Tôhoku Math. Journ.
35 (1983), 155-172.

GENERIC PROPERTIES OF THE EIGENVALUE OF THE LAPLACIAN FOR COMPACT RIEMANNIAN MANIFOLDS

SHIGETOSHI BANDO AND HAJIME URAKAWA*

(Received December 4, 1980, revised April 8, 1982)

Introduction. In this paper, we discuss generic properties of the eigenvalues of the Laplacian for compact Riemannian manifolds without boundary.

Throughout this paper, let M be an arbitrary fixed connected compact C^∞ manifold of dimension n without boundary, and $\mathscr{M}$ the set of all C^∞ Riemannian metrics on M. For $g \in \mathscr{M}$, let Δ_g be the Laplacian (cf. (2.1)) of (M, g) acting on the space $C^\infty(M)$ of all C^∞ real valued functions on M and

$$0 = \lambda_0(g) < \lambda_1(g) \leqq \lambda_2(g) \leqq \cdots \uparrow \infty$$

the eigenvalues of the Laplacian Δ_g counted with their multiplicities. We regard each eigenvalue $\lambda_k(g)$, $k = 0, 1, 2, \cdots$, as a function of g in $\mathscr{M}$. Let us consider the following problem: "*Does each eigenvalue $\lambda_k(g)$ depend continuously on g in $\mathscr{M}$ with respect to the C^∞ topology?*"

The continuous dependence of the eigenvalues of the Dirichlet problem upon variations of domains is well known (cf. [CH, p. 290]). Variations of coefficients of elliptic differential operators were dealt with by Kodaira-Spencer [KS] who gave a proof of the continuity of eigenvalues. In this paper, we give a simple proof of the above problem.

To answer the above problem, in § 1, we introduce a complete distance ρ on $\mathscr{M}$ which gives the C^∞ topology. Then, in § 2, we assert that each $\lambda_k(g)$, $k = 1, 2, \cdots$, depends continuously on $g \in \mathscr{M}$ with respect to the topology on $\mathscr{M}$ induced by the distance ρ. More precisely, we have

THEOREM 2.2. *For each positive number δ and each $g, g' \in \mathscr{M}$, the inequality $\rho(g, g') < \delta$ implies that*

$$\exp(-(n+1)\delta) \leqq \lambda_k(g)/\lambda_k(g') \leqq \exp((n+1)\delta) ,$$

for each $k = 1, 2, \cdots$ (where $n = \dim M$).

* Partly supported by the Grant-in-Aid for Encouragement of Young Scientists (No. 8015-56740005), the Ministry of Education, Science and Culture, Japan.

That is, if two Riemannian metrics g and g' are close to each other with respect to the distance ρ, then the ratio $\lambda_k(g)/\lambda_k(g')$ is close to one *uniformly* in $k = 1, 2, \cdots$. Thus we have immediately the following corollary. A similar result was obtained by [KS].

COROLLARY 2.3. *The multiplicity* $m_k(g)$ *of each eigenvalue* $\lambda_k(g)$, *i.e.,* $m_k(g) = \#\{i;\ \lambda_i(g) = \lambda_k(g)\}$, *depends upper semi-continuously on* $g \in \mathscr{M}$: *For each* $g \in \mathscr{M}$ *and* $k = 0, 1, 2, \cdots$, *there exists a positive number* δ *such that* $\delta(g, g') < \delta$ *implies* $m_k(g') \leqq m_k(g)$.

These results are useful in investigating generic properties of Riemannian metrics. As one of these applications, we give a simple and constructive proof of the following theorem of Uhlenbeck (cf [U], [T]):

THEOREM 3.1. *Let* M *be a compact connected* C^∞ *manifold of dimension not less than two. Then the set* $\mathscr{S} = \{g \in \mathscr{M};$ *all eigenvalues* $\lambda_k(g)$, $k = 0, 1, 2, \cdots$, *have multiplicity one*$\}$ *is a residual set in the complete metric space* $(\mathscr{M}, \rho)$, *i.e., a countable intersection of open dense subsets.*

Therefore $\mathscr{S}$ is a subset of the second category and dense in $\mathscr{M}$, i.e., for most Riemannian metrics, all the eigenvalues of the Laplacian have multiplicity one. A similar result was obtained by Bleecker-Wilson [BW]. They showed that, for each Riemannian metric g, there exists a residual set of f in $C^\infty(M)$ for which all the eigenvalues of the Riemannian metric $\exp(f)g$ have multiplicity one. Their result implies the density of $\mathscr{S}$ in $\mathscr{M}$, but it does not necessarily imply that $\mathscr{S}$ is residual in $\mathscr{M}$.

Secondly, we show the following proposition.

PROPOSITION 3.4. *Let* M *be a compact connected* C^∞ *manifold of dimension not less than two. If a Riemannian metric* g *belongs to the set* $\mathscr{S}$, i.e., *if all the eigenvalues of the Laplacian* Δ_g *have multiplicity one, then the group of all isometries of* (M, g) *is discrete.*

Combining this with Theorem 3.1, we have:

COROLLARY 3.5. *Let* M *be a compact connected* C^∞ *manifold of dimension not less than two. Then the set of all elements* g *in* $\mathscr{M}$ *with discrete isometry group contains a residual subset of* $\mathscr{M}$.

That is, for most Riemannian metrics of a compact connected C^∞ manifold of dimension not less than two, the isometry groups are trivial. This corollary was obtained by Ebin (cf. [E_1, Proposition 8.3]) in a different manner.

We express our thanks to Professors P. Bérard, T. Kotake, M. Tani-

kawa and S. Tanno for their advice and criticism during the preparation of this paper.

1. Complete distance on the set of Riemannian metrics. Let M be a compact n-dimensional C^∞ manifold without boundary. Let $S(M)$ be the space of all C^∞ symmetric covariant 2-tensors on M and $\mathscr{M}$ the set of all C^∞ Riemannian metrics on M. In this section, we define a complete distance on $\mathscr{M}$.

1.1. *Fréchet space* $S(M)$. Following [E₂] and [GG], we introduce a Fréchet norm $|\cdot|$ on $S(M)$. We fix a finite covering $\{U_\lambda\}_{\lambda\in\Lambda}$ of M such that the closure of U_λ is contained in the open coordinate neighborhood V_λ. For $h \in S(M)$, we denote by h_{ij} the components of h with respect to coordinates $(x_1, \cdots, x_n)$ on V_λ, $\lambda\in\Lambda$. For every non-negative integer k and $\lambda\in\Lambda$, put

$$|h|_{\lambda,k} = \sup_{U_\lambda} \sum_{|\alpha|\leq k} \sum_{i,j=1}^{n} |\partial^{|\alpha|}(h_{ij})/\partial(x_1)^{\alpha_1} \cdots \partial(x_n)^{\alpha_n}| ,$$

where $\alpha = (\alpha_1, \cdots, \alpha_n)$ denotes an n-tuple of non-negative integers α_i and $|\alpha| = \alpha_1 + \cdots + \alpha_n$. Define a norm $|\cdot|_k$ on $S(M)$ by $|h|_k = \sum_{\lambda\in\Lambda}|h|_{\lambda,k}$, $h \in S(M)$, and a Fréchet norm $|\cdot|$ on $S(M)$ by

$$|h| = \sum_{k=0}^{\infty} 2^{-k}|h|_k(1 + |h|_k)^{-1} , \quad h \in S(M) .$$

We can define a distance ρ' on $S(M)$ by $\rho'(h_1, h_2) = |h_1 - h_2|$, $h_1, h_2 \in S(M)$. Then it is well-known that $S(M)$ is a Fréchet space, that is, the metric space $(S(M), \rho')$ is complete.

1.2. *Complete distance of* $\mathscr{M}$. For each point x in M, let P_x (resp. S_x) be the set of all symmetric positive definite (resp. merely symmetric) bilinear forms on $T_xM \times T_xM$, where T_xM is the tangent space of M at $x \in M$. We define a distance ρ''_x on P_x, $x \in M$, by

$$\rho''_x(\varphi, \psi) = \inf\{\delta > 0; \exp(-\delta)\varphi < \psi < \exp(\delta)\varphi\} ,$$

where, for φ, ψ in S_x, $\varphi < \psi$ means that $\psi - \varphi \in S_x$ is positive definite on $T_xM\times T_xM$. In fact, ρ''_x defines clearly a distance on P_x. Let G_x, $x \in M$, be the group of all non-singular linear mappings of T_xM onto itself. For $A \in G_x$ and $\varphi \in S_x$, put $\varphi^A(u, v) = \varphi(A(u), A(v))$ for $u, v \in T_xM$. We fix a basis $\{e_i\}_{i=1}^n$ of T_xM and identify S_x with the set $S(n)$ of all real symmetric matrices of degree n by $S_x \ni \varphi \mapsto (\varphi(e_i, e_j))_{1\leq i,j\leq n} \in S(n)$. Denote by Φ this identification of S_x with $S(n)$. Let $P(n)$ be the set of all positive definite matrices in $S(n)$. Then we have the following lemma immediately.

LEMMA 1.1. (i) $\rho''_x(\varphi^A, \psi^A) = \rho''_x(\varphi, \psi)$ *for every* $A \in G_x$ *and* $\varphi, \psi \in P_x$.

(ii) *Let $\varphi_0 \in P_x$ be the element such that $\Phi(\varphi_0)$ is the identity matrix. Then we have*

$$\rho''_x(\varphi, \varphi_0) = \|\log \Phi(\varphi)\| , \quad \varphi \in P_x .$$

Here we denote by $\log A$, $A \in P(n)$, the inverse image of the exponential mapping of $S(n)$ onto $P(n)$ and by $\|H\|$, $H \in S(n)$, the operator norm of H, that is, $\|H\| = \sup\{\|H(x)\|; x \in \boldsymbol{R}^n$ and $\|x\| = 1\}$, where $\|\cdot\|$ is the Euclidean norm of $\boldsymbol{R}^n$.

(iii) *The metric space (P_x, ρ''_x) is complete.*

(iv) *Let $\{\varphi_j\}_{j=1}^{\infty}$ be a sequence in P_x which converges to an element φ in P_x with respect to the distrance ρ''_x. Then $\lim_{j\to\infty} \varphi_j(u, v) = \varphi(u, v)$ for every $u, v \in T_xM$.*

DEFINITION. We define a distance ρ'' on $\mathscr{M}$ by

$$\rho''(g_1, g_2) = \sup_{x \in M} \rho''_x((g_1)_x, (g_2)_x) , \quad g_1, g_2 \in \mathscr{M},$$

and a distance ρ on $\mathscr{M}$ by

$$\rho(g_1, g_2) = \rho'(g_1, g_2) + \rho''(g_1, g_2) , \quad g_1, g_2 \in \mathscr{M}.$$

Then, by Lemma 1.1, we have:

PROPOSITION 1.2. *The metric space $(\mathscr{M}, \rho)$ is complete.*

PROOF. We prove this in the usual manner. Let $\{g_j\}_{j=1}^{\infty}$ be a Cauchy sequence in $(\mathscr{M}, \rho)$. Then it is also a Cauchy sequence in both metric spaces $(S(M), \rho')$ and $(\mathscr{M}, \rho'')$. Since the metric space $(S(M), \rho')$ is complete, there exists an element g in $S(M)$ such that $\lim_{j\to\infty} \rho'(g_j, g) = 0$. In particular, for each $x \in M$ and $u, v \in T_xM$ we have

$$\lim_{j\to\infty} (g_j)_x(u, v) = g_x(u, v) . \tag{1.1}$$

On the other hand, because of $\lim_{i,j\to\infty} \rho''(g_i, g_j) = 0$, for every $\varepsilon > 0$, there exists a positive number N such that

$$\rho''_x((g_i)_x, (g_j)_x) \leqq \rho''(g_i, g_j) < \varepsilon \tag{1.2}$$

for every $i, j \geqq N$ and $x \in M$. Then the sequence $\{(g_j)_x\}_{j=1}^{\infty}$ is a Cauchy sequence in the complete metric space (P_x, ρ''_x), hence it converges to an element $\tilde{g}_x$ in P_x with respect to ρ''_x. By Lemma 1.1 (iv), we have $\lim_{j\to\infty} (g_j)_x(u, v) = \tilde{g}_x(u, v)$, $u, v \in T_xM$, so we obtain $g = \tilde{g} \in \mathscr{M}$. Therefore, combining this with the inequalities (1.2), we have $\rho''_x((g_i)_x, g_x) \leqq \varepsilon$ for all $x \in M$. Thus we obtain $\rho''(g_i, g) \leqq \varepsilon$ for $i \geqq N$, that is, $\lim_{i\to\infty} \rho''(g_i, g) = 0$. Therefore the sequence $\{g_i\}_{i=1}^{\infty}$ converges to $g \in \mathscr{M}$ with respect to the distance ρ. q.e.d.

2. Continuity of eigenvalues. 2.1. *Preliminaries.* For every g in $\mathscr{M}$, let $-\Delta_g$ be the Laplace-Beltrami operator acting on the space $C^\infty(M)$ of all real valued C^∞ functions on M, that is,

$$-\Delta_g = \sum_{i,j=1}^{n} g^{ij}(\partial^2/\partial x_i \partial x_j - \sum_{k=1}^{n} \Gamma_{ij}^k \partial/\partial x_k) . \tag{2.1}$$

Here (g^{ij}) is the inverse matrix of the component matrix (g_{ij}) of the Rimannian metric g with respect to a local coordinate $(x_1, \cdots, x_n)$ on M, and Γ_{ij}^k is the Christoffel symbol:

$$\Gamma_{ij}^k = \frac{1}{2}\sum_{m=1}^{n} g^{km}(\partial g_{mi}/\partial x_j + \partial g_{mj}/\partial x_i - \partial g_{ji}/\partial x_m) . \tag{2.2}$$

Let $(\ ,\ )_g$ be the inner product on $C^\infty(M)$ given by

$$(f_1, f_2)_g = \int_M f_1(x) f_2(x) dv_g(x) , \quad f_1, f_2 \in C^\infty(M) , \tag{2.3}$$

and put $\|f\|_g = ((f, f)_g)^{1/2}$ for $f \in C^\infty(M)$. Here $dv_g(x)$ is the canonical measure of (M, g) given locally by

$$dv_g(x) = (\det(g_{ij}))^{1/2} dx_1 \cdots dx_n \quad \text{(cf. [BGM, p. 10])} . \tag{2.4}$$

Define as usual the inner product $(\ ,\ )_g$ on the space $A^1(M)$ of all real valued C^∞ 1-forms on M by

$$(\omega_1, \omega_2)_g = \int_M \langle \omega_1, \omega_2 \rangle_g(x) dv_g(x) , \quad \omega_1, \omega_2 \in A^1(M) , \tag{2.5}$$

and put $\|\omega\|_g = ((\omega, \omega)_g)^{1/2}$ for $\omega \in A^1(M)$. The pointwise inner product $\langle \omega_1, \omega_2 \rangle_g(x)$ of $\omega_i \in A^1(M)$, $i = 1, 2$, is given by

$$\langle \omega_1, \omega_2 \rangle_g(x) = \sum_{i,j=1}^{n} g^{ij}(x) a_{1i}(x) a_{2i}(x) , \quad x \in M , \tag{2.6}$$

where $\{a_{ki}(x)\}_{i=1}^n$, $k = 1, 2$, are the components of the cotangent vectors $(\omega_k)_x$, $k = 1, 2$, with respect to the local coordinate $(x_1, \cdots, x_n)$.

2.2. *Max-mini principle.* Since M is compact, the spectrum of the Laplacian Δ_g is a discrete set of non-negative eigenvalues with finite multiplicities. We arrange the eigenvalues as

$$0 = \lambda_0(g) < \lambda_1(g) \leqq \lambda_2(g) \leqq \cdots \leqq \lambda_k(g) \leqq \cdots \uparrow \infty .$$

Here the eigenvalues are counted repeatedly as many times as their multiplicities. For example if the multiplicity of $\lambda_1(g)$ is h and $k \leqq h$, then the k-th eigenvalue $\lambda_k(g)$ of (M, g) is $\lambda_1(g)$, i.e., $\lambda_2(g) = \cdots = \lambda_h(g) = \lambda_1(g)$. Then we have the following useful Max-mini principle.

PROPOSITION 2.1. *For $g \in \mathscr{M}$, the k-th eigenvalue $\lambda_k(g)$ of the Laplacian*

Δ_g is given as follows: For every $(k+1)$-dimensional subspace L_{k+1} in $C^\infty(M)$, put

$$\Lambda_g(L_{k+1}) = \sup\{\|df\|_g^2/\|f\|_g^2;\ 0\neq f\in L_{k+1}\}\ .$$

Then we have

$$\lambda_k(g) = \inf_{L_{k+1}} \Lambda_g(L_{k+1})\ ,$$

where L_{k+1} varies over all $(k+1)$-dimensional subspaces of $C^\infty(M)$.

REMARK. The usual Mini-max principle is of the following type: For k-dimensional subspace L_k of $C^\infty(M)$, put

$$\tilde{\Lambda}_g(L_k) = \inf\{\|df\|_g^2/\|f\|_g^2;\ 0\neq f\in C^\infty(M) \quad \text{and} \quad f\perp L_k\}\ ,$$

where $f\perp L_k$ means that f is orthogonal to each element in L_k with respect to the inner product $(\ ,\ )_g$. Then $\lambda_k(g)$ is given by

$$\lambda_k(g) = \sup_{L_k} \tilde{\Lambda}_g(L_k)\ .$$

Here L_k runs over all k-dimensional subspaces of $C^\infty(M)$. Notice that the orthogonality of f to L_k depends on the Riemannian metric g. So we can not use this Mini-max principle to prove Theorem 2.2.

PROOF OF PROPOSITION 2.1. For completeness, we give here a proof of Proposition 2.1. We take a complete orthonormal basis $\{u_k\}_{k=0}^\infty$ of $C^\infty(M)$ with respect to $(\ ,\ )_g$ so that each u_k is an eigenfunction of Δ_g with the eigenvalue $\lambda_k(g)$, $k=0,1,2,\cdots$. Each $f\in C^\infty(M)$ can be expanded as $f=\sum_{i=0}^\infty x_i(f)u_i$, $x_i(f)\in \boldsymbol{R}$, in the sense of the uniform convergence or the L^2-convergence with respect to $(\ ,\ )_g$. In the following we omit the subscript g and simply denote $\Lambda(L_{k+1})=\Lambda_g(L_{k+1})$, $\|\cdot\|=\|\cdot\|_g$, etc.

Let L_{k+1}° be the $(k+1)$-dimensional subspace of $C^\infty(M)$ generated by $\{u_i\}_{i=0}^k$. Then, since $\Lambda(L_{k+1}^\circ)=\lambda_k$, we have $\lambda_k \geqq \inf_{L_{k+1}}\Lambda(L_{k+1})$. Suppose that $\lambda_k > \inf_{L_{k+1}}\Lambda(L_{k+1})$. Then there exists a $(k+1)$-dimensional subspace L_{k+1} of $C^\infty(M)$ such that $\lambda_k > \Lambda(L_{k+1})$. Then by definition each $f\in L_{k+1}$ satisfies $\Lambda(L_{k+1})\cdot\sum_{i=0}^\infty x_i(f)^2 \geqq \sum_{i=0}^\infty \lambda_i x_i(f)^2$. Thus we have

$$\sum_{\Lambda(L_{k+1})\geqq\lambda_i} (\Lambda(L_{k+1})-\lambda_i)x_i(f)^2 \geqq \sum_{\Lambda(L_{k+1})<\lambda_i} (\lambda_i-\Lambda(L_{k+1}))x_i(f)^2\ . \tag{2.7}$$

Now let $m=\max\{i;\ \lambda_i\leqq \Lambda(L_{k+1})\}$. Define a linear mapping Φ of L_{k+1} into $C^\infty(M)$ by

$$\Phi(f) = \sum_{i=0}^m x_i(f)u_i \quad \text{for} \quad f=\sum_{i=0}^\infty x_i(f)u_i\in L_{k+1}\ .$$

Then the dimension of the image of L_{k+1} under Φ is smaller than $k+1$. Indeed, for each $i=0,\cdots,m$, the fact that $\lambda_i\leqq \Lambda(L_{k+1})<\lambda_k$ implies that

$\dim \Phi(L_{k+1}) \leqq m+1 < k+1$. Therefore there exists a non-zero element f_0 in L_{k+1} such that $\Phi(f_0)=0$, that is, $x_i(f_0)=0$ for i with $\lambda_i \leqq \Lambda(L_{k+1})$. We apply (2.7) to this f_0 in L_{k+1}. If the left hand side of (2.7) is equal to zero, then each term on the right hand side is zero. Thus $x_i(f_0)=0$ for i with $\lambda_i > \Lambda(L_{k+1})$. Therefore we obtain $f_0 = \sum_{i=0}^{\infty} x_i(f_0) u_i = 0$, which is a contradiction. q.e.d.

2.3. *Proof of Theorem* 2.2. In this subsection, we show Theorem 2.2. For each positive number δ and $g \in \mathscr{M}$, we denote by $U_\delta(g)$ (resp. $V_\delta(g)$) the set $\{g' \in \mathscr{M};\ \rho(g', g) < \delta\}$ (resp. $\{g' \in \mathscr{M};\ \rho''(g', g) < \delta\}$). We note $U_\delta(g) \subset V_\delta(g)$.

THEOREM 2.2. *Let δ be a positive number and let g be in $\mathscr{M}$. Then*

(2.8) *$g' \in V_\delta(g)$ implies* $\exp(-(n+1)\delta) \leqq \lambda_k(g)/\lambda_k(g') \leqq \exp((n+1)\delta)$, *for each $k = 1, 2, \cdots$. Thus*

(2.9) *$g' \in V_\delta(g)$ implies* $|\lambda_k(g') - \lambda_k(g)| \leqq (\exp((n+1)\delta)-1)\lambda_k(g)$, *for each $k = 0, 1, 2, \cdots$.*

By Theorem 2.2, we have the following:

COROLLARY 2.3. *The multiplicity $m_k(g)$ of each eigenvalue $\lambda_k(g)$, that is, $m_k(g) = \#\{i;\ \lambda_i(g) = \lambda_k(g)\}$ depends upper semi-continuously on $g \in \mathscr{M}$: For each $g \in \mathscr{M}$ and $k = 0, 1, 2, \cdots$, there exists a positive number δ such that*

$$g' \in V_\delta(g) \quad \textit{implies} \quad m_k(g') \leqq m_k(g)\,.$$

PROOF OF THEOREM 2.2. *Let* $(x_1, \cdots, x_n)$ be a local coordinate on an open set U of M. For each $\delta > 0$ and $g' \in V_\delta(g)$, the component matrices (g_{ij}), (g'_{ij}) of g, g' satisfy

$$(\exp(-\delta) g'_{ij}) < (g_{ij}) < (\exp(\delta) g'_{ij})$$

as symmetric matrices on U by the definition of the distance ρ''. Then we have

$$\exp((-n/2)\delta)(\det(g'_{ij}))^{1/2} < (\det(g_{ij}))^{1/2} < \exp((n/2)\delta)(\det(g'_{ij}))^{1/2}$$

and

$$(\exp(-\delta) g'^{ij}) < (g^{ij}) < (\exp(\delta) g'^{ij})\,.$$

Hence, for each $f \in C^\infty(M)$ and $\omega \in A^1(M)$ with support contained in U, we obtain

$$\exp((-n/2)\delta)\|f\|_{g'}^2 \leqq \|f\|_g^2 \leqq \exp((n/2)\delta)\|f\|_{g'}^2 \tag{2.10}$$

and

$$\exp\left(-\left(\frac{n}{2}+1\right)\delta\right)\|\omega\|_{g'}^2 \leqq \|\omega\|_g^2 \leqq \exp\left(\left(\frac{n}{2}+1\right)\delta\right)\|\omega\|_{g'}^2 \tag{2.11}$$

by the definitions of the inner products on $C^\infty(M)$ and $A^1(M)$ and by the above inequalities. Making use of the partition of unity, we have (2.10) and (2.11) for every $f \in C^\infty(M)$ and $\omega \in A^1(M)$. Thus we have

$$\exp(-(n+1)\delta)\|df\|_{g'}^2/\|f\|_{g'}^2 \leqq \|df\|_g^2/\|f\|_g^2 \leqq \exp((n+1)\delta)\|df\|_{g'}^2/\|f\|_{g'}^2$$

for every non-zero element f in $C^\infty(M)$. Therefore, by Proposition 2.1, we obtain

$$\exp(-(n+1)\delta)\lambda_k(g') \leqq \lambda_k(g) \leqq \exp((n+1)\delta)\lambda_k(g') . \qquad \text{q.e.d.}$$

REMARK. From the above proof, for each $g, g' \in \mathscr{M}$, if g' is close to g with respect to the C^0-topology, then the ratio $\lambda_k(g)/\lambda_k(g')$ is close to one for each $k = 1, 2, \cdots$. But notice that the coefficients of the first order terms of the Laplacians Δ_g and $\Delta_{g'}$ are not in general close to each other (cf. 2.1)).

3. Genericity of eigenvalues with multiplicity one. 3.1. *Uhlenbeck's theorem.* A subset S of a topological space X is residual if S is a countable intersection of open dense subsets of X. A topological space X is called a Baire space if any residual subset of X is dense in X. It is well known that a complete metric space (X, ρ) is a Baire space and a residual set in the complete metric space is a subset of the second category. Under these terminologies, we can state Uhlenbeck's theorem:

THEOREM 3.1 (cf. [U] and [T]). *Let M be a compact connected C^∞ manifold of dimension not less than two. Let $\mathscr{M}$ be the set of all C^∞ Riemannian metrics on M and ρ the complete distance on $\mathscr{M}$ as in §1. Let $\mathscr{S}$ be the set of all elements g in $\mathscr{M}$ all of whose eigenvalues of Δ_g have multiplicity one, that is,*

$$\mathscr{S} = \{g \in \mathscr{M};\ \lambda_0(g) < \lambda_1(g) < \lambda_2(g) < \cdots < \lambda_k(g) < \cdots\} .$$

Then $\mathscr{S}$ is a residual set in $(\mathscr{M}, \rho)$.

The proof of Theorem 3.1 can be carried out as follows: Let $\mathscr{S}_k$ be the set of all elements in $\mathscr{M}$ of which the first k eigenvalues have multiplicity one, that is,

$$\mathscr{S}_k = \{g \in \mathscr{M};\ \lambda_0(g) < \lambda_1(g) < \cdots < \lambda_{k-1}(g) < \lambda_k(g)\} ,$$

for each $k = 1, 2, \cdots$. Then we have

$$\mathscr{M} = \mathscr{S}_1 \supset \mathscr{S}_2 \supset \cdots \supset \mathscr{S}_k \supset \cdots \supset \mathscr{S} \quad \text{and} \quad \mathscr{S} = \bigcap_{k=1}^{\infty} \mathscr{S}_k .$$

Then it remains to prove the following two theorems.

THEOREM 3.2. *Each* $\mathscr{S}_k$, $k = 1, 2, \cdots$, *is open in* $(\mathscr{M}, \rho)$.

THEOREM 3.3. *Let M be a compact connected* C^∞ *manifold of dimension not less than two. Then each* $\mathscr{S}_{k+1}$, $k = 1, 2, \cdots$, *is dense in* $\mathscr{S}_k$ *with respect to the topology induced by* $(\mathscr{M}, \rho)$.

3.2. *The isometry group.* Before going into the proof of Theorems 3.2 and 3.3, we discuss the genericity of Riemannian metrics with trivial isometry group.

For $g \in \mathscr{M}$, we denote the eigenvalues of Δ_g by

$$0 < \lambda_1(g) = \cdots = \lambda_{j_1}(g) < \lambda_{j_1+1}(g) = \cdots = \lambda_{j_2}(g) < \cdots, \text{ etc}.$$

Put $\lambda_{j_0}(g) = \lambda_0(g) = 0$. Let V_k be the eigenspace of Δ_g with the eigenvalue $\lambda_{j_k}(g)$, $k = 0, 1, 2, \cdots$. Notice that $\dim V_k = j_k - j_{k-1}$. Let $\{u_i\}_{i=0}^\infty$ be a complete basis of $C^\infty(M)$ such that $\Delta_g u_i = \lambda_i(g)u_i$ and $(u_i, u_j)_g = \delta_{ij}$, $i, j = 0, 1, 2, \cdots$. Take a large integer r so that the mapping $\iota\colon M \ni x \mapsto \iota(x) = (u_0(x), u_1(x), \cdots, u_{i_{N-1}}(x)) \in \boldsymbol{R}^N$, $N = 1 + j_1 + \cdots + j_r$, is an embedding of M into $\boldsymbol{R}^N$. The Lie group G of all isometries of (M, g) acts on $C^\infty(M)$ by $\Phi^*u(x) = u(\Phi^{-1}(x))$, $x \in M$, $u \in C^\infty(M)$ and $\Phi \in G$. Then $\Phi^*, \Phi \in G$, are linear mappings of $C^\infty(M)$ into itself and satisfy the conditions $(\Phi^*u, \Phi^*v)_g = (u, v)_g$ and $\Phi_1^* \circ \Phi_2^* = (\Phi_1 \circ \Phi_2)^*$ for $u, v \in C^\infty(M)$ and $\Phi, \Phi_1, \Phi_2 \in G$. Moreover, since $\Delta_g(\Phi^*u) = \Phi^*(\Delta_g u)$, we see that Φ^* maps each eigenspace V_k, $k = 0, 1, 2, \cdots, r$, into itself. Then we obtain a Lie group homomorphism ι^* of G into the orthogonal group $O(V)$ of the Euclidean space $(V, (\ ,\ )_g)$, $V = \sum_{k=0}^r V_k$, by $G \mapsto \Phi^* \in O(V)$. Note that the homomorphism ι^* is one to one since so is ι. Now, if $g \in \mathscr{S}$, then each V_k, $k = 0, 1, 2, \cdots$, is one dimensional. Thus the Lie subgroup $\iota^*(G)$ of $O(V)$ is discrete. Since ι^* is injective, G itself is discrete. Therefore we have:

PROPOSITION 3.4. *If* $g \in \mathscr{S}$, *that is, if all the eigenvalues of* Δ_g *have multiplicity one, then the group of all isometries of* (M, g) *is discrete.*

Combining this with Theorem 3.1, we have:

COROLLARY 3.5. *Let M be a compact connected* C^∞ *manifold of dimension not less than two. Let* $\mathscr{M}$ *be the set of all* C^∞ *Riemannian metrics on M and* ρ *the complete distance on* $\mathscr{M}$ *as in* §1. *Then the set of all elements g in* $\mathscr{M}$ *with discrete isometry group contains a residual subset of* $\mathscr{M}$.

REMARK. The above corollary was obtained in [E_1, Proposition 8.3, p. 35] in a different manner.

3.3. *Proof of Theorem* 3.2. Let g be an arbitrary element in $\mathscr{S}_k$, $k=0, 1, 2, \cdots$. We prove that there exists a positive number δ such that $V_\delta(g)$ is contained in $\mathscr{S}_k$. Let $\varepsilon = \min\{\lambda_{j+1}(g) - \lambda_j(g);\ j=0, 1, \cdots, k-1\} > 0$. We choose $\delta > 0$ so small that $\varepsilon(2\lambda_k(g))^{-1} > \exp((n+1)\delta) - 1$. Then, for $g' \in V_\delta(g)$ and $j = 0, 1, \cdots, k-1$, we have

$$\begin{aligned}\varepsilon &\leqq \lambda_{j+1}(g) - \lambda_j(g)\\ &\leqq |\lambda_{j+1}(g) - \lambda_{j+1}(g')| + |\lambda_{j+1}(g') - \lambda_j(g')| + |\lambda_j(g') - \lambda_j(g)|\\ &\leqq (\exp((n+1)\delta) - 1)(\lambda_{j+1}(g) + \lambda_j(g)) + |\lambda_{j+1}(g') - \lambda_j(g')|\\ &\qquad\qquad\text{(by Theorem 2.2)}\\ &\leqq 2\lambda_k(g)(\exp((n+1)\delta) - 1) + |\lambda_{j+1}(g') - \lambda_j(g')|\,.\end{aligned}$$

Thus we obtain

$$0 < \varepsilon - 2\lambda_k(g)(\exp((n+1)\delta) - 1) \leqq |\lambda_{j+1}(g') - \lambda_j(g')|\,,$$

$j = 0, 1, \cdots, k-1$, which implies $g' \in \mathscr{S}_k$. We have $V_\delta(g) \subset \mathscr{S}_k$. q.e.d.

4. **Density of $\mathscr{S}_k$ in $\mathscr{M}$.** 4.1. *Preparations.* In this subsection, we prove some lemmas concerning a deformation $g(t)$ of g in $\mathscr{M}$. They will be used in the proof of Theorem 3.3.

LEMMA 4.1 (cf. [B, Lemma 3.15]). *For $g \in \mathscr{M}$ and $h \in S(M)$, let $g(t) = g + th \in \mathscr{M}$, $|t| < \varepsilon$. Let λ be an eigenvalue of Δ_g with multiplicity l. Then there exist $\Lambda_i(t) \in \boldsymbol{R}$* and *$u_i(t) \in C^\infty(M)$, $i = 1, \cdots, l$, such that*

(i) *$\Lambda_i(t)$ and $u_i(t)$ depend real analytically on t, $|t| < \varepsilon$, for each $i = 1, \cdots, l$,*

(ii) *$\Delta_{g(t)}u_i(t) = \Lambda_i(t)u_i(t)$, for each $i = 1, \cdots, l$ and t,*

(iii) *$\Lambda_i(0) = \lambda$, $i = 1, \cdots, l$, and*

(iv) *$\{u_i(t)\}_{i=1}^l$ is orthonormal with respect to $(\ ,\ )_{g(t)}$ for each t.*

For a proof, see [B, p. 137] and also Appendix.

REMARK. Lemma 4.1 does not necessarily imply Theorem 2.2, since the positive number ε may depend on $h \in S(M)$ in general.

LEMMA 4.2. *Let $g \in \mathscr{M}$ and let $a \in C^\infty(M)$ be a positive real valued function on M. Then the Laplacian Δ_{ag} corresponding to the Riemannian metric ag on M is given by*

$$\Delta_{ag} = a^{-1}\Delta_g + (1 - n/2)a^{-2}\nabla_g(a)\,,$$

where $n = \dim M$ and $\nabla_g(a)$ is the gradient vector field of the function $a \in C^\infty(M)$ with respect to the Riemannian metric g.

PROOF. Making use of (2.1) and (2.2), we may prove this by a straightforward calculation.

LEMMA 4.3. *For every* $g \in \mathcal{M}$, *we have the following:*

(i) *For* σ, f_1 *and* $f_2 \in C^\infty(M)$, *we have*

$$(\nabla_g(\sigma)f_1, f_2)_g = (\sigma, \delta(f_2 df_1))_g ,$$

where $\delta; A^1(M) \to C^\infty(M)$ *is the codifferential operator with respect to* g.

(ii) $\delta(f_2 df_1) = -\langle df_1, df_2\rangle_g + f_2 \Delta_g f_1$, $f_1, f_2 \in C^\infty(M)$, *where* $\langle \cdot, \cdot \rangle_g$ *is the pointwise inner product in* $A^1(M)$ *relative to* g.

(iii) *Let* V_λ *be the eigenspace of* Δ_g *belonging to the eigenvalue* λ. *For every* u *and* v *in* V_λ, *we have* $\delta(udv) = \delta(vdu)$.

PROOF. (i) Since $\nabla_g(\sigma)f_1 = \langle d\sigma, df_1\rangle_g$, we have $(\nabla_g(\sigma)f_1, f_2)_g = (d\sigma, f_2 df_1)_g = (\sigma, \delta(f_2 df_1))_g$. (ii) For $\omega = \sum_{j=1}^n \omega_j dx_j \in A^1(M)$, $\delta\omega = -\sum_{i,j=1}^n g^{ij}\nabla_i \omega_j$, where $\nabla_i \omega_j$ is the covariant derivative with respect to g of the 1-form ω by the derivative $\partial/\partial x_i$ relative to the coordinate x_i, $i = 1, \cdots, n$. Then we have

$$\begin{aligned}\delta(f_2 df_1) &= -\sum_{i,j=1}^n g^{ij}\nabla_i(f_2 df_1)_j = -\sum_{i,j=1}^n g^{ij}(\partial f_2/\partial x_i)(\partial f_1/\partial x_j) - \sum_{i,j=1}^n g^{ij} f_2 \nabla_i (df_1)_j \\ &= -\langle df_1, df_2\rangle_g + f_2 \Delta_g f_1 .\end{aligned}$$

(iii) $\delta(udv) = -\langle du, dv\rangle_g + u\Delta_g v = -\langle du, dv\rangle_g + v\Delta_g u = \delta(vdu)$, for u, $v \in V_\lambda$. q.e.d.

4.2. *Splitting the eigenvalues.* In the following, we consider a deformation $g(t)$ of $g \in \mathcal{M}$ given by

$$g(t) = g + t\sigma g , \quad \text{for} \quad \sigma \in C^\infty(M) . \tag{4.1}$$

For small enough $\varepsilon(\sigma) > 0$, we have $g(t) \in \mathcal{M}$ for all t with $|t| < \varepsilon(\sigma)$.

Now let λ be a non-zero eigenvalue of Δ_g with multiplicity l and let $\{u_j\}_{j=1}^l$ be an orthonormal system with respect to $(\, , \,)_g$ such that $\Delta_g u_j = \lambda u_j$, $j = 1, \cdots, l$. Applying Lemma 4.1 to $g(t)$, we obtain $\Lambda_j(t) \in \boldsymbol{R}$ and $u_j(t) \in C^\infty(M)$, $j = 1, \cdots, l$, satisfying the conditions (i)~(iv) in Lemma 4.1. By (i) in Lemma 4.1 (see also Theorem A.3 in Appendix), we can express $\Lambda_j(t)$ and $u_j(t)$, $j = 1, \cdots, l$, as follows:

$$\Lambda_j(t) = \lambda + t\alpha_j + t^2\beta_j(t) \quad \text{for} \quad |t| < \varepsilon(\sigma) , \tag{4.2}$$

where α_j is a real constant and $\beta_j(t)$ is a real analytic real valued function in t.

$$(u_j(t), v)_g \ \text{ are real analytic functions in } t, \ |t| < \varepsilon(\sigma) , \tag{4.3}$$

for every $v \in C^\infty(M)$. Then we have the following:

LEMMA 4.4. *Let* λ *be a non-zero eigenvalue of* Δ_g *with multiplicity* l *and let* $\{u_j\}_{j=1}^l$ *be an orthonormal system with respect to* $(\, , \,)_g$ *such that*

$\Delta_g u_j = \lambda u_j$ *for each* $j = 1, \cdots, l$. *For* $\sigma \in C^\infty(M)$, *let* $g(t)$ *be a deformation of* $g \in \mathscr{M}$ *given by* (4.1). *Let* $\{\alpha_j\}_{j=1}^l$ *be the real constants given by* (4.2). *Then we have*

$$(((1 - n/2)\nabla_g(\sigma) - \lambda\sigma)u_j, u_i)_g = \alpha_j \delta_{ij}, \quad 1 \leqq i, j \leqq l .$$

PROOF. We apply Lemma 4.2 to $g(t) = a(t)g$ with $a(t) = 1 + t\sigma > 0$ for $|t| < \varepsilon(\sigma)$. Then we have, for every $v \in C^\infty(M)$,

$$(a(t)\Delta_g u_j(t) + (1 - n/2)t\nabla_g(\sigma)u_j(t) - \Lambda_j(t)a(t)^2 u_j(t), v)_g = 0 ,$$

by $\Delta_{g(t)} u_j(t) = \Lambda_j(t)u_j(t)$, $j = 1, \cdots, l$, $|t| < \varepsilon(\sigma)$. Differentiating both sides of the above equality at $t = 0$, we obtain by (4.2) and (4.3)

$$((\Delta_g - \lambda)v_j + ((1 - n/2)\nabla_g(\sigma) - \lambda\sigma - \alpha_j)u_j, v)_g = 0 , \quad j = 1, \cdots, l .$$

Thus, for an eigenfunction v of Δ_g belonging to the eigenvalue λ, we have

$$\begin{aligned}(((1 - n/2)\nabla_g(\sigma) - \lambda\sigma - \alpha_j)u_j, v)_g &= -((\Delta_g - \lambda)v_j, v)_g \\ &= -(v_j, (\Delta_g - \lambda)v)_g = 0 . \qquad \text{q.e.d.}\end{aligned}$$

PROPOSITION 4.5. *Assume* $\dim M \geqq 2$. *In the situation of Lemma* 4.4, *there exists a function* σ *in* $C^\infty(M)$ *such that, at least two of* $\{\alpha_i\}_{i=1}^l$ *in* (4.2) *are distinct.*

PROOF. Let P be the orthogonal projection of $C^\infty(M)$ onto the eigenspace V_λ belonging to the eigenvalue λ of Δ_g. For $\sigma \in C^\infty(M)$, define an endomorphism G_σ of V_λ into itself by

$$G_\sigma f = P \circ ((1 - n/2)\nabla_g(\sigma) - \lambda\sigma)f , \quad f \in V_\lambda .$$

Let $\{u_i\}_{i=1}^l$ be an arbitrary fixed orthonormal basis of V_λ with respect to $(\, , \,)_g$. Then we have

$$(G_\sigma u_j, u_i)_g = (((1 - n/2)\nabla_g(\sigma) - \lambda\sigma)u_j, u_i)_g = \alpha_j \delta_{ij} ,$$

by Lemma 4.4. Thus the endomorphism G_σ can be expressed as a diagonal matrix with respect to $\{u_i\}_{i=1}^l$ whose diagonal entries are α_i, $i = 1, \cdots, l$.

Assume that $\alpha_1 = \cdots = \alpha_l$. Then G_σ can be expressed as a constant multiple of the identity matrix with respect to this basis and hence with respect to any basis of V_λ. Therefore, in order to prove Proposition 4.5, we have only to find $\sigma \in C^\infty(M)$ so that $(G_\sigma u_1, u_2)_g \neq 0$.

For $\sigma \in C^\infty(M)$, we have

$$\begin{aligned}(G_\sigma u_1, u_2)_g &= (((1 - n/2)\nabla_g(\sigma) - \lambda\sigma)u_1, u_2)_g \\ &= (\sigma, (1 - n/2)\delta(u_2 du_1) - \lambda u_1 u_2)_g .\end{aligned}$$

Case 1. $(1 - n/2)\delta(u_2 du_1) - \lambda u_1 u_2 \not\equiv 0$. In this case, putting $\sigma =$

$(1-n/2)\delta(u_2du_1)-\lambda u_1u_2$, we have $(G_\sigma u_1, u_2)_g \neq 0$.

Case 2. $(1-n/2)\delta(u_2du_1)-\lambda u_1u_2\equiv 0$. In this case, we have

$$u_1u_2\equiv 0\,. \tag{4.4}$$

In fact, we have

$$\begin{aligned}((1-n/2)\Delta_g-2\lambda)(u_1u_2) &= (1-n/2)\delta d(u_1u_2)-2\lambda u_1u_2\\ &= (1-n/2)\delta(u_1du_2+u_2du_1)-2\lambda u_1u_2\\ &= ((1-n/2)\delta(u_1du_2)-\lambda u_1u_2)+((1-n/2)\delta(u_2du_1)-\lambda u_1u_2)\\ &= 0\,,\end{aligned}$$

by Lemma 4.3 (iii) and the assumption. Since $2-n<0$, if $u_1u_2\not\equiv 0$, then Δ_g would have a negative eigenvalue, which is a contradiction. (4.4) is thus proved.

We take, as an orthonormal basis of V_λ with respect to $(\ ,\ )_g$,

$$f_1=2^{-1}(u_1+u_2)\,,\quad f_2=2^{-1}(u_1-u_2)\,,\quad f_3=u_3,\ \cdots\,,\quad f_l=u_l\,.$$

Put $\sigma=(1-n/2)\delta(f_2df_1)-\lambda f_1f_2$. Then we have

$$(G_\sigma f_1, f_2)_g=\int_M \sigma^2 dv_g\,.$$

So we have only to prove $\sigma\not\equiv 0$. Otherwise, we have

$$\begin{aligned}0\equiv 2\sigma &= (1-n/2)\delta((u_1-u_2)d(u_1+u_2))-\lambda(u_1+u_2)(u_1-u_2)\\ &= (1-n/2)(\delta(u_1du_1)-\delta(u_2du_2))-\lambda(u_1^2-u_2^2)\quad \text{(by Lemma 4.3)}\\ &= (4^{-1}(2-n)\delta d-\lambda)(u_1^2-u_2^2)\,.\end{aligned}$$

Thus, since $2-n\leqq 0$, we have $u_1^2-u_2^2\equiv 0$. Therefore we obtain

$$0=\int_M (u_1^2-u_2^2)^2 dv_g=\int_M (u_1^4-2u_1u_2+u_2^4)dv_g=\int_M (u_1^4+u_2^4)dv_g\,,$$

by (4.4), which is a contradiction. We thus obtain $\sigma\not\equiv 0$. q.e.d.

4.3. *Proof of Theorem* 3.3. Let $\dim M\geqq 2$. We show $\mathscr{S}_k$ is dense in $\mathscr{S}_{k+1}$. To prove this, we construct, for each $g\in\mathscr{S}_k$, an element g' in $\mathscr{S}_{k+1}$ which is arbitrarily close to g.

Let $g\in\mathscr{S}_k$, that is, $\lambda_0(g)<\lambda_1(g)<\cdots<\lambda_k(g)$. Assume that the k-th eigenvalue $\lambda_k(g)$ has multiplicity l, i.e.,

$$\begin{aligned}&\lambda_k(g)=\cdots=\lambda_{k+l-1}(g)=\lambda \quad\text{and}\\ &\lambda_0(g)<\lambda_1(g)<\cdots<\lambda_{k-1}(g)<\lambda<\lambda_{k+l}(g)\leqq\cdots\,.\end{aligned}$$

Consider a deformation $g(t)=g+t\sigma g\in\mathscr{M}$ of g, $|t|<\varepsilon(\sigma)$, of the type (4.1). Let $\Lambda_j(t)$, $j=1,\cdots,l$, be such eigenvalues of $\Delta_{g(t)}$ as (4.2).

We apply Proposition 4.5 to the eigenvalue $\lambda = \lambda_k(g)$. Noting that

$$g' \in V_{1/2}(g) \quad \text{implies} \quad \exp(-(n+1)/2)\lambda_m(g) \leqq \lambda_m(g'), \quad m = 0, 1, 2, \cdots,$$

by (2.8), we may assume

$$\exp((n+1)/2)\cdot(2\lambda) \leqq \lambda_m(g) \quad \text{implies} \quad 2\lambda \leqq \lambda_m(g(t)),$$

for each $m = 0, 1, 2, \cdots$, and $|t| < \varepsilon(\sigma)$. We apply Theorem 2.2 to a finite number of eigenvalues of Δ_g which are smaller than $\exp((n+1)/2)\cdot(2\lambda)$. Then there exists a positive number $\varepsilon'(\sigma) \leqq \varepsilon(\sigma)$ such that

$$\lambda_0(g(t)) < \lambda_1(g(t)) < \cdots < \lambda_{k-1}(g(t)) < \Lambda_j(t) < \lambda_{k+l}(g(t)) \leqq \cdots,$$

for each $|t| < \varepsilon'(\sigma)$ and $j = 1, \cdots, l$.

Now, by Proposition 4.5, we can choose $\sigma \in C^\infty(M)$ in such a way that, at least two of $\{\alpha_j\}_{j=1}^l$ in (4.2) are distinct. Let $\alpha_i \neq \alpha_j$, $1 \leqq i$, $j \leqq l$. For this $\sigma \in C^\infty(M)$, we may choose a positive number $\varepsilon''(\sigma) \leqq \varepsilon'(\sigma)$ in such a way that $\Lambda_i(t) \neq \Lambda_j(t)$ for all $0 < |t| < \varepsilon''(\sigma)$. Therefore all the first k eigenvalues of $\Delta_{g(t)}$, $|t| < \varepsilon''(\sigma)$, have multiplicity one and the k-th eigenvalue $\lambda_k(g(t))$ has multiplicity at most $l - 1$. Repeating this process, we can choose $g' \in \mathscr{S}_{k+1}$ as close to g as one wants. q.e.d.

Appendix. In this appendix, we give a proof of Lemma 4.1. The proof given in [B] was based on Kato's perturbation theory [K, p. 375] (See also [RN, p. 373]). In its proof, it was claimed (cf. [B, p. 138]) that the family of the operators $\Delta_{G(\kappa)}$ is of type (A) in the sense of Kato (cf. [K, p. 375]) and $\Delta_{G(\kappa)}$ are self-adjoint. But if we choose the domain of $\Delta_{G(\kappa)}$ as the Sobolev space $H_2(M)$ for a fixed Riemannian metric γ on M, then $\Delta_{G(\kappa)}$ are not self-adjoint with respect to the inner product $(\ ,\ )_\gamma$ in $H_2(M)$. If we require the self-adjointness of $\Delta_{G(\kappa)}$, then we have to choose the inner product $(\ ,\ )_{G(\kappa)}$ on $H_2(M)$. Since the domains of $\Delta_{G(\kappa)}$ vary as Hilbert spaces, the family of $\Delta_{G(\kappa)}$ is not of type (A). Its proof should be modified accordingly.

First we list some notations. Throughout this appendix, let M be an n-dimensional compact connected C^∞ manifold without boundary. Let $C_C^\infty(M)$ be the space of all complex valued C^∞ functions on M. For a fixed Riemannian metric γ on M, let Δ_γ be its Laplacian and $(\ ,\ )_\gamma$ be the inner product on $C_C^\infty(M)$ defined by

$$(\phi, \psi)_\gamma = \int_M \phi(x)\overline{\psi(x)}dv_\gamma, \quad \phi, \psi \in C_C^\infty(M),$$

where dv_γ is the canonical measure of (M, γ) (cf. [BGM, p. 10]). For every non-negative integer s, let $H_s(M)$ be the Sobolev space on M (cf. [G, p. 35]) which is the completion of $C_C^\infty(M)$ with respect to the following inner product $[\ ,\ ]_s$:

(A. 1) $$[\phi, \psi]_s = ((I + \Delta_r)^s \phi, \psi)_r\ , \quad \phi, \psi \in C_C^\infty(M)\ .$$

Here I is the identity operator and $(I + \Delta_r)^s$ is the s-ple iteration of the operator $I + \Delta_r$. Put $\|\phi\|_s = [\phi, \phi]_s^{1/2}$, $\phi \in H_s(M)$.

We define the notions of the real analytic families of vectors or bounded operators (cf. [K, p. 365]).

DEFINITION A. 1. Let X, Y be complex Banach spaces. Let D be a domain in $\boldsymbol{R}$. A family of vectors x_t, $t \in D$, in X is said to be *real analytic* if it can be expanded as a convergent power series, i.e., for an arbitrary fixed $t_0 \in D$, there exist elements x_α, $\alpha = 0, 1, 2, \cdots$, in X such that

$$x_t = \sum_{\alpha=0}^{\infty} x_\alpha (t - t_0)^\alpha\ , \quad \text{for every} \quad t \in D,\ |t - t_0| < \varepsilon\ ,$$

where the series converges in the sense of the strong topology of X (cf. [Y, p. 30]). A family of bounded operators A_t, $t \in D$, of X into Y is said to be *real analytic* if it can be expanded as a convergent power series of bounded operators, i.e., for an arbitrary fixed $t_0 \in D$, there exist bounded operators C_α, $\alpha = 0, 1, 2, \cdots$, of X into Y such that

$$A_t = \sum_{\alpha=0}^{\infty} C_\alpha (t - t_0)^\alpha\ , \quad \text{for every} \quad t \in D, |t - t_0| < \varepsilon\ ,$$

where the series converges in the uniform topology (cf. [Y, pp. 111–112]).

Then we have:

THEOREM A. 2. *Let D be a small bounded domain in $\boldsymbol{R}$ containing the origin* 0. *Let $s_1 > s_0$ be non-negative integers. Let A_t, $t \in D$, be a real analytic family of bounded operators of $H_{s_1}(M)$ into $H_{s_0}(M)$. Assume that*

(1) *each operator A_t, $t \in D$, is self-adjoint with the domain $H_{s_1}(M)$ contained in $H_{s_0}(M)$ with respect to the inner product $[\ ,\ ]_{s_0}$* (cf. [Y, p. 197]), *and*

(2) *A_0 is bounded below, i.e., there exists a positive constant C such that $[A_0(x), x]_{s_0} \geqq C[x, x]_{s_0}$ for all $x \in H_{s_1}(M)$.*

Let λ be an eigenvalue of the operator A_0. Then

(I) *the kernel of $A_0 - \lambda I$ is finite dimensional.*

(II) *Put $l = \dim \ker(A_0 - \lambda I)$. Then there exists a subdomain D' in D containing the origin and l real analytic families of vectors ϕ_t^i, $i = 1, \cdots, l$, in $H_{s_1}(M)$ and l real analytic real valued functions λ_t^i, $i = 1, \cdots, l$, in $t \in D'$ such that*

(3) $A_t\phi_t^i = \lambda_t^i\phi_t^i$, $i = 1, \cdots, l$, $t \in D'$,
(4) $[\phi_t^i, \phi_t^j]_{s_0} = \delta_{ij}$, $i, j = 1, \cdots, l$, $t \in D'$ *and*
(5) $\lambda_0^i = \lambda$, $i = 1, \cdots, l$.

The assertion (I) is well known since the bounded self-adjoint operator A_0 is bounded below. The similar assertion as (II) was stated in [RN, p. 376, Theorem], [K, p. 392, Theorem 3.9] and [R, p. 57, Theorem 1, p. 74, Theorem 3]. It can be proved by the similar way, so we omit its proof.

We apply Theorem A. 2 to prove Lemma 4.1. Let g_t, $|t| < \varepsilon$, be a one-parameter family of Riemannian metrics on M depending real analytically on the parameter t. In the following, we denote merely by Δ_t (resp. $(\ ,\ )_t$) the Laplacian Δ_{g_t} (resp. the inner product $(\ ,\ )_{g_t}$ on $C_C^\infty(M)$) of (M, g_t). Then we have:

THEOREM A. 3. *Let g_t, $|t| < \varepsilon$, be the one-parameter family of Riemannian metrics on M depending real analycally on the parameter t. For any eigenvalue λ of Δ_0 with multiplicity l, there exist l families of $\phi_t^i \in C_C^\infty(M)$, $i = 1, \cdots, l$, which are real analytic in $H_0(M)$, and l real analytic real valued functions λ_t^i, $i = 1, \cdots, l$, in t such that*
(6) $\Delta_t\phi_t^i = \lambda_t^i\phi_t^i$, $i = 1, \cdots, l$, *and* t,
(7) $(\phi_t^i, \phi_t^j)_t = \delta_{ij}$, $i, j = 1, \cdots, l$, *and* t, *and*
(8) $\lambda_0^i = \lambda$, $i = 1, \cdots, l$.

For the proof of Theorem A. 3, we need the following:

LEMMA A. 4. *Let L_t, $|t| < \varepsilon$, be differential operators of order m which can be expressed locally as*

$$L_t = \sum_{|\alpha| \leqq m} a_\alpha(t, x) D_x^\alpha \,.$$

Here $D_x^\alpha = \partial^{|\alpha|}/\partial(x_1)^{\alpha_1} \cdots \partial(x_n)^{\alpha_n}$ and $|\alpha| = \alpha_1 + \cdots + \alpha_n$ for an n-tuple $\alpha = (\alpha_1, \cdots, \alpha_n)$ of non-negative integers, and $a_\alpha(t, x)$ is real analytic in t, $|t| < \varepsilon$, where x belongs to the local coordinate open subset. Then the family of bounded operators L_t of $H_m(M)$ into $H_0(M)$ is real analytic.

PROOF. By assumption, $a_\alpha(t, x)$ can be expressed as $a_\alpha(t, x) = \sum_{k=0}^\infty a_{\alpha,k}(x)t^k$, where $a_{\alpha,k}(x)$ satisfy the following inequalities:

$$|a_{\alpha,k}(x)| \leqq Cr^k \quad \text{for all} \quad \alpha\,, \quad |\alpha| \leqq m\,, \quad k = 0, 1, 2, \cdots\,, \quad \text{and} \quad x\,.$$

Here the positive constants C and r do not depend on α, k and x. Using the partition of unity, define differential operators L_k, $k = 0, 1, 2, \cdots$, of order m which can be expressed locally as $L_k = \sum_{|\alpha| \leqq m} a_{\alpha,k}(x) D_x^\alpha$. Since L_k satisfy the inequalities

$$\|L_k f\|_0 \leqq m^n C' r^k \|f\|_m , \quad f \in H_m(M) ,$$

for a certain constant C', they are bounded operators of $H_m(M)$ into $H_0(M)$ and the series $\sum_{k=0}^{\infty} L_k t^k$ converges to L_t in the uniform topology.

q.e.d.

PROOF OF THEOREM A. 3. For a function f on M and for t with $|t| < \varepsilon$, put

$$(U_t f)(x) = (\det(g_{tij}(x))/\det(g_{0ij}(x)))^{1/4} f(x) , \quad x \in M ,$$

where g_{tij}, $|t|<\varepsilon$, are the components of g_t with respect to the local coordinate $(x_1, \cdots, x_n)$ around x. Then the operators U_t, $|t|<\varepsilon$, define a real analytic family of bounded operators of $H_s(M)$ into itself for every non-negative integer s. By definition they are isometries of the Hilbert space $(H_0(M), (\ , \ )_t)$ into the Hilbert space $(H_0(M), (\ , \ )_0)$. Since the Laplacian Δ_t, $|t| < \varepsilon$, are self-adjoint operators of $H_0(M)$ with respect to the inner product $(\ , \ )_t$, the operators $\tilde{\Delta}_t$ defined by the composition $U_t \circ \Delta_t \circ U_t^{-1}$ are self-adjoint with respect to the inner product $(\ , \ )_0$. Moreover by Lemma A. 4 the family of $A_t = \tilde{\Delta}_t + I$, $|t| < \varepsilon$, is a real analytic family of bounded operators of $H_2(M)$ into $H_0(M)$ and satisfies (1), (2) of Theorem A. 2. Therefore by Theorem A. 2 there exist l real analytic families of vectors $\tilde{\phi}_t^i$, $i = 1, \cdots, l$, in $H_2(M)$ and l real analytic real valued functions $1 + \lambda_t^i$, $i = 1, \cdots, l$, in t satisfying (3), (4) and (5). Then the vectors ϕ_t^i, $i = 1, \cdots, l$, in $H_0(M)$ defined by $\phi_t^i = U_t^{-1}\tilde{\phi}_t^i$ satisfy $\Delta_t \phi_t^i = \lambda_t^i \phi_t^i$ in the sense of distribution and the condition (7) in Theorem A. 3. By hypoellipticitiy of Δ_t (cf. [G, p. 30]), ϕ_t^i belong to $C_C^\infty(M)$ and satisfy (6). Theorem A. 3 is proved.

REFERENCES

[A] J. H. ALBERT, Genericity of simple eigenvalues for elliptic PDE's, Proc. Amer. Math. Soc. 48 (1975), 413–418.

[B] M. BERGER, Sur les premières valeurs propres des variétés riemanniennes, Compositio Math. 26 (2) (1973), 129–149.

[BGM] M. BERGER, P. GAUDUCHON AND E. MAZET, Le spectre d'une variété riemannienne, Lecture Notes in Math. 194, Springer-Verlag, Berlin-Heidelberg-New York, 1970.

[BW] D. D. BLEECKER AND L. C. WILSON, Splitting the spectrum of a Riemannian manifold, SIAM J. Math. Anal. 11 (5) (1980), 813–818.

[CH] R. COURANT AND D. HILBERT, Methods of Mathematical Physics, Vol. II, Interscience Publishers, New York-London, 1966.

[E1] D. G. EBIN, The manifold of Riemannian metrics, Proceeding Symp. in pure Math., Amer. Math. Soc. 15 (1968), 11–40.

[E2] D. G. EBIN, Espace des metriques riemanniennes et mouvement des fluides via les variétés d'applications, Publ. Centre Math. Ecole Polytech., 1971.

[G] P. B. GILKEY, The Index Theorem and the Heat Equation, Publish or Perish, Boston, 1974.

[GG] M. GOLUBITSKY AND V. GUILLEMIN, Stable Mapping and their Singularities, Springer-Verlag, Berlin-Heidelberg-New York, 1973.
[K] T. KATO, Perturbation Theory for Linear Operators, 2nd ed., Springer-Verlag, Berlin-Heidelberg-New York, 1976.
[KS] K. KODAIRA AND D. C. SPENCER, On deformations of complex analytic structures, III, Stability theorems for complex structures, Ann. Math. 71 (1) (1960), 43-76.
[R] F. RELLICH, Perturbation Theory of Eigenvalue Problems, Gordon and Breach Scienc. Publ., New York-London-Paris, 1969.
[RN] F. RIESZ AND B. SZ. NAZY, Functional Analysis (translation), Frederick Ungar Publ. Co., New York, 1955.
[T] M. TANIKAWA, The spectrum of the Laplacian and smooth deformation of the Riemannian metric, Proc. Japan Acad. 55, Ser. A (1979), 125-127.
[U] K. UHLENBECK, Generic properties of eigenfunctions, Amer. J. Math. 98 (1976), 1059-1078.
[Y] K. YOSIDA, Functional Analysis, 4th. ed., Springer-Verlag, Berlin-Heidelberg-New York, 1974.

MATHEMATICAL INSTITUTE
TOHOKU UNIVERSITY
SENDAI, 980
JAPAN

AND

DEPARTMENT OF MATHEMATICS
COLLEGE OF GENERAL EDUCATION
TOHOKU UNIVERSITY
SENDAI, 980
JAPAN