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GENERIC PROPERTIES OF THE ROTATION NUMBER
OF ONE-PARAMETER DIFFEOMORPHISMS OF THE CIRCLE

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In this paper, we consider C^r ($2 \leq r \leq \infty$) maps $f: P \times R^1 \rightarrow R^1$, where P is a one-dimensional manifold and for every fixed $\mu \in P$ the map $f_\mu: R^1 \rightarrow R^1$ defined as $f_\mu(x) = f(\mu, x)$ is a diffeomorphism satisfying

$$(1) \quad f_\mu(x + 1) = f_\mu(x) + 1.$$

The space of such maps, endowed by the C^r Whitney topology we denote by \mathcal{F} .

The maps f from \mathcal{F} are obtained as liftings of one-parameter diffeomorphisms of S^1 induced by the covering of S^1 by R^1 . By \hat{f} we shall denote the map the lifting of which is f ; similarly, we shall mark by hats the projections of points and subsets of R^1 to S^1 .

Our main subject of interest will be the behavior of the rotation number of f for varying μ and its stability under small perturbation of f in \mathcal{F} . In addition, we obtain some results about the nature and stability of the loci of periodic points of f . Related problems (dependence of the rotation number on the changes of the diffeomorphism, problems of structural stability and topological classification of the diffeomorphisms of S^1) have been studied by several authors (cf. [1]–[5]). The understanding of the topology of the parameter-dependent diffeomorphisms of the circle is important for the problem of structural stability of one-parameter two-dimensional flows (cf. [6]) as well as for the problem of bifurcation of periodic problems of one-parameter flows (cf. [7], [8]).

We shall carry out the proofs only for the case $P = R^1$, since the case $P = S^1$ requires only minor adjustments of the proofs and R^1 and S^1 are the only possible connected components of one-dimensional manifolds.

For $f \in \mathcal{F}$, we denote by $\sigma_f: P \rightarrow R^1$ the function assigning to every $\mu \in P$ the rotation number of f_μ . We shall call (μ, y) an l -periodic point of f , if \hat{y} is an l -periodic point of \hat{f}_μ in the usual sense, (i.e. $f_\mu^l(y) - y$ is integer, but $f_\mu^j(y) - y$ is not integer

for $0 < j < l$). The set of l -periodic points of f in $P \times R^1$ will be denoted by $Z_l(f)$. Given f , by the orbit of a point (μ, y) we shall understand the set of points $\{(\mu, f_\mu^j(y)) \mid j \text{ integer}\}$.

We shall restrict ourselves to the study of typical (or *generic*) properties of σ_f and Z_l . We shall call a property of maps from \mathcal{F} *generic*, if it is valid for every f from some residual subset of \mathcal{F} (i.e. a set which is a countable intersection of open dense subsets of \mathcal{F}).

For convenience, we conclude this section by listing some of the well known properties of diffeomorphisms of S^1 and their rotation numbers, which we shall need in the sequel.

Let f be a C^2 diffeomorphism $R^1 \rightarrow R^1$ which is a lifting of a diffeomorphism $\hat{f}: S^1 \rightarrow S^1$, i.e. f satisfies $f(y + 1) = f(y) + 1$ for all y . Denote by $\varrho(f)$ the rotation number of f . Then

- (i) $\varrho(f)$ is a topological invariant of f and is determined by \hat{f} up to an additive integer,
- (ii) if $\varrho(f)$ is irrational, then f is topologically equivalent to the shift $y \mapsto y + \varrho(f)$,
- (iii) if $\varrho(f) = kl^{-1}$, where k, l are relatively prime integers and $l > 0$ (if we express a rational as a fraction we shall always implicitly assume this), then there exists at least one point y such that $f^l(y) = y + k$ and there is no j -periodic point with $0 < j < l$ of f ,
- (iv) for any $n, \varepsilon_n + \delta_n < 1$, and $\varepsilon_n \geq 0, \delta_n \geq 0$, where $\varepsilon_n = \max_y [f^n(y) - y] - n\varrho(f)$, $\delta_n = n\varrho(f) - \min_y (f^n(y) - y)$.

The proof of these properties can be found e.g. in [9].

Further, we note that since f is a diffeomorphism,

- (v) f' does not change sign and if it is negative, $\varrho(f)$ must be an integer,
- (vi) if f' is positive and $\varrho(f) = kl^{-1}$, then all periodic points of f are l -periodic.

To prove (vi) assume that there exists a point y which is m -periodic, $m \neq l$. Since y is not l -periodic, we have $f^l(y) \neq y$. Assume e.g. $f^l(y) < y + k$. Since f^l is increasing, we have thus $f^{ml}(y) - y < mk$ and since $y \in Z_m(f)$, $f^{ml}(y) - y$ is integer. Therefore, $f^{ml}(y) - y \leq mk - 1$, which contradicts property (iv).

Note also that since always $(f^2)' > 0$, it follows from property (vi) that if $f' < 0$ all periodic points of f are 1- or 2-periodic.

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We know from [1] and [2] that σ_f is continuous but not Lipschitz continuous in general (later we prove that generically it is of bounded variation). From the results of [10] it follows that, generically, there is an open dense subset \mathcal{F}'_{1L} of \mathcal{F} such that for every $f \in \mathcal{F}'_{1L}$, and $l \leq L$, $Z_l(f)$ are one-dimensional imbedded C^r submanifolds

of $P \times R^1$, intersecting every set $\Sigma_\mu = \{\mu\} \times R^1$ in isolated points. Furthermore, the points $(\mu, y) \in Z_l(f)$ at which $Z_l(f)$ does not intersect Σ_μ transversally, are isolated and satisfy $(f_\mu^l)''(y) \neq 0$, $(\partial/\partial\mu)(f_\mu^l)(y) \neq 0$.*

Lemma 1. *Let I, J be intervals of R^1 , J compact. Let $f: I \times J \rightarrow R^1$ be C^1 . Denote $\varphi(\mu) = \min_{y \in J} f(\mu, y)$ for $\mu \in I$. Then, at every $\mu \in \text{int } I$ both the right derivative $\varphi'^+(\mu)$ and the left derivative $\varphi'^-(\mu)$ exist and are equal to $\min \{(\partial f/\partial\mu)(\mu, y) \mid y \in M(\mu)\}$ and $\max \{(\partial f/\partial\mu)(\mu, y) \mid y \in M(\mu)\}$, respectively, where $M(\mu) = \{y \mid f(\mu, y) = \varphi(\mu)\}$.*

Proof. Denote $\psi(\mu) = \min_{y \in M(\mu)} (\partial f/\partial\mu)(\mu, y)$, $N(\mu) = \{y \in M(\mu) \mid (\partial f/\partial\mu)(\mu, y) = \psi(\mu)\}$. Both $M(\mu)$ and $N(\mu)$ are compact.

We prove first that if $h \rightarrow 0+$, $y_h \in M(\mu + h)$, then $y_h \rightarrow N(\mu)$. Assume the contrary. Then, there exists a sequence $h_n \rightarrow 0$, $y_{h_n} \in M(\mu + h_n)$ such that $y_{h_n} \rightarrow y^* \notin N(\mu)$. Obviously, $y^* \in M(\mu)$ which implies that for sufficiently large n , there exists a positive $\eta > 0$ such that $(\partial f/\partial\mu)(\mu, y_{h_n}) > \psi(\mu) + \eta$. Consequently, for sufficiently large n we have for any $y_0 \in N(\mu)$

$$f(\mu + h_n, y_{h_n}) - f(\mu + h_n, y_0) = h_n \left[\frac{\partial f}{\partial\mu}(\mu + v_1 h_n, y_{h_n}) - \frac{\partial f}{\partial\mu}(\mu + v_2 h_n, y_0) \right] > 0$$

($0 \leq v_1, v_2 \leq 1$), which contradicts our assumption.

Now we have for sufficiently small $h > 0$:

$$\begin{aligned} \varphi(\mu + h) - \varphi(\mu) &= \min_{y \in J} f(\mu + h, y) - \min_{y \in J} f(\mu, y) = f(\mu + h, y_h) - f(\mu, y_0) = \\ &= f(\mu + h, y_h) - f(\mu, y_h) + f(\mu, y_h) - f(\mu, y_0) \geq \frac{\partial f}{\partial\mu}(\mu + v h, y_h) h, \end{aligned}$$

where $y_0 \in N(\mu)$, $y_h \in M(\mu + h)$, $v \in [0, 1]$, and, therefore, $y_h \rightarrow N(\mu)$ as $h \rightarrow 0$.

Consequently,

$$(2) \quad \liminf_{h \rightarrow 0+} h^{-1}(\varphi(\mu + h) - \varphi(\mu) - \psi(\mu) h) \geq 0.$$

On the other hand we have

$$\begin{aligned} \varphi(\mu + h) - \varphi(\mu) &= f(\mu + h, y_h) - f(\mu, y) = \\ &= f(\mu + h, y_h) - f(\mu + h, y_0) + f(\mu + h, y_0) - f(\mu, y_0) \leq \frac{\partial f}{\partial\mu}(\mu, y_0) h + o(h) \end{aligned}$$

*) Actually, in [10, I] the openness of \mathcal{F}'_{1L} is not explicitly stated for $L > 1$ (cf. Theorem 1 and the following Remark 2). However, the openness of \mathcal{F}'_{1L} (which is true in general), follows easily in our case, since by properties 5, 6 of §1, Z_l are closed and isolated from $\bigcup_{j < l} Z_j$.

from which

$$\limsup_{h \rightarrow 0^+} h^{-1}(\varphi(\mu + h) - \varphi(\mu) - \psi(\mu)h) \leq 0$$

which together with (2) proves $\psi(\mu) = \varphi'^+(\mu)$. The corresponding statement for the left derivative is obtained by replacing μ by $-\mu$.

For $f \in \mathcal{F}$, denote $f^l(\mu, x) = f_\mu^l(x)$. Obviously, $f^l \in \mathcal{F}$ for all l . In several proofs we shall use the following fact which we shall formulate as

Lemma 2. *Given $f \in \mathcal{F}$ and neighbourhoods W_l of f^l for $0 \leq l \leq L$, there exists a neighbourhood W of f such that $f^l \in W_l$ for $0 \leq l \leq L$ as soon as $f \in W$.*

The proof follows from the fact that the operation of composition is continuous on \mathcal{F} .

A closed interval of R^1 will be called non-trivial if it has a non-empty interior.

Lemma 3. *Let $f \in \mathcal{F}'_{1L}$. Then for every $\varrho = kl^{-1}$ with $l \leq L$, $\sigma_f^{-1}(\varrho)$ is disjoint union of non-trivial closed intervals only finitely many of which intersect any compact interval I . If μ_0 is a right (left) endpoint of an interval of $\sigma_f^{-1}(\varrho)$, and $(\mu_0, y_0) \in Z_l(f)$, then $(f_{\mu_0}^l)'(y_0) = 1$ and $(\partial f / \partial \mu)(f_{\mu_0}^l)(y_0)$ and $(f_{\mu_0}^l)''(y_0)$ are distinct from zero and have the same (the opposite) sign.*

Proof. First we prove that the number connected components of $\sigma_f^{-1}(\varrho)$ intersecting I is finite. If this was not true, there would exist a sequence of points (μ_n, y_n) from mutually disjoint components of $Z_l(f)$ such that $\mu_n \in I$ and $\mu_n \rightarrow \mu^*$. Because of compactness of S^1 we can assume $(\mu_n, y_n) \rightarrow (\mu^*, y^*) \in \hat{Z}_l(f)$, which contradicts to the fact that $\hat{Z}_l(f)$ are imbedded submanifolds of $P \times S^1$.

Let $\mu \in \sigma_f^{-1}(\varrho)$. Then, $Z_l(f) \cap \Sigma_\mu \neq \emptyset$ and, since $Z_l(f)$ is a one-dimensional manifold, the intersection of which with Σ_μ consists of isolated points, it must intersect $\Sigma_{\mu'}$ for all μ' from some right or left neighbourhood U of μ , which implies $\sigma_f(\mu') = \varrho$ for $\mu' \in U$. Since σ_f is continuous and, thus, $\sigma_f^{-1}(\varrho)$ is closed, this proves the first part of the lemma.

The second part of the lemma follows from the fact that if μ_0 is a right endpoint of some interval of $\sigma_f^{-1}(\varrho)$, the components of $Z_l(f)$ intersecting Σ_{μ_0} all lie to the left of it. Consequently, any $(\mu_0, y_0) \in Z_l(f)$ is a collapstation point (in the terminology of [10]) and the inequalities follows from [10, Lemma 3]. The proof for the left endpoint is similar.

If their modifications to other generically possible situations (left endpoint, different signs of the inequalities) is straightforward, we shall formulate the statements concerning the endpoints of the intervals of $\sigma_f^{-1}(\varrho)$ only for those right ones, satisfying $(\partial/\partial\mu)(f_{\mu_0}^l)(y_0) > 0$, $(f_{\mu_0}^l)''(y_0) > 0$.

Lemma 4. *Let $f \in \mathcal{F}'_{1L}$, μ_0 be a right endpoint of an interval of $\sigma_f^{-1}(kl^{-1})$ and let $(\partial/\partial\mu)(f_{\mu_0}^l)(y_0) > 0$, $(f_{\mu_0}^l)''(y_0) > 0$ for some periodic point (μ_0, y_0) . Then,*

$(\partial/\partial\mu)f_{\mu_0}^l(y) > 0$, $(f_{\mu_0}^l)''(y) > 0$ for every periodic point (μ_0, y) and $f_{\mu_0}^l(y) \geq k$ for all y and all $\mu > \mu_0$ sufficiently close to μ_0 .

Proof. Let $f \in \mathcal{F}'_{1L}$. If there was a periodic point (μ_0, y^*) with $(f_{\mu_0}^l)''(y^*) < 0$, y^* would be a local maximum of the function $\varphi(y) = f_{\mu_0}^l(y) - y$, while the point (μ_0, y_0) is a local minimum of φ . Since φ has both a local maximum and a local minimum of value k , there must also be a point y_1 with $\varphi(y_1) = k$ which is not a local extremum of φ . Thus, (μ_0, y_1) would be a periodic point not satisfying lemma 3.

Since by lemma 3, $\text{sign}(\partial/\partial\mu)(f_{\mu_0}^l)(y)$ has to be equal to $\text{sign}(f_{\mu_0}^l)''(y)$ for every periodic point (μ_0, y) , the lemma is proven.

Lemma 5. *The subset \mathcal{F}'_{1L} of \mathcal{F}'_{1L} of those $f \in \mathcal{F}'_{1L}$ which for every $\varrho = kl^{-1}$ with $l \leq L$ satisfy*

- (i) *if μ_0 is an interior point of $\sigma_f^{-1}(\varrho)$, then $\min_y [f_{\mu_0}^l(y) - y] < k < \max_y [f_{\mu_0}^l(y) - y]$,*
- (ii) *if μ_0 is a boundary point of $\sigma_f^{-1}(kl^{-1})$, then the projections (μ_0, \hat{y}_i) of the periodic points (μ_0, y_i) belong to one orbit of \hat{f}*

is open dense in \mathcal{F} .

For the proof of this lemma the following lemma will be useful:

Lemma 6. *Let $f \in \mathcal{F}'_{1L}$, $\sigma_f(\mu_0) = kl^{-1}$. Then $\min_y f_{\mu_0}^l(y) - y < k < \max_y f_{\mu_0}^l(y) - y$ if and only if there exists an y_0 such that $Z_l(f)$ intersects Σ_{μ_0} transversally at (μ_0, y) (i.e. $(f_{\mu_0}^l)'(y) \neq 1$ - cf. [10, I, p. 561]).*

The proof follows easily from the fact that at points $(\mu_0, y) \in Z_l(f)$ with $(f_{\mu_0}^l)'(y) = 1$, $(f_{\mu_0}^l)''(y)$ has to be different from zero and, therefore, $f_{\mu_0}^l(y) - y - k$ cannot change signs at such points.

Corollary 1. *If $f \in \mathcal{F}'_{1L}$, $\mu \in \sigma_f^{-1}(kl^{-1})$, then $\min_y f_{\mu}^l(y) - y < k < \max_y f_{\mu}^l(y) - y$ if and only if μ is an interior point of the projection of some component of $Z_l(f)$ into P .*

Proof of lemma 5. Openness. Let $f \in \mathcal{F}'_{1L}$. Then, by Corollary 1, the interiors of the projections into P of the components of $Z_l(f)$ cover the interior of $\sigma_f^{-1}(kl^{-1})$ and the components of $\hat{Z}_l(\hat{f})$ whose projections contain the same boundary point of $\sigma_f^{-1}(kl^{-1})$, belong to one orbit. The manifolds $Z_l(f)$, $l \leq L$ are obtained as pre-images of points of intersection of certain maps associated with the maps f_{μ}^l (cf. [10, I, Lemma 1]) with the diagonal in R^2 , which for $f \in \mathcal{F}'_{1L}$ is transversal. Therefore, in virtue of lemma 2, the transversal isotopy theorem [11, 20. 2]¹⁾ implies that for

¹⁾ The compactness of X in [11, 20, 2] is not needed if the Whitney topology is used.

every \tilde{f} from some neighbourhood of W of f , $Z_l(\tilde{f})$ are isotopic to $Z_l(f)$ for $l \leq L$. Also, it follows from the proof of [11, 20, 2] that $Z_l(\tilde{f})$ can be made arbitrary close to $Z_l(f)$, $l \leq L$ provided W is chosen small enough. In particular, W can be chosen so small that $W \in \mathcal{F}'_{1L}$ and that the interiors of the projections of those components of $Z_l(\tilde{f})$, which are isotopic to the components of $Z_l(f)$ the projections of which cover some interval J of $\sigma_f^{-1}(kl^{-1})$, cover an interval \tilde{J} in such a way that each of the endpoints of \tilde{J} belongs to the projection of components of $\hat{Z}_l(\tilde{f})$ belonging to one orbit. Thus, $W \in \mathcal{F}_{1L}$.

Density. The density of \mathcal{F}_{1L} in \mathcal{F} will be proven if we show that for any $f \in \mathcal{F}_{1L}$, if μ_0 is a point not satisfying (i) or (ii), then by an arbitrarily small perturbation of f with an arbitrary small support a map from \mathcal{F} can be obtained satisfying (i), (ii) for all μ_0 contained in the projection of this support into P .

Let μ_0 be an interior point of $\sigma_f^{-1}(kl^{-1})$ not satisfying (i). Let e.g. $f_{\mu_0}^l(y) \geq k$ for all y . Let $\varphi : P \rightarrow R^1$ be a bump function equal 1 in some neighbourhood of μ_0 , the support of which contains no boundary points of $\sigma_f^{-1}(kl^{-1})$ as well as no point except of μ_0 in which (i) would not be satisfied (note that by Corollary 1 the points not satisfying (i) are isolated). For sufficiently small $\varepsilon > 0$, the function $f - \varepsilon\varphi$ will be in \mathcal{F}'_{1L} and satisfy (i), (ii) for $\mu \in U$.

Let now μ_0 be a right endpoint of an interval of $\sigma_f^{-1}(kl^{-1})$ satisfying $f_{\mu_0}^l(y) \geq k$ for all y , which implies

$$(3) \quad (f_{\mu_0}^l)''(y) > 0, \quad \frac{\partial}{\partial \mu} (f_{\mu_0}^l)(y) > 0$$

for all periodic points (μ_0, y) . Let $\{(\mu_0, \hat{z}_i)\}$ be a periodic orbit. Choose neighbourhoods U of μ_0 , \hat{V} of \hat{z}_0 and an $\varepsilon > 0$ so small that $\sigma_{f+\varepsilon}(\mu)$ is not rational with denominator less than L for $\mu \in U$, $\mu > \mu_0$ and $\{\mu_0\} \times \hat{V}$ does not contain any other periodic point except of (μ_0, \hat{z}_0) . This is possible due to Corollary 1, the continuous and monotonic dependence of $\sigma_{f+\varepsilon}$ on ε and the fact that by lemma 4 and (3), $\min_y f_{\mu_0}^l(y) - y > k$ for $\mu > \mu_0$ sufficiently close to μ_0 . Further, let $\varphi : P \times R^1 \rightarrow R^1$ be a bump function which is 1-periodic, equal to 1 in some neighbourhood of (μ_0, z_0) and 0 outside $U \times V$. Denote $\tilde{f} = f + \varepsilon\varphi$. From (4) it follows that for all points $(\mu, y) \in U \times V$ with $\mu \geq \mu_0$ we have $\tilde{f}_{\mu}^l(y) > k$. In this way, we can destroy any periodic orbit of \hat{f}_{μ_0} and leave only one to suit (ii).

From this lemma we obtain

Proposition 1. *There is a residual set \mathcal{F}_1 of \mathcal{F} such that for $f \in \mathcal{F}_1$ the following is valid:*

For any $q = kl^{-1}$, $\sigma_f^{-1}(q)$ is a disjoint union of closed non-trivial intervals only finite of them intersecting any compact subset of P . If μ_0 is an interior point of $\sigma_f^{-1}(q)$, then $\min_y f_{\mu_0}^l(y) - y < k < \max_y f_{\mu_0}^l(y) - y$. If μ_0 is a right (left) endpoint of an interval of $\sigma_f^{-1}(q)$, then there is exactly one orbit of periodic points of \hat{f}_{μ_0} , for every point δ_0

of which

$$(f_{\mu_0}^l)'(y_0) = 1, \quad \frac{\partial}{\partial \mu} f_{\mu_0}^l(y_0) \neq 0, \quad (f_{\mu_0}^l)''(y_0) \neq 0$$

and the latter two quantities have the same (the opposite) sign.

Remark 1. In case $f' < 0$ it follows from 5, §1 and the continuity of σ_f , that σ_f is constant over P .

Next we shall examine the behavior of the rotation number in the neighbourhood of a boundary point μ of $\sigma_f^{-1}(\varrho)$, ϱ rational, outside of $\sigma_f^{-1}(\varrho)$. The difficulty here is that in any neighbourhood of μ , σ_f can have rational values with an arbitrarily large denominator. We prove the following

Proposition 2. Let $f \in \mathcal{F}$, $\sigma_f(\mu_0) = kl^{-1}$ and let the set of periodic points of f with $\mu = \mu_0$ form an increasing sequence (μ_0, y_i) such that (μ_0, \hat{y}_i) belong to one orbit of \hat{f} . Let

$$(4) \quad (f_{\mu_0}^l)'(y_i) = 1, \quad \frac{\partial}{\partial \mu} f_{\mu_0}^l(y_i) > 0, \quad (f_{\mu_0}^l)''(y_i) > 0$$

for all i . Then, there is a neighbourhood U of μ_0 in which σ_f is non-decreasing. Furthermore, σ_f is not constant on any neighbourhood of μ_0 and $\sigma_f^{-1}(\varrho) \cap U$ is at most a one point set for ϱ irrational.

The proof of this proposition is based upon the following lemma, which appears to be crucial for the results of this paper:

Lemma 7. Let f, μ_0 satisfy the assumptions of proposition 2. Then, there are $\eta > 0, n$ such that for all $\nu > n, \mu_0 \leq \mu_1 < \mu_2 \leq \mu_0 + \eta, y \in \mathbb{R}^1$,

$$(5) \quad f_{\mu_1}^{\nu l}(y) < f_{\mu_2}^{\nu l}(y).$$

Remark 2. It is easy to check that if (4) is satisfied for some i , then it is satisfied for all i .

Proof. Denote $g_{\mu}(x) = f_{\mu}^l(x) - k$. Obviously, g_{μ_0} has rotation number 0 and g has a sequence of periodic points (μ_0, y_i) , exactly l of which lie in $[0, 1)$. Further, we have

$$(6) \quad \frac{\partial}{\partial \mu} g(\mu_0, y_i) > 0, \quad g'_{\mu_0}(y_i) = 1, \quad g''_{\mu_0}(y_i) > 0.$$

Rewriting (5) in terms of g , we have the inequality $g_{\mu_1}^{\nu}(y) < g_{\mu_2}^{\nu}(y)$. Below in the proof, if the index i occurs in some statement or formula, it should always be read "for all i ".

1° Denote $N = 2 \max_y [g'_{\mu_0}(y)]$, $m = \frac{1}{2} \min_y [g'_{\mu_0}(y)]$. From (6) it follows that $m > 0$ and that y_i are minima of the function $g_{\mu_0}(y) - y$. Consequently, $g_{\mu_0}(y) > 0$ for

$y \neq y_i$ from which it follows

$$(7) \quad \lim_{v \rightarrow \infty} g_{\mu_0}^v(y) = y_i \quad \text{for all } y \in (y_{i-1}, y_i].$$

2° We prove that for every q positive integer, there exist $\zeta_0 > 0$ and $\eta > 0$ such that if $\zeta \leq \zeta_0$, $\mu_0 \leq \mu \leq \mu_0 + \eta$ then

$$(8) \quad g_{\mu}^q(y_i - \zeta) < y_i - \frac{1}{2}\zeta$$

$$(9) \quad g_{\mu}^q(y_i + \zeta/2) < y_i + \zeta.$$

Since $g_{\mu_0}(y_i) = 1$, there is a ζ_0 such that for $|y - y_i| \leq \zeta_0$, $g'_{\mu_0}(y) - 1 > -(2q)^{-1}$. If $\zeta \leq \zeta_0$, we have

$$\begin{aligned} & g^q(y_i - \zeta) - (y_i - \zeta) - \sum_{j=0}^{q-1} [g^{j+1}(y_i - \zeta) - g^j(y_i - \zeta)] = \\ & = \sum_{j=0}^{q-1} [g(g^j(y_i - \zeta)) - g^j(y_i - \zeta)] \leq (2q)^{-1} \sum_{j=0}^{q-1} [y_i - g^j(y_i - \zeta)] < \frac{1}{2}\zeta. \end{aligned}$$

The inequality (9) for $\mu = \mu_0$ is established similarly. Because of continuous dependence of g_{μ} on μ , inequalities (9), (8), remain valid for $|\mu - \mu_0| \leq \eta$ provided η is sufficiently small.

3° Since g is C^1 in μ , for $|\mu - \mu_0| \leq \eta$ there exists a $\Gamma > -\infty$ such that

$$(i) \quad g_{\mu_2}(y) - g_{\mu_1}(y) > \Gamma(\mu_2 - \mu_1) \quad \text{for all } y.$$

It follows from (6) that we can take $\eta > 0$, $0 < \xi < \frac{1}{2} \min(y_i - y_{i-1})$, $\xi > 0$ so small that

$$\begin{aligned} (ii) \quad & g_{\mu_2}(y) - g_{\mu_1}(y) > \gamma(\mu_2 - \mu_1) \quad \text{for all } y \in [y_i - \xi, y_i + \xi] \\ & g'_{\mu}(y) < \kappa < 1 \quad \text{for } y \in [y_i - \xi, y_i - \frac{1}{2}\xi] \\ & g'_{\mu}(y) > 1 \quad \text{for } y \in [y_i - \frac{1}{2}\xi, y_i + \xi] \\ & g_{\mu}(y_{i-1} + \xi) < y_i - \xi. \end{aligned}$$

From (7) it follows that there is a q_1 such that

$$(iii) \quad g_{\mu}^{q_1}(y_{i-1} + \xi) > y_i - \xi$$

and that for sufficiently small ζ we have

$$(iv) \quad g^{q_1}(y_i - \xi) < y_i - \zeta$$

for all $|\mu - \mu_0| < \eta$, provided $\eta > 0$ is sufficiently small.

According to 2°, η and ζ can be chosen so small that

$$\begin{aligned} (v) \quad & g_{\mu}^{q_2}(y_i - \xi) < y_i - \frac{1}{2}\zeta, \\ & g_{\mu}^{q_3}(y_i + \frac{1}{2}\xi) < y_i + \zeta \end{aligned}$$

where $q_2 > \gamma^{-1}(N-1)^{-1}(N^{q_1}-1)|\Gamma|$ and $q_3 > \gamma^{-1}m^{-q_1}(m-1)^{-1}(m^{q_1-1})|\Gamma|$ are fixed.

It follows from (ii) that

(vi) $g_{\mu_2}(y_2) \geq g_{\mu_1}(y_1)$ as soon as $y_2 \in [y_i - \xi, y_i + \xi]$, $y_1 \leq y_2$ and $\mu_0 \leq \mu_1 < \mu_2 \leq \mu_0 + \eta$.

By lemma 4 we have $\min_y g_\mu(y) - y > 0$ for $\mu_0 < \mu \leq \mu_0 + \eta$ from which it follows

(vii) $\lim_{y \rightarrow \infty} g_\mu^y(x) = \infty$ for all $\mu_0 < \mu \leq \mu_0 + \eta$ provided η is suitably restricted.

Throughout the rest of this proof we shall assume that η, ξ, ζ are chosen so as to suit (i)-(vii) and that $\mu_0 \leq \mu_1 < \mu_2 \leq \mu_0 + \eta$; for the purpose of later reference, we shall sometimes write $\eta = \eta(f, \mu_0)$.

4° Let $y \in [y_i - \xi, y_i + \frac{1}{2}\zeta]$. We prove that there exists a q such that

$$(10) \quad y_{i+1} - \xi \geq g_{\mu_2}^q(y) \geq y_i + \xi$$

and if q^* is the smallest q satisfying (10) then

$$(11) \quad g_{\mu_2}^{q^*}(y) > g_{\mu_1}^{q^*}(y) + (m-1)^{-1}m^{-q_1}(m^{q_1}-1)|\Gamma|(\mu_2 - \mu_1).$$

The existence of a q satisfying (5) follows from (ii) and (vii). Let q_4 be such that $g_{\mu_2}^{q_4}(y) \geq y_i + \frac{1}{2}\zeta$, $g_{\mu_2}^{q_4-1} \leq y_i + \frac{1}{2}\zeta$. Then, from (v) it follows $q^* - q_4 \geq q_3$ and by (vi) we have $g_{\mu_2}^{q^*}(y) - g_{\mu_1}^{q^*}(y) \geq g_{\mu_2}^{q_3}(y) - g_{\mu_1}^{q_3}(x)$, where $x = g_{\mu_2}^{q_4}(y)$. But according to (ii) we have for all $v_1, v_2 \in [y_i + \frac{1}{2}\zeta, y_i + \xi]$

$$g_{\mu_2}(v_2) - g_{\mu_1}(v_1) \geq g_{\mu_2}(v_2) - g_{\mu_1}(v_2) + g_{\mu_1}(v_2) - g_{\mu_1}(v_1) \geq \gamma(\mu_2 - \mu_1) + v_2 - v_1$$

from which we obtain by induction

$$g_{\mu_2}^{q^*}(y) - g_{\mu_1}^{q^*}(y) \geq q_3\gamma(\mu_2 - \mu_1)$$

from which (11) follows by (v).

5° Let $v_2 \in [y_{i-1} + \xi, y_i - \xi]$, $v_2 \geq v_1 + (m-1)^{-1}m^{q_1}(m^{q_1}-1)|\Gamma|(\mu_2 - \mu_1)$. Then, we prove $g_{\mu_2}^q(v_2) > g_{\mu_1}^q(v_1)$ for all $q \geq 0$.

In virtue of 4° and (vi), it suffices to prove that

$$(12) \quad g_{\mu_2}^q(v_2) \geq g_{\mu_1}^q(v_1) \quad \text{for all } q \leq q'_1,$$

where $q'_1 \leq q_1$ is such that $g_{\mu_2}^{q'_1}(v_2) \geq y_i - \xi > g_{\mu_2}^{q'_1-1}(v_2)$ (the existence of such a q'_1

follows from (iii)). However, (12) follows by induction from the validity of

$$\begin{aligned} g_{\mu_2}(w_2) - g_{\mu_1}(w_1) &\geq g_{\mu_2}(w_2) - g_{\mu_1}(w_2) + g_{\mu_1}(w_2) - g_{\mu_1}(w_1) \geq \\ &\geq -|\Gamma|(\mu_2 - \mu_1) + m(w_2 - w_1) \quad \text{for any } w_2 \geq w_1, \\ w_2 &\in [y_{i-1} + \xi, y_i - \xi]. \end{aligned}$$

6° From 4° and 5° it follows that for $y \in [y_i - \xi, y_i + \frac{1}{2}\xi]$, $g_{\mu_2}^q(y) \geq g_{\mu_1}^q(y)$ for all $q > 0$.

7° Let $y \in [y_{i-1} + \xi, y_i - \xi]$. Then, there is a $q'_1 \leq q_1$ such that $g_{\mu_2}^{q'_1}(y) - g_{\mu_1}^{q'_1}(y) \geq (N-1)^{-1}(N^{q_1} - 1)\Gamma(\mu_2 - \mu_1)$, $y_i - \xi \leq g_{\mu_j}^{q'_1}(y) \leq y_i - \xi$ for $j = 1, 2$. This follows by induction from (iii), (iv) and the validity of

$$\begin{aligned} g_{\mu_2}(v_2) - g_{\mu_1}(v_1) &= g_{\mu_2}(v_2) - g_{\mu_2}(v_1) + g_{\mu_2}(v_1) - g_{\mu_1}(v_1) \geq \\ &\geq N(v_2 - v_1) + \Gamma(\mu_2 - \mu_1) \end{aligned}$$

for all $v_1, v_2 \in [y_{i-1} + \xi, y_i - \xi]$, $v_2 \leq v_1$.

8° Let y, q'_1 be as in 7°. We prove that then $g_{\mu_2}^{q'_1+q_2}(y) > g_{\mu_1}^{q'_1+q_2}(y)$. It follows from (ii) that if $v_1 \geq v_2 \in [y_i - \xi, y_i - \frac{1}{2}\xi]$, then

$$\begin{aligned} g_{\mu_2}(v_2) - g_{\mu_1}(v_1) &\geq g_{\mu_2}(v_2) - g_{\mu_2}(v_1) + g_{\mu_2}(v_1) - g_{\mu_1}(v_1) \geq \\ &\geq \varkappa(v_2 - v_1) + \gamma(\mu_2 - \mu_1) \end{aligned}$$

from which we obtain by induction and (v)

$$\begin{aligned} g_{\mu_2}^{q'_1+q_2}(y) - g_{\mu_1}^{q'_1+q_2}(y) &\geq (N-1)^{-1}(N^{q_1} - 1)\Gamma(\mu_2 - \mu_1)\varkappa^{q_2} + \\ &+ (\varkappa - 1)\varkappa^{q_2-1}\gamma(\mu_2 - \mu_1) \geq \varkappa^{q_2}[(N-1)^{-1}(N^{q_1} - 1) + \gamma q_2](\mu_2 - \mu_1) > 0. \end{aligned}$$

9° It follows now from 6° and 8° that

$$(13) \quad g_{\mu_2}^q(x) \geq g_{\mu_1}^q(x) \geq 0 \quad \text{for all } x \in [y_{i-1} + \xi, y_i + \frac{1}{2}\xi]$$

and $q > q_1 + q_2$. From (ii) it follows that $g_{\mu}(y) - y \geq \lambda > 0$ for $\mu_0 \leq \mu \leq \mu_0 + \eta$, $y \in [y_i + \frac{1}{2}\xi, y_i + \xi]$ and some positive λ . Therefore, there exists a $q_5 > 0$ such that for every $\mu_0 \leq \mu \leq \mu_0 + \eta$ and $y \in [y_i - \frac{1}{2}\xi, y_i + \xi]$,

$$(14) \quad g_{\mu}^{q_5}(y) \geq y_i + \xi.$$

Let q'_5 be the smallest q_5 satisfying (13). Then, by (ii), $g_{\mu_1}^{q'_5}(y) \leq g_{\mu_2}^{q'_5}(y)$ and applying (13) to $x = g_{\mu_2}^{q'_5}(y)$ we obtain $g_{\mu_1}^q(y) \leq g_{\mu_2}^q(y)$ for all $q > q_1 + q_2 + q_5 > q_1 + q_2 + q'_5$. This completes the proof.

Remark 3. Going carefully through the estimates of 1°–3° one can check that for every $\varepsilon > 0$ there is a neighbourhood W of f such that for every \tilde{f} in W , the

quantity $\eta(\tilde{f}, \mu_0)$ (defined so as to satisfy the analogues of (i)–(vii) for \tilde{f}, μ_0) can be chosen so that $\eta(\tilde{f}, \mu_0) > \eta(f, \mu_0) - \varepsilon$. For any f, μ_0 we can define $\eta^-(f, \mu_0)$ as the quantity for which the analogues of (i)–(vii) are satisfied in $[\mu_0, \mu_0 - \eta^-]$; the analogy of the first part of this remark for η^- is straightforward.

Proof of Proposition 2. From lemma 4 it follows that for $\mu_0 \leq \mu_1 \leq \mu_2 \leq \mu_0 + \eta$

$$\sigma_f(\mu_1) = \lim_{n \rightarrow \infty} n^{-1} f_{\mu_1}^n(y) = \lim_{v \rightarrow \infty} (v!)^{-1} f_{\mu_1}^{v!}(y) \leq \lim_{v \rightarrow \infty} (v!)^{-1} f_{\mu_2}^{v!}(y) = \sigma_f(\mu_2).$$

Furthermore, σ_f is constant in some left neighbourhood of μ_0 . Thus, σ_f is non-decreasing in some neighbourhood of μ_0 . From Lemma 4 it follows that σ_f is not constant in any (right) neighbourhood of μ_0 .

Now, let ϱ be irrational and $\sigma_f(\mu) = \varrho$ for some $\mu \in U$. Then, by (i), (ii), §1 there exists an 1-periodic homeomorphism $h : \mathbb{R}^1 \rightarrow \mathbb{R}^1$ such that $F_{\mu}(y) = h^{-1} \circ f_{\mu} \circ h(y) = y + \varrho$ for all y . From lemma 7 it follows that for all $\mu' > \mu, \mu' \in U$ it holds $F_{\mu'}^v(y) = h^{-1} \circ f_{\mu'}^v \circ h(y) > y + v\varrho$ for all y and $v \geq n$. Because of the periodicity of F , there exists a rational, pq^{-1} , such that $F_{\mu'}^{n!}(y) > y + pq^{-1} > y + n!q = F_{\mu}^{n!}(y)$. This implies $F_{\mu'}^{n!q}(y) - y > p > F_{\mu}^{n!q}(y) - y$ and, by (i), §1, $\sigma_f(\mu') = \sigma_F(\mu') > p(n!q)^{-1} > \sigma_F(\mu) = \sigma_f(\mu)$. In a similar way we can prove $\sigma_f(\mu') < \sigma_f(\mu)$ for $\mu' < \mu$.

Proposition 2 allows us to establish some generic global properties of the behavior of the rotation number as well as some stability results. For a more transparent formulation of these we introduce the following notation:

A closed interval J will be called a plateau of a function $\sigma : P \rightarrow \mathbb{R}^1$ if f is constant on J but on no other interval which contains J properly. A plateau will be called extremal (upper, lower) if its endpoints are local extrema of the same kind (local maxima, local minima). $\sigma(J)$ will be called the value of the plateau.

Theorem 1. *For every $f \in \mathcal{F}_1$, σ_f has at most a countable number of extremal plateaus, only finitely many of them intersecting any compact subset of P . The values of all non-trivial plateaus of σ_f are rational.*

Proof. From proposition 2 it follows that the intervals $[\mu_1 - \eta^-(f, \mu_1), \mu_2 + \eta(f, \mu_2)]$ cover P , if $[\mu_1, \mu_2]$ runs over all plateaus with rational values. From this covering we can extract a locally finite subcovering. From proposition 2 it follows further that there is at most one extremal plateau contained in every element $[\mu_1 - \eta^-(f, \mu_1), \mu_2 + \eta(f, \mu_2)]$ of this covering, namely $[\mu_1, \mu_2]$.

For the next theorem, we shall need two lemmas.

Lemma 8. *Let $f \in \mathcal{F}_1$, f, μ_0 satisfy the assumptions of Proposition 2. Then, for every choice of sufficiently small neighbourhoods U of μ_0 , \hat{V}_i of \mathcal{V}_i , there is a neighbourhood W of f such that for every $\tilde{f} \in W$:*

1. there is exactly one l -periodic point $(\tilde{\mu}, \tilde{y}_i)$ in $U \times V_i$ such that

$$(15) \quad (\tilde{f}_\mu^l)'(y) = 1$$

is satisfied for $\mu = \tilde{\mu}$, $y = \tilde{y}_i$, where $\tilde{\mu}$ is the same for all i , and \tilde{y}_i belong to one orbit;

2. there are no other l -periodic points in $U \times R^1$ with $\mu > \tilde{\mu}$ except of the points $(\tilde{\mu}, \tilde{y}_i)$, while $\sigma_f(\mu) = \varrho$ for $\mu \leq \tilde{\mu}$, $\mu \in U$;

3. it holds

$$(16) \quad \frac{\partial}{\partial \mu} \tilde{f}_\mu^l(y) > 0, \quad (\tilde{f}_\mu^l)''(y) > 0$$

for $\mu = \tilde{\mu}$ and $y = \tilde{y}_i$.

Proof. Let us choose U and V_i so that f has no l -periodic points in $H = \bar{U} \times (R^1 \setminus \bigcup_i V_i)$, no periodic points satisfying (15) in $\bar{U} \times R^1$ except of (μ_0, y_i) , and (16) is valid for $f = \tilde{f}$ and all $(\mu, y) \in G = \bar{U} \times \bigcup_i \bar{V}_i$. Since \hat{H} and \hat{G} are compact there exists a neighbourhood W_1 of f such that every $\tilde{f} \in W_1$ is without periodic points in H , satisfies (18) in G and

$$(17) \quad \max_y (\tilde{f}_\mu^l(y) - y) < k + 1.$$

Denote $\tilde{F}(\mu, y) = \tilde{f}_\mu^l(y) - k$.

Then, the set of l -periodic points of f in G satisfying (15) can be written as $(J^1 \tilde{F})^{-1}(Q) \cap G$, where $J^1 \tilde{F}$ means the 1-jet of \tilde{F} in $J^1(P \times R^1, R^1)$ and by Q we have denoted the submanifold of points $(\mu, y, y, 0)$ in $J^1(P \times R^1, R^1) \simeq (P \times R^1 \times R^1 \times J^1(1, 1))$ (in the notation of [12]). From (16) it follows that F intersects Q transversally in G at the points (μ_0, y_i) . By [12, 8. 2, Proposition 1], there is a neighbourhood $W \subset W_1$ of f such that for every $\tilde{f} \in W$, $J^1 \tilde{F}$ will intersect Q transversally in G . Because of our choice of W_1 , (16) will be satisfied at these points. The unicity of the points of intersection (which we denote by (μ_i, y_i)) in $U \times V_i$ follows from the validity of (16) in $U \times V_i$ and lemma 1. Because of unicity, the points (μ_i, y_i) must belong to one orbit, which implies the equality of μ_i for all i .

The equality $\sigma_f(\mu) = kl^{-1}$ for $\mu \leq \tilde{\mu}$ in U follows from the unicity of the periodic points of f satisfying (15) in $U \times R^1$ (there should be a periodic point satisfying (15) with $\mu = \mu_1$, if μ_1 is a left endpoint of an interval of $\sigma_f(kl^{-1})$). From (16) and (17) it follows that there are no l -periodic points of \tilde{f} in $U \times V_i$ with $\mu \geq \tilde{\mu}$. Since \tilde{f} has no periodic points in H as well this completes the proof.

Lemma 9. *Let f, μ_0 satisfy the assumptions of proposition 2. Then, for every $\varepsilon > 0$, there exists a neighbourhood W of f such that for every $\tilde{f} \in W$, there is a $\tilde{\mu}_0 \in (\mu_0 - \varepsilon, \mu_0 + \varepsilon)$ such that $\sigma_{\tilde{f}}(\mu) = kl^{-1}$ for $\mu \in (\mu_0 - \varepsilon, \tilde{\mu}_0)$ and $\eta(\tilde{f}, \tilde{\mu}_0) > \eta(f, \mu_0) - \varepsilon$.*

This lemma is a consequence of Lemma 8 and Remark 3.

By the profile of a real function σ with a countable number of extremal plateaus we shall understand the sequence $\{\sigma_j\}_{j=r}^s$ of values of the extremal plateaus ordered from left to right, where r is 0 or $-\infty$ according to whether the sequence is finite from the left or not (note that in the infinite case the profile is defined up to shift). Two profiles are considered equal, if they can be brought to each other by an appropriate shift.

Theorem 2. *For every $f \in \mathcal{F}_1$, there is an open neighbourhood W of f such that for every $\tilde{f} \in W$ the profiles of σ_f and $\sigma_{\tilde{f}}$ are equal.*

Proof. Let $J_i = [\mu_{i1}, \mu_{i2}]$ be a collection of plateaus of σ_f with rational values k_i/l_i chosen in such a way that only finitely many of them intersect any compact subset of P and the intervals $(\mu_{i1} - \eta(f, \mu_{i1}), \mu_{i2} + \eta(f, \mu_{i2}))$ cover P . Then, by lemma 6, there exists a neighbourhood W'_i of f such that for every $\tilde{f} \in W'_i$, numbers μ_{ij} will exist, corresponding to μ_{ij} as $\tilde{\mu}_0$ to μ_0 in lemma 9 and satisfying

$$\begin{aligned} |\tilde{\mu}_{i1} - \mu_{i1}| &< \min \{ \eta^-(f, \mu_{i1}), \frac{1}{2}(\mu_{i1} + \mu_{i2}) \}, \\ |\tilde{\mu}_{i2} - \mu_{i2}| &< \min \{ \eta(f, \mu_{i2}), \frac{1}{2}(\mu_{i1} + \mu_{i2}) \}, \end{aligned}$$

such that

$$\sigma_{\tilde{f}}(\mu) = \sigma_f(J_i) \quad \text{for } \mu \in [\tilde{\mu}_{i1}, \mu_{i1} + \varepsilon], \quad \mu \in [\mu_{i2} - \varepsilon, \tilde{\mu}_{i2}]$$

for some $\varepsilon > 0$ independent of \tilde{f} . By Lemma 6, $\min [f'_\mu(y) - y]$ and $\max [f''(y) - y]$ must be bounded away from k_i by some positive constant for $\mu \in [\mu_{i1} + \varepsilon, \mu_{i2} - \varepsilon]$. Therefore, there is a neighbourhood $W_i \subset W'_i$ of f such that $\sigma_{\tilde{f}}(\mu) = \sigma_f(J_i)$ for $\mu \in [\mu_{i1} + \varepsilon, \mu_{i2} - \varepsilon]$ and, consequently, for $\mu \in J_i = [\mu_{i1}, \mu_{i2}]$. Since there is only a finite number of J_i 's intersecting any compact subset of P , and W_i can be chosen in such a way that they do not restrict f outside a certain bounded neighbourhood of J_i , it follows from lemma 9 that $\bigcap_i W_i$ will contain a neighbourhood W of f such that for every $\tilde{f} \in W$ the intervals $(\tilde{\mu}_{i1} - \eta(\tilde{f}, \tilde{\mu}_{i1}), \tilde{\mu}_{i2} + \eta(\tilde{f}, \tilde{\mu}_{i2}))$ will cover P . The correspondence $J_i \rightarrow \tilde{J}_i$ renders the equality of profiles of $\sigma_{\tilde{f}}$ and σ_f for $f \in W$.

As a consequence of theorem 2, we obtain

Theorem 3. *There is an open dense subset \mathcal{F}_0 of \mathcal{F} such that for every $f \in \mathcal{F}_0$, σ_f has only finitely many extremal plateaus intersecting any compact subset of P and the values of the extremal plateaus are rational. Moreover, for every $f \in \mathcal{F}_0$ there exists a neighbourhood W of f such that for every $\tilde{f} \in W$, the profiles of $\sigma_{\tilde{f}}$ and σ_f are equal.*

Proof. \mathcal{F}_0 is obtained as the union of the neighbourhoods of the maps $f \in \mathcal{F}_1$, the existence of which is proven in Theorem 2.

Corollary 2. *Generically (for every $f \in \mathcal{F}_0$), σ_f is of bounded variation.*

The second part of theorem 3 expresses a certain stability of behavior of σ_f under small changes of f . This result is refined somewhat by the next theorem.

Theorem 4. *For every $f \in \mathcal{F}_1$ and any $\varepsilon > 0$ there is a neighbourhood W of f such that for every $f \in W$ there is a continuous map $h : P \rightarrow P$ such that $|h(\mu) - \mu| < \varepsilon$ for all μ and $\sigma_{\tilde{f}} \circ h = \sigma_f$; for $\tilde{f} \in W \cap \mathcal{F}_1$, h is a homeomorphism.*

Proof. Let K be any compact subset of P . Take L so large that the length of no interval separating the plateaus of $\sigma_{\tilde{f}}^{-1}(kl^{-1})$ with $l \leq L$ exceeds $\frac{1}{2}\varepsilon$. According to Theorem 3 we can choose a neighbourhood W_1 of f so small that for every $\tilde{f} \in W_1$ the profile of $\sigma_{\tilde{f}}$ is equal to that of σ_f . Since σ_f and $\sigma_{\tilde{f}}$ are monotonic between their extremal plateaus, the correspondence of the extremal plateaus, rendered by the equality of the profiles, can be extended to a unique value and order preserving correspondence of all plateaus. From lemma 9 and lemma 2 it follows that there is a neighbourhood $W \subset W_1$ of f such that for all $f \in W_2$, if $[\mu_1, \mu_2]$ is a plateau intersecting K of value kl^{-1} with $l \leq L$, then the corresponding interval $[\tilde{\mu}_1, \tilde{\mu}_2]$ of $\sigma_{\tilde{f}}^{-1}(kl^{-1})$ satisfies $|\tilde{\mu}_1 - \mu_1| < \frac{1}{4}\varepsilon$, $|\tilde{\mu}_2 - \mu_2| < \frac{1}{4}\varepsilon$. For $f \in W_2$, we construct h on K as follows: h maps every plateau of σ_f linearly and increasingly into the corresponding plateau of $\sigma_{\tilde{f}}$. It is obvious that h has the required properties on K . The fact that h is a homeomorphism $K \rightarrow h(K)$ for $\tilde{f} \in W_1 \cap I$ follows from the non-triviality of the plateaus of \tilde{f} with rational values.

To construct h on P we cover P by a strictly increasing sequence of compacts K_i . It is obvious that having constructed h on K_{i-1} it can be extended to K_i without changing it in K_{i-1} for every \tilde{f} from some neighbourhood W_i which does not impose any new restrictions on f in K_{i-1} . Therefore, $\bigcap_i W_i$ will contain a neighbourhood W of f such that for every $\tilde{f} \in W$ the map h with required properties over all P will exist.

Remark 4. If $f \in \mathcal{F}_0$ but $f \notin \mathcal{F}_1$, h may not exist (since some of the plateaus of f with rational values can be trivial), but the one to one correspondence of the plateaus of f and those of \tilde{f} from some neighbourhood of f can nevertheless be established.

We conclude this section by a partial analogue of proposition 2 for the case of $\sigma_f(\mu_0)$ irrational. This result has certain implications for the problem of bifurcation of periodic orbits (cf. [7], [8]).

Proposition 3. *Let $f_{\mu_0}(y) = y + \alpha$, where α is irrational. Let $\beta_0 = \int_0^1 (\partial f / \partial \mu) \cdot (\mu_0, y) dy \neq 0$. Then, σ_f is increasing or decreasing in some neighbourhood of μ_0 in accordance with the sign of β_0 .*

This Proposition follows, similarly as Proposition 2 from lemma 7, from the following

Lemma 10. *Let the assumptions of Proposition 3 be satisfied and let $\beta_0 > 0$. Then, there exist $\eta > 0$, k such that for all $\mu_1 < \mu_2$ such that $\mu_0 - \eta < \mu_1 <$*

$< \mu_2 < \mu_0 + \eta$ and all $y, n > 0$,

$$f_{\mu_1}^{nk}(y) < f_{\mu_2}^{nk}(y).$$

Proof. Denote $\beta(y) = (\partial f / \partial \mu)(\mu_0, y)$. We have $f_{\mu}^n(y) = y + n\alpha + (\mu - \mu_0) \cdot [\sum_{j=0}^{n-1} \beta(y + j\alpha) + \omega_n(\mu, y)]$ where $\lim_{\mu \rightarrow \mu_0} \omega_n(\mu, y) = 0$ uniformly in y . From the individual ergodic theorem it follows that

$$(18) \quad \lim_{n \rightarrow \infty} n^{-1} [\sum_{j=0}^{n-1} \beta(y + j\alpha)] = \beta_0$$

for a.e.y. Since the derivative of β (and, consequently, that of $n^{-1} [\sum_{j=0}^{n-1} \beta(y + j\alpha)]$ is bounded, (18) is valid uniformly for all y .

Choose k so large that $\sum_{j=0}^{k-1} (y + j\alpha) \geq \frac{1}{2}\beta_0$ and then η so small that $|\omega_k(\mu, y)| \leq \frac{1}{4}\beta_0$ for $|\mu - \mu_0| < \eta$. Then, we have,

$$f_{\mu_2}^k(y) - f_{\mu_1}^k(y) \geq (\mu_2 - \mu_1) (\frac{1}{2}\beta_0 - \frac{1}{4}\beta_0) > 0$$

from which the lemma follows immediately.

Remark 5. Although (cf. [1, §1]) f_{μ_0} is topologically equivalent to the shift provided $\sigma_f(\mu_0)$ is irrational, proposition 1 is not as general as it might seem, because it is not known whether the equivalence homeomorphism is C^2 differentiable in general (cf. [1], [4]).

3

In this section, we apply the results of §2 to establish some generic properties and stability properties of the loci of periodic points.

Proposition 4. *Let f, μ_0 satisfy the assumptions of Proposition 2. Then, for every $\varrho \in \sigma_f(\mu_0, \mu_0 + \eta(f, \mu_0))$, $\varrho = \kappa\lambda^{-1}$ and every plateau J of $\sigma_f^{-1}(\varrho)$ contained in $(\mu_0, \mu_0 + \eta(f, \mu_0)) Z_\lambda \cap (J \times R^1)$ is topologically a line.*

Remark 6. The assertion of proposition 4 is true also under the assumptions of proposition 3, for J contained in a sufficiently small (two sided) neighbourhood of μ_0 . The proof is the same, lemma 10 replacing lemma 7.

Proof of Proposition 4. From proposition 2 it follows that for $\mu \in [\mu_0, \mu_0 + \eta(f, \mu_0)]$ to the right (left) of J , $\sigma_f(\mu) > (< \varrho)$. Therefore, by (iv), §1, for sufficiently large ν , $f_{\mu}^{\nu\lambda}(y) - y > \nu\lambda\kappa (< \nu\lambda\kappa)$. Consequently, for every $y \in R^1$, there is a $\mu^* \in J$ such that $f_{\mu^*}^{\nu\lambda}(y) - y = \nu\lambda\kappa$, from which we obtain by (vi), §1, $(\mu^*, y) \in Z_\lambda(f)$. It

follows from lemma 7 that μ^* is unique. Denote $\mu^* = \chi(y)$. Then, $Z_\lambda(f) \cap (J \times R^1)$ is the graph of χ . Since $Z_\lambda(f)$ is closed, χ is continuous, which proves the lemma.

Theorem 5. *Let $f \in \mathcal{F}_0$. Then, given $\varepsilon > 0$, there is a neighbourhood W of f such that for every $\tilde{f} \in W$ there exists a 1-periodic in y map $g : \bigcup_i Z_i(f) \rightarrow \bigcup_i Z_i(\tilde{f})$ satisfying*

$$(19) \quad |g(y) - y| < \varepsilon$$

for all $y \in \bigcup_i Z_i(f)$, the restriction of which to every $Z_i(f)$ is an isomorphism of $Z_i(f)$ and $Z_i(\tilde{f})$ isotopic to the identity.

Proof. We can assume without loss of generality that $f'_\mu > 0$. The case $f'_\mu < 0$ can be reduced to the former by considering f_μ^2 .

Let $J_i = [\mu_{i1}, \mu_{i2}]$ be plateaus of σ_f with rational values $q_i = k_i l_i^{-1}$ ordered from left to right in such a way that any compact subset of P is intersected by only finitely many of them, the intervals $(\mu_{i1} - \eta(f, \mu_{i1}), \mu_{i2} + \eta(f, \mu_{i2}))$ cover P and

$$(20) \quad |\mu_{i+1,1} - \mu_{i2}| < \frac{1}{2}\varepsilon.$$

For every i there is a finite number of components of Z_i , contained in $J_i \times S^1$. Since for every K compact, the numbers l_i with $J_i \cap K \neq \emptyset$ are bounded, the existence of the restriction of g on $\bigcup_i Z_i(f) \cap \bigcup_i (J_i \times R^1)$ follows from the transversal isotopy theorem [11, 20. 2] similarly as in the proof of lemma 5.

To extend g outside $\bigcup [J_i \times R^1]$ we choose W so that for $\tilde{f} \in W$ the correspondence of plateaus asserted in Remark 4 exists,

$$(21) \quad |\mu' - \mu| < \frac{1}{4}\varepsilon$$

for μ and μ' from corresponding plateaus outside $\bigcup_i J_i$ and the intervals $(\mu_{i1} - \eta^-(f, \mu_{i1}), \mu_{i2} + \eta(f, \mu_{i2}))$ cover P . Then, it follows from proposition 4 that given a plateau J different from J_i , if \tilde{J}_i is the corresponding plateau of \tilde{f} , $Q = \bigcup_i Z_i(f) \cap (J \times R^1)$ and $\tilde{Q} = \bigcup_i Z_i(\tilde{f}) \cap [\tilde{J} \times R^1]$ are lines which can be represented as $Q = \{(\mu, y) \mid \mu = \chi(y)\}$ and, similarly, $\tilde{Q} = \{(\mu, y) \mid \mu = \tilde{\chi}(y)\}$, $\chi, \tilde{\chi}$ continuous. From (20), (21) it follows that if we define $G : [0, 1] \times [\bigcup_i Z_i(f) \cap (J \times R^1)] \rightarrow P \times R^1$ as $G(t, \chi(y), y) = (\chi(y) + t[\tilde{\chi}(y) - \chi(y)], y)$, $g = G(1, \dots)$ will be the required isomorphism, satisfying (19).

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