K consists of all instances of the necessitation rule (\mathbf{N}) and the distribution axiom (\mathbf{D}) in Figure 1. The rule sets that axiomatise the logics T and $K4$ arise by extending this set with the reflexivity axiom (\mathbf{R}) and the ($\mathbf{4}$)-axiom, respectively. We reserve the name (\mathbf{T}) for the reflexivity rule in a cut-free system.

2. The basic conditional logic is the system \mathbf{CK} of [3], axiomatised by the rule set that consists of all instances of (\mathbf{RCEA}) and (\mathbf{RCK}) in Figure 2. The system \mathbf{CK} constitutes a minimal set of properties to be reasonably expected of any conditional, however nonstandard: replacement of equivalents in the left hand argument, and compatibility with conjunction in the right-hand argument. Additional properties are typically imposed when more specific interpretations of \Rightarrow are intended. E.g. the basic properties of \Rightarrow viewed as a default implication are given by Burgess' System \mathcal{S} [2], which is related to the well-known KLM postulates of default reasoning [11]. A treatment of System \mathcal{S} using methods of the present work and [14] is presented in [19]. Here, we consider several other standard axioms, namely identity (\mathbf{ID}), conditional modus ponens (\mathbf{MP}) and conditional excluded middle (\mathbf{CEM}), also given in Figure 2. We denote the corresponding extensions of \mathbf{CK} by juxtaposition of the respective axioms, e.g. $\mathbf{CKMPCEM}$ contains the rules for \mathbf{CK} and the axioms (\mathbf{MP}) and (\mathbf{CEM}). As indicated above, whether or not these axioms are accepted depends on the intended reading of the conditional. E.g., modus ponens is a reasonable principle for interpretations of the conditional as a relevant implication or as a counterfactual, but not for default implication; conditional excluded middle is a controversially discussed property of the subjunctive conditional [6]. The identity axiom, while accepted for many interpretations of the conditional including as default implication, is typically rejected for causal interpretations [8].

Rules with more than one premise arise through saturation of a given rule set under cut that, e.g. leads to the rules (\mathbf{CK}_g) and (\mathbf{MP}_g) presented in Section 6.

In order to make the mapping between Hilbert-style and Gentzen-style systems easier, we take the derivability predicate of a Hilbert-system to be induced by a set of Λ -rules and read each sequent as the disjunction of its elements. The notion of deduction in modal Hilbert systems then takes the following form.

Definition 3.3. Suppose \mathbf{R} is a set of rules. The set of \mathbf{R} -derivable formulas in the Hilbert-system given by \mathbf{R} is the least set of formulas that

- contains $A\sigma$ whenever A is a propositional tautology and σ is a substitution
- contains B whenever it contains A and $A \rightarrow B$
- contains $\bigvee \Gamma_0$ whenever it contains $\bigvee \Gamma_1, \dots, \bigvee \Gamma_n$ and $\frac{\Gamma_1 \dots \Gamma_n}{\Gamma_0} \in \mathbf{R}$.

We write $\mathbf{HR} \vdash A$ if A is \mathbf{R} -derivable.

In other words, the set of derivable formulas is the least set that contains propositional tautologies, is closed under uniform substitution, modus ponens and application of rules. We will later consider Hilbert systems that induce the same provability predicate based on the following notion of admissibility.

Definition 3.4. A rule set R' is *admissible* in HR if $HR \vdash A \iff H(R \cup R') \vdash A$ for all formulas $A \in \mathcal{F}(\Lambda)$. Two rule sets R, R' are *equivalent* if R is admissible in HR' and R' is admissible in HR .

In words, R' is admissible in HR if adding the rules R' to those of R leaves the set of provable formulas unchanged. We note the following trivial, but useful consequence of admissibility.

Lemma 3.5. *Let R and R' be equivalent, and let $A \in \mathcal{F}(\Lambda)$. Then $HR \vdash A$ iff $HR' \vdash A$.*

The next proposition establishes a rudimentary form of proof normalisation in Hilbert systems and is the key for proving equivalence of Hilbert and Gentzen-type systems. We show that every derivable formula in a Hilbert-system is a propositional consequence of conclusions of rules with provable premises, which stratifies proofs into rule application and propositional reasoning and avoids modus ponens.

Proposition 3.6. *Suppose that S is the least set of formulas that is closed under propositional consequences of rule conclusions, that is, S contains $A \in \mathcal{F}(\Lambda)$ whenever there are rules $\Theta_1/\Gamma_1, \dots, \Theta_n/\Gamma_n \in R$ and substitutions $\sigma_1, \dots, \sigma_n : V \rightarrow \mathcal{F}(\Lambda)$ such that $\bigvee \Delta\sigma_i \in S$ for all $\Delta \in \Theta_i$ ($i = 1, \dots, n$), and $\{\bigvee \Gamma_1\sigma, \dots, \bigvee \Gamma_n\sigma\} \vdash_{\text{PL}} A$.*

Then S coincides with the set of derivable formulas in the Hilbert-calculus induced by R , that is $S = \{A \in \mathcal{F}(\Lambda) \mid HR \vdash A\}$.

Proof. We write $\text{HT}(R) = \{A \in \mathcal{F}(\Lambda) \mid HR \vdash A\}$ for the set of provable formulas in HR . The inclusion $S \subseteq \text{HT}(R)$ is immediate as $\text{HT}(R)$ contains propositional tautologies, is closed under uniform substitution and modus ponens. For the reverse inclusion we show that S is closed under R -derivability as considered in Definition 3.3.

This is clear for all cases (propositional tautologies, uniform substitutions, rule application) except possibly modus ponens. So assume that $HR \vdash A \rightarrow B$ and $HR \vdash A$. By induction hypothesis, there are

- Rules $\Theta_1/\Gamma_1, \dots, \Theta_n/\Gamma_n$ and substitutions $\sigma_1, \dots, \sigma_n$ such that $\{\bigvee \Gamma_1\sigma_1, \dots, \bigvee \Gamma_n\sigma_n\} \vdash_{\text{PL}} A \rightarrow B$
- Rules $\Sigma_1/\Delta_1, \dots, \Sigma_k/\Delta_k$ and substitutions τ_1, \dots, τ_k such that $\{\bigvee \Delta_1\tau_1, \dots, \bigvee \Delta_k\tau_k\} \vdash_{\text{PL}} A$

and moreover $\bigvee \Xi\sigma \in S$ whenever $\Xi \in \Theta_1, \dots, \Theta_n, \Sigma_1, \dots, \Sigma_k$. The claim follows, as $\{\bigvee \Gamma_1\sigma_1, \dots, \bigvee \Gamma_n\sigma_n, \bigvee \Delta_1\tau_1, \dots, \bigvee \Delta_k\tau_k\} \vdash_{\text{PL}} B$. \square

In other words, in a modal Hilbert system, each provable formula is a propositional consequence of rule conclusions with provable premises. This result forms the basis of our comparison of Hilbert and Gentzen systems. The point to note is that in a Hilbert system, provable formulas are propositional consequences of *zero or more* rule conclusions with provable premises. The propositional reasoning that is applied when showing that the set of conclusions implies a formula generally uses the cut-rule. As a consequence, the need for cut vanishes if there is no need to apply propositional reasoning to combine conclusions. This is what our notion of cut-absorption (introduced later in Definition 4.5) formalises: we show that cut elimination essentially amounts to the fact that – in the corresponding

Hilbert system – each valid formula is a consequence of a *at most one* rule conclusion with provable premise.

We now set the stage for sequent systems that we are going to address in the remainder of the paper. As we are dealing with extensions of classical propositional logic, it suffices to work with a right-handed calculus. Our calculus is equipped with explicit negation, and therefore precisely dual to modal tableau calculi [9] that serve as the usual basis for syntactically determining the complexity of the satisfiability problem.

The notion of derivability in the sequent calculus associated with a rule set \mathbf{R} is formulated parametric in terms of a set \mathbf{X} of additional rules that will later be instantiated with relativised versions of cut, weakening, contraction and inversion.

Definition 3.7. Suppose \mathbf{R} and \mathbf{X} are sets of Λ -rules. The set of $\mathbf{GR} + \mathbf{X}$ -derivable sequents in the Gentzen-system given by \mathbf{R} is the least set of sequents that

- contains $A, \neg A, \Gamma$ for all sequents $\Gamma \in \mathcal{S}(\Lambda)$ and formulas $A \in \mathcal{F}(\Lambda)$
- contains $\neg \perp, \Gamma$ for all $\Gamma \in \mathcal{S}(\Lambda)$
- is closed under instances of the rule schemas

$$(\neg \wedge) \frac{\Gamma, \neg A, \neg B}{\Gamma, \neg(A \wedge B)} \quad (\wedge) \frac{\Gamma, A \quad \Gamma, B}{\Gamma, A \wedge B} \quad (\neg) \frac{\Gamma, A}{\Gamma, \neg \neg A}$$

where $A \in \mathcal{F}(\Lambda)$ ranges over formulas and $\Gamma \subseteq \mathcal{F}(\Lambda)$ over multisets of formulas. We call the above rules the *propositional rules* and the formula occurring in the conclusion but not in Γ *principal* in the respective rule.

- is closed under the rules in $\mathbf{R} \cup \mathbf{X}$, i.e. it contains Γ_0 whenever it contains $\Gamma_1, \dots, \Gamma_n$ and $\frac{\Gamma_1 \dots \Gamma_n}{\Gamma_0} \in \mathbf{R} \cup \mathbf{X}$.

We write $\mathbf{GR} + \mathbf{X} \vdash \Gamma$ if Γ can be derived in this way and $\mathbf{GR} \vdash \Gamma$ if $\mathbf{X} = \emptyset$. As for Hilbert-style calculi (Definition 3.4), we call a rule set \mathbf{R}' *admissible* in \mathbf{GR} in case $\mathbf{GR} \vdash \Gamma \iff \mathbf{G}(\mathbf{R} \cup \mathbf{R}') \vdash \Gamma$ for all $\Gamma \in \mathcal{S}(\Lambda)$.

The set \mathbf{X} of extra rules will later be instantiated with a relativised version of the cut rule and additional axioms that locally capture the effect of weakening, contraction and inversion, applied to rule premises. This allows to formulate *local* conditions for the admissibility of cut that can be checked on a per-rule basis.

Many other formulations of sequent systems only permit axioms of the form $\Gamma, p, \neg p$ where $p \in V$ is a propositional atom. The reason for being more liberal here is that this makes it easier to prove admissibility of uniform substitution, at the expense of losing depth-preserving admissibility of structural rules. We come back to this matter in Remark 4.4. The following proposition is readily established by an induction on the provability predicate $\mathbf{HR} \vdash$.

Proposition 3.8. *Suppose $\Gamma \in \mathcal{S}(\Lambda)$ is a sequent. Then $\mathbf{HR} \vdash \bigvee \Gamma$ if $\mathbf{GR} \vdash \Gamma$.*

The remainder of the paper is concerned with the converse of the above proposition, which relies on specific properties of the rule set \mathbf{R} .

4. GENERIC MODAL CUT ELIMINATION

In order to establish the converse of Proposition 3.8 we need to establish that the cut rule is admissible in the Gentzen system \mathbf{GR} defined by the ruleset \mathbf{R} . Clearly, we cannot expect that cut elimination holds in general: it is well known (and easy to check)

that the sequent system arising from the rule set consisting of all instances of (N) and (D), presented in Example 3.2 does *not* enjoy cut elimination. In other words, we have to look for constructions that allow us to transform a given rule set into one for which cut elimination holds. The main result of our analysis is that cut elimination holds if the rule set under consideration satisfies four crucial requirements that are local in the sense that they can be checked on a per-rule basis without the need of carrying out a fully-fledged cut-elimination proof: absorption of weakening, contraction, inversion and cut.

The first three properties can be checked for each rule individually and amount to the admissibility of the respective principle, and the last requirement amounts to the possibility of eliminating cut between a pair of rule conclusions. We emphasise that these properties can be checked locally for the modal rules, and cut elimination will follow automatically. It is not particularly surprising that cut elimination holds under these assumptions. However, isolating the four conditions above provides us with means to convert a modal Hilbert system into an equivalent cut-free sequent calculus. We now introduce relativised versions of the structural rules that will be the main tool in the proof of cut elimination. This can be seen as permutability of structural rules: every derivation of Γ from premises $\Gamma_1, \dots, \Gamma_n$ that ends in weakening, inversion or contraction is applied can be replaced by a derivation of Γ where weakening, inversion and contraction is only applied to the premises $\Gamma_1, \dots, \Gamma_n$.

Definition 4.1. Suppose Γ is a Λ -sequent and let $A(\Gamma)$ consist of the axioms

- Γ, A for all $A \in \mathcal{F}(\Lambda)$
- Δ, A if $\Gamma = \Delta, A, A$ for some $\Delta \in \mathcal{S}(\Lambda), A \in \mathcal{F}(\Lambda)$
- Δ, A if $\Gamma = \Delta, \neg\neg A$ for some $\Delta \in \mathcal{S}(\Lambda), A \in \mathcal{F}(\Lambda)$
- $\Delta, \neg A_1, \neg A_2$ if $\Gamma = \Delta, \neg(A_1 \wedge A_2)$ for some $\Delta \in \mathcal{S}(\Lambda), A_1, A_2 \in \mathcal{F}(\Lambda)$
- Δ, A_i for $i = 1, 2$ if $\Gamma = \Delta, (A_1 \wedge A_2)$ for some $\Delta \in \mathcal{S}(\Lambda), A_1, A_2 \in \mathcal{F}(\Lambda)$

We say that a rule set R *absorbs the structural rules* if

$$\text{GR} + A(\Gamma_1) \cup \dots \cup A(\Gamma_n) \vdash \Gamma$$

for all $\frac{\Gamma_1 \dots \Gamma_n}{\Gamma_0} \in R$ and all $\Gamma \in A(\Gamma_0)$.

In other words, a deduction step that applies weakening, contraction or inversion to a rule conclusion can be replaced by a (possibly different) rule where the corresponding structural rules are applied to the premises. We discuss a number of standard examples before stating that absorption of the structural rules implies their admissibility.

Example 4.2. The rule sets containing all instances of either of the following rule schemas (K), (T), and (K4), respectively,

$$(K) \frac{\neg A_1, \dots, \neg A_n, A_0}{\neg \Box A_1, \dots, \neg \Box A_n, \Box A_0, \Gamma} \quad (T) \frac{\neg A, \neg \Box A, \Gamma}{\neg \Box A, \Gamma} \quad (K4) \frac{\neg A_1, \neg \Box A_1, \dots, \neg A_n, \neg \Box A_n, B}{\neg \Box A_1, \dots, \neg \Box A_n, \Box B, \Gamma}$$

absorb the structural rules. We note that (K) absorbs weakening due to the presence of Γ in the conclusion, and the absorption of contraction in (T) and (K4) is a consequence of the presence of the negated \Box -formulas in the premise. The absorption of inversion in a consequence of the weakening context Γ in (K) and (K4) and implied by duplicating the context Γ in (T). On the other hand, the rule sets defined by

$$\frac{\neg A_1, \dots, \neg A_n, A_0}{\neg \Box A_1, \dots, \neg \Box A_n, \Box A_0} \quad \frac{\neg A, \Gamma}{\neg \Box A, \Gamma}$$

fail to absorb the structural rules: the rule on the left fails to absorb weakening, whereas the right-hand rule does not absorb contraction.

It should be intuitively clear that absorption of structural rules implies their admissibility, which we establish next.

Proposition 4.3. *Suppose \mathbf{R} absorbs the structural rules. Then all instances of the rule schemas of weakening, contraction and inversion*

$$\frac{\Gamma}{\Gamma, A} \quad \frac{\Gamma, A, A}{\Gamma, A} \quad \frac{\Gamma, \neg\neg A}{\Gamma, A} \quad \frac{\Gamma, \neg(A_1 \wedge A_2)}{\Gamma, \neg A_1, \neg A_2} \quad \frac{\Gamma, A_1 \wedge A_2}{\Gamma, A_i} \quad (i = 1, 2)$$

where $\Gamma \in \mathcal{S}(\Lambda)$ and $A, A_1, A_2 \in \mathcal{F}(\Lambda)$ are admissible in \mathbf{GR} .

Proof. Standard induction on proofs in \mathbf{GR} where the case of propositional rules is standard and the inductive case for modal rules immediately follows from absorption. \square

Remark 4.4. 1. The main purpose for introducing the notion of absorption of structural rules (Definition 4.1) is to have a handy criterion that guarantees admissibility of the structural rules (Proposition 4.3). Our definition offers a compromise between generality and simplicity. In essence, a rule set absorbs structural rules if an application of weakening, contraction or inversion can be pushed up one level of the proof tree. A weaker version of Definition 4.1 would require that an application of weakening, contraction or inversion to a rule conclusion can be replaced by a sequence of deduction steps where the structural rule in question can not only be applied to the premises of the rule, but also freely anywhere else, provided that these additional applications are smaller in a well-founded ordering. However, we are presently not aware of any examples where this extra generality would be necessary.

2. In many sequent systems, the statement of Proposition 4.3 can be strengthened to say that weakening, contraction and inversion are depth-preserving admissible, i.e. does not increase the height of the proof tree. This is in general false for the systems considered here as axioms are of the form $A, \neg A, \Gamma$ for $A \in \mathcal{F}(\Lambda)$ and, for instance, $(A \wedge B), \neg(A \wedge B)$ is derivable with a proof of height one (being an axiom), but, e.g. $A \wedge B, \neg A, \neg B$ cannot be established by a proof of depth one (not being an axiom). It is easy to see that weakening, inversion and contraction are in fact depth-preserving admissible if only atomic axioms of the form $p, \neg p, \Gamma$ are allowed, for $p \in V$ a propositional variable. The more general form of axioms adopted in this paper allows us to simplify many constructions as we do not have to consider a congruence rule explicitly which would allow us to prove (rather than to assume as axioms) sequents of the form $\Box A, \neg\Box A, \Gamma$.

Having dealt with the structural rules, we now address our main concern: the admissibility of the cut rule. In contrast to the absorption of structural rules, we need one additional degree of freedom in that we need to allow ourselves to apply cut to a structurally smaller formula.

Definition 4.5. The *size* of a formula $A \in \mathcal{F}(\Lambda)$ is given inductively by $\text{size}(p) = \text{size}(\perp) = 1$, $\text{size}(A \wedge B) = 1 + \text{size}(A) + \text{size}(B)$, $\text{size}(\neg A) = 1 + \text{size}(A)$ and, for the modal case, $\text{size}(\heartsuit(A_1, \dots, A_n)) = 1 + \text{size}(A_1) + \dots + \text{size}(A_n)$.

A ruleset \mathbf{R} *absorbs cut*, if for all rules $(r_1) \frac{\Gamma_1 \dots \Gamma_n}{A, \Gamma_0}$, $(r_2) \frac{\Delta_1 \dots \Delta_k}{\neg A, \Delta_0} \in \mathbf{R}$

$$\mathbf{GR} + \text{Cut}(A, r_1, r_2) \vdash \Gamma_0, \Delta_0$$

where $\text{Cut}(A, r_1, r_2)$ consists of all instances of the rule schemas

$$\frac{\Gamma, C \quad \Delta, \neg C}{\Gamma, \Delta} \quad \frac{\Gamma}{\Gamma, A} \quad \frac{\Gamma, A, A}{\Gamma, A} \quad \frac{\Gamma, \neg\neg A}{\Gamma, A} \quad \frac{\Gamma, \neg(A_1 \wedge A_2)}{\Gamma, \neg A_1, \neg A_2} \quad \frac{\Gamma, A_1 \wedge A_2}{\Gamma, A_i}$$

where $\text{size}(C) < \text{size}(A)$ in the leftmost rule and $i = 1, 2$ in the rightmost schema, together with the axioms $\Gamma_1, \dots, \Gamma_n, \Delta_1, \dots, \Delta_k$ and all sequents of the form Γ, Δ where $\Gamma, \Delta \in \mathcal{S}(\Lambda)$ and, for some $B \in \mathcal{F}(\Lambda)$,

- Γ, B and $\Delta, \neg B \in \{\Gamma_1, \dots, \Gamma_n, \Delta_1, \dots, \Delta_k\}$, or
- $\Gamma, B = \Gamma_0, A$ and $\Delta, \neg B \in \{\Delta_1, \dots, \Delta_k\}$, or
- $\Gamma, B = \Delta_0, \neg A$ and $\Delta, \neg B \in \{\Gamma_1, \dots, \Gamma_n\}$.

A rule set that absorbs structural rules and the cut rule is called *absorbing*.

The intuition behind the above definition is similar to that of absorption of structural rules, but we have two additional degrees of freedom: we can not only apply the cut rule to rule premises, but we can moreover freely use both cut on structurally smaller formulas and the structural rules. This allows us to use the standard argument, a double induction on the structure of the cut formula and the size of the proof tree, to establish cut elimination. This is carried out in the proof of the next theorem.

Theorem 4.6. *Suppose R is absorbing. Then the cut rule*

$$\frac{\Gamma, A \quad \Delta, \neg A}{\Gamma, \Delta}$$

is admissible in GR.

Proof. We use Gentzen's classical method and proceed by a double induction on the size of the cut formula and the size of the proof tree. That is, we prove the statement

$$\forall A \in \mathcal{F}(\Lambda) \forall n \in \omega (n = n_1 + n_2 \ \& \ \vdash_{n_1} \Gamma, A \ \& \ \vdash_{n_2} \Delta, \neg A \implies \vdash \Gamma, \Delta)$$

by induction on $\text{size}(A)$ where, in the inductive step, we use a side induction on the size of proof trees, as indicated by the subscript of the entailment sign. Formally, the relation \vdash_n is defined inductively by $\vdash_1 \Gamma, A, \neg A$ and

$$\frac{\vdash_n A}{\vdash_{n+1} \neg \neg A} \quad \frac{\vdash_n \Gamma, \neg A, \neg B}{\vdash_{n+1} \Gamma, \neg(A \wedge B)} \quad \frac{\vdash_n \Gamma, A \quad \vdash_k \Gamma, B}{\vdash_{n+k+1} \Gamma, A \wedge B} \quad \frac{\vdash_{n_1} F_1 \dots \vdash_{n_k} F_k}{\vdash_{n_1 + \dots + n_{k+1}} F_0}$$

where, in the last rule, $\frac{F_1 \dots F_k}{F_0} \in \mathbf{R}$. We may inductively assume that the statement holds for all cut formulas $C < A$ and to prove the statement for A we have to consider the following cases:

1. cuts that arise between two rule conclusions
2. cuts that arise between a rule conclusion and the conclusion of a propositional rule or axiom
3. cuts that arise between two propositional rules.

We start with item (1), which follows directly from the fact that \mathbf{R} absorbs cut. In more detail, suppose that $\frac{F_1 \dots F_k}{F_0, A} \in \mathbf{R}$ and $\frac{G_1 \dots G_k}{G_0, \neg A} \in \mathbf{R}$ and $\vdash_{n_i} F_i$ ($i = 1, \dots, k$) and $\vdash_{m_j} G_j$ for $j = 1, \dots, l$. As \mathbf{R} absorbs cut, we have that F_0, G_0 is derivable using cuts on formulas $< A$ from the additional assumptions Γ, Δ provided that for some $D \in \mathcal{F}(\Lambda)$ we have that both Γ, D and $\Delta, \neg D$ are among the $F_1, \dots, F_k, G_1, \dots, G_l$. In case $\Gamma, D = \Delta, \neg D$ we have that $\Gamma \subseteq \Gamma, \Delta$ and $\vdash \Gamma, \Delta$ as weakening is admissible in \mathbf{GR} . Assuming that $\vdash_x \Gamma, D$ and $\vdash_y \Delta, \neg D$ for $\Gamma, D \neq \Delta, \neg D$ we have that $x + y < 2 + \sum_i n_i + \sum_j m_j$ and hence $\vdash \Gamma, \Delta$ by (inner) induction hypothesis. The fact that – in the deduction of F_0, G_0 – we may also have to use cuts on formulas $< A$ is discharged by the outer induction hypothesis and possible uses of weakening, contraction and inversion are admissible by Proposition 4.3.

As regards item (2) we only discuss a subset of the cases that showcase the need for contraction, weakening and inversion to be admissible. For the whole discussion, suppose that $\frac{F_1 \dots F_k}{F_0} \in \mathbf{R}$ and $\vdash_{n_i} F_i$ for $i = 1, \dots, k$.

- Suppose that $F_0 = F'_0, A$ and $G_0, \neg A$ is an axiom. In case $A \in G_0$ we have that $F_0 = F'_0, A \subseteq F'_0, G_0$ and $\vdash F'_0, G_0$ follows from $\vdash F'_0, A$ as GR admits weakening. In case $\neg A \notin G_0$ we have that G_0 is an axiom, and hence so is G_0, F'_0 .

- Suppose that $F_0 = F'_0, A$ and $\neg A, G_0$ has been derived using $(\neg\wedge)$. We have to discuss two cases, depending on whether or not $\neg A$ is principal in the application of $(\neg\wedge)$.

Case $A = A' \wedge B'$ and $\vdash_m \neg A', \neg B', G_0$ so that $\vdash_{m+1} \neg A, G_0$. As R absorbs structural rules, we have that $\mathbf{GR} \vdash F'_0, A$ and $\mathbf{GR} \vdash F'_0, B$. As cuts on A' and B' can be eliminated by induction hypothesis, we have $\mathbf{GR} \vdash F'_0, F'_0, G_0$ and therefore $\mathbf{GR} \vdash F'_0, G_0$ as GR admits contraction.

Case $\vdash_m \neg C, \neg D, \neg A, G_0$ so that $\vdash_{m+1} \neg(C \wedge D), \neg A, G_0$. As $m + 1 + \sum_{i=1}^k n_i < m + 1 + 1 + \sum_{i=1}^k n_i$ we may apply the inner induction hypothesis to conclude $\vdash \neg C, \neg D, G_0, F'_0$ and applying $(\neg\wedge)$ gives $\vdash F'_0, \neg(C \wedge D), G_0$.

All the other cases follow exactly the same pattern. We now focus on item (3), that is, we show how cuts between the conclusions of propositional rules and axioms can be eliminated. This is mostly standard and again we only discuss a subset of the cases. Suppose that $\vdash_n F_0, A$ and $\vdash_m G_0, \neg A$.

- If both F_0, A and $G_0, \neg A$ are axioms, then so is F_0, G_0 .
- Suppose that F_0, A has been derived using (\wedge) and $G_0, \neg A$ has been derived using $(\neg\wedge)$.

We distinguish four cases depending on whether or not A is principal in the application of (\wedge) or $(\neg\wedge)$.

Case $A = A' \wedge B'$ and $\vdash_{n_0} F_0, A, \vdash_{n_1} F_0, B'$ so that $n = n_0 + n_1 + 1$ and $\vdash_n A \wedge B, F_0$. If A is principal in the application of $(\neg\wedge)$, we have that $\vdash_{m-1} G_0, \neg A', \neg B'$. By (outer) induction hypothesis, cuts on A' and B' can be eliminated so that we have $\vdash F_0, F_0, G_0$ and it follows from closure under contraction that $\vdash F_0, G_0$.

If A is not principal in the application of $(\neg\wedge)$ we have that $\vdash_{m-1} \neg C, \neg D, \neg(A' \wedge B'), G'_0$ so that $G_0 = \neg(C \wedge D), G'_0$ and $\vdash_m \neg(A \wedge B), G'_0$. As $\vdash_n F_0, A$ and $\vdash_{m-1} \neg C, \neg D, \neg A, G'_0$ and $n + (m - 1) < n + m$ we can apply the inner induction hypothesis to eliminate the cut on A so that $\vdash F_0, \neg C, \neg D, G'_0$ and applying $(\neg\wedge)$ gives $\vdash F_0, \neg(C \wedge D), G'_0 = F_0, G_0$ as required. The two cases where A is not principal in the application of $(\neg\wedge)$ follow exactly the same pattern.

The remaining cases of cuts between propositional rules and axioms are entirely analogous, and therefore omitted. \square

We illustrate the preceding theorem by using it to derive the well-known fact that cut-elimination holds for the modal logics K, K4 and T and use it to derive cut-elimination for various conditional logics in Section 6.

Example 4.7. The rule sets K, K4 and T are absorbing. We have already seen that they absorb weakening, contraction and inversion in Example 4.2 so everything that remains to be seen is that they also absorb cut. For (K), we need to apply cut to a formula of smaller size. For the two instances

$$(r_1) \frac{\neg A_1, \dots, \neg A_n, A_0}{\neg \Box A_1, \dots, \neg \Box A_n, \Box A_0, \Gamma} \quad (r_2) \frac{\neg B_1, \dots, \neg B_k, B_0}{\neg \Box B_1, \dots, \neg \Box B_k, \Box B_0, \Delta}$$

we need to consider, up to symmetry, the cases $A_i = B_0$, $\Box A_i \in \Delta$ and $\neg\Box A_0 \in \Delta$, for $i = 1, \dots, n$. Here, we only treat the first case for $i = 1$ where we have to show that $\neg\Box A_2, \dots, \neg\Box A_n, \Box A_0, \neg\Box B_1, \dots, \neg\Box B_k, \Gamma, \Delta$ is derivable from $\text{GR} + \text{Cut}(\Box A_1, r_1, r_2)$, which follows as the latter system allows us to apply cut on $A_1 = B_0$. The case $\Box A_i \in \Delta$ and $\neg\Box A_0 \in \Delta$ are straight forward.

The argument to show that (K4) is absorbing is similar, and uses an additional (admissible) instance of cut on a formula of smaller size and contraction. For (T) we only consider instances of cut between two conclusions of

$$(r_1) \frac{\neg A, \neg\Box A, \Gamma}{\neg\Box A, \Gamma} \quad (r_2) \frac{\neg B, \neg\Box B, \Delta}{\neg\Box B, \Delta}$$

of the T-rule. We only demonstrate the case $\Box A \in \Delta$. In this case, $\Delta = \Delta', \Box A$ and we have to show that $\neg\Box B, \Gamma, \Delta'$ can be derived in $\text{Cut}(\Box A, r_1, r_2)$. The latter system allows us to cut $\neg\Box A$ between the conclusion of (T) on the left and the premise of the right hand rule, i.e., we have that $\text{Cut}(\Box A, r_1, r_2) \vdash \neg B, \neg\Box B, \Gamma, \Delta'$ and an application of (T) now gives derivability of $\neg\Box B, \Gamma, \Delta'$.

5. EQUIVALENCE OF HILBERT AND GENTZEN SYSTEMS

We now investigate the relationship between provability in a Hilbert-system and provability in the associated Gentzen system. We note the following standard lemmas that we will use later on.

Lemma 5.1. *Suppose that Λ is a modal similarity type and R is a set of Λ -rules.*

- (1) *Let $A \in \mathcal{F}(\Lambda)$ be a propositional tautology. Then $\text{GR} \vdash A$.*
- (2) *Let R be closed under substitution and $\Gamma \in \mathcal{S}(\Lambda)$. Then $\text{GR} \vdash \Gamma\sigma$ whenever $\text{GR} \vdash \Gamma$.*

Remark 5.2. Being able to prove the previous lemma is the main reason for formulating axioms as $A, \neg A, \Gamma$ where $A \in \mathcal{F}(\Lambda)$ rather than $p, \neg p, \Gamma$. Both formulations are equivalent if the modal congruence rule

$$\frac{A_1 \leftrightarrow A'_1 \quad \dots \quad A_n \leftrightarrow A'_n}{\heartsuit(A_1, \dots, A_n) \rightarrow \heartsuit(A'_1, \dots, A'_n)}$$

is admissible. However, Lemma 5.1 can be proved without the assumption that congruence is admissible using axioms of the form $A, \neg A, \Gamma$.

Theorem 5.3. *Suppose R is absorbing and substitution closed. Then $\text{GR} \vdash \Gamma \iff \text{HR} \vdash \bigvee \Gamma$ for all $\Gamma \in \mathcal{S}(\Lambda)$.*

Sketch. We only need to show the direction from right to left. Inductively assume that $\text{HR} \vdash \bigvee \Gamma$ for $\Gamma \in \mathcal{S}(\Lambda)$. By Proposition 3.8 we have that there are rules Θ_i/Γ_i and substitutions σ_i , $i = 1, \dots, n$ such that

- $\text{HR} \vdash \Delta\sigma_i$ whenever $\Delta \in \Theta_i$ ($i = 1, \dots, n$)
- $\{\bigvee \Gamma_1\sigma_1, \dots, \bigvee \Gamma_n\sigma_n\} \vdash_{\text{PL}} \bigvee \Gamma$.

By induction hypothesis, $\text{GR} \vdash \Delta\sigma_i$ for all $i = 1, \dots, n$ and $\Delta \in \Theta_i$. By Lemma 5.1 we have

$$\text{GR} \vdash \bigvee \Gamma_1\sigma_1 \wedge \dots \wedge \bigvee \Gamma_n\sigma_n \rightarrow \bigvee \Gamma.$$

The claim follows by applying cut, contraction and inversion. \square

The construction of an absorbing rule set from a given set of axioms and rules essentially boils down to adding the missing instances of cut, weakening, contraction and inversion to a given rule set. The soundness of this process is witnessed by the following two trivial lemmas (both of which rest on the fact that HR incorporates full propositional reasoning). We use these to derive an absorbing rule set for K in the present section, and to establish cut-elimination for a large range of conditional logics in the next section.

Lemma 5.4. *Suppose $\Gamma_1, \dots, \Gamma_n / \neg A, \Gamma_0$ and $\Delta_1, \dots, \Delta_k / A, \Delta_0 \in \mathbf{R}$. Then the rule $\Gamma_1, \dots, \Gamma_n, \Delta_1, \dots, \Delta_k / \Gamma_0, \Delta_0$ is admissible in HR.*

The same applies to instances of the structural rules of weakening, contraction and inversion. As we wish to extend the rule set while leaving the provability predicate in the Hilbert calculus unchanged, the following formulation is handy for our purposes – in particular it implies that we can freely use structural rules both in the premise and conclusion.

Lemma 5.5. *Let $\Gamma_1, \dots, \Gamma_n / \Gamma_0 \in \mathbf{R}$. If $\Delta_0, \dots, \Delta_k \in \mathbf{S}(\Lambda)$ and both*

$$\{\bigvee \Delta_1, \dots, \bigvee \Delta_k\} \vdash_{\text{PL}} \bigvee \Gamma_i (1 \leq i \leq n) \text{ and } \bigvee \Gamma_0 \vdash_{\text{PL}} \bigvee \Delta_0$$

then the rule $\Delta_1, \dots, \Delta_k / \Delta_0$ is admissible in HR.

This gives us a recipe for constructing rule sets that absorb contraction and cut: simply add more rules according to the lemmas above. This will not change the notion of provability in the Hilbert system, but when this process terminates, the ensuing rule set will be absorbing and gives rise to a cut free sequent calculus.

Example 5.6 (Modal Logic K). In a Hilbert-style calculus, the axiomatisation of K is usually described in terms of the distribution axiom (which we view as a rule with empty premise) and the necessitation rule:

$$(D) \quad \Box(A \rightarrow B) \rightarrow \Box A \rightarrow \Box B \quad (N) \quad \frac{A}{\Box A}$$

We first apply Lemma 5.4 to break the propositional connectives in the distribution axiom. We have that the axiom $\neg\Box(A \rightarrow B), \neg\Box A, \Box B$ is admissible by Lemma 5.5, and applying Lemma 5.4 to this axiom and the instance $A \rightarrow B / \Box(A \rightarrow B)$ of the necessitation rule gives admissibility of the all instances of

$$\frac{\neg A, B}{\neg\Box A, \Box B}$$

with the help of (admissible) propositional reasoning in the premise. The same procedure, applied to the instances

$$\frac{\neg A, B \rightarrow C}{\neg\Box A, \Box(B \rightarrow C)} \quad \neg\Box(B \rightarrow C), \neg\Box B, \Box C$$

gives admissibility of the left hand rule below,

$$\frac{\neg A, \neg B, C}{\neg\Box A, \neg\Box B, \Box C} \quad \frac{\neg A_1, \dots, \neg A_n, A_0}{\neg\Box A_1, \dots, \neg\Box A_n, \Box A_0, \Gamma}$$

and continuing in this way and absorbing weakening, we obtain admissibility of the rule on the right, where $\Gamma \in \mathbf{S}(\Lambda)$ is an arbitrary context. We have shown previously that this rule set is absorbing, and it is easy to see that it is equivalent to the rule set consisting of all instances of (N) and (D).

$$\begin{array}{c}
 (\text{CK}_g) \frac{A_0 = \dots = A_n \quad \neg B_1, \dots, \neg B_n, B_0}{\neg(A_1 \Rightarrow B_1), \dots, \neg(A_n \Rightarrow B_n), (A_0 \Rightarrow B_0), \Gamma} \\
 (\text{CKID}_g) \frac{A_0 = \dots = A_n \quad \neg A_0, \neg B_1, \dots, \neg B_n, B_0}{\neg(A_1 \Rightarrow B_1), \dots, \neg(A_n \Rightarrow B_n), (A_0 \Rightarrow B_0), \Gamma} \\
 (\text{MP}_g) \frac{A, \neg(A \Rightarrow B), \Gamma \quad \neg B, \neg(A \Rightarrow B), \Gamma}{\neg(A \Rightarrow B), \Gamma} \\
 (\text{CKCEM}_g) \frac{A_0 = \dots = A_n \quad B_0, \dots, B_j, \neg B_{j+1}, \neg B_n}{(A_0 \Rightarrow B_0), \dots, (A_j \Rightarrow B_j), \neg(A_{j+1} \Rightarrow B_{j+1}), \dots, \neg(A_n \Rightarrow B_n), \Gamma} \quad (0 \leq j \leq n) \\
 (\text{CKCEMID}_g) \frac{A_0 = \dots = A_n \quad \neg A_0, B_0, \dots, B_j, \neg B_{j+1}, \neg B_n}{(A_0 \Rightarrow B_0), \dots, (A_j \Rightarrow B_j), \neg(A_{j+1} \Rightarrow B_{j+1}), \dots, \neg(A_n \Rightarrow B_n), \Gamma} \\
 (\text{MPCEM}_g) \frac{A, (A \Rightarrow B), \Gamma \quad B, (A \Rightarrow B), \Gamma}{(A \Rightarrow B), \Gamma}
 \end{array}$$

Figure 3: Cut-Free Conditional Sequent Rules

6. APPLICATIONS: SEQUENT CALCULI FOR CONDITIONAL LOGICS

After having seen how the construction of absorbing rule sets gives rise to cut-elimination for a number of well-studied normal modal logics, in this section we construct a cut-free sequent calculus for a number of conditional logics.

Conditional logics [3] are extensions of propositional logic by a non-monotonic conditional $A \Rightarrow B$, read as “ B holds under the condition that A ”. Formulas of the form $A \Rightarrow B$ or $\neg(A \Rightarrow B)$ are called *conditional literals*, and in such a conditional literal, we refer to A as the (*conditional*) *antecedent* and to B as the (*conditional*) *consequent*. The conditional implication is non-monotonic in general, i.e. the validity of $A \Rightarrow B$ does *not* imply that $(A \wedge C) \Rightarrow B$ is also a valid statement.

Axiomatically, the first argument A of the conditional operation $A \Rightarrow B$ behaves like the \Box in neighbourhood frames and only supports replacement of equivalents, whereas the second argument B obeys the rules of K . We recall from Example 3.2 (see also Figure 2) that CK is axiomatised by the rules (RCEA) and (RCK) that we augment with a subset of (ID), (MP) and (CEM). For each system, we apply Lemma 5.5 and Lemma 5.4 to the given rule sets repeatedly to generate new rules that are automatically sound over the original Hilbert system. This procedure leads to the rules summarised in Figure 3 where we have used the following notational shorthand to express the equivalences in the premise of CK:

Notation 6.1. *If $A_0, \dots, A_n \in \mathcal{F}(\Lambda)$ are conditional formulas, we write $A_0 = \dots = A_n$ for the sequence of sequents consisting of $\neg A_0, A_i$ and $\neg A_i, A_0$ for all $1 \leq i \leq n$.*

We now discuss the arising system in detail, and start with those not containing (CEM) and then proceed to add (CEM) as an additional principle. For each system, we show cut-free completeness and develop the format of the respective rules as we go along. In summary, we obtain the following cut-free sequent calculi for extensions of (CK) summarised in Figure 4.

Hilbert System	Sequent System	Hilbert System	Sequent System
CK	CK_g	CKCEM	$CKCEM_g$
CKID	$CKID_g$	CKCEMID	$CKCEMID_g$
CKMP	$CK_g + MP_g$	CKCEMMP	$CKCEM_g + MP_g + MPCEM_g$
CKMPID	$CKID_g + MP_g$	CKCEMMPID	$CKCEMID_g + MP_g + MPCEM_g$

Figure 4: Summary of Cut-Free Sequent Systems

6.1. Cut Elimination for Extensions of CK without CEM. We first treat extensions of the basic conditional logic CK with axioms ID and MP, but not including CEM and discuss CEM later, as the effect of adding CEM leads to a more general form of the CK rule.

If we absorb cuts using Lemmas 5.4 and 5.5 we see that all instances of

$$(CK_g) \frac{A_0 = \dots = A_n \quad \neg B_1, \dots, \neg B_n, B_0}{\neg(A_1 \Rightarrow B_1), \dots, \neg(A_n \Rightarrow B_n), (A_0 \Rightarrow B_0), \Gamma}$$

are admissible in HCK. It is easy to see that the rule set CK_g is actually absorbing:

Theorem 6.2. *The rule set CK_g is absorbing and equivalent to CK. As a consequence, GCK_g has cut-elimination and $GCK_g \vdash A$ iff $HCK \vdash A$ whenever $A \in \mathcal{F}(\Lambda)$.*

Proof. Using Lemmas 5.4 and Lemma 5.5 it is immediate that the rule set CK_g is admissible in HCK. The argument that shows that CK_g is absorbing is analogous to that for the modal logic K (Example 4.7), and the result follows from Theorem 5.3. \square

The logic CKID arises from CK by adding the identity axiom $A \Rightarrow A$ to the rule set CK_H that axiomatises standard conditional logic. Applying Lemma 5.4 to the two rule instances

$$\frac{A = A \quad \neg A, B}{\neg(A \Rightarrow A), (A \Rightarrow B)} \quad \frac{}{A \Rightarrow A}$$

yields the (admissible) rule

$$\frac{\neg A, B}{A \Rightarrow B}$$

Again applying the same lemma, this time to a general instance of (CK) and the rule that we just derived, that is,

$$\frac{A_0 = C = A_1 \dots = A_n \quad \neg D, \neg B_1, \dots, \neg B_n, B_0}{\neg(C \Rightarrow D), \neg(A_1 \Rightarrow B_1), \dots, \neg(A_n \Rightarrow B_n)} \quad \frac{\neg C, D}{C \Rightarrow D}$$

now gives the rule

$$(CKID_g) \frac{A_0 = \dots = A_n \quad \neg A_0, \neg B_1, \dots, \neg B_n, B_0}{\neg(A_1 \Rightarrow B_1), \dots, \neg(A_n \Rightarrow B_n), (A_0 \Rightarrow B_0), \Gamma}$$

that can be again seen to be admissible by Lemma 5.5. It is easy to see that both (CK) and (ID) are derivable under (CKID), and we note that $(CKID_g)$ is admissible by construction. If we denote the rule set consisting of all instances of CKID by $CKID_g$, we obtain:

Proposition 6.3. *The rule set $CKID_g$ is absorbing and equivalent to CKID.*

Proof. It is easy to see that $CKID_g$ absorbs the structural rules, and that CKID is equivalent to $CKID_g$.

To see that CKID_g absorbs cut, we consider two instances of (CKID_g) , say

$$(r_1) \frac{A_0 = \dots = A_n \quad \neg B_1, \dots, \neg B_n, B_0}{\neg(A_1 \Rightarrow B_1), \dots, \neg(A_n \Rightarrow B_n), (A_0 \Rightarrow B_0), \Gamma}$$

and

$$(r_2) \frac{C_0 = \dots = C_k \quad \neg D_1, \dots, \neg D_k, D_0}{\neg(C_1 \Rightarrow D_1), \dots, \neg(C_k \Rightarrow D_k), (C_0 \Rightarrow D_0), \Delta}$$

and assume that the cut happens on $F \in \mathcal{F}(\Lambda)$. The case where $F \in \Gamma, \Delta$ is straightforward, so assume without loss of generality that $F = (A_0 \Rightarrow B_0) = (C_1 \Rightarrow D_1)$. By converting the equalities in the premise, and repeatedly applying cut on $A_0 \equiv C_1$ we obtain

$$\frac{C_0, \neg C_i \quad A_0, \neg A_i}{C_0, \neg A_i} \text{Cut}(A_0 \equiv C_1) \quad \frac{\neg C_0, C_i \quad \neg A_0, A_i}{\neg C_0, A_i} \text{Cut}(A_0 \equiv C_i)$$

so that we obtain the derivability of

$$\Sigma_1 = C_0 = A_1 = \dots = A_n = C_2 = \dots = C_k$$

in $\text{GCKID}_g + \text{Cut}(F, r_1, r_2)$ (recall Notation 6.1). The derivation

$$\frac{\frac{\frac{\neg A_0, \neg B_1, \dots, \neg B_n, B_0 \quad \neg C_0, \neg D_1, \dots, \neg D_k, D_0}{\neg A_0, \neg B_1, \dots, \neg B_n, \neg C_0, \neg D_2, \dots, \neg D_k, D_0} \text{Cut}(B_0 \equiv D_1)}{\frac{\neg C_0, \neg C_0, \neg B_1, \dots, \neg B_n, \neg D_2, \dots, \neg D_k, D_0}{\neg C_0, \neg B_1, \dots, \neg B_n, \neg D_2, \dots, \neg D_k, D_0}} \text{Cut}(C_1 \equiv A_0)}{\neg C_0, C_1} \text{Cut}(C_1 \equiv A_0)$$

where contraction on (C_0) was applied in the last step, shows that

$$\Sigma_2 = \neg C_0, \neg B_1, \dots, \neg B_n, \neg D_2, \dots, \neg D_k, D_0$$

is derivable in $\text{GCKID}_g + \text{Cut}(F, r_1, r_2)$, and applying CKID_g to Σ_1 and Σ_2 gives (cut-free) derivability of the desired sequent $\neg(A_1 \Rightarrow B_1), \dots, \neg(A_n \Rightarrow B_n), \neg(C_2 \Rightarrow D_2), \dots, \neg(C_k \Rightarrow D_k), (C_0 \Rightarrow D_0), \Gamma, \Delta$. This completes the case distinction on F and hence the proof of the proposition. \square

Before we move to the next system, we briefly demonstrate the derivation of the identity axiom in CKID_g .

Example 6.4. It is easy to say that $\text{GCKID}_g \vdash A \Rightarrow A$ for all $A \in \mathcal{F}(\Lambda)$: we pick $n = 0$ to obtain the following instance of CKID_g

$$\frac{\neg A, A}{A \Rightarrow A}$$

and note that the premise is in fact an axiom.

The logic CKMP arises by augmenting the logic CK with the additional axiom $(A \Rightarrow B) \rightarrow (A \rightarrow B)$. We briefly sketch the construction of the additional axiom that gives rise to the rule (MP_g) that we will use to establish cut-free completeness.

We consider a cut between an instance of (CK_g) and (MP) , that is, we have the derivation

$$\frac{\frac{A_0 = A_1 \quad \neg B_1, B_0}{\neg(A_1 \Rightarrow B_1), (A_0 \Rightarrow B_0)} \quad \neg(A_0 \Rightarrow B_0), A_0 \rightarrow B_0}{\neg(A_1 \Rightarrow B_1), A_0 \rightarrow B_0} \text{Cut}(A_0 \Rightarrow B_0)$$

that leads to the rule

$$\frac{A_0 = A_1 \quad \neg B_1, B_0 \quad C = (A_0 \rightarrow B_0)}{(A_1 \Rightarrow B_1), C}$$

by putting $C = A_0 \rightarrow B_0$. By Lemma 5.4, this rule is admissible, and by Lemma 5.5 so is the rule

$$\frac{A_1, C \quad \neg B_1, C}{\neg(A_1 \Rightarrow B_1), C}$$

and absorbing the structural rules (in particular contraction on $A \Rightarrow B$ and inversion) leads to the general form

$$(\text{MP}_g) \frac{A, \neg(A \Rightarrow B), \Gamma \quad \neg B, \neg(A \Rightarrow B), \Gamma}{\neg(A \Rightarrow B), \Gamma}$$

where we have elided the subscripts. The effect of adding (MP) is similar to that of enriching the modal logic \mathbf{K} with the (T)-axiom. We denote the rule set consisting of all instances of CK_g and MP_g by CKMP_g . Our cut elimination theorem then takes the following form:

Proposition 6.5. *The rule set CKMP_g is absorbing and equivalent to CKMP .*

Proof. It is clear that both (CK_g) and (MP_g) absorb the structural rules. For cut, we first consider cuts between two instances of (MP_g) , say

$$(r_1) \frac{A, \neg(A \Rightarrow B), \Gamma \quad \neg B, \neg(A \Rightarrow B), \Gamma}{\neg(A \Rightarrow B), \Gamma} \quad (r_2) \frac{C, \neg(C \Rightarrow D), \Delta \quad \neg D, \neg(C \Rightarrow D), \Delta}{\neg(C \Rightarrow D), \Delta}$$

where the cut happens on $F \in \mathcal{F}(\Lambda)$. We distinguish several cases:

Case $F = (A \Rightarrow B)$ and $F \in \Delta$. Then $\Delta = (A \Rightarrow B), \Delta'$ for some $\Delta' \in \mathcal{S}(\Lambda)$. To eliminate the cut on C , we note that the following two derivations

$$\frac{\frac{A, \neg(A \Rightarrow B), \Gamma \quad \neg B, \neg(A \Rightarrow B), \Gamma}{\neg(A \Rightarrow B), \Gamma} (\text{MP}) \quad C, \neg(C \Rightarrow D), (A \Rightarrow B), \Delta'}{C, \neg(C \Rightarrow D), \Gamma, \Delta'} (\text{cut } (F))$$

and

$$\frac{\frac{A, \neg(A \Rightarrow B), \Gamma \quad \neg B, \neg(A \Rightarrow B), \Gamma}{\neg(A \Rightarrow B), \Gamma} (\text{MP}) \quad \neg D, \neg(C \Rightarrow D), (A \Rightarrow B), \Delta'}{\neg D, \neg(C \Rightarrow D), \Gamma, \Delta'} (\text{cut } (F))$$

witness that we can use both $C, \neg(C \Rightarrow D), \Gamma, \Delta'$ and $\neg D, \neg(C \Rightarrow D), \Gamma, \Delta'$ as axioms in $\text{CKMP}_g + \text{Cut}(F, r_1, r_2)$ as the cuts occur between the premises of (r_2) and conclusions of (r_1) . Applying (MP_g) to these axioms, we obtain that $\text{CKMP}_g + \text{Cut}(F, r_1, r_2) \vdash \neg(C \Rightarrow D), \Gamma, \Delta'$.

Case $F = (C \Rightarrow D)$ and $F \in \Gamma$. This is symmetric to the case above.

Case $F \in \Gamma$ and $\neg F \in \Delta$. Then $\Gamma = \Gamma', F$ and $\Delta = \Delta', \neg F$. We have to show that

$$\neg(A \Rightarrow B), \neg(C \Rightarrow D), \Gamma', \Delta'$$

is derivable in $\text{Cut}(F, r_1, r_2)$. We note that the deduction

$$\frac{A, \neg(A \Rightarrow B), F, \Gamma' \quad C, \neg(C \Rightarrow D), \neg F, \Delta'}{A, \neg(A \Rightarrow B), C, \neg(C \Rightarrow D), \Gamma', \Delta'} (\text{Cut}(F))$$

witnesses that we may use $A, \neg(A \Rightarrow B), C, \neg(C \Rightarrow D), \Gamma', \Delta'$ as an axiom in the system $\text{CKMP}_g + \text{Cut}(F, r_1, r_2)$ as the cut on F has occurred between premises of r_1 and r_2 . The same deduction, with C replaced by $\neg D$ throughout, witnesses that this is also the case for $A, \neg(A \Rightarrow B), \neg D, \neg(C \Rightarrow D), \Gamma', \Delta'$. An application of (MP_g) now yields $\text{CKMP}_g + \text{Cut}(F, r_1, r_2) \vdash \neg(C \Rightarrow D), A, \neg(A \Rightarrow B), \Gamma', \Delta'$.

By the symmetric argument (replace A by $\neg B$) we obtain that also $\text{CKMP}_g + \text{Cut}(F, r_1, r_2) \vdash \neg(C \Rightarrow D), \neg B, \neg(A \Rightarrow B), \Gamma', \Delta'$ and an application of (MP_g) now yields $\text{CKMP}_g + \text{Cut}(F, r_1, r_2) \vdash \neg(C \Rightarrow D), \neg(A \Rightarrow B), \Gamma', \Delta'$ as required.

What is left is to consider cuts, say on $F \in \mathcal{F}(\Lambda)$, between the conclusions of the rules

$$(r_1) \frac{A_0 = \dots = A_n \quad \neg B_1, \dots, \neg B_n, B_0}{\neg(A_1 \Rightarrow B_1), \dots, \neg(A_n \Rightarrow B_n), (A_0 \Rightarrow B_0), \Gamma}$$

$$(r_2) \frac{C, \neg(C \Rightarrow D), \Delta \quad \neg D, \neg(C \Rightarrow D), \Delta}{\neg(C \Rightarrow D), \Delta}$$

As before, we need to discuss several cases.

Case $F \in \Gamma$ or $\neg F \in \Gamma$. Trivial, as the conclusion of the cut can be derived using a different weakening context Γ .

Case $F = (A_i \Rightarrow B_i)$ for some $1 \leq i \leq n$. We assume without loss of generality that $i = 1$ and have that $F = (A_1 \Rightarrow B_1) \in \Delta$ so that $\Delta = \Delta', F$. To replace the cut on F , we consider the deduction

$$\frac{\frac{A_0 = \dots = A_n \quad \neg B_1, \dots, \neg B_n, B_0}{\neg(A_1 \Rightarrow B_1), \dots, \neg(A_n \Rightarrow B_n), (A_0 \Rightarrow B_0), \Gamma} \text{ (CK)} \quad C, \neg(C \Rightarrow D), (A_1 \Rightarrow B_1), \Delta'}{\neg(A_2 \Rightarrow B_2), \dots, \neg(A_n \Rightarrow B_n), (A_0 \Rightarrow B_0), C, \neg(C \Rightarrow D), \Gamma, \Delta'} \text{ (cut (F))}$$

which witnesses that we may use

$$\Sigma_1 = \neg(A_2 \Rightarrow B_2), \dots, \neg(A_n \Rightarrow B_n), (A_0 \Rightarrow B_0), C, \neg(C \Rightarrow D), \Gamma, \Delta'$$

as an axiom in $\text{GCKMP}_g + \text{Cut}(F, r_1, r_2)$. The above deduction, with C replaced by $\neg D$ throughout, witnesses that the same is true for

$$\Sigma_2 = \neg(A_2 \Rightarrow B_2), \dots, \neg(A_n \Rightarrow B_n), (A_0 \Rightarrow B_0), \neg D, \neg(C \Rightarrow D), \Gamma, \Delta'$$

and applying (MP_g) with premises Σ_1 and Σ_2 yields $\text{GCKMP}_g + \text{Cut}(F, r_1, r_2) \vdash \neg(A_2 \Rightarrow B_2), \dots, \neg(A_n \Rightarrow B_n), (A_0 \Rightarrow B_0), \neg(C \Rightarrow D), \Gamma, \Delta'$ as required.

Case $F = (A_0 \Rightarrow B_0) = (C \Rightarrow D)$. We have to give a derivation of $\neg(A_1 \Rightarrow B_1), \dots, \neg(A_n \Rightarrow B_n), \Gamma, \Delta$ in $\text{Cut}(F, r_1, r_2)$. The deduction

$$\frac{\frac{A_0 = \dots = A_n \quad \neg B_1, \dots, \neg B_n, B_0}{\neg(A_1 \Rightarrow B_1), \dots, \neg(A_n \Rightarrow B_n), (A_0 \Rightarrow B_0), \Gamma} \text{ (CK)} \quad \neg B_0, \neg(A_0 \Rightarrow B_0), \Delta}{\neg(A_1 \Rightarrow B_1), \dots, \neg(A_n \Rightarrow B_n), \neg B_0, \Gamma, \Delta} \text{ (Cut (F))}$$

witnesses that we may use

$$\Sigma_1 = \neg(A_1 \Rightarrow B_1), \dots, \neg(A_n \Rightarrow B_n), \neg B_0, \Gamma, \Delta$$

as an axiom in $\text{GCKMP}_g + \text{Cut}(F, r_1, r_2)$ as the cut on F occurs between a conclusion of (r_1) and a premise of (r_2) . The same derivation, with $\neg B_0$ replaced by A_0 shows that the same is true for

$$\Sigma_2 = \neg(A_1 \Rightarrow B_1), \dots, \neg(A_n \Rightarrow B_n), A_0, \Gamma, \Delta.$$

We therefore have the two derivations

$$\frac{\Sigma_1 \quad \neg B_1, \dots, \neg B_n, B_0}{\neg(A_1 \Rightarrow B_1), \dots, \neg(A_n \Rightarrow B_n), \neg B_1, \dots, \neg B_n, \Gamma, \Delta} \text{ (Cut } (B_0))$$

and

$$\frac{\Sigma_2 \quad \frac{\neg A_0, A_1}{\neg A_0, A_1, B_2, \dots, B_n} \text{ (w)}}{\neg(A_1 \Rightarrow B_1), \dots, \neg(A_n \Rightarrow B_n), A_1, B_2, \dots, B_n, \Gamma, \Delta} \text{ (Cut } (A_0))$$

in $\text{GCKMP}_g + \text{Cut}(F, r_1, r_2)$. Applying (MP_g) to the conclusions of both yields that $\text{GCKMP}_g + \text{Cut}(F, r_1, r_2) \vdash \Sigma_3$ where

$$\Sigma_3 = \neg(A_1 \Rightarrow B_1), \dots, \neg(A_n \Rightarrow B_n), \neg B_2, \dots, \neg B_n, \Gamma, \Delta$$

as $\text{GCKMP}_g + \text{Cut}(F, r_1, r_2)$ contains the contraction rule. We now iterate the same scheme, where we use weakening on a successively smaller subset of B_2, \dots, B_n . First, we note that

$$\frac{\Sigma_2 \quad \frac{\neg A_0, A_2}{\neg A_0, A_2, \neg B_3, \dots, \neg B_n} \text{ (w)}}{\neg(A_1 \Rightarrow B_1), \dots, \neg(A_n \Rightarrow B_n), A_2, \neg B_3, \dots, \neg B_n, \Gamma, \Delta} \text{ (Cut } (A_0))$$

is a derivation in $\text{GCKMP}_g + \text{Cut}(F, r_1, r_2)$ and applying (MP_g) to Σ_3 and its conclusion yields $\text{GCKMP}_g + \text{Cut}(F, r_1, r_2) \vdash \Sigma_4$ where

$$\Sigma_4 = \neg(A_1 \Rightarrow B_1), \dots, \neg(A_n \Rightarrow B_n), \neg B_3, \dots, \neg B_n$$

Iterating this scheme, we finally obtain that $\text{GCKMP}_g + \text{Cut}(F, r_1, r_2) \vdash \neg(A_1 \Rightarrow B_1), \dots, \neg(A_n \Rightarrow B_n), \Gamma, \Delta$. Note that weakening and cuts on formulas of size $< \text{size}(F)$ is admissible in $\text{GCKMP}_g + \text{Cut}(F, r_1, r_2)$.

Case $F = (A_0 \Rightarrow B_0)$ and $\neg F \in \Delta$. We have that $\Delta = \neg(A_0 \Rightarrow B_0), \Delta'$ and the deduction

$$\frac{\frac{A_0 = \dots = A_n \quad \neg B_1, \dots, \neg B_n, B_0}{\neg(A_1 \Rightarrow B_1), \dots, \neg(A_n \Rightarrow B_n), (A_0 \Rightarrow B_0), \Gamma} \text{ (CK)} \quad C, \neg(C \Rightarrow D), \neg(A_0 \Rightarrow B_0), \Delta'}{\neg(A_1 \Rightarrow B_1), \dots, \neg(A_n \Rightarrow B_n), C, \neg(C \Rightarrow D), \Gamma, \Delta'} \text{ (Cut}(F))$$

witnesses that we may use

$$\Sigma_2 = \neg(A_1 \Rightarrow B_1), \dots, \neg(A_n \Rightarrow B_n), C, \neg(C \Rightarrow D), \Gamma, \Delta'$$

as an axiom in $\text{GCKMP}_g + \text{Cut}(F, r_1, r_2)$. The same derivation, with C replaced by $\neg D$, shows that the same is true for

$$\Sigma_2 = \neg(A_1 \Rightarrow B_1), \dots, \neg(A_n \Rightarrow B_n), \neg D, \neg(C \Rightarrow D), \Gamma, \Delta'$$

and applying (MP_g) with premises Σ_1 and Σ_2 yields the claim $\text{GCKMP}_g + \text{Cut}(F, r_1, r_2) \vdash \neg(A_1 \Rightarrow B_1), \dots, \neg(A_n \Rightarrow B_n), \Gamma, \Delta'$. This finishes our analysis of cuts that may arise between conclusions of the (CK_g) and the (MP_g) -rule, and hence the proof. \square

As an example, we give a derivation of (MP) in the system CKMP_g (in fact, a single application of (MP_g) suffices).

Example 6.6. If expressed solely in terms of \wedge and \neg , conditional modus ponens takes the form $\neg((A \Rightarrow B) \wedge A \wedge \neg B)$. The following derivation establishes that (MP) is derivable in the above form

$$\frac{A, \neg(A \Rightarrow B), \neg A, B \quad \neg B, \neg(A \Rightarrow B), \neg A, B}{\neg(A \Rightarrow B), \neg A, B} (\text{MP}_g)$$

$$\frac{\neg(A \Rightarrow B), \neg A, B}{\neg(A \Rightarrow B), \neg A, \neg\neg B} (\neg\neg)$$

$$\frac{\neg(A \Rightarrow B), \neg A, \neg\neg B}{\neg(A \Rightarrow B), \neg(A \wedge \neg B)} (\neg\wedge)$$

$$\frac{\neg(A \Rightarrow B), \neg(A \wedge \neg B)}{\neg((A \Rightarrow B) \wedge A \wedge \neg B)} (\neg\wedge)$$

so that (MP) is derivable in CKMP_g .

We now consider the logic that arises by adding both conditional modus ponens ($A \Rightarrow B \rightarrow (A \rightarrow B)$) and the identity axiom $A \Rightarrow A$ to the logic CK. In line with our naming conventions, this logic is called CKMPID . To obtain a cut-free axiomatisation of this logic, we consider the rule set CKMPID_g containing all instances of CKID_g and MP_g . A close inspection of the proof of Proposition 6.5 gives that CKMPID_g is absorbing, and therefore cut-free complete.

Proposition 6.7. *The rule set CKMPID_g is absorbing and equivalent to CKMPID .*

Proof. We follow the same strategy (and consider the same cases) as in the proof of Proposition 6.5 where we note that the conclusions of CK_g and CKID_g are identical, the only difference being that in premise displayed on the far right in CK_g and CKID_g there is one additional (negative) literal in where CKID_g . The proof of Proposition 6.5 can now be repeated literally by adding this extra literal to all instances of CK_g , thus turning every instance of CK_g in the proof of Proposition 6.5 into an instance of CKID_g . \square

6.2. Cut Elimination for Extensions of CKCEM. To construct an absorbing rule set for conditional logic plus the axiom

$$(\text{CEM})(A \Rightarrow B) \vee (A \Rightarrow \neg B)$$

we start from the admissible rule set for CK and close under cuts that arise with (CEM). Repeated applications of Lemma 5.4 and Lemma 5.5 lead to the rule set

$$(\text{CKCEM}_g) \frac{A_0 = \dots = A_n \quad B_0, \dots, B_j, \neg B_{j+1}, \neg B_n}{(A_0 \Rightarrow B_0), \dots, (A_j \Rightarrow B_j), \neg(A_{j+1} \Rightarrow B_{j+1}), \dots, \neg(A_n \Rightarrow B_n), \Gamma}$$

for $0 \leq j \leq n$.

Proposition 6.8. *The rule set CKCEM_g is absorbing and equivalent to CKCEM.*

Proof. Again, it suffices to check that the rule set CKCEM_g is absorbing, where the absorption of structural rules is clear. It therefore suffices to treat instances of cuts between conclusions of rules of CKCEM. Owing to the form of the CKCEM_g -rule, our argument is

very similar to that used for CK_g . We consider the following two instances

$$(r_1) \frac{A_0 = \dots = A_n \quad B_0, \dots, B_i, \neg B_{i+1}, \dots, \neg B_n}{(A_0 \Rightarrow B_0), \dots, (A_i \Rightarrow B_i), \neg(A_{i+1} \Rightarrow B_{i+1}), \dots, \neg(A_n \Rightarrow B_n), \Gamma}$$

$$(r_2) \frac{C_0 = \dots = C_m \quad D_0, \dots, D_j, \neg D_{j+1}, \dots, \neg D_m}{(C_0 \Rightarrow D_0), \dots, (C_j \Rightarrow D_j), \neg(C_{j+1} \Rightarrow D_{j+1}), \dots, \neg(C_m \Rightarrow D_m), \Delta}$$

and assume that the conclusions permit an instance of cut on $F \in \mathcal{F}(\Lambda)$. As usual, we distinguish several cases, where the cases $F \in \Gamma, F \in \Delta, \neg F \in \Gamma$ and $\neg F \in \Delta$ are trivial.

Case $F = (A_k \Rightarrow B_k) = (C_l \Rightarrow D_l)$ for $k > i$ and $0 \leq l \leq j$. Without loss of generality we assume that $k = n$ and $l = 0$ and get $A_n = C_0$ and $B_n = D_0$. Denote the sequent that arises from applying cut on F to the conclusions of r_1 and r_2 by Σ_0 and notice that, using cuts on $A_n \equiv C_0$, we have that

$$\Sigma = A_0 = A_1 = \dots = A_{n-1} = C_1 = \dots = C_m$$

is derivable in $\text{GCKCEM}_g + \text{Cut}(F, r_1, r_2)$. This feeds into the derivation

$$\frac{\frac{B_0, \dots, B_i, \neg B_{i+1}, \dots, \neg B_n \quad D_0, \dots, D_j, \neg D_{j+1}, \dots, \neg D_m}{B_0, \dots, B_i, D_1, \dots, D_j, \neg B_{i+1}, \dots, \neg B_{n-1}, \neg D_{j+1}, \dots, \neg D_m} (\text{Cut } (B_n))}{\Sigma_0} \Sigma \quad (\text{CKCEM})$$

which establishes that $\text{Cut}(F, r_1, r_2) \vdash \Sigma_0$ as desired.

The case $F = (A_k \Rightarrow B_k) \equiv (B_l \Rightarrow D_l)$ for $k > i$ and $1 \leq l \leq j$ is symmetric, which finishes the proof. \square

As a consequence, cut elimination holds in CKCEM_g . We show of the (CEM) can be derived before moving on to the next calculus.

Example 6.9. If we spell out the abbreviations of \vee in terms of \neg and \wedge , the axiom of conditional excluded middle takes the form $\neg(\neg(A \Rightarrow B) \wedge \neg(A \Rightarrow \neg B))$. The following derivation in CKCEM_g shows that this form of the axiom is derivable.

$$\frac{\frac{\frac{A = A \quad B, \neg B}{(A \Rightarrow B), (A \Rightarrow \neg B)} (\text{CKCEM}_g)}{\neg\neg(A \Rightarrow B), (A \Rightarrow \neg B)} (\neg\neg)}{\neg\neg(A \Rightarrow B), \neg\neg(A \Rightarrow \neg B)} (\neg\neg)}{\neg(\neg(A \Rightarrow B) \wedge \neg(C \Rightarrow \neg B))} (\neg\wedge)$$

In this derivation, we have chosen $j = n = 1$ and when applying (CKCEM_g) and choosing $j = 0$ yields an instance of (CK_g) .

We now consider the extension of CK with both conditional excluded middle $(A \Rightarrow B) \vee (A \Rightarrow \neg B)$ and the identity axiom $(A \Rightarrow A)$ and denote the ensuing logic by CKCEMID . As in the construction of the rule set CKID_g , we construct a rule set by applying Lemma 5.4 and 5.5 by considering cuts between an instance of (CKCEM_g) (left) and a rule arising from a cut between (CK_g) and the identity axiom (right)

$$(\text{CKCEM}_g) \frac{A_0 = \dots = A_n \quad B_0, \dots, B_j, \neg B_{j+1}, \neg B_n}{(A_0 \Rightarrow B_0), \dots, (A_j \Rightarrow B_j), \neg(A_{j+1} \Rightarrow B_{j+1}), \dots, \neg(A_n \Rightarrow B_n), \Gamma} \quad \frac{\neg C, D}{C \Rightarrow D}$$

(where $1 \leq j \leq n$) leads to the (admissible) rule schema

$$(\text{CKCEMID}_g) \frac{A_0 = \dots = A_n \quad \neg A_0, B_0, \dots, B_j, \neg B_{j+1}, \neg B_n}{(A_0 \Rightarrow B_0), \dots, (A_j \Rightarrow B_j), \neg(A_{j+1} \Rightarrow B_{j+1}), \dots, \neg(A_n \Rightarrow B_n), \Gamma}$$

that provides a cut-free axiomatisation of CKCEMID, as we now show.

Proposition 6.10. *The rule set CKCEMID_g is equivalent to CKCEMID and absorbing.*

Proof. As in the proof of Proposition 6.8 we consider cuts between two instances of (CKCEMID_g) and note that the premises of (CKCEMID_g) and (CKCEMID_g) only differ by a negative literal that is added to the premise on the far right in (CKCEMID_g) . We consider precisely the same cases as in the proof of Proposition 6.8. Using the same notation, we note that the equality $A_0 = \dots = A_n$ in particular gives $\neg A_n, A_0$ as a premise, and we use an additional cut on A_0 , followed by an instance of contraction, immediately prior to the application of (CKCEMID_g) (that we replace by an instance of (CKCEMID_g)) to show absorption of cut. \square

We now consider extending CK with both conditional modus ponens and conditional excluded middle, but have to take care of the cuts arising between MP_g and CKCEM_g , which leads to the new rule

$$(\text{MPCEM}_g) \frac{A, (A \Rightarrow B), \Gamma \quad B, (A \Rightarrow B), \Gamma}{(A \Rightarrow B), \Gamma}$$

that was obtained in the same way as (MP_g) . If we denote the extension of CKCEM_g with MP_g and MPCEM_g by CKCEMMP_g , we obtain:

Proposition 6.11. *CKCEMMP_g is absorbing and equivalent to CKCEMMP.*

Proof. It is clear that the rule set CKCEMMP_g absorbs the structural rules and it is easy to see that it is equivalent to CKCEMMP. We have to show that it absorbs cut.

Cuts between the conclusions of two instances of MP_g have already been treated in the proof of Theorem 6.5, and the proof translates verbatim to cuts between instances of MPCEM_g . We consider cuts between two instances

$$(r_1) \frac{A, (A \Rightarrow B), \Gamma \quad B, (A \Rightarrow B), \Gamma}{(A \Rightarrow B), \Gamma} \quad (r_2) \frac{C, \neg(C \Rightarrow D), \Delta \quad \neg D, \neg(C \Rightarrow D), \Delta}{\neg(C \Rightarrow D), \Delta}$$

where the cut is performed on $F \in \mathcal{F}(\Delta)$, say. The cases where either $F \in \Gamma$ and $\neg F$ in Δ or $F \in \Delta$ and $\neg F \in \Gamma$ are straightforward.

Case $F = (A \Rightarrow B) = (C \Rightarrow D)$. The derivation

$$\frac{\frac{A, (A \Rightarrow B), \Gamma \quad B, (A \Rightarrow B), \Gamma}{(A \Rightarrow B), \Gamma} (\text{MPCEM}) \quad \frac{\neg B, \neg(A \Rightarrow B), \Delta}{\neg B, \Gamma, \Delta} (\text{Cut } (A \Rightarrow B))}{\neg B, \Gamma, \Delta}$$

witnesses that we may use $\Sigma_1 = \neg B, \Gamma, \Delta$ as an axiom in $\text{GCKCEMMP} + \text{Cut}(F, r_1, r_2)$. Similarly, the derivation

$$\frac{\frac{A, \neg(A \Rightarrow B), \Delta \quad \neg B, \neg(A \Rightarrow B), \Delta}{\neg(A \Rightarrow B), \Delta} (\text{MP}) \quad \frac{B, (A \Rightarrow B), \Gamma}{B, \Gamma, \Delta} (\text{Cut } (A \Rightarrow B))}{B, \Gamma, \Delta}$$

shows that the same is true for $\Sigma_2 = B, \Gamma, \Delta$: note that in both cases, the cut was performed between an axiom and a conclusion of both rules. As $\text{size}(B) < \text{size}(A \Rightarrow B)$ we may now use cut on B to establish that $\text{GCKCEMMP}_g + \text{Cut}(F, r_1, r_2) \vdash \Gamma, \Delta$.

Case $F = (A \Rightarrow B)$ and $\neg F \in \Delta$. We have that $\Delta = \neg(A \Rightarrow B), \Delta'$. The derivation

$$\frac{\frac{A, (A \Rightarrow B), \Gamma \quad B, (A \Rightarrow B), \Gamma}{(A \Rightarrow B), \Gamma} (\text{MPCEM}) \quad C, \neg(C \Rightarrow D), \neg(A \Rightarrow B), \Delta'}{C, \neg(C \Rightarrow D), \Gamma, \Delta'} (\text{Cut } (F))$$

witnesses that we may use

$$\Sigma_1 = C, \neg(C \Rightarrow D), \Gamma, \Delta'$$

as an axiom in $\text{GCKCEMMP} + \text{Cut}(F, r_1, r_2)$. The same derivation, with C replaced by $\neg D$ shows that the same is true for

$$\Sigma_2 = \neg D, \neg(C \Rightarrow D), \Gamma, \Delta'$$

and an application of MP_g yields derivability of $\neg(C \Rightarrow D), \Gamma, \Delta'$.

Case $F = (C \Rightarrow D)$ and $\neg F \in \Gamma$. Analogous by interchanging the role of MP_g and MPCEM_g .

This leaves to consider cuts between two instances of CKCEM_g and MP_g and between CKCEM_g and MPCEM_g . We first consider the rules

$$(r_1) \frac{A, (A \Rightarrow B), \Gamma \quad B, (A \Rightarrow B), \Gamma}{(A \Rightarrow B), \Gamma}$$

$$(r_2) \frac{A_0 = \dots = A_n \quad B_0, \dots, B_j, \neg B_{j+1}, \dots, \neg B_n}{A_0 \Rightarrow B_0, \dots, (A_j \Rightarrow B_j), \neg(A_{j+1} \Rightarrow B_{j+1}), \dots, \neg(A_n \Rightarrow B_n), \Delta}$$

In this setting, all cases except the case $F = (A \Rightarrow B) = (A_i \Rightarrow B_i)$ with $i > j$ are entirely analogous to those considered in the proof of Theorem 6.5 where applications of CK_g need to be replaced by applications of CKCEM_g . For the case $F = (A \Rightarrow B) = (A_i \Rightarrow B_i)$ with $i > j$ we assume without loss of generality that $i = n$ and argue that, as in the proof of Theorem 6.5, that both

$$\Sigma_1 = (A_0 \Rightarrow B_0), \dots, (A_j \Rightarrow B_j), \neg(A_{j+1} \Rightarrow B_{j+1}), \dots, \neg(A_{n-1} \Rightarrow B_{n-1}), B_n, \Gamma, \Delta$$

and

$$\Sigma_2 = (A_0 \Rightarrow B_0), \dots, (A_j \Rightarrow B_j), \neg(A_{j+1} \Rightarrow B_{j+1}), \dots, \neg(A_{n-1} \Rightarrow B_{n-1}), A_n, \Gamma, \Delta$$

are axioms of $\text{GCKCEMMP}_g + \text{Cut}(F, r_1, r_2)$, leading to deductions ending in $\Sigma_1, B_0, \dots, B_j, \neg B_{j+1}, \dots, \neg B_{n-1}$ and $\Sigma_2, \neg A_0, B_1, \dots, B_j, \neg B_{j+1}, \dots, \neg B_{n-1}$, respectively. An application of MP_g now yields derivability of $(A_0 \Rightarrow B_0), \dots, (A_j \Rightarrow B_j), \neg(A_{j+1} \Rightarrow B_{j+1}), \dots, \neg(A_{n-1} \Rightarrow B_{n-1}), B_1, \dots, B_j, \neg B_{j+1}, \dots, \neg B_{n-1}$. Iterating the same schema, where MPCEM_g is used instead of MP_g to eliminate occurrences of $\neg B_i$ for $i > j$ finally yields that $(A_0 \Rightarrow B_0), \dots, (A_j \Rightarrow B_j), \neg(A_{j+1} \Rightarrow B_{j+1}), \dots, \neg(A_{n-1} \Rightarrow B_{n-1}), \Gamma, \Delta$ is derivable in $\text{GCKCEMMP}_g + \text{Cut}(F, r_1, r_2)$.

To see that cuts between conclusions of CKCEM_g and MPCEM_g can be eliminated, one uses the same reasoning as above, with MPCEM_g and MP_g interchanged. \square

We note that the latter theorem was left as an open problem for the sequent system presented in [13]. To complete the treatment of conditional logics, we now turn to the system CKCEMMPID that arises by extending CK with the axioms corresponding to conditional excluded middle, conditional modus ponens and identity. It follows by construction that the rule set CKCEMMPID_g, that we take as containing all instances of CKCEMID_g, (MP_g) and (MPCEM_g) induces a calculus that is equivalent to CKCEMMPID and we just have to establish absorption.

Proposition 6.12. *The rule set CKCEMMPID_g is equivalent to CKCEMMPID and absorbing.*

Proof. It is clear that CKCEMMPID_g absorbs the structural rules (this was established before for each rule schema). To see that cut is absorbed, we proceed as in the proof of Proposition 6.11 where we replace every occurrence of (CKCEM_g) by the corresponding instance of (CKCEMID_g). The additional literal in the rightmost premise of (CKCEMID_g) is treated in the same way as in the proof of Proposition 6.3. \square

In summary, we obtain the following results about extensions of the conditional logic CK.

Theorem 6.13. *Suppose that L is a combination of ID, MP, CEM. Then $GL_g \vdash A$ whenever $HL \vdash A$ for all $A \in \mathcal{F}(\Lambda)$. Moreover, cut elimination holds in GL.*

The theorem follows, in each of the cases, from Theorem 4.6 and Theorem 5.3 together with the fact that the rule set L and L_g are equivalent and the latter is absorbing.

7. COMPLEXITY OF PROOF SEARCH

It is comparatively straightforward to extract complexity bounds for provability of the logics considered above by analysing the complexity of proof search under suitable strategies in the cut-free sequent systems obtained. Clearly, in those cases where all modal rules peel off exactly one layer of modal operators, the depth of proofs is polynomial in the nesting depth of modal operators in the target formula, and therefore, proof search is in PSPACE under mild assumptions on the branching width of proofs [18, 14]. Besides improving Ladner’s classical result for K [12], we thus have

Theorem 7.1. *Provability in CK and CKID is in PSPACE.*

This reproves known complexity bounds originally shown in [13] (alternative short proofs using coalgebraic semantics are given in [17]). For CKCEM, the bound can be improved to coNP using dynamic programming in the same style as in [20]. This concept has to be handled carefully when dealing with coNP bounds, however, as in nondeterministic programs we cannot actually pretend that during the execution of stage n we have the results of the stages up to $n - 1$ stored in memory — otherwise, we could, e.g., just negate these results and arrive at proving NP=coNP. Rather, dynamic programming should be regarded as a metaphor for merging identical computations on a non-deterministic machine; in particular, we need to take care to use results of previous stages only *positively* (as done in [20]).

The point in our decision procedure where these considerations become relevant is that we will wish to apply rule (CKCEM_g) deterministically to subsequents that are as large as possible; i.e. we are interested in collecting *maximal* sets of conditional literals with provably

equivalent antecedents, where the latter equivalences are supposed to have been computed in previous stages. Here, the maximality condition carries the danger of negative use of previous results. The solution to this problem lies in the following key lemma.

Lemma 7.2. *Let $\Gamma = (A_0 \Rightarrow B_0), \dots, (A_j \Rightarrow B_j), A_{j+1} \Rightarrow B_{j+1}, \dots, A_n \Rightarrow B_n$ be a sequent consisting of conditional literals. Then Γ is provable in CKCEM_g iff for every decomposition of $\{0, \dots, n\}$ into disjoint sets I_0, \dots, I_k ($k \geq 0$), one of the following conditions holds.*

- (1) *There exists l such that $\{B_i \mid i \in I_l, 0 \leq i \leq j\} \cup \{\neg B_i \mid i \in I_l, j+1 \leq i \leq n\}$ is provable.*
- (2) *There exist $l \neq r$ and $i \in I_l, p \in I_r$ such that $A_i = A_p$ is provable.*

Proof. Only if: Since Γ consists of conditional literals, any proof of Γ must end in an application of rule CKCEM_g . Thus, there exists $I \subseteq \{0, \dots, n\}$ such that $A_i = A_p$ is provable for all $i, p \in I$ and $\{B_i \mid i \in I, 0 \leq i \leq j\} \cup \{\neg B_i \mid i \in I, j+1 \leq i \leq n\}$ is provable. Now let I_0, \dots, I_k be as in the statement. Then we have the following two cases:

- (1) There exists l such that $I \subseteq I_l$. In this case, the first alternative of the claim holds.
- (2) We have $i, p \in I$ and $r \neq l$ such that $i \in I_r, p \in I_l$. In this case, the second alternative of the claim holds.

If: Define an equivalence relation on $\{0, \dots, n\}$ by taking i and p to be equivalent if $A_i = A_p$ is provable, and let I_1, \dots, I_k be the induced disjoint decomposition of $\{0, \dots, n\}$ into equivalence classes. By construction, this decomposition does not satisfy the second alternative of the claim, hence it satisfies the first, which implies that Γ is provable by applying rule CKCEM_g . \square

This lemma now enables us to prove the announced coNP upper bound:

Theorem 7.3. *Provability in CKCEM and in CKIDCEM is in coNP.*

Proof. Since some aspects of our algorithm are more easily understood in NP style, we prove that *unprovability* of a sequent Γ can be decided in NP. We use dynamic programming as in [20]: we proceed in stages; at stage i , we decide unprovability of all sequents of the form $A, \neg B$ where A and B are subformulas of Γ with nesting depth of conditionals at most i . We perform such stages up to $i = m - 1$, where m is the maximal nesting depth of conditionals in Γ . In a further, final stage, we then check unprovability of Γ . As there are at most linearly many stages, it suffices to show that each stage can be performed in NP, and since there are at most quadratically many candidate sequents in each stage, it suffices that unprovability of a single candidate sequent can be checked in NP at each stage.

To this end, observe that proofs may generally be normalised to proceed as follows: first apply the propositional rules as long as possible, thus decomposing target sequents into sequents over *conditional literals*, i.e. literals of the form $A \Rightarrow B$ or $\neg(A \Rightarrow B)$, in the various branches of the proof, and only apply (CKCEM_g) when no more propositional rules are applicable (since all propositional rules monotonically increase the set of conditional literals when moving from the conclusion to the premises, it is clear that their – backwards – application never obstructs a possible application of (CKCEM_g)). The existential branching that arises from the conjunction rule ($A \wedge B, \Delta$ is unprovable if either A, Δ or B, Δ is unprovable) is handled non-deterministically. It is clear that one can apply only linearly many propositional rules in any given branch of the computation.

The application of rule (CKCEM_g) after exhaustion of the propositional rules is then handled according to Lemma 7.2: to check that a sequent of the form $\Delta = A_0 \Rightarrow$

$B_0, \dots, A_j \Rightarrow B_j, \neg(A_{j+1} \Rightarrow B_{j+1}), \dots, \neg(A_n \Rightarrow B_n)$ is unprovable, guess a disjoint decomposition I_1, \dots, I_k of $\{0, \dots, n\}$ and check that it violates Conditions (1) and (2) in the Lemma; the negation of these conditions introduces universal quantifiers over polynomial-sized ranges, which we check deterministically. Here, checking violation of Condition (2) amounts to using unprovability of quadratically many sequents checked in previous stages, which from the perspective of the present stage can be done in polynomial time. Checking violation of Condition (1) is more problematic, as it involves a recursive check of unprovability of sequents $\Delta_l = \{B_i \mid i \in I_l, 0 \leq i \leq j\} \cup \{\neg B_i \mid i \in I_l, j+1 \leq i \leq n\}$ for all $1 \leq l \leq k$; specifically, we have to ensure that this recursion, whose depth is limited by the nesting depth of conditionals in Γ , does not lead to exponentially long computation paths.

To this end, we note that the breadth of the part of the proof tree that we explore in one computation is given by a function $f(\Delta)$ that obeys a recursive equation of the form

$$f(\Delta) = \sum_{l=0}^k f(\Delta_l)$$

where

$$\text{size}(\Delta) \geq \sum_{l=0}^k \text{size}(\Delta_l) \tag{7.1}$$

because I_0, \dots, I_k is a disjoint decomposition of $\{0, \dots, n\}$. It follows easily that $f(\Delta)$ is at most linear in Δ , and hence the overall size of the part of the proof tree explored in one computation is at most quadratic. This finishes the proof for the case of CKCEM. The proof for CKIDCEM is entirely analogous, noting that although the main premise in rule (CKIDCEM_g) is by one literal $\neg A_0$ larger than that in the case of (CKCEM_g), estimate (7.1) remains true. \square

More interesting are those cases where some of the modal operators from the conclusion remain in the premise, such as T, K4, CKMP, and CKMPCEM (where the difference between non-iterative logics, i.e. ones whose Hilbert-axiomatisation does not use nested modalities, such as T, CKMP, and CKMPCEM, and iterative logics such as K4 is surprisingly hard to spot in the sequent presentation). For K4, the standard approach is to consider proofs of minimal depth, which therefore never attempt to prove a sequent repeatedly, and analyse the maximal depth that a branch of a proof can have without repeating a sequent. For T, a different strategy is used, where the (T) rule is limited to be applied at most once to every formula of the form $\neg \Box A$ in between two applications of (K) [10]. A similar strategy works for the conditional logics CKMP and CKMPCEM, which we explain in some additional detail for CKMP.

We let CKMP_g⁰ and CKMP_g¹ denote restricted sequent systems, defined as follows.

- In CKMP_g⁰, a formula $\neg(A \Rightarrow B)$ is *marked* on a branch as soon as the rule (MP_g) has been applied to it (backwards) and unmarked only at the next application of rule (CK_g). Rule (MP_g) applies only to unmarked formulas.
- In CKMP_g¹, we instead impose a similar restriction where rule (MP_g) applies to a sequent $\neg(A \Rightarrow B), \Gamma$ only in case Γ does not contain a propositional descendant of either A or $\neg B$. Here, a sequent Δ is called a *propositional descendant* of a formula A if it can be generated from A by applying propositional sequent rules backwards. Formally, the set \mathcal{D} of propositional descendants of A is the closure of

$\{A\}$ under the inversion rules. (E.g. the propositional descendants of $(\neg(A \wedge B) \wedge C)$ are $(\neg(A \wedge B) \wedge C)$; $\neg(A \wedge B)$; C ; and $\neg A, \neg B$.)

Our goal is to show that CKMP_g , CKMP_g^0 , and CKMP_g^1 prove the same sequents. Here, two inclusions are easy to show:

Lemma 7.4. *Every sequent that is provable in CKMP_g^1 is provable in CKMP_g^0 , and every sequent that is provable in CKMP_g^0 is provable in CKMP_g .*

Proof. The second implication is trivial, since CKMP_g^0 explicitly restricts CKMP_g . The first implication follows from the fact that whenever an occurrence of a formula $\neg(A \Rightarrow B)$ is marked in a sequent $\neg(A \Rightarrow B), \Gamma$ in a CKMP_g^0 proof, then Γ must contain a propositional descendant of either A or $\neg B$. \square

Next, we observe:

Lemma 7.5. *The system CKMP_g^1 admits inversion.*

Proof. The inductive proof for CKMP_g can just be copied due to the fact that absorption of inversion by CKMP_g never involves the introduction of additional applications of (MP_g) , and the conclusion of instances of inversion never introduces additional propositional descendants (unlike, e.g., in the case of weakening). \square

This enables us to prove the missing inclusion:

Lemma 7.6. *Every sequent that is provable in CKMP_g is provable in CKMP_g^1 .*

Proof. By Lemma 7.5, it suffices to prove that we can replace backwards applications of (MP_g) to sequents $\neg(A \Rightarrow B), \Gamma$ with Γ containing a propositional descendant of either A or $\neg B$, with subproofs using inversion. This is clear: e.g. if Γ contains a propositional descendant of A , then $\neg(A \Rightarrow B), \Gamma$ can be proved from $\neg(A \Rightarrow B), A, \Gamma$ alone by repeated application of inversion. \square

Corollary 7.7. *The systems CKMP_g and CKMP_g^0 prove the same sequents.*

This determines the complexity of proof search in CKMP :

Corollary 7.8. *Provability in CKMP is in PSPACE.*

Proof. By Corollary 7.7, it suffices to show that proof search in CKMP_g^0 is in PSPACE. The latter is shown analogously to Theorem 7.1, as proofs in CKMP_g^0 are easily seen to have at most polynomial depth. \square

The same line of reasoning applies essentially without change to CKIDMP , so that *provability in CKIDMP is in PSPACE*.

The same approach works for logics that include (CEM) , with the only actual modification being that in the systems CKMPCEM_g^0 and CKIDMPCEM_g^0 , backwards application of both (MP_g) and (MPCEM_g) is restricted to unmarked formulas. Equivalence of the restricted systems to the full systems is shown in the same manner as for CKMP . We then have, in analogy to Theorem 7.3

Theorem 7.9. *Provability in CKMPCEM and in CKIDMPCEM is in PSPACE.*

We note that the complexity of CKMPCEM was explicitly left open in [13]. We also note that regrettably we were not able to reproduce our claim from [18] that CKMPCEM is in coNP, the problem being that the estimate (7.1) breaks down in the presence of (MP_g) and $(MPCEM)_g$; since no better lower bound than coNP is currently known for CKMPCEM and CKIDMPCEM, this means that the exact complexity of these logics remains open.

8. CONCLUSIONS

We have established a generic method of cut elimination in modal sequent system based on absorption of cut and structural rules by sets of modal rules. We have applied this method in particular to various conditional logics, thus obtaining cut-free unlabelled sequent calculi that complement recently introduced labelled calculi [13]. In at least one case, the conditional logic CKMPCEM with modus ponens and conditional excluded middle, our calculus seems to be the first cut-free calculus in the literature, as cut elimination for the corresponding calculus in [13] was explicitly left open. We have applied these calculi to obtain complexity bounds on proof search in conditional logics; in particular we have reproved known upper complexity bounds for CK, CKID, CKMP [13] and improved the bound for CKCEM and CKIDCEM from PSPACE to coNP using dynamic programming techniques following [20]. Moreover, we have obtained an upper bound PSPACE for CKMPCEM, for which no bound has previously been published; a strong suspicion remains, however, that this logic is actually in coNP. We conjecture that our general method can also be applied to other base logics, e.g. intuitionistic propositional logic or first-order logic; this is the subject of further investigations.

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