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GENERIC PROPERTIES OF PARAMETRIZED VECTORFIELDS I

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This paper is concerned with vectorfields depending on a parameter. Similar problems have been studied by P. BRUNOVSKÝ [1], [2], whose works deal with one-parameter families of diffeomorphisms. These problems for parametrized vectorfields have been studied by V. I. ARNOLD [3], too.

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1. INTRODUCTION

We shall refer to [4] for some basic definitions and notations. Let X be a C^{r+1} manifold ($r \geqq 0$) and $\tau_X : T(X) \to X$ the C^r vector bundle ([4, § 6]). Denote by $\Gamma^r(\tau_X)$ the set of C^r sections of τ_X. Let A be a C^{r+1} manifold ($r \geqq 0$) and $\xi : A \times X \to T(X)$ a C^r mapping. We say that ξ is a parametrized C^r vectorfield on X (depending on a parameter in A) if for every $a \in A$, $\xi_a \in \Gamma^r(\tau_X)$, where $\xi_a(x) = \xi(a, x)$ for every $x \in X$. Let $\varphi : A \times X \times R \to X$ be a C^r mapping. Then φ is called a C^r *parametrized* flow of ξ if φ_a is the flow of ξ_a for every $a \in A$, where $\varphi_a : X \times R \to X$, $\varphi_a(x, t) = \varphi(a, x, t)$ for $(x, t) \in X \times R$. A point $x \in X$ will be called a *critical* point of a vectorfield $\eta \in \Gamma^r(\tau_X)$ if $\eta(x) = O_x$, where O_x denotes the zero of the space T_xX. The point x will be called *regular* if it is not critical.

We assume that A is an 1-dimensional C^{r+1} compact manifold and X is an n-dimensional C^{r+1} compact manifold ($r \geqq 0$).

Let us denote by $G^r(A, X)$ the set of all parametrized C^r vectorfields on $A \times X$. If $k_1, k_2 \in R$, $\xi, \eta \in G^r(A, X)$, we can define $(k_1\xi + k_2\eta)(a, x) = k_1\xi(a, x) + k_2\eta(a, x)$. Then $G^r(A, X)$ has linear structure. Let us define the mapping $\omega : G^r(A, X) \to \Gamma^r(\tau_{A \times X})$, $\omega(\xi)(a, x) = (O_a, \xi(a, x))$ for $\xi \in G^r(A, X)$, $(a, x) \in A \times X$, where O_a denotes the zero in T_aA. The mapping ω is a linear injection with closed image. By [4, Theorem 12.2] $\Gamma^r(\tau_{A \times X})$ is a second-countable Banach space. The C^r topology on $G^r(A, X)$ is the topology induced by the injection ω($N \subset G^r(A, X)$ is an open set in $G^r(A, X)$ if and only if $\omega(N) \subset \Gamma^r(\tau_{A \times X})$ is an open set in $\Gamma^r(\tau_{A \times X})$).

2. CRITICAL POINTS AT WHICH THE LINEARIZATION OF THE VECTORFIELD HAS AN EIGENVALUE 0

Let $(TX)_0 = \{O_x \in T(X) \mid x \in X\}$, where O_x denotes the zero in T_xX. $(TX)_0$ is a closed submanifold of $T(X)$. Define the set $G_0^r(A, X) = \{\xi \in G^r(A, X) \mid \xi \mathrel{\bar{\pitchfork}} (TX)_0\}$.

Lemma 1. *The set $G_0^r(A, X)$ is open and dense in $G^r(A, X)$.*

Proof. Define the mapping $\varrho : G^r(A, X) \to C^r(A \times X, T(X))$, $\varrho(\xi) = \xi$ for $\xi \in G^r(A, X)$. The mapping ϱ is a C^r representation [4, § 18]. $A \times X$ is a compact manifold and $(TX)_0$ is a closed submanifold of $T(X)$, so by [4, Theorem 18.2] the set $G_0^r(A, X) = \{\xi \in G^r(A, X) \mid \varrho(\xi) \mathrel{\bar{\pitchfork}} (TX)_0\}$ is an open set in $G^r(A, X)$. It remains to prove the density. $(TX)_0$ is diffeomorphic to X, hence codim $(TX)_0 = n$. The conditions (1), (2), (3) from [4, Theorem 19.1] are satisfied. We have to verify the condition (4) of this theorem.

The mapping $ev_\varrho : G^r(A, X) \times A \times X \to T(X)$ is such that $ev_\varrho(\xi, a, x) = \xi(a, x)$ for $\xi \in G^r(A, X)$, $(a, x) \in A \times X$. We shall prove that for every $\xi \in G^r(A, X)$, $a \in A$, $x \in X$ it is $ev_\varrho \mathrel{\bar{\pitchfork}}_{(\xi,a,x)} (TX)_0$. We have to prove that if $\xi(a, x) \in (TX)_0$, then

$$T_{(\xi,a,x)} ev_\varrho(T_\xi G^r(A, X) \times T_aA \times T_xX) \oplus T_{\xi(a,x)}(TX)_0 = T_{\xi(a,x)}\, T(X)\,.$$

It suffices to prove that for every $\dot{y} \in T_{O_x}(TX)$ there exist $\eta \in G^r(A, X)$, $\dot{a} \in T_aA$, $\dot{x} \in T_xX$, $\dot{x}_1 \in T_{O_x}(TX)_0$ such that $T_{(\xi,a,x)} ev_\varrho(\eta, \dot{a}, \dot{x}) + \dot{x}_1 = \dot{y}$. It suffices to put $\dot{a} = O_a$, where O_a denotes the zero in T_aA, $\dot{x} = O_x$ and we can choose $\eta \in G^r(A, X)$ such that $\eta(a, x) = \dot{y} - \dot{x}_1$ if $\dot{x}_1$ is chosen arbitrarily. So all assumptions from [4, Theorem 19.1] are satisfied. By this theorem the set $G_0^r(A, X)$ is dense in $G^r(A, X)$.

Define the set $K(\xi, 0) = \{(a, x) \in A \times X \mid \xi(a, x) \in (TX)_0\}$ for $\xi \in G^r(A, X)$.

Proposition 1. *If $\xi \in G_0^r(A, X)$, then $K(\xi, 0)$ is a closed, 1-dimensional C^r submanifold of $A \times X$.*

Proof. The proposition follows immediately from [4, Theorem 17.2].

If $\xi \in G^r(A, X)$, $(a, x) \in K(\xi, 0)$, then $T_{(a,x)}\xi : T_aA \times T_xX \to T_{O_x} T(X) = T_{O_x}(TX)_0 \oplus T_{O_x}(T_xX)$. Since $T_{O_x}(T_xX)$ is isomorphic to T_xX, we can identify them. Let $\pi_2 : T_{O_x}(TX) \to T_xX$ be the projection onto the second summand. We can define the mapping $\dot{\xi}(a, x) : T_aA \times T_xX \to T_xX$ by $\dot{\xi}(a, x) = \pi_2 T_{(a,x)}\xi$.

Proposition 2. *Let $\xi \in G^r(A, X)$ and $(a, x) \in K(\xi, 0)$. Then $\xi \mathrel{\bar{\pitchfork}}_{(a,x)} (TX)_0$ if and only if the mapping $\dot{\xi}(a, x)$ is surjective.*

Proof. If $(a, x) \in K(\xi, 0)$, then $\xi \mathrel{\bar{\pitchfork}}_{(a,x)} (TX)_0$ if and only if $T_{(a,x)}\xi(T_aA \times T_xX) \oplus T_{O_x}(TX)_0 = T_{O_x}(TX)$ and since $T_{O_x}(TX) = T_{O_x}(TX)_0 \oplus T_xX$, the proposition is proved.

If $(a, x) \in K(\xi, 0)$, then we can define the Hessian of ξ_a at x by $\dot{\xi}_a(x) : T_xX \to T_xX$ [4, § 22], where $\xi_a \in \Gamma^r(\tau_X)$, $\xi_a(y) = \xi(a, y)$ for $y \in X$. Denote $X_1(\xi) = \{(a, x) \in K(\xi, 0) \mid \dot{\xi}_a(x)$ is not surjective$\}$.

Let M and N be C^r manifolds and $C^r(M, N)$ the set of all C^r differentiable mappings from M into N. Let $f \in C^r(M, N)$ and $x \in M$. Denote by $J^k(f)(x)$ the k-jet from M into N of the mapping f at the point x. $J^k(M, N)$ denotes the set of all k-jets from M into N.

The mapping $\pi_1 : J^1(M, N) \to M \times N$ defined by $\pi_1(J^1(f)(x)) = (x, f(x))$ is a C^r vector bundle. If (U, α_0) is a chart on M at x and (V, β_0) is a chart on N at $f(x)$, then $(\alpha, \alpha_0 \times \beta_0, U \times V)$ is a chart on $J^1(M, N)$ at $J^1(f)(x)$, where $\alpha : \pi_1^{-1}(U \times V) \to (\alpha_0 \times \beta_0)(U \times V) \times A(n, n)$, $A(n, n)$ is the set of all $n \times n$ matrices. The set $J^1(M, N)$ is a C^{r-1} manifold of dimension $m + n + mn$, where $m = \dim M$, $n = \dim N$.

If $f \in C^r(M, N)$, $k \leqq r$, then the mapping $J^k(f) : M \to J^k(M, N)$ defined by $x \to J^k(f)(x)$ is called the *k-prolongation* of f.

Let $S_k(m, n) \subset A(m, n)$ be the set of all matrices with rank $q - k$, where $q = \min(m, n)$, $0 \leqq k \leqq q$. By [5] $S_k(m, n)$ is a submanifold of $A(m, n)$, where $A(m, n)$ denotes the set of all matrices with the differential structure induced by its natural identification with R^{mn}.

$$A(m, n) = \bigcup_{i=0}^{q} S_i(m, n), \quad \bar{S}_k(m, n) = \bigcup_{i=0}^{q-k} S_{k+i}(m, n),$$

codim $S_k(m, n) = (m - q + k)(n - q + k)$ for $0 \leqq k \leqq q$.

Denote $S_k(M, N) = \{J^1(f)(x) \in J^1(M, N) \mid D(\beta \circ f \circ \alpha^{-1})(y) \in S_k(m, n)\}$, where (U, α) is a chart on M at x, $\alpha(x) = y$ and (V, β) is a chart on N at $f(x)$. Obviously, the definition of $S_k(M, N)$ is independent of the choice of charts. $S_k(M, N)$ is a submanifold of $J^1(M, N)$ of codimension $(m - q + k)(n - q + k)$, where $q = \min(m, n)$, $0 \leqq k \leqq q$.

$$J^1(M, N) = \bigcup_{i=0}^{q} S_i(M, N), \quad \bar{S}_k(M, N) = \bigcup_{i=0}^{q-k} S_{k+i}(M, N) \quad \text{for} \quad 0 \leqq k \leqq q.$$

If $\xi \in G_0^r(A, X)$, then by Proposition 1 the set $K(\xi, 0)$ is an 1-dimensional C^r submanifold of $A \times X$. Therefore, $S_k(K(\xi, 0), A)$, $k = 0, 1$ are submanifolds of $J^1(K(\xi, 0), A)$.

Let $j = j_A \times j_X : K(\xi, 0) \to A \times X$ be the imbedding of $K(\xi, 0)$ into $A \times X$. Let $J^1(j_A) : K(\xi, 0) \to J^1(K(\xi, 0), A)$ be the 1-prolongation of the mapping j_A.

Proposition 3. *If $\xi \in G_0^r(A, X)$, then*

$$X_1(\xi) = [J^1(j_A)]^{-1} (S_1(K(\xi, 0), A)).$$

Proof. Let $(a_0, x_0) \in X_1(\xi)$. By Proposition 2 the mapping $\dot{\xi}(a_0, x_0)$ is surjective. Let (U, α) be a chart on $A \times X$ at (a_0, x_0), $\alpha(a_0, x_0) = (\mu_0, y_0)$ and (μ, y) are coordinates of the point $(a, x) \in U$. The local representation of the mapping $\dot{\xi}(a_0, x_0)$ with respect to the chart (U, α) is $D\xi_\alpha(\mu_0, y_0) = (D_\mu \xi_\alpha(\mu_0, y_0), D_y \xi_\alpha(\mu_0, y_0))$, where ξ_α is the principal part of the local representation of ξ with respect to (U, α) and D_μ, D_y denote the derivatives with respect to μ and y, respectively. $D_y \xi_\alpha(\mu_0, y_0)$ is the local representation of the mapping $\dot{\xi}_{a_0}$. Since $(a_0, x_0) \in X_1(\xi)$, so $\dot{\xi}_{a_0}$ is not a surjective mapping and therefore rank $[D_y \xi_\alpha(\mu_0, y_0)] < n$. Since $\xi \in G_0^r(A, X)$, so rank $[D\xi_\alpha(\mu_0, y_0)] = n$. Therefore, the matrix $D\xi_\alpha(\mu_0, y_0)$ has n linearly independent columns. Assume that the first n are linearly independent. Let $y_0 = (y_1^0, \ldots, y_1^0)$. Since $\xi_\alpha(\mu_0, y_1^0, \ldots, y_n^0) = 0$, it follows by implicit function theorem that there is an open neighborhood J of the point y_n^0 in R and C^r functions $\psi_i : J \to R$, $i = 0, 1, \ldots$ $\ldots, n - 1$ such that $\psi_i(y_n^0) = y_i^0$ for $i = 1, 2, \ldots, n - 1$, $\psi_0(y_n^0) = \mu_0$ and $\xi_\alpha(\psi_0(y_n), \ldots$ $\ldots, \psi_{n-1}(y_n), y_n) = 0$ for $y_n \in J$. Since det $D\xi_{\mu_0}(y_0) = 0$ so $(\mathrm{d}/\mathrm{d}y_n)\, \psi_0(y_n^0) = 0$, where $\xi_{\mu_0}(y) = \xi_\alpha(\mu_0, y)$. Therefore $J^1(j_A)\,(a_0, x_0) \subset S_1(K(\xi, 0), A)$. It has been proved that $X_1(\xi) \subset [J^1(j_A)]^{-1}\,(S_1(K(\xi, 0), A))$.

Assume $(a_0, x_0) \subset [J^1(j_A)]^{-1}\,(S_1(K(\xi, 0), A))$. Let $(a_0, x_0) \notin X_1(\xi)$. Then rank $[D_y \xi_\alpha(\mu_0, y_0)] = n$. From the implicit function theorem it follows that there is an open neighborhood J of μ_0 in R and C^r functions φ_i, $i = 1, 2, \ldots, n$ on J such that $\varphi_i(\mu_0) = y_i^0$ for $i = 1, 2, \ldots, n$ and $\xi_\alpha(\mu, \varphi_1(\mu), \ldots, \varphi_n(\mu)) = 0$ for $\mu \in J$. Therefore, there is a chart (W_1, β_1) on A at x_0 and a chart (W_2, β_2) on X at x_0 such that

$$(\beta_1 \times \beta_2)\,[(W_1 \times W_2) \cap K(\xi, 0)] = \{(\mu, y) \mid (\mu, y) = (\mu, \varphi_1(\mu), \ldots, \varphi_n(\mu))\}\,.$$

Therefore rank $[D(\beta_1 \circ j_A \circ \beta^{-1})\,(\mu_0, y_0) \neq 0$ and this contradicts the assumption. Therefore $(a_0, x_0) \in X_1(\xi)$ and so $[J^1(j_A)]^{-1}\,(S_1(K(\xi, 0), A)) \subset X_1(\xi)$.

Lemma 2. *Let $\xi \in G_0^r(A, X)$, $r \geqq 2$ and let $K_0 \subset K(\xi, 0)$ be a compact set. Then the set*

$$V(\xi) = \{f \in C^r(K(\xi, 0), A) \mid J^1(f) \mathrel{\bar{\pitchfork}} S_1(K(\xi, 0), A) \quad on \quad K_0\}$$

is open and dense in $C^r(K(\xi, 0), A)$.

Proof. Since $\bar{S}_k(K(\xi, 0), A) = \bigcup_{i=0}^{1-k} S_{k+i}(K(\xi, 0), A)$, $k = 0, 1$, so $\bar{S}_1(K(\xi, 0), A) = S_1(K(\xi, 0), A)$. By [5, Theorem 1, II. § 7] the set $\{f \in C^r(K(\xi, 0), A) \mid J^1(f) \mathrel{\bar{\pitchfork}} S_1(K(\xi, 0), A)\}$ is dense in $C^r(K(\xi, 0), A)$ and so the set $V(\xi)$ is dense in $C^r(K(\xi, 0), A)$. Since K_0 is compact, openness follows from [5, Lemma 1, II § 7].

For $\xi \in G_0^r(A, X)$ denote by $j = j_{A,\xi} \times j_{X,\xi}$ the imbedding of $K(\xi, 0)$ into $A \times X$ and let

$$G_{01}^r(A, X) = \{\xi \in G_0^r(A, X) \mid J^1(j_{A,\xi}) \mathrel{\bar{\pitchfork}} S_1(K(\xi, 0), A)\}\,.$$

Lemma 3. *The set $G_{01}^r(A, X)$ $(r \geqq 2)$ is open and dense in $G_0^r(A, X)$.*

To prove this lemma, we first prove the following lemma and a proposition.

Lemma 4. *Let $\xi \in G_0^r(A, X)$ $(r \geqq 2)$, $(a_0, x_0) \in K(\xi, 0)$. Let (W, h) be a chart on $A \times X$ at (a_0, x_0) such that $W = U \times V$, $h = h_1 \times h_2$, where (U, h_1) is a chart* *on A at a_0, (V, h_2) is a chart on X at x_0, $h_1(U) = B_1(\sigma)$, $h_2(V) = B_n(\delta)$, $\sigma, \delta > 0$, $h(a_0, x_0) = (0, 0)$ $(B_s(\varepsilon) = \{x \in R^s \mid |x| < \varepsilon,\ \varepsilon > 0,\ s$ is an integer and $|.|$ is the Euclidean norm in $R^s)$. Denote $W_i = U_i \times V_i = h_1^{-1}[B_1(\sigma . i/3) \times h_2^{-1}[B_n(\delta . i/3)]$, $i = 1, 2$. Then, in any neighborhood of ξ there is a $\tilde{\xi} \in G_0^r(A, X)$ such that $\tilde{\xi} = \xi$ outside W_2 and $J^1(j_{A,\tilde{\xi}}) \pitchfork S_1(K(\tilde{\xi}, 0), A)$ on the set $K(\tilde{\xi}, 0) \cap \overline{W}_1$.*

Proof. By Lemma 2 there exists a $g \in C^r(K(\xi, 0), A)$ arbitrarily C^r-close to $j_{A,\xi}$ such that $J^1(g) \pitchfork S_1(K(\xi, 0), A)$ on the set $K_0 = K(\xi, 0) \cap \overline{W}_1$. By [8, Theorem 7.2], there exists a tubular neighborhood of K_0 in $A \times X$, i.e. there is an open subset Z of $A \times X$ with a submersion $\pi : Z \to K_0$ such that π is a C^r vector bundle and $K_0 \subset Z$ is the zero section of this vector bundle. Let ψ be a C^r function on $A \times X$ such that $\psi = 1$ on W_1 and $\psi = 0$ outside W_2. Define

$$\tilde{\xi}(a, x) = \xi(h_1^{-1}(h_1(a, x) + \psi(a, x)\,[h_1 g\pi(a, x) - h_1 j_{A,\xi}\pi(a, x)], x)$$

for $(a, x) \in W$ and $\tilde{\xi}(a, x) = \xi(a, x)$ for $(a, x) \in A \times X - W$. Obviously, $K(\tilde{\xi}, 0) \cap \cap W = (g \times j_{A,\xi})(K(\xi, 0) \cap W)$ and $K(\tilde{\xi}, 0) - K(\tilde{\xi}, 0) \cap W = K(\xi, 0) - K(\xi, 0) \cap \cap W$.

Proposition 4. *Let $\xi \in G_{01}^r(A, X)$ and $(a_0, x_0) \in X_1(\xi)$. Then there exists a chart (W, h) on $A \times X$ at (a_0, x_0) such that $h(K(\xi, 0)\ W) = \{(\mu, y_1, \ldots, y_n) \in R^{n+1} \mid \mu = \varphi_0(y_n),\ y_i = \varphi_i(y_n),\ y_n \in J\}$, where $\varphi_i \in C^r$ on J for $i = 0, 1, \ldots, n - 1$, J is an open interval, $0 \in J$ and $(\mathrm{d}^2\varphi_0/\mathrm{d}y_n^2)\ \varphi_0 \neq 0$.*

Proof. Since $\xi \in G_{01}^r(A, X)$, so $J^1(j_{A,\xi}) \pitchfork_{(a_0,x_0)} S_1(K(\xi, 0), A)$. The proposition follows from the coordinate representation of the last transversality condition.

Proof of Lemma 3. Openness. Let $\xi \in G_{01}^r(A, X)$. Since the set $K(\xi, 0)$ is compact, we can cover it by a finite number of charts on $A \times X$. We can choose a covering (W_k, h_k), $k = 1, 2, \ldots, s$, $W_k = U_k \times V_k$, $h_k = h_{k1} \times xh_{k2}$, where (U_k, h_{k1}) is a chart on A, (V_k, h_{k2}) is a chart on X such that

$$h_k(W_k \cap K(\xi, 0)) =$$
$$= \{(\mu, y_1, \ldots, y_n) \mid \mu = \varphi_0^{(k)}(t),\ y_i = \varphi_i^{(k)}(t),\ i = 1, \ldots, n,\ t \in J_k\},$$

where $\varphi_i^{(k)}$ are C^r functions on J_k for $i = 0, 1, \ldots, n$. We can find the last charts by using the implicit function theorem as in the proof of Proposition 3. If ξ_{h_k} is the principal part of the local representation of ξ with respect to the chart (V_k, h_k), then $\xi_{h_k}(\varphi_0^{(k)}(t), \ldots, \varphi_n^{(k)}(t)) = 0$ for $t \in J_k$. If $(a, x) \in A \times X$ is such that $\dot{\xi}_a(x)$ is a surjective mapping, then we can choose $\varphi_0^{(k)}(t) \equiv t$ for $t \in J_k$. $\varphi_i^{(k)}(t) \equiv t$ for some $i \neq 0$ if $\dot{\xi}_a(x)$ is not surjective. If $(a_0, x_0) \notin X_1(\xi)$ and $h_k(a_0, x_0) = (\varphi_0^{(k)}(t_0), \ldots \ldots, \varphi_n^{(k)}(t_0))$, then $(\mathrm{d}\varphi_0^{(k)}/\mathrm{d}t)\,(t_0) \neq 0$. If $(a_0, x_0) \in X_1(\xi)$, then by Proposition 4 we can choose (W_k, h_k) such that $\mathrm{d}^2\varphi_0^{(k)}(t_0)/\mathrm{d}t^2 \neq 0$. Denote

$$\pi_{k,\xi}(t) = \left(\frac{\mathrm{d}\varphi_0^{(k)}(t)}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}^2\varphi_0^{(k)}(t)}{\mathrm{d}t^2}\right)^2$$

for $t \in J_k$. Then $\pi_{k,\xi}(t) \neq 0$ for every $t \in J_k$. If $\tilde{\xi}$ is close enough to ξ, $K(\tilde{\xi}, 0)$ will be contained in $\bigcup_{k=1}^{s} W_k$ and $\pi_{k,\tilde{\xi}}(t) \neq 0$ for $t \in J_k$. This follows from the implicit function theorem and from [6, Theorem 3]. Consequently, $K(\tilde{\xi}, 0)$ will satisfy the transversality condition and the openness is proved. We have to prove the density of the set $G^r_{01}(A, X)$. Let $\xi \in G^r_0(A, X)$. We can cover the set $K(\xi, 0)$ by finite number of charts (W_k, h_k), $k = 1, \dots, s$, where $W_k = U_k \times V_k$, $h_k = h_{k1} \times h_{k2}$, (U_k, h_{k1}) is a chart on A, (V_k, h_{k2}) is a chart on X, $h_{k1}(U_k) = B_1(\sigma_k)$, $h_{k2}(V_k) = B_n(\delta_k)$, $\sigma_k, \delta_k > 0$. We can choose (W_k, h_k), $k = 1, 2, \dots, s$ such that $W_{1,k} \cap W_{1,k+1} \neq \emptyset$ for $k = 1, 2, \dots, s - 1$, $W_{1,k} \cap W_{1,k+2} = \emptyset$ for $k = 1, 2, \dots, s - 2$, where $W_{1,k} = h_k^{-1}[B_1(\sigma_k/3) \times B_n(\delta_k/3)]$, $k = 1, 2, \dots, s$. By Lemma 4 we can find an approximation $\tilde{\xi}_k$ of ξ such that $J^1(j_{A,\tilde{\xi}_k}) \mathrel{\bar{\pitchfork}} S_1(K(\tilde{\xi}_k, 0), A)$ on the set $W_{1,k} \cap K(\tilde{\xi}_k, 0)$, choosing ξ_k for $k > 1$ close enough to $\tilde{\xi}_{k-1}$ so that $J^1(j_{A,\tilde{\xi}_k}) \mathrel{\bar{\pitchfork}} S_1(K(\tilde{\xi}_k, 0), A)$ on the set $(\bigcup_{j=1}^{k-1} \overline{W}_{1,j}) \cap K(\tilde{\xi}_n, 0)$. By such construction we can get a $\tilde{\xi} \in G^r_{01}(A, X)$ arbitrarily close to ξ.

Proposition 5. *If $\xi \in G^r_{01}(A, X)$, then the set $X_1(\xi)$ is finite.*

Proof. Since $J^1(j_{A,\xi}) \mathrel{\bar{\pitchfork}} S_1(K(\xi, 0), A)$ and codim $S_1(K(\xi, 0), A) = 1$, so $X_1(\xi) = [J^1(j_{A,\xi})]^{-1} (S_1(K(\xi, 0), A)$ is a submanifold of $K(\xi, 0)$ of codimension 0. Since the set $K(\xi, 0)$ si compact, the set $X_1(\xi)$ is finite.

Let $\xi \in G^r_{01}(A, X)$, $(a_0, x_0) \in X_1(\xi)$ and let (W, h) be a chart on $A \times X$ at (a_0, x_0), $h(a_0, x_0) = (0, 0, \dots, 0)$. Then the principal part ξ_h of the local representation of ξ has the form $\xi_h(\mu, x_1, y) = (\alpha\mu + \beta x_1^2 + \omega(\mu, x_1, y),\ By + \chi(\mu, x_1, y))$, where B is an $(n - 1) \times (n - 1)$ matrix, $y = (x_2, x_3, \dots, x_n)$, $\omega, \chi \in C^r$, $\chi(0, 0, 0) = 0$, $d\chi(0, 0, 0) = 0$, $\omega(\mu, x_1, 0)$ contains only μ^2, μx_1 and terms of orders higher than 2. Let $G^r_{02}(A, X)$ be the subset of $G^r_{01}(A, X)$ such that for all $\xi \in G^r_{01}(A, X)$ the matrix B from the expression for ξ_h has no eigenvalue with zero real part. This set is open and dense in $G^r_{01}(A, X)$. The openness is obvious. To prove density we assume $\xi \in G^r_{01}(A, X)$. We change ξ into $\tilde{\xi}$ by changing the term By in the local representation ξ_h of ξ into $(B + \psi(\mu, x_1, y)\,\delta E)\,y$, where E is the unit matrix, ψ is a C^r bump function vanishing outside $h(W)$ and equal to 1 at $(0, 0, 0)$ and $0 < \delta$ is a real number such that $B + \delta E$ has no eigenvalue with zero real part. By the choice of a sufficiently small δ, $\tilde{\xi}$ can be made sufficiently close to ξ.

We shall prove that β in the expression for ξ_h is different from zero. Suppose $\beta = 0$. Since $(a_0, x_0) \in X_1(\xi)$, there are C^r functions $\varphi_i(x_1)$, $i = 0, 2, \dots, n$ such that $\alpha\,\varphi_0(x_1) + \omega(\varphi_0(x_1), x_1, \varphi_2(x_1), \dots, \varphi_n(x_1)) = 0$ for $x_1 \in J$, where J is an open neighborhood of 0. Then

$$\frac{\alpha\, d^2\varphi_0(0)}{dx_1^2} + \frac{d^2\tilde{\omega}(0)}{dx_1^2} = 0\,,$$

where $\tilde{\omega}(x_1) = \omega(\varphi_0(x_1), \ldots, \varphi_n(x_1))$. By Proposition 4, $\mathrm{d}^2\varphi_0(0)/\mathrm{d}x_1^2 \neq 0$. This implies that $\alpha = 0$, but this is impossible because rank $(D\xi_h(0, 0, \ldots, 0)) = n$.

Assume $\xi \in G^r_{02}(A, X)$ and $(a_0, x_0) \in X_1(\xi)$. Let (W, h) be a chart on $A \times X$ at (a_0, x_0) such that $h(a_0, x_0) = (0, 0)$ and $h(K(\xi, 0) \cap W) = \{(\mu, x_1, \ldots, x_n) \mid \mu = \varphi_0(x_1), y_i = \varphi_i(x_1), x_1 \in J\}$, where J is an open interval in R, $0 \in J$, $\varphi_i : J \to R$ are C^r functions on J for $i = 0, 1, \ldots, n$, $\varphi_0(0) = 0$, $\mathrm{d}\varphi_0(0)/\mathrm{d}x_1 = 0$, $\mathrm{d}^2\varphi_0(0)/\mathrm{d}x_1^2 \neq 0$. It is possible to find such a chart using the implicit function theorem. By [4, Appendix C] we can assume that the principal part of the local representation of ξ with respect to the chart (W, h) has the form

$$\xi_h(\mu, x_1, y, z) =$$
$$= (\alpha\mu + \beta x_1^2 + \omega(\mu, x_1, y, z), Ay + \chi(\mu, x_1, y, z), Bz + \theta(\mu, x_1, y, z)),$$

where $\omega, \chi, \theta \in C^r$, $\chi(\mu, x_1, 0, z) = 0$, $\theta(\mu, x_1, y, 0) = 0$, $\mathrm{d}\omega(0, 0, 0, 0) = 0$, $\mathrm{d}\chi(0, 0, 0, 0) = 0$, $\omega(\mu, x_1, 0, 0)$ contains only μ^2, μx_1 and terms of orders higher than 2, A has only eigenvalues with real part <0 and B has only eigenvalues with real part >0. If $\beta/\alpha < 0$, then $\mathrm{d}^2\varphi_0(0)/\mathrm{d}x_1^2 > 0$. The other case can be transformed to the above one by a suitable change of coordinates. If $\varphi_0(0) = 0$, $\mathrm{d}\varphi_0(0)/\mathrm{d}x_1 = 0$, $\mathrm{d}^2\varphi_0(0)/\mathrm{d}x_1^2 > 0$, then there is no critical point for $\mu < 0$ and there are exactly two critical points $(\mu, x_1(\mu), 0, 0)$, $(\mu, x_2(\mu), 0, 0) \in h(K(\xi, 0) \cap W)$ such that $x_1(\mu) > 0$ and $x_2(\mu) < 0$. Denote $\xi'_h(\mu, x_1) = \alpha\mu + \beta x_1^2 + \omega(\mu, x_1, 0, 0)$. Then

$$\frac{\mathrm{d}\xi'_h(\mu, x_1(\mu))}{\mathrm{d}x_1} = 2\beta\, x_1(\mu) + o(x_1(\mu)) > 0\,,$$

$$\frac{\partial\xi'_h(\mu, x_2(\mu))}{\partial x_1} = 2\beta\, x_2(\mu) + o(x_2(\mu)) < 0$$

for small μ.

Theorem 1. *Assume $r \geqq 3$. Then there is a set $G^r_{02}(A, X)$ open and dense in $G^r(A, X)$ with the following properties:*

(1) *For $\xi \in G^r_{02}(A, X)$, $K(\xi, 0)$ is a closed 1-dimensional submanifold of $A \times X$.*

(2) *For fixed $a \in A$, the set $\{x \in X \mid (a, x) \in K(\xi, 0)\}$ consists of isolated points.*

(3) *The set $X_1(\xi)$ is finite.*

(4) *For every $(a_0, x_0) \in K(\xi, 0) - X_1(\xi)$ there is a chart (W, h) on $A \times X$ at (a_0, x_0), $h(W) = U \times V$, $h(a_0, x_0) = (0, 0)$ and a C^r mapping $\varphi : U \to V$ such that $h(K(\xi, 0) \cap W) = \{(\mu, y) \mid y = \varphi(\mu), \mu \in U\}$.*

(5) *For every $(a_0, x_0) \in X_1(\xi)$ there is a chart (W, h) on $A \times X$ at (a_0, x_0), $h(a_0, x_0) = (0, 0)$ such that*

(a) $h(K(\xi, 0) \cap W) = \{(\mu, y_1, \dots, y_n) \mid \mu = \varphi_0(y_1),\ y_i = \varphi_i(y_1),\ i = 2, 3, \dots, n,$ $\mu \in J\}$, *where* J *is an open interval,* $0 \in J$, $\varphi_0(0) = 0$, $\mathrm{d}\varphi_0(0)/\mathrm{d}y_1 = 0$, $\mathrm{d}^2\varphi_0(0)/\mathrm{d}y_1^2 > 0$.

(b) *If* $\mu > 0$ *then there are exactly two numbers* $y_1 > 0$, $z_1 < 0$ *such that* $(a_1, x_1) = h^{-1}(\mu, y_1, 0, 0) \in K(\xi, 0)$, $(a_1, x_2) = h^{-1}(\mu, z_1, 0, 0) \in K(\xi, 0)$ *and the following is true: If s is the number of real eigenvalues of the mapping* $\dot{\xi}_{a_1}(x_1)$ *greater than* 0, *then the number of real eigenvalues of the mapping* $\dot{\xi}_{a_1}(x_2)$ *greater than* 0 *is* $s - 1$.

(6) *If* $(a, x) \in X_1(\xi)$, *then the mapping* $\dot{\xi}_a(x)$ *has exactly one eigenvalue equal to* 0.

(7) $W - K(\xi, 0)$ *contains no invariant set.*

We say that a property $G(\xi)$ of parametrized vectorfield is generic in $G^r(A, X)$ if the set $H^r(A, X) = \{\xi \in G^r(A, X) \mid G(\xi)\}$ contains a residual set in $G^r(A, X)$.

The properties (1)–(7) from Theorem 1 are generic in $G^r(A, X)$.

3. CRITICAL POINTS AT WHICH THE LINEARIZATION OF THE VECTORFIELD HAS COMPLEX EIGENVALUE WITH ZERO REAL PART

Let $\eta \in \Gamma^r(\tau_X)$ and let $x \in X$ be a critical point of η. We say that x is a *nonelementary critical point* of multiplicity k, if the mapping $\dot{\eta}(x)$ has a complex eigenvalue with zero real part of multiplicity k.

Denote by $G^r_{11}(A, X)$ the set of all $\xi \in G^r(A, X)$ such that if for $a \in A$ the vectorfield ξ_a has a nonelementary critical point, then it has multiplicity 1.

Lemma 6. *The set* $G^r_{11}(A, X)$ $(r \geqq 1)$ *is open and dense in* $G^r(A, X)$.

For the proof of this lemma we shall need another lemma. For this reason consider $A_1 = \{(B, \lambda_1, \lambda_2) \in A(n, n) \times R^2 \mid \lambda_1 = 0,\ P_1(\lambda_1, \lambda_2) = P_2(\lambda_1, \lambda_2) = P'_1(\lambda_1, \lambda_2) = P'_2(\lambda_1, \lambda_2) = 0\}$, where $P(\lambda) = P_1(\mathrm{Re}\,\lambda, \mathrm{Im}\,\lambda) + i\,P_2(\mathrm{Re}\,\lambda, \mathrm{Im}\,\lambda)$ is the characteristic polynomial of B and $P'_1 + i\,P'_2 = \partial P/\partial\lambda$. By [7], $A_1 = \bigcup_{j=1}^{r_1} A_{1j}$, where A_{1j}, $j = 1, 2, \dots, r_1$ are disjoint submanifolds of $A(n, n) \times R^2$ of strictly decreasing dimensions and $\bigcup_{j=\varrho_0}^{r_1} A_{1j}$ is a closed set for $0 < \varrho_0 \leqq r_1$.

Lemma 7. codim $A_{1j} \geqq 4$ *for* $j = 1, 2, \dots, r_1$.

The proof of this lemma is analogous to that of [2, Lemma 1].

Proof of Lemma 6. Let $\xi, \eta \in G^r(A, X)$, $(a_1, x_1), (a_2, x_2) \in A \times X$ and let (W, h) be a chart on X. Let ξ_1, η_1 be the principal part of the local representation of ξ_{a_1}, η_{a_2} respectively, with respect to the chart (W, h). We say that (ξ, a_1, x_1) is k-equivalent to (ξ, a_2, x_2) if and only if $a_1 = a_2$, $x_1 = x_2$ and $(\xi_1(h(x_1)), \ldots, D^k\xi_1(h(x_1))) = (\eta_1(h(x_2)), \ldots, D^k\eta_1(h(x_2))).$ Obviously, k-equivalence is an equivalence. Let $J^k\xi(a, x)$ denote the class of triples equivalent to the triple (ξ, a, x). Denote by $J^k(\tau_X, A)$ the set of all classes $J^k\xi(a, x)$. The mapping $\pi^1 : J^1(\tau_X, A) \to A \times X$, $\pi^1(j^1\xi(a, x)) = (a, x)$ is a C^r vector bundle. If $(U \times V, \alpha_0 \times \beta_0)$ is a chart on $A \times X$, then $(\beta, \alpha_0 \times \beta_0, U \times V)$ is a chart on $J^1(\tau_X, A)$, where $\beta : [\pi^1]^{-1}(U) \to (\alpha_0 \times \beta_0) . (U \times V) \times R^n \times A(n, n)$, $\beta(j^1\xi(a, x)) = (\alpha_0(a), \beta_0(x), \xi'_a(x), D\,\xi'_a(x))$, where ξ'_a is the principal part of the local representation of ξ_a. For $\xi \in G^r(A, X)$, define the mapping $\varrho_\xi : A \times X \to J^1(\tau_X, A)$, $\varrho_\xi(a, x) = j^1\xi(a, x)$ for $(a, x) \in A \times X$. Now, define the mapping $\tilde{\varrho}_\xi : A \times X \times R^2 \to J^1(\tau_X, A) \times R^2$, $\tilde{\varrho}_\xi = \varrho_\xi \times \mathrm{id}$, where id is the identical mapping of R^2 onto R^2. The mapping $\varrho : G^r(A, X) \to C^{r-1}(A \times X \times R^2, J^1(\tau_X, A) \times R^2)$, $\varrho(\xi) = \tilde{\varrho}_\xi$ for $\xi \in G^r(A, X)$ is a C^{r-1} representation. It is easy to prove that $ev_\varrho \mathrel{\bar{\pitchfork}} W$ for every submanifold W of $J^1(\tau_X, A) \times R^2$. Let $(\alpha, \alpha_0 \times \beta_0, U \times V)$ be a natural chart on π^1. Let $W \subset J^1(\tau_X, A) \times R^2$ be the set of $(p, \lambda_1, \lambda_2) \in J^1(\tau_X, A) \times R^2$ such that $(\alpha(p), \lambda_1, \lambda_2) = (\mu, y, 0, B, \lambda_1, \lambda_2)$, $\mu \in R$, $y \in R^n$, $(B, \lambda_1, \lambda_2) \in A_1$. It is easy to prove that this definition is independent of the coordinates. Since $A_1 = \bigcup_{j=1}^{r_1} W_j$, where the sets A_{1j} have the properties as before, $W = \bigcup_{j=1}^{r_1} W_j$, where W_j are disjoint submanifolds of $J^1(\tau_X, A)$ of strictly decreasing dimension, $\bigcup_{j=\varrho_0}^{r_1} W_j$ is a closed set for $0 < \varrho_0 \leqq r_1$. Lemma 7 implies codim $W_j \geqq n + 4$ for every j. Let $\xi \in G^r_{11}(A, X)$ and let $(\beta, \alpha_0 \times \beta_0, U \times V)$ be a natural chart on π^{-1} as in the definition of W. $\beta(J^1\xi(a, x)) = (\alpha_0(a), \beta_0(x), \xi'_a(x), D\,\xi'_a(x))$. There is a neighborhood $N(\xi)$ of ξ in $G^r(A, X)$ and a number $q > 0$ such that for every $\eta \in N(\xi)$, $(a, x) \in A \times X$, every eigenvalue $\lambda(\eta, a, x)$ of $D\,\eta'_a(x)$ is such that $|\lambda(\eta, a, x)| < q$, where $\beta(J^1\eta(a, x)) = (\alpha_0(a), \beta_0(x), \eta'_a(x), D\,\eta'_a(x))$. Therefore, for $\eta \in N(\xi)$, $\varrho(\eta) \mathrel{\bar{\pitchfork}} W$ if and only if $\varrho_0(\eta) \mathrel{\bar{\pitchfork}} W$, where $\varrho_0(\eta) = \varrho(\eta)/A \times X \times [-q, q]$. Denote $\psi_i = \{\eta \in N(\xi) \mid \varrho_0(\eta) \mathrel{\bar{\pitchfork}} \bigcap_{j=r_1-i+1}^{r_1} W_j\}$ for $i = 1, 2, \ldots, r_1$. From [4, Theorem 18.2] it follows that the set ψ_i, $i = 1, 2, \ldots, r_1$ are open in $N(\xi)$. Since codim $W_j \geqq n + 4$ for all j, $\varrho_0(\eta) \mathrel{\bar{\pitchfork}} W$ means that $\varrho_0(\eta)(A \times X \times [-q, q]) \cap W = \emptyset$ and so the set $G^r_{11}(A, X)$ is open in $G^r(A, X)$. Density: Let $\xi \in G^r(A, X)$ and let $N(\xi)$ be a neighborhood of ξ as before. We shall prove that the sets ψ_i, $i = 1, 2, \ldots, r_1$ are dense in $N(\xi)$. Denote $\tilde{\psi}_1 = \{\eta \in N(\xi) \mid \varrho(\eta) \mathrel{\bar{\pitchfork}} W_{r_1}\}$. By [4, Theorem 19.1] the set $\tilde{\psi}_1$ is dense in $N(\xi)$ and therefore the set ψ_1 is dense in $N(\xi)$, too. Suppose the sets ψ_i, $i = 1, 2, \ldots, k$ are dense in $N(\xi)$. We shall prove that the set ψ_{k+1} is dense, too. The assumptions together with the openness of ψ_i, $i = 1, 2, \ldots, r_1$ imply that the set $\psi = \bigcap_{i=1}^{k} \psi_i$ is open and dense in $N(\xi)$. Since $\overline{W}_{r_1-k} \subset \bigcap_{i=0}^{k} W_{r_1-i}$, it is $\varrho_0(\eta) \mathrel{\bar{\pitchfork}} \overline{W}_{r_1-k}$ for $\eta \in \psi$ if and only if $\varrho_0(\eta) \mathrel{\bar{\pitchfork}} W_{r_1-k}$. Denote by ϱ' the restriction of ϱ on the set ψ. By [4, Theorem 19.1] the set $\psi_{k+1} = \{\eta \in \psi \mid \varrho'(\eta) \mathrel{\bar{\pitchfork}} W_{r_1-k}\}$ is open and dense in ψ and so the sets ψ_i, $i = 1, 2, \ldots, r_1$ are open and dense in $N(\xi)$. Therefore the set

$\bigcap_{i=1}^{r_1} \psi_i$ is open and dense in $N(\xi)$. The set $\bigcap_{i=1}^{r_1} \psi_i$ is a subset of the set $\{\eta \in N(\xi) \mid \eta \in G^r_{11}(A, X)\}$ and therefore the set $G^r_{11}(A, X)$ is dense in $G^r(A, X)$.

Consider the set $A_2 = \{(B, \lambda_1, \lambda_2) \in A(n, n) \times R^2 \mid P_1(\lambda_1, \lambda_2) = P_2(\lambda_1, \lambda_2) = \lambda_1 = 0\}$. By [7] $A_2 = \bigcup_{j=1}^{r_2} A_{2j}$, where $A_{2j}, j = 1, 2, \ldots, r_2$ are disjoint submanifolds of $A(n, n) \times R^2$ of strictly decreasing dimensions and the set $\bigcup_{j=\varrho_0}^{r_2} A_{2j}$ is closed for $0 < \varrho_0 \leqq r_2$.

Lemma 8. codim $A_{21} = 3$.

The proof of Lemma 8 is analogous to that of [2, Lemma 5].

Let $\pi^1 : J^1(\tau_X, A) \to A \times X$ be the mapping defined as before and let $(\alpha, \alpha_0 \times \beta_0, U \times V)$ be a natural chart on π^1. Let $W' \subset J^1(\tau_X, A) \times R^2$ be the set of $(p, \lambda_1, \lambda_2) \in J^1(\tau_X, A) \times R^2$ such that $(\alpha(p), \lambda_1, \lambda_2) = (\mu, y, 0, B, \lambda_1, \lambda_2)$, $\mu \in R$, $y \in R^n$, $(B, \lambda_1, \lambda_2) \in A_2$. Since $A_2 = \bigcup_{j=1}^{r_2} A_{2j}$, where the sets A_{2j} have the same properties as before, it is $W' = \bigcup_{j=1}^{r_2} W'_j$, where W'_j are disjoint submanifolds of $J^1(\tau_X, A) \times R^2$ of strictly decreasing dimensions, $\bigcup_{j=\varrho_0}^{r_2} W'_j$ is closed for $0 < \varrho_0 \leqq r_2$. Lemma 8 implies codim $W'_j \geqq n + 4$ for $j > 1$ and codim $W'_1 = n + 3$. Let $\varrho : G^r(A, X) \to C^{r-1}(A \times X \times R^2, J^1(\tau_X, A) \times R^2)$ be the mapping from the proof of Lemma 7. Let $G^r_{12} = \{\xi \in G^r(A, X) \mid \varrho(\xi) \mathrel{\bar{\cap}} W'\}$. Analogously to the case of the set $G^r_{11}(A, X)$, we can prove

Lemma 9. *The set $G^r_{12}(A, X)$ is open and dense in $G^r(A, X)$.*

Denote $G^r_{13}(A, X) = G^r_0(A, X) \cap G^r_{11}(A, X) \cap G^r_{12}(A, X)$. Let $\xi \in G^r_{13}(A, X)$, $(a_0, x_0) \in K(\xi, 0)$ and let (V, β) be a chart on $A \times X$ at (a_0, x_0). Let ξ_β be the principal part of the local representation of ξ. Denote $F(t) = D_y \xi_\beta(t)$ for $t \in I = \beta(V \cap K(\xi, 0))$, where $D_y \xi_\beta$ is the derivative of $\xi_\beta(\mu, y)$ with respect to y. Denote $T = \{(s, z) \in R^2 \mid s = 0\}$.

Proposition 6. [2, Lemma 6]. *Let λ_0 be a simple eigenvalue of $F(t_0)$, where $t_0 \in I$. Then there is a neighborhood N of t_0 in I and a unique function $\lambda : N \to C$ such that $\lambda(t_0) = \lambda_0$ and $\lambda(t)$ is an eigenvalue of $F(t)$ for $t \in N$. Further, there is a non-singular C^r matrix $C(t)$ on N such that $C^{-1}FC = B$, where the first column of B is the transpose of $(\lambda(t), 0, \ldots, 0)$.*

Let $\lambda(t) = \lambda_1(t) + i\,\lambda_2(t)$. Define the mapping $\hat{\lambda} : N \to R^2$, $\hat{\lambda}(t) = (\lambda_1(t), \lambda_2(t))$. Obviously, $\hat{\lambda} \in C^r(N, R^2)$. Similarly to [2, Proposition 3] we can prove

Proposition 7. *Let the assumptions be the same as in Proposition 6 and let $\xi \in G^r_{13}(A, X)$. Then $\hat{\lambda} \mathrel{\bar{\cap}} T$.*

For $\xi \in G^r(A, X)$ denote by $X_2(\xi)$ the set of points $(a, x) \in K(\xi, 0)$ for which x is a nonelementary critical point of ξ_a.

Corollary of Proposition 7. If $\xi \in G^r_{13}(A, X)$, then the set $X_2(\xi)$ is finite.

Let $G^r_1(A, X)$ be the set of all $\xi \in G^r(A, X)$ such that

(1) $\xi \in G^r_{13}(A, X)$.

(2) If $(a, x) \in X_2(\xi)$, then the mapping $\dot{\xi}_a(x)$ has exactly one pair of conjugate complex eigenvalues with zero part real.

Lemma 10. *The set $G^r_1(A, X)$ $(r \geqq 1)$ is open and dense in $G^r(A, X)$.*

Proof. The openness of $G^r_1(A, X)$ is obvious. To prove the density of $G^r_1(A, X)$, it suffices to prove the density of $G^r_1(A, X)$ in $G^r_{13}(A, X)$, because the set $G^r_{13}(A, X)$ is dense in $G^r(A, X)$. Let $\xi \in G^r_{13}(A, X)$, $(a_0, x_0) \in X_2(\xi)$, let $(U \times V, \alpha \times \beta)$ be a chart on $A \times X$ at (a_0, x_0) and $\xi_{\alpha\times\beta}$ the principal part of the local representation of ξ. Assume that the chart is chosen so that the set $(U \times V) \cap K(\xi, 0)$ is the graph of a mapping $\varphi : U \to V$. Let (μ, y) be the coordinates in the chart. Then in the coordinates $(a, x) \to (\mu, z)$, $z = y - \beta\,\varphi(a)$, ξ can be represented by $\xi'(\mu, z) = A(\mu)\, z + Y(\mu, z)$, where $Y(\mu, 0) = 0$, $\mathrm{d}Y(\mu, 0) = 0$, $A : \alpha(U) \to A(n, n)$ is a C^r mapping such that $A(\mu_0)$ $(\mu_0 = \alpha(a_0))$ has complex eigenvalues with zero real part of multiplicity 1 while $A(\mu)$ for $\mu \neq \mu_0$ has no complex eigenvalues with zero real part. Assume that $\xi_{\alpha\times\beta}$ has the the same form as ξ'. Let $A(\mu_0)$ have k pairs of conjugate eigenvalues $\lambda^0_j, \overline{\lambda^0_j}$, $j = 1, 2, \ldots, k$ with zero real parts. Let $\alpha_0 > 0$ be a number such that there are C^r functions λ_j, $j = 1, \ldots, k$ defined on $N = \alpha(U) \cap [\mu_0 - \alpha_0, \mu_0 + \alpha_0]$, where $\lambda_j(\mu)$, $\mu \in N$ is an eigenvalue of $A(\mu)$ and $\lambda_j(\mu_0) = \lambda^0_j$. Existence of such functions follows from [2, Lemma 6]. There is a nonsingular C^r matrix $C(\mu)$ on N such that $C^{-1}(\mu)\, A(\mu)\, C(\mu) = B(\mu)$ has the form

$$B(\mu) = \operatorname{diag}\left\{\begin{pmatrix} \lambda_{11}(\mu) & \lambda_{12}(\mu) \\ -\lambda_{12}(\mu) & \lambda_{11}(\mu)\end{pmatrix}, \ldots, \begin{pmatrix} \lambda_{k1}(\mu) & \lambda_{k2}(\mu) \\ -\lambda_{k2}(\mu) & \lambda_{k1}(\mu)\end{pmatrix}, B_1\right\},$$

where $\lambda_j = \lambda_{j1} + i\lambda_{j2}$. Choose an $\varepsilon < \frac{1}{2}\alpha_0$ and τ_j, $j = 1, 2, \ldots, k$ such that $|\tau_j| < \varepsilon$, $\tau_i \neq \tau_j$ for $i \neq j$; $i, j = 1, 2, \ldots, k$. Let $\chi : N \to R$ be a C^r function such that $\chi(\mu) = 0$ outside $K = \alpha(U) \cap [\mu_0 - \frac{1}{3}\alpha_0, \mu_0 + \frac{1}{3}\alpha_0]$ and $\chi(\mu) = 1$ for $t \in K_0 = \alpha(U) \cap [\mu_0 - \frac{1}{2}\alpha_0, \mu_0 + \frac{1}{2}\alpha_0]$. Define $\hat{\lambda}_j(\mu) = \lambda_j(\mu + \tau_j\,\chi(\mu)) = \hat{\lambda}_{j1} + i\hat{\lambda}_{j2}$, $j = 1, 2, \ldots, k$,

$$\hat{B}(\mu) = \operatorname{diag}\left\{\begin{pmatrix} \hat{\lambda}_{11}(\mu) & \hat{\lambda}_{12}(\mu) \\ -\hat{\lambda}_{12}(\mu) & \hat{\lambda}_{11}(\mu)\end{pmatrix}, \ldots, \begin{pmatrix} \hat{\lambda}_{k1}(\mu) & \hat{\lambda}_{k2}(\mu) \\ -\hat{\lambda}_{k2}(\mu) & \hat{\lambda}_{k1}(\mu)\end{pmatrix}, B_1\right\},$$

$$\hat{A}(\mu) = \begin{cases} A(\mu) & \text{for} \quad \mu \notin K \\ C(\mu)\, \hat{B}(\mu)\, C^{-1}(\mu) & \text{for} \quad \mu \in K\,. \end{cases}$$

Let $W_1, W_2 \subset \alpha(U) \times \beta(V)$ be open sets in R^{n+1} such that $\overline{W}_1 \subset W_2$, $\overline{W}_2 \subset \alpha(U) \times \beta(V)$, $(\mu_0, 0) \in W_1$ and let $\psi : \alpha(U) \times \beta(V) \to R$ be a C^r function such that $\psi = 0$ outside $\overline{W}_2$ and $\psi = 1$ on W_1. Define $\xi''(\mu, z) = [A(\mu) + \psi(\mu, z)\,(\hat{A}(\mu) - A(\mu))]\, z +$

$+ Y(\mu, z)$. Let $\tilde{\xi}$ be a parametrized vectorfield, which is equal to ξ outside $(\alpha \times \beta)^{-1}(\overline{W}_2)$ and which has the principal part of the local representation on $(\alpha \times \beta)^{-1}(W_1)$ equal to ξ''. If ε is chosen small enough, $\tilde{\xi}$ will be arbitrarily close to ξ. Since $G^r_{13}(A, X)$ is open, so if $\tilde{\xi}$ is close enough to ξ, then $\tilde{\xi} \in G^r_{13}(A, X)$ and $\tilde{\xi} \in G^r_1(A, X)$.

Let $\xi \in G^r_1(A, X)$, $(a_0, x_0) \in X_2(\xi)$. There is a chart $(U \times V, \alpha \times \beta)$ on $A \times X$ at (a_0, x_0) such that $\alpha(a_0) = 0$, $\beta(x_0) = 0$ and the local representation ξ' of ξ has the form

$$\xi_1(\mu, x_1, x_2, y, z) = a(\mu)\, x_1 + b(\mu)\, x_2 + \omega_1(\mu, x_1, x_2, y, z)\,,$$

$$\xi_2(\mu, x_1, x_2, y, z) = c(\mu)\, x_1 + d(\mu)\, x_2 + \omega_2(\mu, x_1, x_2, y, z)\,,$$

$$\xi_3(\mu, x_1, x_2, y, z) = B(\mu)\, y + \omega_3(\mu, x_1, x_2, y, z)\,,$$

$$\xi_4(\mu, x_1, x_2, y, z) = C(\mu)\, z + \omega_4(\mu, x_1, x_2, y, z)\,,$$

where $a(0) + d(0) = 0$, $a(0)\, d(0) - b(0)\, c(0) > 0$, all eigenvalues of $B(\mu)$ have real parts <0 for every μ, all eigenvalues of $C(\mu)$ have real parts >0 for every μ, $\omega_i \in C^r$, $i = 1, 2, 3, 4$; $a, b, c, d \in C^r$. By [3, Appendix C] we may assume that $\omega_i(\mu, x_1, x_2, y, z) = o(|\mu| + |x_1| + |x_2| + |y| + |z|)$ for $i = 1, 2$, $\omega_3(\mu, x_1, x_2, 0, z) = 0$, $\omega_4(\mu, x_1, x_2, y, 0) = 0$, $d\omega_i(0, 0, 0, 0) = 0$ for $i = 1, 2, 3, 4$. Let $\varphi = (\varphi_1, \varphi_2, \varphi_3, \varphi_4)$ be the parametrized flow of ξ'. If $\bar{y} \neq 0$ or $\bar{z} \neq 0$, then $\varphi(\bar{\mu}, \bar{x}_1, \bar{x}_2, \bar{y}, \bar{z}, t) \notin V'$ for sufficiently large t, where $V' \subset V$ is a neighborhood of 0. Therefore, if for $\mu \in \alpha(U)$ there exists an invariant set of φ in $\beta(V)$, then it must be part of the manifold $y = 0$, $z = 0$. We therefore consider the restriction of ξ' to the manifold $y = 0$, $z = 0$, the representation of which is given by

$$(\mu) \qquad \begin{aligned} x_1' &= a(\mu)\, x_1 + b(\mu)\, x_2 + \chi_1(\mu, x_1, x_2)\,,\\ x_2' &= c(\mu)\, x_1 + d(\mu)\, x_2 + \chi_2(\mu, x_1, x_2)\,,\end{aligned}$$

where $\chi_i(\mu, x_1, x_2) = \omega_i(\mu, x_1, x_2, 0, 0)$, $i = 1, 2$, $\chi_1 = P_2 + P_3 + P^*$, $\chi_2 = Q_2 + Q_3 + Q^*$, where

$$P_2(\mu, x_1, x_2) = a_{20}(\mu)\, x_1^2 + a_{11}(\mu)\, x_1 x_2 + a_{02}(\mu)\, x_2^2\,,$$

$$P_3(\mu, x_1, x_2) = a_{30}(\mu)\, x_1^3 + a_{12}(\mu)\, x_1 x_2^2 + a_{21}(\mu)\, x_1^2 x_2 + a_{03}(\mu)\, x_2^3\,,$$

$$Q_2(\mu, x_1, x_2) = b_{20}(\mu)\, x_1^2 + b_{11}(\mu)\, x_1 x_2 + b_{02}(\mu)\, x_2^2\,,$$

$$Q_3(\mu, x_1, x_2) = b_{30}(\mu)\, x_1^3 + b_{12}(\mu)\, x_1 x_2^2 + b_{21}(\mu)\, x_1^2 x_2 + b_{03}(\mu)\, x_2^3\,,$$

where $a_{ik}, b_{ik} \in C^r$ for $i, k = 0, 1, 2, 3$, $P^*, Q^* \in C^r$, $P^*(0, 0) = 0$, $Q^*(0, 0) = 0$. Let $d : [0, r_0) \to R$ be a function as in [6, IX] defined with respect to the critical point $(0, 0)$ of the system (μ). $d'''(0) = 3!\, \alpha_3$, where α_3 is expressed by the formula (76) from [6, IX]. From this formula it is easy to see that α_3 depends continuously on ξ. Let $G^r_{03}(A, X) \subset G^r_1(A, X)$ be the set of $\xi \in G^r_1(A, X)$ such that if $(a_0, x_0) \in \in X_2(\xi)$, then $\alpha_3 \neq 0$.

Lemma 11. The set $G^r_{03}(A, X)$ is open and dense in $G^r_1(A, X)$.

Proof. Openness is obvious. To prove the density, assume $\xi \in G^r_1(A, X)$, $(a_0, x_0) \in \in X_2(\xi)$ and the local representation of ξ in the form (μ). From the form of α_3 it follows that there are C^r functions $\hat{a}_{ik}$, $\hat{b}_{ik}$ arbitrarily close to a_{ik} and b_{ik}, respectively, such that if we put $\hat{a}_{ik}$, $\hat{b}_{ik}$ instead of a_{ik}, b_{ik} into the expression of α_3, then $\alpha_3 \neq 0$. Now, it is obvious that we can construct $\bar{\xi} \in G^r(A, X)$ arbitrarily close to ξ, for which $\alpha_3 \neq 0$. Since $X_2(\bar{\xi})$ is compact for $\bar{\xi}$ close enough to ξ, Lemma 11 have been proved.

As a corollary of the previous lemmas and [6, p. 274] we obtain

Theorem 2. *There exists an open and dense set $G^r_{03}(A, X)$ in $G^r(A, X)$ $(r \geqq 3)$ such that for every $\xi \in G^r_{03}(A, X)$ the following is true:*

(I) *The set $X_2(\xi)$ is finite.*

(II) *If $(a_0, x_0) \in X_2(\xi)$, then*

(1) *the mapping $\dot{\xi}_{a_0}(x_0)$ has exactly one pair of conjugate complex eigenvalues with zero real part;*

(2) *there is a chart $(U \times V, \alpha \times \beta)$ on $A \times X$ at (a_0, x_0) such that the point (a_0, x_0) divides $K(\xi, 0) \cap (U \times V)$ into two components K_1 and K_2, where*

(a) *for $(a, x) \in K_1$ there is no closed orbit of ξ_a in V,*

(b) *for $(a, x) \in K_2$ there exists a closed orbit of ξ_a in V.*

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