GENERIC SPLITTING FIELDS OF CENTRAL SIMPLE ALGEBRAS: GALOIS COHOMOLOGY AND NON-EXCELLENCE

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Abstract. A field extension L/F is called *excellent*, if for any quadratic form ϕ over F the anisotropic part $(\phi_L)_{\mathrm{an}}$ of ϕ over L is defined over F; L/F is called *universally* excellent, if $L \cdot E/E$ is excellent for any field extension E/F. We study the excellence property for a generic splitting field of a central simple F-algebra. In particular, we show that it is universally excellent if and only if the Schur index of the algebra is not divisible by 4. We begin by studying the torsion in the second Chow group of products of Severi-Brauer varieties and its relationship with the relative Galois cohomology group $H^3(L/F)$ for a generic (common) splitting field L of the corresponding central simple F-algebras.

Let F be a field and let $A_1, \dots, A_n$ be central simple F-algebras. A *splitting field* of this collection of algebras is a field extension L of F such that all the algebras $(A_i)_L \stackrel{\mathrm{def}}{=} A_i \otimes_F L$ $(i = 1, \dots, n)$ are split. A splitting field L/F of $A_1, \dots, A_n$ is called *generic* (cf. [41, Def. C9]), if for any splitting field L'/F of $A_1, \dots, A_n$, there exists an F-place of L to L'.

Clearly, generic splitting field of $A_1, \dots, A_n$ is not unique, however it is uniquely defined up to *equivalence over F* in the sense of [24, §3]. Therefore, working on problems (Q1) and (Q2) discussed below, we may choose any particular generic splitting field L of the algebras $A_1, \dots, A_n$ (cf. [24, Prop. 3.1]) and we choose the function field of the product of the Severi-Brauer varieties $\mathrm{SB}(A_1) \times \cdots \times \mathrm{SB}(A_n)$.

Let $A_1, \dots, A_n$ be F-algebras and let L be their generic splitting field. Behaviour of different algebraic structures (algebras, quadratic forms, Brauer group and Witt ring, cohomology groups, etc.) under the base change L/F is a very critical question. Among basic results concerning algebras, we point out the following two:

(A1) An algebra A splits over L if and only if the Brauer class of A belongs to the subgroup of $\mathrm{Br}(F)$ generated by the classes of the algebras $A_1, \dots, A_n$. In other words, the relative Brauer group $\mathrm{Br}(L/F)$ is generated by $[A_1], \dots, [A_n]$.

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(A2) For an arbitrary central simple F-algebra A, the division part of the algebra A_L is defined over F. In other words, there exists an F-algebra D such that D_L is a division algebra Brauer-equivalent to A_L.

(these assertions are e.g. straight-forward consequences of the van den Bergh-Schofield index reduction formula [45, Th. 3.1], see also [41, Th. on p. 268]).

This article concerns similar problems in the theory of quadratic forms:

(Q1) Which quadratic F-forms split over the field L? In other words, one likes to determine the relative Witt group $W(L/F)$.
(Q2) Is it true that for any quadratic F-form ϕ, the anisotropic part of the L-form ϕ_L is defined over F? In other words, does there always exist an anisotropic F-form τ such that τ_L is anisotropic and Witt equivalent to ϕ_L?

These questions are very significant both for the theory of central simple algebras and for the theory of quadratic forms, especially in the case where the algebras $A_1, \dots, A_n$ are of exponent 2.

The problem of computation of $W(L/F)$ seems to be very complicated. In view of bijectivity of the generalized Arason invariants $e^i : I^i(F)/I^{i+1}(F) \to H^i(F)$,[1] a natural first step in study of the relative Witt group $W(L/F)$ is the study of the relative cohomology groups $H^i(L/F)$.

There is no problem in the case $i = 2$: since the group $H^2(L/F)$ coincides with the 2-torsion part of the relative Brauer group $\mathrm{Br}(L/F)$, the group $H^2(L/F)$ can be described by means of assertion (A1).

The case $i = 3$ is much more complicated. Suppose that the algebras $A_1, \dots, A_n$ are of exponent 2. Clearly, the group $H^3(L/F)$ contains the subgroup $[A_1]H^1(F) + \dots [A_n]H^1(F)$. One can ask if the equality $H^3(L/F) = [A_1]H^1(F) + \dots [A_n]H^1(F)$ always holds. In [36] E. Peyre constructs a counterexample to this question. He also constructs an injective homomorphism of the quotient $H^3(L/F)/([A_1]H^1(F) + \dots + [A_n]H^1(F))$ into the torsion part of the Chow group $\mathrm{CH}^2(\mathrm{SB}(A_1) \times \dots \times \mathrm{SB}(A_n))$. In Appendix A we show that this homomorphism is bijective in the case $n \le 2$. Thus, computation of $H^3(F(A_1, \dots, A_n)/F)$ is closely related to computation of the torsion part of the group $\mathrm{CH}^2(\mathrm{SB}(A_1) \times \dots \times \mathrm{SB}(A_n))$.

In Part I of this paper we study the group $\mathrm{Tors}\mathrm{CH}^2(\mathrm{SB}(A_1) \times \dots \times \mathrm{SB}(A_n))$ for an arbitrary collection of algebras $A_1, \dots, A_n$. It is a finite abelian group. For any given indexes $\mathrm{ind}(A_1^{\otimes i_1} \otimes \dots A_n^{\otimes i_n})$, we develop a machinery for getting the precise upper bound for the group $\mathrm{Tors}\mathrm{CH}^2(\mathrm{SB}(A_1) \times \dots \times \mathrm{SB}(A_n))$.

In Part II we study question (Q2) in the case of one algebra (i.e. $n = 1$). The answer is known to be positive for a quaternion algebra; assertion (A2) also give a motivation to expect the positive answer to (Q2). It turns out, however, that the answer is negative in general (see Theorem 3.9).

[1]Here we use the notation $H^i(F)$ for the group $H^i_{\text{ét}}(F, \mathbb{Z}/2\mathbb{Z})$. Bijectivity (and existence) of e^i for all i was recently proved by V. Voevodsky [46].

For further introduction, we refer the reader to the beginnings of Parts I and II. We like to mention that the results of Part I are obtained by the second-named author, while the results of Part II are obtained by the first-named author.

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Contents

Part I. **Cohomology**

In this Part, we study the Chow group of 2-codimensional cycles on products of n Severi-Brauer varieties ($n \geq 2$). We analyze more detailed

- the product of a biquaternion variety and a conic;
- the product of two Severi-Brauer surfaces.

0. Introduction

In [22], the Chow group CH^2 for one Severi-Brauer variety is studied. Here, the same group for a direct product of Severi-Brauer varieties is studied by using the same methods. The motivation for doing this work is given by a result of E. Peyre establishing a connection between CH^2 and a 3-d Galois cohomology group ([36, Th. 4.1], see also Theorem 1.4.1 of Part II). In fact, it is the cohomology group we are interested in. The information on it obtained here is then applied in Part II for investigation of the excellence property for the function fields of Severi-Brauer varieties.

The main and general result of this Part is Theorem 4.5 (with Corollary 4.6). We apply it to products of two small-dimensional varieties (Theorems 5.1 and 6.1); this way we obtain, in particular, new examples of torsion in CH^2 (an example of product of three conics with torsion in CH^2 was obtained in [36, Rem. 6.1]).

In this Part, we use the following terminology and notation. By saying "A is an algebra", we always mean that A is a central simple algebra (CS algebra for short) over a field. For an algebra A over a field F, we denote by $[A]$ its class in the Brauer group $\mathrm{Br}(F)$ of F; $\exp A$ stays for the exponent, $\deg A$ for the degree and $\mathrm{ind} A$ for the index of A.

The Severi-Brauer variety of an algebra A is denoted by $\mathrm{SB}(A)$. A *variety* is always a smooth projective algebraic variety over a field; a *sheaf* over X is an $\mathcal{O}_X$-module. The Grothendieck ring of a variety X is denoted by $K(X)$;

$$K(X) = \Gamma^0 K(X) \supset \Gamma^1 K(X) \supset \dots \quad \text{and} \quad K(X) = \mathrm{T}^0 K(X) \supset \mathrm{T}^1 K(X) \supset \dots$$

are respectively the gamma-filtration and the topological filtration on $K(X)$; we use the notation $\mathrm{G}^*\Gamma K(X)$ and $\mathrm{G}^*\mathrm{T} K(X)$ for the adjoint graded rings of these filtrations. There are certain relations between $\mathrm{G}^*\Gamma K(X)$, $\mathrm{G}^*\mathrm{T} K(X)$, and the Chow ring $\mathrm{CH}^*(X)$ we use here; they can be found in [22, §2].

1. Grothendieck group of product of Severi-Brauer varieties

Let $A_1, \dots, A_n$ be algebras over a field F, let $X_1, \dots, X_n$ be their Severi-Brauer varieties, and $X = X_1 \times \dots \times X_n$. Fix a separable closure $\bar{F}$ of F and put $\bar{X}_i = (X_i)_{\bar{F}}$ for each i. The varieties $\bar{X}_i$ are (isomorphic to) projective spaces; denote by ξ_i the class in $K(\bar{X})$ of the tautological sheaf of the projective space bundle

$$\bar{X} \to \prod_{j \neq i} \bar{X}_j \ .$$

The ring $K(\bar{X})$ is generated by the elements $\xi_1, \dots, \xi_n$ subject to the relations

$$(\xi_1 - 1)^{\deg A_1} = \dots = (\xi_n - 1)^{\deg A_n} = 0 \; .$$

Consider the restriction $K(X) \to K(\bar{X})$ which is a ring homomorphism.

Theorem 1.1. *The homomorphism $K(X) \to K(\bar{X})$ is injective; its image is additively generated by the elements*

$$\operatorname{ind}(A_1^{\otimes j_1} \otimes \dots \otimes A_n^{\otimes j_n}) \cdot \xi_1^{j_1} \cdots \xi_n^{j_n}$$

with $0 \le j_1 < \deg A_1$, ... , $0 \le j_n < \deg A_n$.

Proof. Use a generalized Peyre's version [36, Prop. 3.1] of Quillen's computation of K-theory for Severi-Brauer schemes [38, Th. 4.1 of §8] n times. □

Corollary 1.2. *For algebras $A_1, \dots, A_n$ of fixed degrees, the ring $K(X)$ with the gamma-filtration (and in particular the group $\operatorname{TorsG}^2\Gamma K(X)$) depend only on the numbers $\operatorname{ind}(A_1^{\otimes j_1} \otimes \dots \otimes A_n^{\otimes j_n})$.*

Proof. By the theorem, the numbers determine $K(X)$ completely as a subring in $K(\bar{X})$. The Chern classes with values in K ([22, Def. 2.1]) for X, which determine the gamma-filtration ([22, Def. 2.6]), are the restrictions of the Chern classes for $\bar{X}$. □

2. Disjoint varieties and disjoint algebras

Definition 2.1. Let $X_1, \dots, X_n$ be arbitrary varieties over a field. We say that they are *disjoint* if the ring homomorphism

$$K(X_1) \otimes \dots \otimes K(X_n) \to K(X_1 \times \dots \times X_n) \; ,$$

induced by the pull-back homomorphisms

$$pr_i^* \colon K(X_i) \to K(X_1 \times \dots \times X_n)$$

with respect to the projections $pr_i \colon X_1 \times \dots \times X_n \to X_i$, is an isomorphism.

Proposition 2.2. *Let $X_1, \dots, X_n$ be disjoint varieties. The gamma-filtration on $K(X_1 \times \dots \times X_n)$ coincides with the filtration induced by the gamma-filtrations on $K(X_1), \dots, K(X_n)$.*

Proof. Denote by X the product $X_1 \times \dots \times X_n$ and by $\tilde{\Gamma}$ the induced filtration, where for each $l \ge 0$, the term $\tilde{\Gamma}^l K(X)$ is going to be the subgroup of $K(X)$ generated by the products

$$pr_1^* \Gamma^{l_1} K(X_1) \cdots pr_n^* \Gamma^{l_n} K(X_n)$$

for all $l_1, \dots, l_n \ge 0$ with $l_1 + \dots + l_n \ge l$. Since a pull-back homomorphism respects the gamma-filtration, one has an inclusion $\tilde{\Gamma}^l K(X) \subset \Gamma^l K(X)$. Let us prove the inverse inclusion. Since the gamma-filtration Γ on $K(X)$ is the smallest ring filtration having the properties $\Gamma^0 K(X) = K(X)$ and $c^l(x) \in$

$\Gamma^l K(X)$ for all $x \in K(X)$ and $l \geq 1$, where c^l is the l-th Chern class with values in K ([22, Def. 2.1]), it suffices to show that

$$(*) \qquad c^l(x) \in \tilde{\Gamma}^l K(X)\ .$$

Since the varieties $X_1, \ldots, X_n$ are disjoint, the additive group of $K(X)$ is generated by the products

$$(**) \qquad x = pr_1^*(x_1) \cdots pr_n^*(x_n)$$

where $x_i \in K(X_i)$ is the class of a locally free sheaf. Therefore it suffices to check the inclusion $(*)$ only for x of the form $(**)$. Since c^l commutes with pr_i^*, one has

$$c^l\big(\, pr_i^*(x_i)\big) \in \tilde{\Gamma}^l K(X)\ ,$$

and the last step of the proof is

Lemma 2.3. *Let $n, m, l \geq 0$. There exists a $\mathbb{Z}$-polynomial $f_l\big((\sigma_i), (\tau_j)\big)$, where $\sigma_1, \ldots, \sigma_n$ and $\tau_1, \ldots, \tau_m$ are variables, having two following properties:*

- *if $x, y \in K(X)$ are classes of locally free sheaves over a variety X, the Chern class $c^l(x \cdot y)$ is equal to $f_l\big(c^i(x), c^j(y)\big)$;*
- *if one puts $\deg \sigma_i = i$ and $\deg \tau_j = j$, the degree of every monomial of f_l is at least l.*

Proof. By the splitting principle ([31, Prop. 5.6]), it suffices to consider the case where

$$x = \xi_1 + \cdots + \xi_n, \quad y = \eta_1 + \cdots + \eta_m$$

with the classes of invertible sheaves ξ_i, η_j. For the total Chern class c_t ([22, Def. 2.1]), one has

$$c_t(x) = c_t(\sum_{i=1}^{n} \xi_i) = \prod_{i=1}^{n} \big(1 + (\xi_i - 1)t\big) = \prod_{i=1}^{n} (1 + a_i t) \ \text{ where } a_i = \xi_i - 1;$$

$$c_t(y) = c_t(\sum_{j=1}^{m} \eta_j) = \prod_{j=1}^{m} \big(1 + (\eta_j - 1)t\big) = \prod_{j=1}^{m} (1 + b_j t) \ \text{ where } b_j = \eta_j - 1;$$

$$c_t(xy) = c_t(\sum_{i,j} \xi_i \eta_j) = \prod_{i,j} \big(1 + (\xi_i \eta_j - 1)t\big) = \prod_{i,j} \big(1 + (a_i b_j + a_i + b_j)t\big)\ .$$

The class $c^l(xy)$ is (by definition) the coefficient of t^l in $c_t(xy)$. This coefficient is evidently a polynomial in $a_1, \ldots, a_n$ and $b_1, \ldots, b_m$ symmetric with respect to the variables (a_i) and also symmetric with respect to the variables (b_j) (notice that the degree of each monomial is at least l). Consequently, by the main theorem on the symmetric polynomials, $c^l(xy) = f_l\big((\sigma_i), (\tau_j)\big)$ for a polynomial f_l, where $(\sigma_i)_{i=1}^n$ are the standard symmetric polynomials for (a_i) (σ_i is a homogeneous polynomial of degree i) and $(\tau_j)_{j=1}^m$ are the standard symmetric polynomials for (b_j). The assertion of the lemma concerning the degree is evidently satisfied. Finally, note that $\sigma_i = c^i(x)$ and $\tau_j = c^j(y)$. $\square$

$\square$

Corollary 2.4. *Let $X_1, \dots, X_n$ be varieties with finitely generated Grothendieck groups (for instance, Severi-Brauer varieties). If the varieties are disjoint and the groups $\mathrm{G}^*\Gamma K(X_1)$, $\dots$, $\mathrm{G}^*\Gamma K(X_n)$ are torsion-free, then the group $\mathrm{G}^*\Gamma K(X_1 \times \dots \times X_n)$ is torsion-free as well.*

Proof. According to the proposition, the natural homomorphism

$$\mathrm{G}^*\Gamma K(X_1) \otimes \dots \otimes \mathrm{G}^*\Gamma K(X_n) \to \mathrm{G}^*\Gamma K(X_1 \times \dots \times X_n)$$

is surjective. By our assumption, the group on the left-hand side is finitely generated and torsion-free; so, it is a free abelian group of finite rank. This rank coincides with the rank of the group on the right-hand side, because the varieties are disjoint. □

No we are going to understand what the condition of being disjoint means for Severi-Brauer varieties.

Definition 2.5. Let $A_1, \dots, A_n$ be algebras over a field. We say that they are *disjoint* if

$$\mathrm{ind}(A_1^{\otimes j_1} \otimes \dots \otimes A_n^{\otimes j_n}) = \mathrm{ind} A_1^{\otimes j_1} \cdots \mathrm{ind} A_n^{\otimes j_n} \quad \text{for all } j_1, \dots, j_n \geq 0.$$

Proposition 2.6. *Algebras $A_1, \dots, A_n$ are disjoint if and only if their Severi-Brauer varieties are disjoint.*

Proof. Since, for an arbitrary algebra A, there is a canonical isomorphism $K(A) = \mathrm{ind} A \cdot \mathbb{Z}$, where $K(A)$ denotes the Grothendieck group of the algebra, the algebras are disjoint if and only if the maps

$$K(A_1^{\otimes j_1}) \otimes \dots \otimes K(A_n^{\otimes j_n}) \to K(A_1^{\otimes j_1} \otimes \dots \otimes A_n^{\otimes j_n})$$

are isomorphisms for all $0 \leq j_1 < \deg A_1, \dots, 0 \leq j_n < \deg A_n$. Taking the direct sum over all $j_1, \dots, j_n$, we obtain the map

$$\left(\coprod_{j_1=0}^{\deg A_1 - 1} K(A_1^{\otimes j_1}) \right) \otimes \dots \otimes \left(\coprod_{j_n=0}^{\deg A_n - 1} K(A_n^{\otimes j_n}) \right) \longrightarrow$$
$$\longrightarrow \coprod_{j_1=0}^{\deg A_1 - 1} \dots \coprod_{j_n=0}^{\deg A_n - 1} K(A_1^{\otimes j_1} \otimes \dots \otimes A_n^{\otimes j_n}) \, .$$

Identifying the factors of the product on the left-hand side with

$$K\big(\mathrm{SB}(A_1)\big), \dots, K\big(\mathrm{SB}(A_n)\big)$$

and the direct sum on the right-hand side with

$$K\big(\mathrm{SB}(A_1) \times \dots \times \mathrm{SB}(A_n)\big)$$

by Theorem 1.1, one obtains on the place of the arrow the homomorphism of Definition 2.1. □

3. "Generic" varieties

Definition 3.1. Let us say that a variety X is *"generic"*, if the gamma-filtration on $K(X)$ coincides with the topological filtration.

Lemma 3.2. *If* $\mathrm{Tors}\mathrm{G}^*\Gamma K(X) = 0$ *(for an arbitrary variety X), then X is "generic".*

Proof. To see that the filtrations coincide, it suffices to show that the homomorphism

$$\alpha\colon \mathrm{G}^*\Gamma K(X) \to \mathrm{G}^*\mathrm{T}K(X)\ ,$$

induced by the inclusion of the filtrations, is injective. Since $\alpha \otimes \mathbb{Q}$ is bijective ([10, Prop. 5.5 of Chap. VI]), the kernel of α contains only elements of finite order. Therefore, α is really injective if the group $\mathrm{G}^*\Gamma K(X)$ has no torsion. □

Lemma 3.3. *Let $\mathcal{G} \to X$ be a grassmanian bundle. If X is "generic", the variety $\mathcal{G}$ is "generic" as well.*

Proof. Since $\mathcal{G}$ is a grassmanian bundle over X, the $\mathrm{CH}^*(X)$-algebra $\mathrm{CH}^*(\mathcal{G})$ is generated by the Chern classes (with values in CH^*) (see [9, Prop. 14.6.5] or [26, Th. 3.2]). Using the natural epimorphism $\mathrm{CH}^* \to \mathrm{G}^*\mathrm{T}K$, one obtains the same result for $\mathrm{G}^*\mathrm{T}K$: the $\mathrm{G}^*\mathrm{T}K(X)$-algebra $\mathrm{G}^*\mathrm{T}K(\mathcal{G})$ is generated by the Chern classes (with values in $\mathrm{G}^*\mathrm{T}K$). Since X is "generic", the ring $\mathrm{G}^*\mathrm{T}K(X)$ itself is generated by the Chern classes ([22, Rem. 2.17]). Consequently, $\mathrm{G}^*\mathrm{T}K(\mathcal{G})$ is generated by the Chern classes not only as algebra but also as a ring. That means $\mathcal{G}$ is "generic" ([22, Rem. 2.17]). □

Lemma 3.4. *Let $X \to Y$ be a smooth morphism of varieties and let $\tilde{X}$ be its generic fiber. If X is "generic", the variety $\tilde{X}$ (it is a variety over the function field of Y) is also "generic".*

Proof. The morphism (of schemes) $\tilde{X} \to X$ induces a homomorphism of Grothendieck groups $K(X) \to K(\tilde{X})$, respecting the both filtrations, and a homomorphism of Chow groups $\mathrm{CH}^*(X) \to \mathrm{CH}^*(\tilde{X})$ which is surjective ([23, Th. 3.1]). Consequently, the homomorphism

$$\mathrm{G}^*\mathrm{T}K(X) \to \mathrm{G}^*\mathrm{T}K(\tilde{X})$$

is also surjective, and therefore, for every l, the group $\mathrm{T}^lK(X)$ is mapped surjectively onto $\mathrm{T}^lK(\tilde{X})$. Since $\mathrm{T}^lK(X) = \Gamma^lK(X)$, it follows that $\mathrm{T}^lK(\tilde{X}) \subset \Gamma^lK(\tilde{X})$. The inverse inclusion is always true. □

Corollary 3.5. *Let X and Y be varieties over a field F such that the projection $X \times Y \to X$ is a grassmanian bundle. If X is "generic", then $X_{F(Y)}$ is also "generic".*

Proof. The variety $X \times Y$ is "generic" according to Lemma 3.3; therefore the variety $X_{F(Y)}$ is "generic" by Lemma 3.4. □

4. "Generic" algebras

Proposition 4.1. *Let A be a primary algebra (i.e. $\deg A$ is a power of a prime). Suppose that*

- *either* $\mathrm{ind} A = \exp A$
- *or* $\mathrm{ind} A = 2^n$ *and* $\mathrm{ind} A^{\otimes 2^{n-2}} = 4$ *($n \geq 2$)*

(an example of such A is a biquaternion algebra). Then the group $\mathrm{G}^*\Gamma\big(\mathrm{SB}(A)\big)$ is torsion-free.

Proof. For algebras of the first type see [22, Prop. 3.3 and Cor. 3.6]; for the second type see the proof of [22, Prop. 4.9]. □

Corollary 4.2. *Let $A_1, \ldots, A_n$ be disjoint algebras and suppose that each A_i satisfies the condition of Proposition 4.1. Then for the product X of their Severi-Brauer varieties, one has: $\mathrm{Tors}\mathrm{G}^*\Gamma K(X) = 0$; in particular,*

$$\mathrm{Tors}\mathrm{CH}^2(X) = 0 \ .$$

Proof. It is a straightforward consequence of the proposition with Corollary 2.4 and Proposition 2.6. □

For an algebra B and an integer $r \geq 0$, denote by $\mathrm{SB}(r, B)$ the *generalized Severi-Brauer variety* of rank r right ideals in B ([3, §2]). In particular, $\mathrm{SB}(1, B) = \mathrm{SB}(B)$.

Proposition 4.3. *Let $A_1, \ldots, A_n$ and B be algebras over a field, let $X = \mathrm{SB}(A_1) \times \cdots \times \mathrm{SB}(A_n)$ and let $Y = \mathrm{SB}(r, B)$ with certain $r \geq 0$.*

If the Brauer class $[B]$ of the algebra B belongs to the group generated by $[A_1], \ldots, [A_n]$, then the projection $X \times Y \to X$ is an r-grassmanian.

Proof. We may assume that

$$B \simeq A_1^{\otimes j_1} \otimes \cdots \otimes A_n^{\otimes j_n}$$

with some $j_1, \ldots, j_n \geq 0$. Consider the cartesian square

$$\begin{array}{ccc} X \times Y & \longrightarrow & T \times Y \\ \big\downarrow & & \big\downarrow \\ X & \longrightarrow & T \end{array}$$

where $T = \mathrm{SB}(B)$ and where the morphism $X \to T$ is given by tensor product of ideals. The arrow on the right-hand side (that is the projection $T \times Y \to T$) is an r-grassmanian by [22, Prop. 6.3]. Therefore, the projection $X \times Y \to Y$ (that is the left-hand side arrow) is an r-grassmanian as well. □

Definition 4.4. We call a collection of algebras $\tilde{A}_1, \ldots, \tilde{A}_n$ *"generic"*, if it can be obtained by the following procedure. One starts with disjoint algebras $A_1, \ldots, A_n$ over a field F such that each A_i satisfies the condition of Proposition 4.1. Then one takes F-algebras $B_1, \ldots, B_m$ such that their classes in $\mathrm{Br}(F)$ belong to the subgroup generated by $[A_1], \ldots, [A_n]$. Finally, one takes as Y a direct product of some generalized Severi-Brauer of algebras $B_1, \dots, B_m$ and one puts $\tilde{A}_i = (A_i)_{F(Y)}$ for all $i = 1, \dots, n$.

Theorem 4.5. *If a collection of algebras $\tilde{A}_1, \dots, \tilde{A}_n$ is "generic", then the product $\tilde{X}$ of their Severi-Brauer varieties is a "generic" variety (Definition 3.1); in particular, the epimorphism*

$$\mathrm{Tors}\mathrm{G}^2\Gamma K(\tilde{X}) \to \mathrm{Tors}\mathrm{CH}^2(\tilde{X})$$

is bijective in this case.

Proof. Let $A_1, \dots, A_n$ be algebras used in construction of our "generic" collection (Definition 4.4). Put $X_i = \mathrm{SB}(A_i)$ for $i = 1, \dots, n$ and let $X = X_1 \times \cdots \times X_n$. According to Corollary 4.2, the group $\mathrm{G}^*\Gamma K(X)$ is torsion-free. In particular, the variety X is "generic" (Lemma 3.2).

Now, let Y be the direct product of generalized Severi-Brauer varieties, used in the construction of our generic collection. By Proposition 4.3, the projection $X \times Y \to X$ is a fiber product (over X) of grassmanians. Therefore, using Corollary 3.5 m times, one proves that the variety $\tilde{X} = X_{F(Y)}$ is "generic". □

Corollary 4.6. *Let $A_1, \dots, A_n$ be arbitrary algebras and let X be the product of their Severi-Brauer varieties. Let $\tilde{A}_1, \dots, \tilde{A}_n$ be a "generic" collection of algebras such that* $\deg \tilde{A}_i = \deg A_i$ *and*

$$\mathrm{ind}(\tilde{A}_1^{\otimes j_1} \otimes \cdots \otimes \tilde{A}_n^{\otimes j_n}) = \mathrm{ind}(A_1^{\otimes j_1} \otimes \cdots \otimes A_n^{\otimes j_n})$$

for all i and all $j_1, \dots, j_n$. Then the group $\mathrm{Tors}\mathrm{CH}^2(X)$ *is isomorphic to a factorgroup of* $\mathrm{Tors}\mathrm{CH}^2(\tilde{X})$.

Proof. By the theorem, there is an isomorphism

$$\mathrm{Tors}\mathrm{CH}^2(\tilde{X}) \simeq \mathrm{Tors}\mathrm{G}^2\Gamma K(\tilde{X}) \ ;$$

by Corollary 1.2, one has

$$\mathrm{Tors}\mathrm{G}^2\Gamma K(\tilde{X}) \simeq \mathrm{Tors}\mathrm{G}^2\Gamma K(X) \ ;$$

finally, we always have a surjection ([22, Cor. 2.15])

$$\mathrm{Tors}\mathrm{G}^2\Gamma K(X) \twoheadrightarrow \mathrm{Tors}\mathrm{CH}^2(X) \ .$$

□

Proposition 4.7. *Let $A_1, \dots, A_n$ and $B_1, \dots, B_m$ be algebras over a field F such that the subgroups in* $\mathrm{Br}(F)$ *generated by $[A_1]$, ..., $[A_n]$ and by $[B_1]$, ..., $[B_m]$ coincide. Then*

$$\mathrm{Tors}\mathrm{CH}^2\big(\mathrm{SB}(A_1) \times \cdots \times \mathrm{SB}(A_n)\big) \simeq \mathrm{Tors}\mathrm{CH}^2\big(\mathrm{SB}(B_1) \times \cdots \times \mathrm{SB}(B_m)\big) \ .$$

Proof. Set $X = \mathrm{SB}(A_1) \times \cdots \times \mathrm{SB}(A_n)$, $Y_1 = \mathrm{SB}(B_1)$. It suffices to show that

$$\mathrm{Tors}\mathrm{CH}^2(X) \simeq \mathrm{Tors}\mathrm{CH}^2(X \times Y_1) \ .$$

Since $X \times Y_1 \to X$ is a projective space bundle (Proposition 4.3), one has ([12, §2 of App. A])

$$\mathrm{CH}^2(X \times Y_1) \simeq \mathrm{CH}^2(X) \oplus \cdots \oplus \mathrm{CH}^{2-\dim Y_1}(X) \ .$$

The last observation is: for all $i < 2$, the group $\mathrm{CH}^i(X)$ has no torsion (for $i = 1$ see [42, Lemme 6.3(i)]). $\square$

Let p be a prime. For an algebra A as well as for an abelian group A, we are going to denote by $A\{p\}$ the p-primary part of A.

Proposition 4.8. *Let $A_1, \dots, A_n$ be algebras over a field. One has*

$$\mathrm{CH}^2\big(\mathrm{SB}(A_1) \times \cdots \times \mathrm{SB}(A_n)\big)\{p\} \simeq \mathrm{Tors}\mathrm{CH}^2\big(\mathrm{SB}(A_1\{p\}) \times \cdots \times \mathrm{SB}(A_n\{p\})\big) .$$

Proof. For $n = 1$, the assertion is proved in [22, Prop. 1.3]. The same proof works for $n > 1$. $\square$

5. Biquaternion variety times conic

A Severi-Brauer variety of a biquaternion algebra is called *biquaternion variety* here.

Theorem 5.1. *Let X be a biquaternion variety, Y be a conic (over the same field) and A, B be the corresponding algebras (B is a quaternion algebra).*

1. *The torsion in the group $\mathrm{CH}^2(X \times Y)$ is either trivial, or of order 2.*
2. *If the torsion is non-trivial, then*

$$(*) \qquad \mathrm{ind}A = \mathrm{ind}(A \otimes B) = 4 \quad \textit{and} \quad \mathrm{ind}B = 2 .$$

3. *If the collection A, B is "generic" (Definition 4.4) and satisfies the condition $(*)$, then the torsion is not trivial.*

Proof. If $\mathrm{ind}B \neq 2$, i.e. if B is split, then we know from Proposition 4.7 that $\mathrm{Tors}\mathrm{CH}^2(X \times Y) \simeq \mathrm{CH}^2(X)$; hereby the latter group is torsion-free ([21, Cor.]).

If $\mathrm{ind}A \neq 4$, than A is Brauer-equivalent to a quaternion algebra A'; denoting by X' its Severi-Brauer variety, one gets (Proposition 4.7)

$$\mathrm{Tors}\mathrm{CH}^2(X \times Y) \simeq \mathrm{Tors}\mathrm{CH}^2(X' \times Y) .$$

Since $\dim(X' \times Y) = 2$, the group

$$\mathrm{G}^2\Gamma K(X' \times Y) = \Gamma^2 K(X' \times Y) \subset K(X' \times Y)$$

has no torsion. It follows that in this case $\mathrm{Tors}\mathrm{CH}^2(X \times Y) = 0$ as well.

Let C be a division algebra Brauer-equivalent to the product $A \otimes B$; $T = \mathrm{SB}(C)$. Using Proposition 4.7 once again, we have

$$\mathrm{Tors}\mathrm{CH}^2(X \times Y) \simeq \mathrm{Tors}\mathrm{CH}^2(T \times Y) .$$

If $\mathrm{ind}(A \otimes B) \leq 2$, then $\dim T \times Y \leq 2$ and we are done in the same way as above.

If $\mathrm{ind}(A \otimes B) = 8$, then the algebras A, B are disjoint and Corollary 4.2 shows that $\mathrm{Tors}\mathrm{CH}^2(X \times Y) = 0$.

The rest is served by

Proposition 5.2. *Suppose that a biquaternion algebra A and a quaternion algebra B are division algebras and* $\mathrm{ind}(A \otimes B) = 4$. *For X, Y as above, one has:* $\mathrm{Tors}\mathrm{G}^2\Gamma K(X \times Y) \simeq \mathbb{Z}/2$.

Proof. Put $K = K(X \times Y)$, $\bar K = K(\bar X \times \bar Y)$. The commutative ring $\bar K$ is generated by elements ξ, η subject to the relations $(\xi-1)^4 = 0 = (\eta-1)^2$ (see §1). In particular, the additive group of $\bar K$ is a free abelian group generated by the elements $\xi^i\eta^j$, $i = 0,1,2,3$, $j = 0,1$. We are also going to use another system of generators: f^ig^j, $i = 0,1,2,3$, $j = 0,1$, where $f = \xi - 1$, $g = \eta - 1$.

For each l, the l-th term $\Gamma^l\bar K$ of the gamma-filtration on $\bar K$ is generated by the products f^ig^j with $i+j \geq l$. In particular, $\mathrm{G}^l\Gamma\bar K$ is an abelian group freely generated by the residue classes of the products f^ig^j with $i+j = l$.

Lemma 5.3. *The subring $K \subset \bar K$ is additively generated by the elements*

$$1,\ 4\xi,\ \xi^2,\ 4\xi^3,\ 2\eta,\ 4\xi\eta,\ 2\xi^2\eta,\ 4\xi^3\eta\ .$$

Proof. It is a particular case of Theorem 1.1. □

Lemma 5.4. *The following elements are also generators of the additive group of K:*

$$1,\ 2f-f^2,\ 2g,\ 2f^2,\ 4fg,\ 4f^3,\ \boxed{2f^2g},\ 4f^3g$$

(the singled out element is going to produce the torsion — see Corollary 5.9).

Proof. A straightforward verification. □

Lemma 5.5. *There are the following inclusions:*

$$\Gamma^1K \ni 2f-f^2,\ 2g\ ;$$
$$\Gamma^2K \ni 2f^2,\ 4fg,\ \boxed{2f^2g}\ ;$$
$$\Gamma^3K \ni 4f^3,\ 2\cdot\boxed{2f^2g}\ ;$$
$$\Gamma^4K \ni 4f^3g\ .$$

Proof. The assertion on Γ^1K is evident.

Since $2f^2,\ 4fg \in K \cap \Gamma^2\bar K$ and the restriction homomorphism $\mathrm{G}^1\Gamma K \to \mathrm{G}^1\Gamma\bar K$ is injective ([42, Lemme 6.3(i)]), the assertion on Γ^2K holds (a direct verification (see the rest of the proof) is also easy).

Finally, one has:

$$c_t(4\xi) = (1+ft)^4 \quad\Rightarrow\quad c^3(4\xi) = 4f^3 \quad\Rightarrow\quad 4f^3 \in \Gamma^3K\ ;$$
$$2f^2 \in \Gamma^2K \text{ and } 2g \in \Gamma^1K \quad\Rightarrow\quad 4f^2g = (2f^2)\cdot(2g) \in \Gamma^3K\ ;$$

$$c^4(4\xi\eta) = (\xi\eta-1)^4 = \big((f+1)(g+1)-1\big)^4 = 4f^3g \in \Gamma^4K\ .$$

□

Corollary 5.6. *Denote by α^* the restriction homomorphism*

$$\mathrm{G}^*\Gamma K \to \mathrm{G}^*\Gamma\bar K\ .$$

For all $i > 0$, one has: $\operatorname{Im}\alpha^i \subset 2\mathrm{G}^i\Gamma\bar K$.

Proof. According to the lemma, the group $\mathrm{G}^1\Gamma K$ is generated by the residue classes of the elements $2f-f^2$ and $2g$; their images in $\mathrm{G}^1\Gamma\bar{K}$ are really divisible by 2. So,the assertion of the corollary for $i=1$ is proved.

Since the elements of $\Gamma^2 K$, $\Gamma^3 K$ and $\Gamma^4 K$, listed in the lemma, generate $\Gamma^2 K$ and are divisible by 2 in $\bar{K}$, we obtain the assertion for $i\geq 2$ (use the absence of torsion in $\mathrm{G}^*\Gamma\bar{K}$). □

Corollary 5.7. $\#(\mathrm{Tors}\mathrm{G}^*\Gamma K)\leq 2$.

Proof. Since the group $\mathrm{G}^*\Gamma\bar{K}$ is torsion-free, $\mathrm{Tors}\mathrm{G}^*\Gamma K\subset\operatorname{Ker}\alpha^*$. We are going to show that $\#(\operatorname{Ker}\alpha^*)\leq 2$, using the following formula ([20, Prop.]):

$$\#(\operatorname{Ker}\alpha^*)=\#(\operatorname{Coker}\alpha^*)/\#(\bar{K}/K)\ .$$

It is easy to calculate that $\#(\bar{K}/K)=2^{10}$. According to the lemma,

$$\#(\operatorname{Coker}\alpha^*)\leq 2^{11}\ .$$

□

Lemma 5.8. $2f^2g\notin\Gamma^3 K$.

Proof. It suffices to show that $\operatorname{Im}\alpha^3\subset 4\mathrm{G}^3\Gamma\bar{K}$.

The group $\operatorname{Im}\alpha^3$ is generated by the subgroup $\operatorname{Im}\alpha^1\cdot\operatorname{Im}\alpha^2$ and by the subset $\alpha^3(c^3K)$, where c^3 is the 3d Chern class with values in $\mathrm{G}^*\Gamma K$ ([22, Def. 2.7]). Since $\operatorname{Im}\alpha^i\subset 2\mathrm{G}^i\Gamma\bar{K}$ for $i>0$ by Corollary 5.6, one has: $\operatorname{Im}\alpha^1\cdot\operatorname{Im}\alpha^2\subset 4\mathrm{G}^3\Gamma\bar{K}$. Therefore, it suffices to verify that $\alpha^3\big(c^3(S)\big)\subset 4\mathrm{G}^3\Gamma\bar{K}$ for a system S of generators of the additive group of K. The verification is trivial if we take as S the system of generators of Lemma 5.3. □

Corollary 5.9. *The residue of* $2f^2g$ *in* $\mathrm{G}^2\Gamma K$ *has order 2 and generates the torsion subgroup.*

Proof. The residue is of order 2 by Lemmas 5.8 and 5.5. It generates the whole torsion subgroup (not only in $\mathrm{G}^3\Gamma K$ but also in $G^*\Gamma K$) by Corollary 5.7. □

The proofs of the theorem and of the proposition are complete. □

□

Remark 5.10. In the condition of the theorem, denote the base field by F and suppose that there exists a quadratic extension L/F (or, more generally, an extension of degree not divisible by 4) such that the algebra A_L is no more a division algebra and the algebra B_L is split. In this case, $f^2g\in\mathrm{T}^3K(X_L\times Y_L)$; using the norm map, we obtain: $2f^2g\in\mathrm{T}^3K(X\times Y)$, i.e. $\mathrm{Tors}\mathrm{CH}^2(X\times Y)=0$.

Therefore, if A,B are such that $\mathrm{Tors}\mathrm{CH}^2(X\times Y)\neq 0$ (for example, if A,B form a "generic" collection (Theorem 5.1)), there are no extensions like that. The first example of this phenomenon is constructed in [30].

6. Product of two Severi-Brauer surfaces

A *Severi-Brauer surface* is a Severi-Brauer variety of dimension 2.

Theorem 6.1. *Let X, Y be Severi-Brauer surfaces over a field and let A, B be the corresponding algebras.*

1. *The torsion in the group $\mathrm{CH}^2(X \times Y)$ is either trivial, or of order 3.*
2. *If the torsion is not trivial, then*

$$(*) \qquad \mathrm{ind} A = \mathrm{ind} B = \mathrm{ind}(A \otimes B) = \mathrm{ind}(A \otimes B^{\mathrm{o}}) = 3$$

   where B^{o} is the algebra opposite to B.

3. *If the collection A, B is "generic" (Definition 4.4) and satisfies the condition $(*)$, then the torsion is not trivial.*

Proof. If at least one of the algebras A, B, $A \otimes B$, $A \otimes B^{\mathrm{o}}$ is split, then there exists an algebra C of degree 3 such that its class $[C]$ in the Brauer group generates the same subgroup as $[A]$ and $[B]$ (together). According to Proposition 4.7, in this case, the group $\mathrm{TorsCH}^2(X \times Y)$ is isomorphic to the group $\mathrm{TorsCH}^2\big(\mathrm{SB}(C)\big)$ which is trivial by [20, Cor.], or also by [17, Lemma 2.4].

If $\mathrm{ind}(A \otimes B) = \mathrm{ind}(A \otimes B^{\mathrm{o}}) = 9$, then the algebras A, B are disjoint and one can use Corollary 4.2.

Put $Y^{\mathrm{o}} = \mathrm{SB}(B^{\mathrm{o}})$. Since by Proposition 4.7

$$\mathrm{TorsCH}^2(X \times Y) \simeq \mathrm{TorsCH}^2(X \times Y^{\mathrm{o}}) ,$$

it suffices to consider only one of the two following cases:

- $\mathrm{ind}(A \otimes B) = 3$ and $\mathrm{ind}(A \otimes B^{\mathrm{o}}) = 9$;
- $\mathrm{ind}(A \otimes B) = 9$ and $\mathrm{ind}(A \otimes B^{\mathrm{o}}) = 3$.

Lemma 6.2. *If $\mathrm{ind} A = \mathrm{ind} B = \mathrm{ind}(A \otimes B) = 3$ and $\mathrm{ind}(A \otimes B^{\mathrm{o}}) = 9$, then $\mathrm{TorsG}^2\Gamma K(X \times Y) = 0$.*

Proof. Put $K = K(X \times Y)$, $\bar{K} = K(\bar{X} \times \bar{Y})$. The commutative ring $\bar{K}$ is generated by elements ξ, η subject to the relations $(\xi - 1)^3 = 0 = (\eta - 1)^3$ (see §1). In particular, the additive group of $\bar{K}$ is an abelian group freely generated by the elements $\xi^i \eta^j$, $i, j = 0, 1, 2$. We also are going to use another system of generators: $f^i g^j$, $i, j = 0, 1, 2$, where $f = \xi - 1$, $g = \eta - 1$.

For every l, the l-th term $\Gamma^l \bar{K}$ of the gamma-filtration on $\bar{K}$ is generated by the products $f^i g^j$ with $i + j \geq l$.

The condition of the lemma implies that

$$\mathrm{ind} A^{\otimes 2} = \mathrm{ind} B^{\otimes 2} = \mathrm{ind}(A^{\otimes 2} \otimes B^{\otimes 2}) = 3 \ \text{ and}$$
$$\mathrm{ind}(A \otimes B^{\otimes 2}) = \mathrm{ind}(A^{\otimes 2} \otimes B) = 9 .$$

So, according to Theorem 1.1, the subring $K \subset \bar{K}$ is additively generated by

$$1,\ 3\xi,\ 3\xi^2,\ 3\eta,\ 3\xi\eta,\ 9\xi^2\eta,\ 3\eta^2,\ 9\xi\eta^2,\ 3\xi^2\eta^2 .$$

We are also going to use another system of generators:

$$1,\ 3f,\ 3g,\ 3f^2,\ 3fg,\ 3g^2,\ 9f^2g,\ 3f^2g+3fg^2+6f^2g^2,\ 9f^2g^2\ .$$

Now it is evident that the intersection $K\cap\Gamma^3\bar{K}$ is generated by

$$9f^2g,\ 3f^2g+3fg^2+6f^2g^2,\ \text{ and }\ 9f^2g^2\ .$$

To prove that the group $\mathrm{G}^2\Gamma K$ is torsion-free, it suffices to verify that these three elements belong to Γ^3K.

Since $3f^2,\ 3g^2\in\Gamma^2K$, and $3g\in\Gamma^1K$, one has:

$$9f^2g=(3f^2)\cdot(3g)\in\Gamma^3K,\ \ 9f^2g^2=(3f^2)\cdot(3g^2)\in\Gamma^4K\ .$$

The last element coincides with a 3-d Chern class:

$$c^3(3\xi\eta)=(\xi\eta-1)^3=\big((f+1)(g+1)-1\big)^3=(fg+f+g)^3=\\ 3fg(f+g)^2+(f+g)^3=6f^2g^2+3f^2g+3fg^2\ .$$

□

We finish the proof of the theorem by

Proposition 6.3. *If* $\mathrm{ind}A=\mathrm{ind}B=\mathrm{ind}(A\otimes B)=\mathrm{ind}(A\otimes B^{\mathrm{o}})=3$, *then* $\mathrm{TorsG}^2\Gamma K(X\times Y)\simeq\mathbb{Z}/3$.

Proof. We use the notation introduced in the beginning of the proof of the last lemma.

Lemma 6.4. *The subring $K\subset\bar{K}$ is now generated by 1 and $3\bar{K}$. Moreover,*

$$\begin{aligned}&\Gamma^1K=3\Gamma^1\bar{K}\ ;\\&\Gamma^2K=3\Gamma^2\bar{K}\ ;\\&\Gamma^3K\ni 3f^2g-3fg^2,\ 3f^2g+3fg^2+6f^2g^2\ ;\\&\Gamma^4K\ni 9f^2g^2\ .\end{aligned}$$

Proof. The assertion about Γ^1K is trivial. The assertion about Γ^2K is caused by injectivity of the restriction homomorphism $\mathrm{G}^1\Gamma K\to\mathrm{G}^1\Gamma\bar{K}$ ([42, Lemme 6.3(i)]); $9f^2g^2\in\Gamma^4K$ because $3f^2,3g^2\in\Gamma^2K$.

To prove the assertion about Γ^3K, let us compute the 3d Chern class

$$c^3(\xi^2\eta)=(\xi^2\eta-1)^3=\big((f+1)^2(g+1)-1\big)^3=27f^2g^2+12f^2g+6fg^2\ .$$

Since $9f^2g,9fg^2\in\Gamma^3K$, we conclude that $3f^2g-3fg^2\in\Gamma^3K$.

Finally, as we have already computed in the proof of Lemma 6.2,

$$3f^2g+3fg^2+6f^2g^2=c^3(3\xi\eta)\in\Gamma^3K\ .$$

□

Corollary 6.5. $\#(\mathrm{TorsG}^*\Gamma K)\leq 3$.

Proof. Analogously to Corollary 5.7. Now one has (Lemma 6.4):

$$\#(\bar{K}/K) = 3^8 \quad\text{and}\quad \#(\operatorname{Coker}\alpha^*) \leq 3^9 \, .$$

□

Lemma 6.6. $3f^2g^2 \notin \Gamma^3 K$.

Proof. Let us define a homomorphism $\phi_9 : \bar{K} \to \mathbb{Z}/9$ as follows: write an arbitrary element $x \in \bar{K}$ as a linear combination

$$x = \sum_{i,j=0}^{2} a_{ij} f^i g^j \quad \text{with } a_{ij} \in \mathbb{Z},$$

put $\phi(x) = a_{21} + a_{12} - a_{22}$ and define $\phi_9(x)$ as the residue of $\phi(x)$ modulo 9.

Since $\phi_9(3f^2g^2) \neq 0$, it suffices to show that $\phi_9(\Gamma^3 K) = 0$.

A priori, the group $\Gamma^3 K$ is generated by $\Gamma^1 K \cdot \Gamma^2 K$, $c^3(S)$ et $c^4(S)$ where

$$S = 1,\ 3\xi,\ 3\xi^2,\ 3\eta,\ 3\xi\eta,\ 3\xi^2\eta,\ 3\eta^2,\ 3\xi\eta^2,\ 3\xi^2\eta^2 \, .$$

Hereby, $c^4(s) = 0$ for all $s \in S$; thus one can eliminate $c^4(S)$ from the list of generators.

Since $\Gamma^1 K \cdot \Gamma^2 K \subset \Gamma^1 K \cdot \Gamma^1 K \subset 9\bar{K}$ (Lemma 6.4), $\phi_9(\Gamma^1 K \cdot \Gamma^2 K) = 0$.

It remains $c^3(S)$. For $s = 1$, 3ξ, $3\xi^2$, 3η, and $3\eta^2$, the value $\phi(s)$ is already 0. The following calculations show that $\phi_9\big(c^3(s)\big) = 0$ for the other four elements $s \in S$ as well:

$$c^3(3\xi\eta) = (\xi\eta - 1)^3 = \big((f+1)(g+1) - 1\big)^3 = \big(fg + (f+g)\big)^3 =$$
$$= 3fg(f+g)^2 + (f+g)^3 = \boxed{6f^2g^2 + 3f^2g + 3fg^2}\, ;$$

$$c^3(3\xi^2\eta^2) = (\xi^2\eta^2 - 1)^3 =$$
$$= \big((f+1)^2(g+1)^2 - 1\big)^3 = \big((f^2 + 4fg + g^2) + 2(f+g)\big)^3 =$$
$$= 12(f^2 + 4fg + g^2)(f+g)^2 + 8(f+g)^3 = \boxed{120f^2g^2 + 24f^2g + 24fg^2}\, ;$$

$$c^3(3\xi^2\eta) = (\xi^2\eta - 1)^3 = \big((f+1)^2(g+1) - 1\big)^3 = \big(f(f+2g) + (2f+g)\big)^3 =$$
$$= 3f(f+2g)(2f+g)^2 + (2f+g)^3 = \boxed{27f^2g^2 + 12f^2g + 6fg^2}\, ;$$

$$c^3(3\xi\eta^2) = \boxed{27f^2g^2 + 6f^2g + 12fg^2}\, .$$

□

According to Lemma 6.4, we have $3f^2g^2 \in \Gamma^2 K$. The residue class of the element $3f^2g^2$ in $\mathrm{G}^2\Gamma K$ has order 3 by Lemmas 6.4 and 6.6. Therefore, by Corollary 6.5, it generates the whole torsion subgroup of $\mathrm{G}^2\Gamma K$. So, the proof of Proposition 6.3 is complete. □

Proposition 6.3 completes the proof of Theorem 6.1. □

Part II. Non-excellence

Let F be a field of characteristic different from 2. A field extension L/F is called *excellent* if for any quadratic form ϕ over F the anisotropic part $(\phi_L)_{\text{an}}$ of ϕ over L is defined over F. In this Part, we study the excellence property for the function fields of Severi-Brauer varieties.

0. Introduction

Let F be a field of characteristic different from 2 and let ϕ be a non-degenerate quadratic form over F. It is an important problem to study the behavior of the anisotropic part of forms over F under a field extension L/F. A field extension L/F is called *excellent* if for any quadratic form ϕ over F the anisotropic part $(\phi_L)_{\text{an}}$ of ϕ over L is defined over F (this means that there exists a form ξ over F such that $(\phi_L)_{\text{an}} \cong \xi_L$).

Any quadratic extension is excellent. Since any anisotropic quadratic form ψ over F is still anisotropic over the field of rational functions $F(t)$, every purely transcendental field extension is excellent.

Let $F(X)$ be the field of rational functions on a geometrically integral variety X. One of the important problems is to find conditions on X so that the field extension $F(X)/F$ is excellent. We say that $F(X)/F$ is *universally excellent* if for any extension K/F the extension $K(X)/K$ is excellent. The following varieties are most important in the algebraic theory of quadratic forms: quadric hypersurfaces, Severi-Brauer varieties, varieties of totally isotropic flags, and products of such varieties.

If X is rational (or unirational) then $F(X)/F$ is purely transcendental (respectively, unirational), and it follows from Springer's theorem that $F(X)/F$ is excellent and moreover that it is universally excellent.

In the case of a hypersurface $X = X_q$ defined by the equation $q = 0$ where q is a non-degenerate quadratic form, the following results are known: 1) if q is isotropic, then $F(X_q)/F$ is universally excellent (because X_q is rational in this case); 2) if the field extension $F(X_q)/F$ is excellent and q is anisotropic, then q is a Pfister neighbor (see [24]); 3) if $\dim q \leq 3$ (or $\dim q = 4$ and $\det q = 1$), then X_q is universally excellent (see [2] or [40], [29]); 4) if q is anisotropic, then $F(X_q)/F$ is universally excellent if and only if q is a Pfister neighbor of dimension ≤ 4 (see [16] or [14]). Thus, the problem whether the field extension $F(X)/F$ is universally excellent is completely solved in the case where X is a quadric surface X_q.

In this paper we study the function field of a Severi-Brauer variety $X = \mathrm{SB}(A)$. If X is the Severi-Brauer variety of a quaternion algebra $A = (a, b)$, the field extension $F(X)/F$ is excellent. Indeed, in this case the variety X coincides with the quadric hypersurface X_ϕ, where $\phi = \langle 1, -a, -b\rangle$.

The next interesting case is the case of a biquaternion division algebra A. In Section 3 we prove that the field extension $F(\mathrm{SB}(A))/F$ is not universally excellent for any biquaternion division F-algebra A. Moreover, we construct an unirational field extension E/F such that $E(\mathrm{SB}(A))/E$ is not excellent (see

Theorem 3.2). Applying this result, we find a condition on a central simple algebra A under which $F(\mathrm{SB}(A))/F$ is universally excellent. Theorem 3.9 asserts that $F(\mathrm{SB}(A))/F$ is universally excellent only in the following two cases: 1) the index of A is odd; 2) the algebra A has the form $Q \otimes_F D$, where Q is a quaternion algebra and D is of odd index. In addition, we show that the field extension $F(\mathrm{SB}(A))/F$ is not excellent for an arbitrary algebra A of index 8 and exponent 2 (see Theorem 3.10).

In our proof of the main result of Section 3 we apply a deep result of E. Peyre concerning the groups

$$\ker\bigl(H^3(F,\mathbb{Z}/2\mathbb{Z}) \to H^3(F(X),\mathbb{Z}/2\mathbb{Z})\bigr) \quad \text{and} \quad \mathrm{Tors}_2\mathrm{CH}^2(X),$$

where X is a product of Severi-Brauer varieties of algebras of exponent 2 (see [36]). In Section 2 and Appendix A we prove some results concerning Chow groups and Galois cohomology. In particular, in Appendix A we prove the following

Theorem. *Let A and B be central simple algebras of exponent 2 over F. Let $X = \mathrm{SB}(A) \times \mathrm{SB}(B)$. Then the homomorphism*

$$\frac{\ker\left(H^3(F) \to H^3(F(X))\right)}{[A] \cup H^1(F) + [B] \cup H^1(F)} \xrightarrow{\bar{\varepsilon}_2} \mathrm{Tors}_2\mathrm{CH}^2(X).$$

is an isomorphism. Here $H^*(F) = H^*(F,\mathbb{Z}/2\mathbb{Z})$ and the homomorphism $\bar{\varepsilon}_2$ is induced by the homomorphism $\varepsilon : H^3(F(X)/F,\mathbb{Q}/\mathbb{Z}(2)) \to \mathrm{CH}^2(X)$ defined in [44].

This theorem plays an important part of the proof that the function fields of the Severi-Brauer varieties of biquaternion division algebras are not universally excellent.

In Section 4 we prove the following statement: for any central simple F-algebra A, the field extension $F(\mathrm{SB}(A))/F$ is 5-excellent (this means that if $\dim\phi \le 5$ then $(\phi_{F(\mathrm{SB}(A))})_{\mathrm{an}}$ is defined over F). In Section 5 we construct explicit examples of a biquaternion division algebra A such that the field extension $F(\mathrm{SB}(A))/F$ is not excellent. [2] In particular, we prove that the biquaternion algebra $A = (a,b) \otimes (c,d)$ over the field of rational functions in 4 variables $F(a,b,c,d)$ yields such an example (see Proposition 5.10). In Appendix B we study the excellence property for generic (partial) splitting fields of quadratic forms. In particular, we find a criterion of universal excellence for the function fields of integral varieties of totally isotropic subspaces (see Theorem B.21).

1. Main notation and facts

1.1. **Quadratic forms and central simple algebras.** By $\phi \perp \psi$, $\phi \cong \psi$, and $[\phi]$ we denote respectively orthogonal sum of forms, isometry of forms, and the class of ϕ in the Witt ring $W(F)$ of the field F (sometimes the Witt

[2] Another example (a little more complicated than ours) was independently constructed by A. Sivatskij.

class of ϕ is denoted simply ny ϕ). The maximal ideal of $W(F)$ generated by the classes of even-dimensional forms is denoted by $I(F)$. We say that ϕ is similar to ψ (and write $\phi \sim \psi$) if there exists $k \in F^*$ such that $k\phi \cong \psi$. The anisotropic part of ϕ is denoted by ϕ_{an} and $i_W(\phi)$ denotes the Witt index of ϕ. We denote by $\langle\langle a_1, \dots, a_n\rangle\rangle$ the n-fold Pfister form $\langle 1, -a_1\rangle \otimes \cdots \otimes \langle 1, -a_n\rangle$ and by $P_n(F)$ the set of all n-fold Pfister forms. The set of all forms similar to n-fold Pfister forms is denoted by $GP_n(F)$. The fundamental Arason-Pfister Hauptsatz (APH for short) states that if $\phi \in I^n(F)$ and $\dim \phi < 2^n$ then $[\phi] = 0$; if $\phi \in I^n(F)$ and $\dim \phi = 2^n$ then $\phi \in GP_n(F)$. For any field extension L/F we put $\phi_L = \phi \otimes L$, $W(L/F) = \ker(W(F) \to W(L))$, and $I^n(L/F) = \ker(I^n(F) \to I^n(L))$.

Let ϕ be a quadratic form such that $\dim \phi \geq 2$ and $\phi \not\cong \mathbb{H}$. The function field $F(\phi)$ of the form ϕ over F is the function field of the projective variety X_ϕ given by equation $\phi = 0$. If $\dim \phi \leq 1$ or $\phi \cong \mathbb{H}$, we set $F(\phi) \overset{\mathrm{def}}{=} F$.

In this Part, notations and conventions concerning F-algebras are the same in Part I.

We recall that two field extensions E/F and K/F are stably isomorphic if and only if there exist indeterminates $x_1, \dots, x_s, y_1, \dots, y_r$ and an isomorphism $E(x_1, \dots, x_r) \cong K(y_1, \dots, y_s)$ over F. We will write $E/F \overset{\mathrm{st}}{\sim} K/F$ if E/F is stably isomorphic to K/F. If $[A] = [A']$ in $\mathrm{Br}(F)$ then the field extensions $F(\mathrm{SB}(A))/F$ and $F(\mathrm{SB}(A'))/F$ are stably isomorphic.

Let ϕ be a quadratic form. We denote by $C(\phi)$ the Clifford algebra of ϕ. If $\phi \in I^2(F)$ then $C(\phi)$ is a CS algebra. Hence, we get a well defined element $[C(\phi)]$ of $\mathrm{Br}_2(F)$ which we will denote by $c(\phi)$.

1.2. **Cohomology groups.** Let F be a field of characteristic $\neq 2$. By $H^*(F)$ we denote the graded ring of Galois cohomology $H^*(\mathrm{Gal}(F_{sep}/F), \mathbb{Z}/2\mathbb{Z})$. We use the standard canonical isomorphisms $H^0(F) = \mathbb{Z}/2\mathbb{Z}$, $H^1(F) = F^*/F^{*2}$, and $H^2(F) = \mathrm{Br}_2(F)$. Thus, any element $a \in F^*$ determines an element of $H^1(F)$ which is denoted by (a). The cup product $(a_1) \cup \cdots \cup (a_n)$ is denoted by $(a_1, \dots, a_n)$. For any CS algebra A of exponent 2 we get an element $[A]$ of the group $H^2(F) = \mathrm{Br}_2(F)$.

Let L/F be a field extension. The relative Galois cohomology group

$$\ker(H^*(F) \to H^*(L))$$

is sometimes also denoted by $H^*(L/F)$.

For $n = 0, 1, 2$ there is a homomorphism $e^n : I^n(F) \to H^n(F)$ defined as follows: $e^0(\phi) = \dim \phi \pmod 2$, $e^1(\phi) = \det_\pm \phi$, and $e^2(\phi) = c(\phi)$. Moreover, there exists a homomorphism $e^3 : I^3(F) \to H^3(F)$ which is uniquely determined by the condition $e^3(\langle\langle a_1, a_2, a_3\rangle\rangle) = (a_1, a_2, a_3)$ (see [1]). The homomorphism e^n is surjective and $\ker e^n = I^{n+1}(F)$ for $n = 0, 1, 2, 3$ (see [32], [35], and [39]). Thus, for $n = 0, 1, 2, 3$ we have isomorphism $\bar{e}^n : I^n(F)/I^{n+1}(F) \to H^n(F)$ which is uniquely determined by the condition $e^n(\langle\langle a_1, \dots, a_n\rangle\rangle) = (a_1, \dots, a_n)$.

1.3. **The group** $\Gamma(F; A_1, \dots, A_k)$**.** Let $A_1, \dots, A_k$ be CS algebras of exponent 2 over F. We define the group $\Gamma(F; A_1, \dots, A_k)$ by the following formula

$$\Gamma(F; A_1, \dots, A_k) = \frac{\ker\left(H^3(F) \to H^3(F(\mathrm{SB}(A_1) \times \cdots \times \mathrm{SB}(A_k)))\right)}{[A_1]H^1(F) + \cdots + [A_k]H^1(F)}.$$

The group $\Gamma(F; A_1, \dots, A_k)$ depends only on the subgroup $\langle [A_1], \dots, [A_k] \rangle$ of $\mathrm{Br}_2(F)$ generated by $[A_1], \dots, [A_k]$. In particular, for any algebras A_1, A_2, and B with $[A_1] + [A_2] + [B] = 0$, we have

$$\Gamma(F; A_1, A_2, B) = \Gamma(F; A_1, A_2) = \Gamma(F; A_1, B) = \Gamma(F; A_2, B). \tag{1.3.1}$$

Theorem 1.3.2 (see [1], [36])**.** *If* $\mathrm{ind}(A) \leq 4$ *and* $\exp(A) = 2$*, then the group* $\Gamma(F; A)$ *is trivial. In other words,* $\ker\left(H^3(F) \to H^3(F(\mathrm{SB}(A)))\right) = [A]H^1(F)$*.*

Applying this theorem and the injectivity of the homomorphism $\bar{e}^3$, we get the following

Corollary 1.3.3. *Let A be a biquaternion algebra and let q be a corresponding Albert form. Then* $I^3(F(\mathrm{SB}(A))/F) \subset [q]I(F) + I^4(F)$*.* □

1.4. **Chow groups.** For any smooth projective variety X, a homomorphism ε_X of the group $\ker\left(H^3(F, \mathbb{Q}/\mathbb{Z}(2)) \to H^3(F(X), \mathbb{Q}/\mathbb{Z}(2))\right)$ into $\mathrm{CH}^2(X)$ was constructed in [44, §23]. We need the following

Theorem 1.4.1 (see [36, Th. 4.1])**.** *Let $A_1, \dots, A_k$ be CS algebras over F. Let* $X = \mathrm{SB}(A_1) \times \cdots \times \mathrm{SB}(A_k)$*.*

1) *The homomorphism ε induces an isomorphism*
$$\frac{\ker\left(H^3(F, \mathbb{Q}/\mathbb{Z}(2)) \to H^3(F(X), \mathbb{Q}/\mathbb{Z}(2))\right)}{[A_1]H^1(F, \mathbb{Q}/\mathbb{Z}) + \cdots + [A_k]H^1(F, \mathbb{Q}/\mathbb{Z})} \xrightarrow{\sim} \mathrm{Tors}\mathrm{CH}^2(X),$$
which we will denote by $\bar{\varepsilon}_X$ or $\bar{\varepsilon}$.
2) *If all the algebras $A_1, \dots, A_k$ have exponent 2 then the homomorphism ε induces a monomorphism*
$$\frac{\ker\left(H^3(F) \to H^3(F(X))\right)}{[A_1]H^1(F) + \cdots + [A_k]H^1(F)} \to \mathrm{Tors}_2\mathrm{CH}^2(X),$$
which we will denote by $\bar{\varepsilon}_{X,2}$ or $\bar{\varepsilon}_2$.

Thus Theorem 1.4.1 shows that for any collection $A_1, \dots, A_k$ of algebras of exponent 2, there exists a natural monomorphism

$$\bar{\varepsilon}_2 : \Gamma(F; A_1, \dots, A_k) \hookrightarrow \mathrm{Tors}_2\mathrm{CH}^2(\mathrm{SB}(A_1) \times \cdots \times \mathrm{SB}(A_k)).$$

The group $\mathrm{Tors}\mathrm{CH}^2(\mathrm{SB}(A_1) \times \cdots \times \mathrm{SB}(A_k))$ was investigated in Part I. In this Part we need the following obvious consequence of Proposition 4.7 of Part I:

Lemma 1.4.2. *Let A_1, A_2, and B be CS algebras such that* $[A_1] + [A_2] + [B] = 0$*. Then*

$$\mathrm{Tors}_2\mathrm{CH}^2(\mathrm{SB}(A_1) \times \mathrm{SB}(A_2) \times \mathrm{SB}(B)) \cong \mathrm{Tors}_2\mathrm{CH}^2(\mathrm{SB}(A_1) \times \mathrm{SB}(B)).$$

□

1.5. **The group $\Gamma(F;q_1,\dots,q_k)$.** Let $q_1,\dots,q_k \in I^2(F)$. Let us define the group $\Gamma(F;q_1,\dots,q_k)$ by the formula $\Gamma(F;q_1,\dots,q_k) = \Gamma(F;C(q_1),\dots,C(q_k))$. By equation (1.3.1), for any $q_1,q_2,q_3 \in I^2(F)$ satisfying $q_1 \perp q_2 \perp q_3 \in I^3(F)$, we have

$$\Gamma(F;q_1,q_2,q_3) = \Gamma(F;q_1,q_2) = \Gamma(F;q_1,q_3) = \Gamma(F;q_2,q_3). \tag{1.5.1}$$

Let $X = \mathrm{SB}(C(q_1)) \times \dots \times \mathrm{SB}(C(q_k))$. We have a well-defined homomorphism

$$I^3(F(X)/F) \xrightarrow{e^3} \ker\big(H^3(F) \to H^3(F(X))\big) \twoheadrightarrow \Gamma(F;q_1,\dots,q_k)$$

We denote this composition by $\tilde{e}^3$. Thus, for any $\phi \in I^3(F(X)/F)$ we get an element $\tilde{e}^3(\phi) \in \Gamma(F;q_1,\dots,q_k)$.

Lemma 1.5.2. *Let $X = \mathrm{SB}(C(q_1) \times \dots \times \mathrm{SB}(C(q_k)))$ and $\phi \in I^3(F(X)/F)$. The following assertions are equivalent:*

1) $\tilde{e}^3(\phi) = 0$ *in* $\Gamma(F;q_1,\dots,q_k)$.
2) $\phi \in [q_1]I(F) + \dots + [q_k]I(F) + I^4(F)$.

Proof. The isomorphism $\bar{e}^3 : I^3(F)/I^4(F) \to H^3(F)$ induces an isomorphism

$$\frac{I^3(F)}{[q_1]I(F) + \dots + [q_k]I(F) + I^4(F)} \to \frac{H^3(F)}{[C(q_1)]H^1(F) + \dots + [C(q_k)]H^1(F)}.$$

□

Lemma 1.5.3. *Let $q_1,\dots,q_k \in I^2(F)$ satisfy the following conditions:*

a) $\dim(q_1),\dots,\dim(q_k) \le 6$,
b) $q_1 \perp \dots \perp q_k \in I^3(F)$.

Let $X = \mathrm{SB}(C(q_1)) \times \dots \times \mathrm{SB}(C(q_k))$. Then $e^3(q_1 \perp \dots \perp q_k) \in H^3(F(X)/F)$. In particular, we get a well-defined element

$$\tilde{e}^3(q_1 \perp \dots \perp q_k) \in \Gamma(F;q_1,\dots,q_k).$$

Proof. Obviously, $(q_1)_{F(X)},\dots,(q_k)_{F(X)} \in I^3(F(X))$. The assumption

$$\dim(q_i) \le 6 \ \ (i = 1,\dots,k)$$

and APH imply that $[(q_1)_{F(X)}] = \dots = [(q_k)_{F(X)}] = 0$. Hence $q_1 \perp \dots \perp q_k \in W(F(X)/F)$. Since $q_1 \perp \dots \perp q_k \in I^3(F)$, we have $q_1 \perp \dots \perp q_k \in I^3(F(X)/F)$. Therefore, $e^3(q_1 \perp \dots \perp q_k) \in H^3(F(X)/F)$. □

2. Special triples

Definition 2.1. Let F be a field of characteristic $\neq 2$.

1) We say that a triple (q_1,q_2,π) of quadratic forms over F is *special* if the following conditions hold:
   a) q_1 and q_2 are Albert forms and π is a 2-fold Pfister form;
   b) $q_1 \perp q_2 \perp \pi \in I^3(F)$.
2) We say that a triple (A_1,A_2,B) of F-algebras is *special* if the following conditions hold:
   a) A_1 and A_2 are biquaternion F-algebras and B is a quaternion algebra;

b) $[A_1] + [A_1] + [B] = 0 \in \mathrm{Br}_2(F)$.

3) We say that a triple (q_1, q_2, π) is *anisotropic* if all the forms q_1, q_2, and π are anisotropic. We say that a special triple of forms (q_1, q_2, π) *corresponds* to a special triple of algebras (A_1, A_1, B) if $c(q_1) = [A_1]$, $c(q_2) = [A_2]$ and $c(\pi) = [B]$.

It is clear that for any special triple of forms (q_1, q_2, π) there exists a unique (up to an isomorphism) special triple of algebras (A_1, A_2, B) which corresponds to (q_1, q_2, π). Conversely, for any special triple of algebras (A_1, A_2, B) there exists a special triple of forms (q_1, q_2, π), which corresponds to the triple (A_1, A_2, B). In the latter case, the quadratic forms q_1, q_2, and π are uniquely defined up to similarity.

In view of Lemma 1.5.3 we have a well-defined element

$$\tilde{e}^3(q_1 \perp q_2 \perp \pi) \in \Gamma(F; q_1, q_2, \pi).$$

Lemma 2.2 (cf. [28])**.** *Let A be a biquaternion algebra and let B be a quaternion algebra over F such that* $\mathrm{ind}(A \otimes B) = 4$. *Then*

$$H^3(F(\mathrm{SB}(A) \times \mathrm{SB}(B))/F) = [A]H^1(F) + [B]H^1(F) + e^3(\phi)H^0(F),$$

where the quadratic form ϕ is defined as follows: $\phi = q \perp q' \perp \pi$, where q and q' are Albert forms corresponding to the algebras A and $A \otimes_F B$, and π is a 2-fold Pfister form corresponding to B.

In other words, the element $e^3(\phi)$ generates the group $\Gamma(F; A, B)$.

Proof. Let $X = \mathrm{SB}(A)$, $Y = \mathrm{SB}(B)$ and $L = F(Y) = F(\mathrm{SB}(B))$. Since $\mathrm{ind}(A) \leq 4$ and $\mathrm{ind}(B) \leq 2$, Theorem 1.3.2 implies that

$$\ker\big(H^3(L) \to H^3(L(X))\big) = [A_L]H^1(L),$$
$$\ker\big(H^3(F) \to H^3(F(Y))\big) = [B]H^1(F).$$

Let $u \in \ker(H^3(F) \to H^3(F(X \times Y)))$. We need to prove that $u \in [A]H^1(F) + [B]H^1(F) + e^3(\phi)H^0(F)$. We have

$$u_L \in \ker\big(H^3(L) \to H^3(L(X))\big) = [A_L]H^1(L).$$

Hence, there is $f \in L^*$ such that $u_L = [A_L] \cup (f) = e^3(q_L \langle\langle f \rangle\rangle)$, where q is an Albert form corresponding to A. Since the homomorphism e^3 is surjective, there exists $\phi \in I^3(F)$ such that $e^3(\phi) = u$. We have

$$e^3(\phi_L) = u_L = [A_L] \cup (f) = e^3(q_L \langle\langle f \rangle\rangle) = e^3(q_L \perp -f \cdot q_L).$$

Hence $\phi_L - q_L + f \cdot q_L \in \ker(I^3(L) \xrightarrow{e^3} H^3(L)) = I^4(L)$. Let $\tau = f \cdot q_{F(Y)}$. Since $L = F(Y)$, we have $\tau = f \cdot q_{F(Y)} \equiv (q \perp -\phi)_{F(Y)} \pmod{I^4(F(Y))}$. Hence for any 0-dimensional point $y \in Y$ we have $\partial_y^2(\tau) \equiv 0 \pmod{I^3(F(y))}$. Since $\dim \tau = 6 < 8$, it follows from APH that $\partial_y^2(\tau) = 0$. Since $\partial_y^2(\tau) = 0$ for each 0-dimensional point y on the projective conic Y, it follows from [4, Lemma 3.1] that the form τ is defined over the field F (see also [11]). This means that there exists a 6-dimensional form $\tilde{q}$ over F such that $\tilde{q}_L = \tau = f \cdot q_L$. Therefore

$c(\tilde{q})_L = c(q)_L = [A_L]$. Hence $c(\tilde{q}) - [A] \in \mathrm{Br}_2(L/F) = \mathrm{Br}_2(F(\mathrm{SB}(B))/F) = \{0, [B]\}$. Therefore $c(\tilde{q}) \in \{[A], [A \otimes B]\}$.

Consider the case $c(\tilde{q}) = [A]$. Since $[A] = c(q)$, we have $c(\tilde{q}) = c(q)$. Thus $\tilde{q} \sim q$. Let $k \in F^*$ be such that $\tilde{q} = kq$. Then $f \cdot q_L = \tilde{q}_L = kq_L$. We have

$$u_L = e^3(q_L \perp -f \cdot q_L) = e^3(q_L \perp -kq_L) = (e^3(q \langle\langle k \rangle\rangle))_L = ([A] \cup (k))_L.$$

Hence $u - [A] \cup (k) \in \ker(H^3(F) \to H^3(F(Y))) = [B]H^1(F)$. Therefore, $u \in [A]H^1(F) + [B]H^1(F)$.

Suppose now that $c(\tilde{q}) = [A \otimes_F B]$. By the assumption of the lemma, $c(q') = [A \otimes_F B]$. We have $c(\tilde{q}) = c(q')$. Hence $\tilde{q} \sim q'$. Choose $k \in F^*$ such that $\tilde{q} = kq'$. Then $fq_L = \tilde{q}_L = kq'_L$. Since $[\pi_L] = 0$, we have

$$\begin{aligned} u_L &= e^3(q_L \perp -fq_L) = e^3(q_L \perp -kq'_L) = e^3((q + q' + \pi) - q' \langle\langle k \rangle\rangle))_L \\ &= (e^3(\phi) - [c(q')] \cup (k))_L = (e^3(\phi) - [A] \cup (k) - [B] \cup (k))_L. \end{aligned}$$

Thus $u + [A] \cup (k) + [B] \cup (k) - e^3(\phi) \in \ker(H^3(F) \to H^3(F(Y))) = [B]H^1(F)$. Therefore $u \in [A]H^1(F) + [B]H^1(F) + e^3(\phi)H^0(F)$. □

Proposition 2.3. *Let (q_1, q_2, π) be a special triple. Then*

1) $\Gamma(F; q_1, q_2, \pi) = \Gamma(F; q_1, q_2) = \Gamma(F; q_1, \pi) = \Gamma(F; q_2, \pi)$,
2) *the group $\Gamma(F; q_1, q_2, \pi)$ is either 0 or $\mathbb{Z}/2\mathbb{Z}$,*
3) *the element $e^3(q_1 \perp q_2 \perp \pi)$ generates the group $\Gamma(F; q_1, q_2, \pi)$,*
4) *the homomorphism*
$$\bar{\varepsilon}_2 : \Gamma(F; q_1, q_2, \pi) \to \mathrm{Tors}_2\mathrm{CH}^2(\mathrm{SB}(C(q_1)) \times \mathrm{SB}(C(q_2)) \times \mathrm{SB}(C(\pi)))$$
is an isomorphism.

Proof. Assertion 1) is a particular case of equation (1.5.1). Assertion 3) follows immediately from Lemma 2.2 since $\Gamma(F; q_1, q_2, \pi) = \Gamma(F; q_1, \pi)$. Obviously 3) implies 2). Assertion 4) is proved in Appendix A (see Corollary A.9). □

Lemma 2.4. *Let (q_1, q_2, π) be a special anisotropic triple over F and let (A_1, A_2, B) be the corresponding triple of algebras. Let $E = F(\mathrm{SB}(A_1))$. Then*

1) *$(q_2)_E$ is isotropic and* $\dim((q_2)_E)_{\mathrm{an}} = 4$,
2) *for any $s \in D_E(((q_2)_E)_{\mathrm{an}})$ we have $((q_2)_E)_{\mathrm{an}} \cong s \cdot \pi_E$,*
3) *if $((q_2)_E)_{\mathrm{an}}$ is defined over F, then there exists $s \in F^*$ such that*
$$((q_2)_E)_{\mathrm{an}} \cong s \cdot \pi_E.$$

Proof. 1). Since $[A_1] + [A_2] = [B] \in \mathrm{Br}_2(F)$ and $[(A_1)_E] = 0 \in \mathrm{Br}_2(E)$, we have $[(A_2)_E] = [B_E]$. Therefore, $(A_2)_E$ is not a division algebra. Hence, its Albert form $(q_2)_E$ is isotropic and $\dim((q_2)_E)_{\mathrm{an}} \leq 4$.

We claim that $\dim((q_2)_E)_{\mathrm{an}} = 4$ (and hence $((q_2)_E)_{\mathrm{an}} \in GP_2(E)$). Otherwise we would have $[(q_2)_E] = 0$, and hence $[(A_2)_E] = 0$. Then $[A_2] \in \mathrm{Br}_2(E/F) = \mathrm{Br}_2(F(\mathrm{SB}(A_1))/F) = \{0, [A_1]\}$. Therefore, either $[A_2] = 0$, or $[B] = [A_1] + [A_2] = 0$, a contradiction.

2). Since $c(q_2)_E = [(A_2)_E] = [B_E] = c(\pi)_E$, it follows that the form $((q_2)_E)_{\mathrm{an}}$ is similar to the 2-fold Pfister form π_E. Since $s \in D_E(((q_2)_E)_{\mathrm{an}})$, we have $((q_2)_E)_{\mathrm{an}} \cong s \cdot \pi_E$.

3). If $((q_2)_E)_{\rm an}$ is defined over F, we can choose s in $D_E(((q_2)_E)_{\rm an}) \cap F^*$. □

Proposition 2.5. *Let (q_1, q_2, π) be a special anisotropic triple over F and let (A_1, A_2, B) be the corresponding triple of algebras. The following conditions are equivalent:*

1) $((q_2)_{F(\mathrm{SB}(A_1))})_{\rm an}$ *is defined over F,*
2) $((q_1)_{F(\mathrm{SB}(A_2))})_{\rm an}$ *is defined over F,*
3) $q_1 \perp q_2 \perp \pi \in [q_1]I(F) + [q_2]I(F) + [\pi]I(F) + I^4(F)$,
4) *there exist $k_1, k_2 \in F^*$ such that*
$$k_1q_1 \perp k_2q_2 \perp \pi \in I^4(F),$$
5) *the group $\Gamma(F; q_1, q_2, \pi)$ is trivial,*
6) *the group $\mathrm{Tors}_2\mathrm{CH}^2(\mathrm{SB}(A_1) \times \mathrm{SB}(A_2) \times \mathrm{SB}(B))$ is trivial.*

Proof. It suffices to prove that 1) ⇒ 3) ⇒ 4) ⇒ 1) and 3) ⟺ 5) ⟺ 6).

1) ⇒ 3). Let $E = \mathrm{SB}(A_1)$. It follows from Lemma 2.4 that there exists $s \in F^*$ such that $[(q_2)_E] = [s\pi_E]$. Hence $(q_2 \perp -s\pi) \in W(E/F)$. Since $q_1 \in W(E/F)$, we have $(q_1 \perp q_2 \perp -s\pi) \in W(E/F)$. Therefore $(q_1 \perp q_2 \perp \pi) \in W(E/F) + [\pi]I(F)$. Since $\phi = q_1 \perp q_2 \perp \pi \in I^3(F)$, we have $\phi \in I^3(E/F) + [\pi]I(F)$. It follows from Corollary 1.3.3 that $I^3(E/F) \subset [q_1]I(F) + I^4(F)$. Hence
$$\phi \in [q_1]I(F) + [\pi]I(F) + I^4(F) \subset [q_1]I(F) + [q_2]I(F) + [\pi]I(F) + I^4(F).$$

3) ⇒ 4). Since $\phi \in [q_1]I(F) + [q_2]I(F) + [\pi]I(F) + I^4(F)$, there exist $\mu_1, \mu_2, \mu_3 \in I(F)$ such that $[\phi]-[q_1\mu_1]-[q_2\mu_2]-[\pi\mu_3] \in I^4(F)$. Let $r_i = \det_\pm \mu_i$ $(i = 1, 2, 3)$. Clearly $\mu_i \equiv \langle\langle r_i\rangle\rangle \pmod{I^2(F)}$. Therefore $[\phi] - [q_1\langle\langle r_1\rangle\rangle] - [q_2\langle\langle r_2\rangle\rangle]-[\pi\langle\langle r_3\rangle\rangle] \in I^4(F)$. Since $[\phi] = [q_1]+[q_2]+[\pi]$, we have $[r_1q_1]+[r_2q_2]+[r_3\pi] \in I^4(F)$. Setting $k_1 = r_1/r_3$ and $k_2 = r_2/r_3$, we have $[k_1q_1]+[k_2q_2]+[\pi] \in I^4(F)$.

4) ⇒ 1). Let $E = \mathrm{SB}(A_1)$. Then $(k_1q_1 \perp k_2q_2 \perp \pi)_E \in I^4(E)$ and $[(q_1)_E] = 0$. Using APH, we have $[(k_1q_1)_E] + [\pi_E] = 0$. Hence $((q_1)_E)_{\rm an} = -k_1\pi_E$ is defined over F.

3) ⟺ 5). Obvious in view of Lemma 1.5.2 and Proposition 2.3.

5) ⟺ 6). See Proposition 2.3. □

3. A criterion of universal excellence for the function fields of Severi-Brauer varieties

In this Section for any biquaternion division algebra A over F we construct a field extension E/F such that the field extension $E(\mathrm{SB}(A))/E$ is not excellent. The construction is based on the following obvious consequence of Propositions 2.3 and 2.5, and Lemma 1.4.2:

Lemma 3.1. *Let (q_1, q_2, π) be an anisotropic special triple over E and let (A_1, A_2, B) be the corresponding triple of E-algebras. The following conditions are equivalent:*

1) *For any $k_1, k_2 \in F^*$ we have $k_1q_1 \perp k_2q_2 \perp \pi \notin I^4(E)$,*

2) *the group* $\Gamma(E; q_1, q_2, \pi) = \Gamma(E; A_1, A_2, B)$ *is not trivial,*
3) $\Gamma(E; q_1, q_2, \pi) = \Gamma(E; A_1, A_2, B) \cong \mathbb{Z}/2\mathbb{Z}$,
4) *the group* $\mathrm{Tors}_2\mathrm{CH}^2(\mathrm{SB}(A_1) \times \mathrm{SB}(B))$ *is not trivial.*

If these conditions hold then the field extension $E(\mathrm{SB}(A_1))/E$ *is not excellent.* $\square$

Theorem 3.2. *Let A be a biquaternion division F-algebra. Then there exists an unirational field extension E/F such that the field extension $E(\mathrm{SB}(A))/E$ is not excellent.*

Proof. Let $K = F(x, y)$ be the field of rational functions in 2 variables. Let B be the quaternion algebra (x, y) over K. Clearly, $\mathrm{ind}(A_K \otimes_K B) = 8$. Let E be the function field $K(Y)$ of the generalized Severi-Brauer variety $Y = \mathrm{SB}(A_K \otimes_K B, 4)$. By Theorem 5.1 of Part I, we have $\mathrm{Tors}_2\mathrm{CH}^2(\mathrm{SB}(A_E) \times_E \mathrm{SB}(B_E)) \cong \mathbb{Z}/2\mathbb{Z}$.

It follows from the properties of the generalized Severi-Brauer varieties [3] that the algebra $A_E \otimes_E B_E$ has the form $M_2(A')$ where A' is a biquaternion E-algebra. Obviously $[A_E] + [A'] + [B_E] = 0 \in \mathrm{Br}_2(E)$. Hence the triple (A_E, A', B_E) is special. By Lemma 3.1, the extension $E(\mathrm{SB}(A))/E$ is not excellent.

Now we need to verify that the field extension E/F is unirational. Let $\tilde{K} = K(\sqrt{x})$. Since $[B_{\tilde{K}}] = (x, y)_{K(\sqrt{x})} = 0$, we see that $\mathrm{ind}((A_K \otimes_K B)_{\tilde{K}}) = \mathrm{ind}(A_{\tilde{K}}) \le 4$. Hence the variety $Y_{\tilde{K}} = \mathrm{SB}((A_K \otimes_K B)_{\tilde{K}}, 4)$ is rational. Therefore the field extension $\tilde{K}E/\tilde{K} = \tilde{K}(Y)/\tilde{K}$ is purely transcendental. Obviously $\tilde{K}/F$ is purely transcendental. Hence $\tilde{K}E/F$ is purely transcendental too. Therefore the field extension E/F is unirational. $\square$

Definition 3.3. We say that the field extensions E_1/F and E_2/F are q-equivalent (and write $E_1/F \overset{\mathrm{q}}{\sim} E_2/F$) if the following conditions hold:

1) For any quadratic form ϕ over F, the form ϕ_{E_1} is isotropic if and only if ϕ_{E_2} is isotropic.
2) $W(E_1/F) = W(E_2/F)$.

We have the following examples of q-equivalent field extensions.

Lemma 3.4. *Field extensions E_1/F and E_2/F are q-equivalent in the following cases:*

(1) *if $E_1 \subset E_2$ and E_2/E_1 is a finite odd extension;*
(2) *if $E_1 \subset E_2$ and E_2/E_1 is a purely transcendental field extension;*
(3) *if E_1/F and E_2/F are stably isomorphic.*

Proof. (1). Obvious in view of Springer's theorem [27, Th. 2.3 of Chap. VII].

(2). Follows from [27, Lemma 1.1 of Chap. IX].

(3). Since E_1/F and E_2/F are stably isomorphic, there is a field K such that K/E_1 and K/E_2 are purely transcendental. By (2) we have $E_1/F \overset{\mathrm{q}}{\sim} K/F \overset{\mathrm{q}}{\sim} E_2/F$. $\square$

Lemma 3.5 (see [7, Lemma 2.6]). *Let E_1/F and E_2/F be some field extensions such that $E_1/F \overset{\mathrm{q}}{\sim} E_2/F$. Then E_1/F is excellent if and only if E_2/F is excellent.*

Lemma 3.6. *Let A_1 and A_2 be CS algebras such that $\operatorname{ind}(A_1 \otimes_F A_2^{\mathrm{op}})$ is odd, where A_2^{op} is the opposite to A_2 algebra. Then*

1) *the field extensions $F(\mathrm{SB}(A_1))/F$ and $F(\mathrm{SB}(A_2))/F$ are q-equivalent,*
2) *the field extension $F(\mathrm{SB}(A_1))/F$ is excellent if and only if $F(\mathrm{SB}(A_2))/F$ is excellent.*

Proof. 1) Let $X_1 = F(\mathrm{SB}(A_1))$ and $X_2 = F(\mathrm{SB}(A_2))$. Since $\operatorname{ind}(A_1 \otimes_F A_2^{\mathrm{op}})$ is odd, there is an odd field extension K/F such that $[(A_1 \otimes_F A_2^{\mathrm{op}})_K] = 0$. Then $[(A_1)_K] = [(A_2)_K]$. Hence the field extensions $K(X_1)/K$ and $K(X_2)/K$ are stably isomorphic. Therefore $K(X_1)/F$ and $K(X_2)/F$ are stably isomorphic too. By Lemma 3.4, we have $K(X_1)/F \overset{\mathrm{q}}{\sim} K(X_2)/F$. Since $[K(X_1) : F(X_1)] = [K(X_2) : F(X_2)] = [K : F]$ is odd, it follows from Lemma 3.4 that $F(X_1)/F \overset{\mathrm{q}}{\sim} K(X_1)/F \overset{\mathrm{q}}{\sim} K(X_2)/F \overset{\mathrm{q}}{\sim} F(X_2)/F$.

2) Obvious in view of Lemma 3.5. □

Corollary 3.7. *Let A and B be CS algebras over F such that $[A] = [B]$ in $\mathrm{Br}(F)$. Then the field extension $F(\mathrm{SB}(A))/F$ is excellent if and only if $F(\mathrm{SB}(B))/F$ is excellent.* □

Corollary 3.8. *Let A be a CS algebra over F and let $A\{2\}$ denote the 2-primary component of A. Then the following conditions are equivalent:*

1) *the field extension $F(\mathrm{SB}(A))/F$ is excellent,*
2) *the field extension $F(\mathrm{SB}(A\{2\}))/F$ is excellent.* □

Theorem 3.9. *Let A be a CS algebra over F. Let $X = \mathrm{SB}(A)$. The following conditions are equivalent:*

1) *$F(X)/F$ is universally excellent,*
2) *$\operatorname{ind}(A)$ is not divisible by 4.*

In other words, the field extension $F(\mathrm{SB}(A))/F$ is universally excellent only in the following two cases: 1) index of A is odd; 2) algebra A has the form $Q \otimes_F D$, where Q is a quaternion algebra and the index of D is odd.

Proof. 1) $\Rightarrow$ 2). Suppose that $\operatorname{ind}(A)$ has the form $\operatorname{ind}(A) = 4k$. Let $Y = \mathrm{SB}(A, 4) \times \mathrm{SB}(A^{\otimes 2})$ and $K = F(Y)$. Obviously $\operatorname{ind}(A_K) \leq 4$ and $2[A_K] = 0$. By Blanchet's index reduction formula (see [3]), we have $\operatorname{ind}(A_K) = 4$. Hence there is a biquaternion division algebra $\tilde{A}$ over K such that $[A_K] = [\tilde{A}]$. It follows from Theorem 3.2, that there is a field extension E/K such that $E(\mathrm{SB}(\tilde{A}))/E$ is not excellent. By Corollary 3.7 the field extension $E(\mathrm{SB}(A))/E$ is not excellent too.

2) $\Rightarrow$ 1). In view of Corollary 3.8, we can suppose that A is a division algebra and $\deg A = 2^n$. Since $\operatorname{ind}(A)$ is not divisible by 4, we see that A is a quaternion algebra or $A = F$. Hence $F(\mathrm{SB}(A))/F$ is universally excellent. □

For algebras of index 8 we have the following

Theorem 3.10. *Let A be a CS algebra of index 8 and exponent 2. Then the field extension $F(\mathrm{SB}(A))/F$ is not excellent.*

Since any algebra of index 8 and exponent 2 is Brauer equivalent to a 4-quaternion algebra, it suffices to prove the following lemma. [3]

Lemma 3.11. *Let $A = (a_1, b_1) \otimes_F (a_2, b_2) \otimes_F (a_3, b_3) \otimes_F (a_4, b_4)$ be a 4-quaternion algebra over F such that $\mathrm{ind}A \geq 8$. Then the field extension $F(\mathrm{SB}(A))/F$ is not excellent.*

In the proof of this lemma we will use the following

Theorem 3.12 (see [8, Cor. 9.3] or [18, Cor. 0.3])**.** *Let ϕ be a quadratic form over F such that $\mathrm{ind}C(\phi) \geq 8$. Let $K = F(\mathrm{SB}(C(\phi)))$. Then $\phi_K \notin I^4(K)$ (and hence $[\phi_{F(\mathrm{SB}(C(\phi)))}] \neq 0$).*

Proof of Lemma 3.11. Let $E = F(\mathrm{SB}(A))$ and $q \in I^2(F)$ be an arbitrary 10-dimensional quadratic form such that $c(q) = [A]$. Since $q_E \in I^3(E)$ and $\dim q_E = 10$, the form q_E is isotropic (see [37]). Hence there is $\gamma \in GP_3(E)$ such that $[q_E] = [\gamma] \in W(E)$. Suppose at the moment that the field extension E/F is excellent. Then γ is defined over F. It follows from Lemma 3.13 below that there is $\alpha \in GP_3(F)$ such that $\gamma = \alpha_E$. We have $[q_E] = [\gamma] = [\alpha_E]$. Let $\phi = q \perp -\alpha$. Then $[\phi_E] = 0$. Since $\alpha \in I^3(F)$, it follows that $c(\phi) = c(q) = [A]$. Therefore the field extension $F(\mathrm{SB}(C(\phi)))/F$ is equivalent to E/F. Hence it follows from $[\phi_E] = 0$ that $[\phi_{F(\mathrm{SB}(C(\phi)))}] = 0$, what provides a contradiction to Theorem 3.12. □

Lemma 3.13. *Let E/F be an excellent field extension and $\gamma \in GP_n(E)$ be a form defined over F. Then there is $\alpha \in GP_n(F)$ such that $\gamma = \alpha_E$.*

Proof. Since γ is defined over F, there is $c \in D_E(\gamma) \cap F^*$. Then the form $\phi = c\gamma$ is an n-fold E-Pfister form which is defined over F. By [7, Prop. 2.10] there is an n-fold F-Pfister form β such that $\phi = \beta_E$. Setting $\alpha = c\beta$, we have $\gamma = \alpha_E$, $\alpha \in GP_n(E)$. □

4. Five-excellence of $F(\mathrm{SB}(A))/F$

Let n be a positive integer. We say that a field extension L/F is n-excellent if for any quadratic form ϕ over F of dimension $\leq n$ the quadratic form $(\phi_L)_{\mathrm{an}}$ is defined over F.

Lemma 4.1. *Let ϕ be an anisotropic 5-dimensional quadratic form and A be a CS algebra over F. Than $(\phi_{F(\mathrm{SB}(A))})_{\mathrm{an}}$ is defined over F*

[3]We adduce here the proof suggested by D. Hoffmann which is essentially shorter than original author's proof.

Proof. Let $E = F(\mathrm{SB}(A))$. We can suppose that ϕ_E is isotropic. Let $s = -\det\phi$ and $q = \phi \perp \langle s\rangle$. If q is isotropic, then ϕ is a 5-dimensional Pfister neighbor. In this case ϕ is an excellent form (see [25]). Thus $(\phi_E)_{\mathrm{an}}$ is defined over F. So, we can suppose that q is an anisotropic Albert form. Hence $\mathrm{ind}(C(q)) = 4$. Since q_E is isotropic, $\mathrm{ind}(C(q)_E) \le 2$. By the Schofield-van den Bergh-Blanchet index reduction formula (see [3] or [45]), there exists an algebra B over F such that $[B_E] = [C(q)_E]$ and $\mathrm{ind}B = \mathrm{ind}C(q)_E$. Thus $\mathrm{ind}B \le 2$. Hence there exist $a, b \in F^*$ such that $[B] = (a, b)$. Let $\tau = s\langle -a, -b, ab\rangle$. We claim that $(\phi_E)_{\mathrm{an}} = (\tau_E)_{\mathrm{an}}$. Indeed, since $(\phi_E)_{\mathrm{an}}$ and $(\tau_E)_{\mathrm{an}}$ are odd-dimensional forms of dimension ≤ 3, it is sufficient to verify that $\det_\pm \phi_E = \det_\pm \tau_E$ and $[C_0(\phi_E)] = [C_0(\tau_E)]$. Both equations are obvious by the definition of τ. Since $\dim\tau \le 3$, it follows that τ is an excellent form. Hence $(\phi_E)_{\mathrm{an}} = (\tau_E)_{\mathrm{an}}$ is defined over F. $\square$

Theorem 4.2. *The field extension $F(\mathrm{SB}(A))/F$ is 5-excellent for any CS algebra A over F.*

Proof. Let $E = F(\mathrm{SB}(A))$ and τ be a quadratic form of dimension ≤ 5 over F. We need to verify that τ_E is defined over F. In view of Lemma 4.1, we can assume that $\dim\tau \le 4$. Since all forms of dimension < 4 are excellent, we can suppose that $\dim\tau = 4$. Let $\phi = \tau_{F(t)} \perp \langle t\rangle$, where t is an indeterminate, and let $\xi = (\tau_E)_{\mathrm{an}}$. We have $\xi_{E(t)} \perp \langle t\rangle = (\tau_{E(t)})_{\mathrm{an}} \perp \langle t\rangle \cong (\phi_{E(t)})_{\mathrm{an}} = (\phi_{F(t)(\mathrm{SB}(A))})_{\mathrm{an}}$. By Lemma 4.1, $(\phi_{F(t)(\mathrm{SB}(A))})_{\mathrm{an}}$ is defined over $F(t)$. Hence $\xi_{E(t)} \perp \langle t\rangle$ is defined over $F(t)$. It follows from Lemma 4.3 below that $\xi = (\tau_E)_{\mathrm{an}}$ is defined over F. $\square$

Lemma 4.3. *Let E/F be a field extension and ξ be an anisotropic form over E. Suppose that $\xi_{E(t)} \perp \langle t\rangle$ is defined over $F(t)$. Then ξ is defined over F.*

Proof. Let γ be a quadratic form over $F(t)$ such that $\xi_{E(t)} \perp \langle t\rangle \cong \gamma_{E(t)}$. We can write $\gamma_{F((t))}$ in the form $\gamma_{F((t))} \cong \lambda_{F((t))} \perp t\lambda'_{F((t))}$ where λ and λ' are quadratic forms over F. Obviously $\xi_{E((t))} \perp t\langle 1\rangle \cong \lambda_{E((t))} \perp t\lambda'_{E((t))}$. Since ξ and $\langle 1\rangle$ are anisotropic, we have $\xi = \lambda_E$, $\langle 1\rangle = \lambda'_E$. Hence ξ is defined over F. $\square$

Proposition 4.4. *Let A be a biquaternion division algebra over F. Then there is a field extension E/F such that A_E is a division algebra and the field extension $E(\mathrm{SB}(A))/E$ is excellent.*

Proof. Let q be an Albert form corresponding to A. By [33] there is a field extension E/F such that: 1) $u(E) = 6$; 2) A_E is a division algebra; 3) all 6-dimensional anisotropic quadratic E-forms are similar to q_E.

Let ϕ be an anisotropic quadratic form over E. We need to prove that $(\phi_{E(\mathrm{SB}(A))})_{\mathrm{an}}$ is defined over E. Since $u(E) = 6$, we have $\dim\phi \le 6$. By Theorem 4.2, we can assume that $\dim\phi = 6$. It follows from property 3) of the field E that $\phi \sim q_E$. Therefore $[\phi_{E(\mathrm{SB}(A))}] = 0$. $\square$

Corollary 4.5. *There exist a field F and a biquaternion division algebra A over F such that the field extension $F(\mathrm{SB}(A))/F$ is excellent.*

5. Examples of non-excellent field extensions $F(\mathrm{SB}(A))/F$

In this Section we give some explicit examples of non-excellent field extensions $F(\mathrm{SB}(A))/F$. The main tool for constructing these examples is the following assertion.

Lemma 5.1. *Let $\mu_1, \mu_2, \mu_3, \mu_1', \mu_2', \mu_3'$ be anisotropic 2-dimensional quadratic forms over a field K. Let $\pi \in GP_2(K)$. Suppose that $\pi_{K(\mu_i)}$ is anisotropic for all $i = 1, 2, 3$. Let $\hat{K} = K((x))((y))$, where x, y are indeterminates, and let $k, k' \in \hat{K}^*$. Then*

$$k(\mu_1 \perp x\mu_2 \perp y\mu_3) \perp k'(\mu_1' \perp x\mu_2' \perp y\mu_3') \perp \pi_{\hat{K}} \notin I^4(\hat{K}).$$

Proof. In view of Springer's theorem we can identify $W(\hat{K})$ with the direct sum $W(K) \oplus xW(K) \oplus yW(K) \oplus xyW(K)$. Moreover we can regard $W(K)$ as a subring of $W(\hat{K})$. Let $\phi = k(\mu_1 \perp x\mu_2 \perp y\mu_3) \perp k'(\mu_1' \perp x\mu_2' \perp y\mu_3')$. Suppose at the moment that $\phi \perp \pi_{\hat{K}} \in I^4(\hat{K})$. Then $\phi \perp \pi_{\hat{K}} \in GP_4(\hat{K})$. Since $(\phi \perp \pi_{\hat{K}})_{\hat{K}(\pi)}$ is isotropic, it is hyperbolic. Hence $\phi_{\hat{K}(\pi)}$ is hyperbolic. Therefore $\phi \in [\pi_{\hat{K}}]W(\hat{K})$.

Let us write $[\phi] \in W(\hat{K})$ in the form $[\phi] = [\tau_1] + x[\tau_2] + y[\tau_3] + xy[\tau_4]$, where τ_i $(i = 1, 2, 3, 4)$ are defined over K. Since all the forms μ_i, μ_i' $(i = 1, 2, 3)$ have dimension 2, we have $\dim \tau_i \leq 4$ $(i = 1, \dots, 4)$. Since

$$[\phi] \in [\pi_{\hat{K}}]W(\hat{K}) \cong [\pi]W(K) \oplus x[\pi]W(K) \oplus y[\pi]W(K) \oplus xy[\pi]W(K),$$

we have $\tau_1, \tau_2, \tau_3, \tau_4 \in [\pi]W(K)$.

Suppose that there exists j such that $[\tau_j] \neq 0$. Since $\dim \tau_j \leq 4$ and $\tau_j \in [\pi]W(K)$, we see that $\tau_j \sim \pi$. By the definition of ϕ, there exists i $(1 \leq i \leq 3)$ such that μ_i is similar to a subform in τ_j. Therefore μ_i is similar to a subform in π and hence the form $\pi_{K(\mu_i)}$ is isotropic; this yields a contradiction (see the assumptions of the lemma).

Therefore $[\tau_i] = 0$ for all $i = 1, 2, 3, 4$. Then $[\phi] = 0$. It follows from $\phi \perp \pi_{\hat{K}} \in I^4(\hat{K})$ that $[\pi_{\hat{K}}] \in I^4(\hat{K})$. Hence $[\pi] \in I^4(K)$. By APH the form π is isotropic, a contradiction. $\square$

Corollary 5.2. *Let $r, s, u, v \in K^*$, and $\pi \in P_2(K)$ satisfy the conditions:*

1) *$c(\pi) = (r, u) + (s, v)$,*
2) *π is anisotropic over the fields $K(\sqrt{u})$, $K(\sqrt{v})$, and $K(\sqrt{uv})$.*

Let $q_1 = \langle\langle uv\rangle\rangle \perp -x\langle\langle u\rangle\rangle \perp -y\langle\langle v\rangle\rangle$ and $q_2 = \langle\langle uv\rangle\rangle \perp -xr\langle\langle u\rangle\rangle \perp -ys\langle\langle v\rangle\rangle$ be quadratic forms over $\tilde{K} = K(x, y)$. Then $(q_1, q_2, \pi_{\tilde{K}})$ is a special triple over $\tilde{K}$ and $\Gamma(\tilde{K}; q_1, q_2, \pi) \cong \mathbb{Z}/2\mathbb{Z}$.

Proof. Obviously q_1 and q_2 are Albert forms. Since $c(q_1 \perp q_2 \perp \pi) = c(-q_1 \perp q_2 \perp \pi) = c(x\langle\langle u, r\rangle\rangle \perp y\langle\langle s, v\rangle\rangle \perp \pi) = (u, r) + (s, v) + c(\pi) = 0$, the triple $(q_1, q_2, \pi_{\tilde{K}})$ is special. The quadratic forms $\mu_1 = \langle\langle uv\rangle\rangle$, $\mu_2 = -\langle\langle u\rangle\rangle$, $\mu_3 = -\langle\langle v\rangle\rangle$, $\mu_1' = \langle\langle uv\rangle\rangle$, $\mu_2' = -s\langle\langle u\rangle\rangle$, $\mu_3' = -r\langle\langle v\rangle\rangle$, and π satisfy the conditions of Lemma 5.1. Hence for any $k_1, k_2 \in \hat{K} = K((x))((y))$ we have $k_1(q_1)_{\hat{K}} \perp$

$k_2(q_2)_{\hat{K}} \perp \pi_{\hat{K}} \notin I^4(\hat{K})$. Therefore for any $k_1, k_2 \in \tilde{K} = K(x,y)$ we have $k_1 q_1 \perp k_2 q_2 \perp \pi_{\tilde{K}} \notin I^4(\tilde{K})$. It follows from Lemma 3.1, that $\Gamma(\tilde{K}; q_1, q_2, \pi_{\tilde{K}}) = \mathbb{Z}/2\mathbb{Z}$. □

Lemma 5.3. *Let $w_1, w_2 \in F^*$ be such that $w_1, w_2, w_1w_2 \notin F^{*2}$. Let $K = F(t)$ be the field of rational functions in one variable t. Let*

$$r = -tw_1, \quad s = -tw_2, \quad u = t + w_1, \quad v = t + w_2, \quad \text{and} \quad \pi = \langle\langle t, w_1w_2 \rangle\rangle .$$

Then $r, s, u, v \in K^*$, and $\pi \in P_2(K)$ satisfy the conditions of Corollary 5.2.

Proof. 1) We have $(r,u) + (s,v) = (-tw_1, t+w_1) + (-tw_2, t+w_2) = (t, w_1) + (t, w_2) = (t, w_1w_2) = c(\pi)$.

2) Let $p(t)$ be equal to one of the polynomials $u = t + w_1$, $v = t + w_2$, or $uv = t^2 + (w_1 + w_2)t + w_1w_2$. We need to verify that π is anisotropic over the field $K(\sqrt{p(t)})$. Suppose that $\pi_{K(\sqrt{p(t)})}$ is isotropic. Then $p(t) \in D_F(-\pi')$ where $\pi' = \langle -t, -w_1w_2, tw_1w_2 \rangle$ is the pure subform of π (see [43, Th. 5.4(ii) of Chap. 4]). Therefore $p(t) \in D_{F(t)}(\langle t, w_1w_2, -tw_1w_2 \rangle)$. By the Cassels-Pfister theorem [4] there are polynomials $p_1(t), p_2(t), p_3(t) \in F[t]$ such that

$$\begin{aligned} p(t) &= tp_1^2(t) + w_1w_2p_2^2(t) - tw_1w_2p_3^2(t) \\ &= t(p_1^2(t) - w_1w_2p_3^2(t)) + w_1w_2p_2^2(t). \end{aligned} \tag{5.4}$$

If $p(t) = t + w_1$, we have $w_1 = p(0) = w_1w_2p_2^2(0) \in w_1w_2F^{*2}$. Therefore $w_2 \in F^{*2}$, a contradiction. If $p(t) = t + w_2$, then $w_2 = p(0) = w_1w_2p_2^2(0) \in w_1w_2F^{*2}$. Then $w_2 \in F^{*2}$, a contradiction.

Let now $p(t) = t^2 + (w_1 + w_2)t + w_1w_2$. Since $w_1w_2 \notin F^{*2}$, it follows that $\deg(t(p_1^2(t) - w_1w_2p_3^2(t)))$ is odd and $\deg(p(t) - w_1w_2p_2^2(t))$ is even. We get a contradiction to the equation (5.4). □

Corollary 5.5. *Let $w_1, w_2 \in F^*$ and assume that $w_1, w_2, w_1w_2 \notin F^{*2}$. Let $E = F(t,x,y)$ be the field of rational functions in 3 variables. Consider the quadratic forms*

$$\begin{aligned} q_1 &= \langle\langle (t+w_1)(t+w_2) \rangle\rangle \perp -x \langle\langle t + w_1 \rangle\rangle \perp -y \langle\langle t + w_2 \rangle\rangle , \\ q_1 &= \langle\langle (t+w_1)(t+w_2) \rangle\rangle \perp xtw_1 \langle\langle t + w_1 \rangle\rangle \perp ytw_2 \langle\langle t + w_2 \rangle\rangle , \\ \pi &= \langle\langle t, w_1w_2 \rangle\rangle \end{aligned}$$

over E. Then (q_1, q_2, π) is a special triple and $\Gamma(E; q_1, q_2, \pi) \cong \mathbb{Z}/2\mathbb{Z}$. □

Proposition 5.6. *Let $w_1, w_2 \in F^*$ and assume that $w_1, w_2, w_2w_2 \notin F^{*2}$. Let $E = F(t,x,y)$ be the field of rational functions in 3 variables. Let $A = (x,y) \otimes (x(t+w_2), y(t+w_1))$. Then the field extension $E(\mathrm{SB}(A))/E$ is not excellent.*

Proof. Let (q_1, q_2, π) be the special triple constructed in Corollary 5.5. Clearly $c(q_1) = (x,y) + (x(t+w_2), y(t+w_1)) = [A]$. Since $\Gamma(E; q_1, q_2, \pi) \cong \mathbb{Z}/2\mathbb{Z}$, it follows from Lemma 3.1 that $E(\mathrm{SB}(A))/E$ is not excellent. □

[4]Note that a strong version of the Cassels-Pfister theorem assumes that the coefficients in a diagonalization of the quadratic form are polynomials of degree ≤ 1.

Corollary 5.7. *Let F be a field such that $|F^*/F^{*2}| \geq 4$. Let $E = F(x, y, t)$ be the field of rational functions in 3 variables. Then there is a biquaternion algebra A over E such that the field extension $E(\mathrm{SB}(A))/E$ is not excellent.* □

Lemma 5.8. *Suppose that a field F satisfies the following condition: there exists $w \in F^*$ such that $w, w+1, w(w+1) \notin F^{*2}$. Let $E = F(a, b, c)$ be the field of rational functions in 3 variables and define a biquaternion algebra A over E as $A = (a, b) \otimes (a+1, c)$. Then the field extension $E(\mathrm{SB}(A))/E$ is not excellent.*

Proof. Let $E' = F(t, x, y)$ be the field of rational functions in 3 variables. Let $w_1 = w$, $w_2 = w+1$ and $A' = (x, y) \otimes (x(t+w_1), y(t+w_2))$. By Proposition 5.6, the field extension $E'(\mathrm{SB}(A'))/E'$ is not excellent. Let us identify the fields $E' = F(t, x, y)$ and $E = F(a, b, c)$ by means of the isomorphism $t \mapsto (a - w)$, $x \mapsto ac$, $y \mapsto b$. We have

$$\begin{aligned}[A'] =&(x, y) + (x(t + w), y(t + w + 1)) \mapsto \\ &\mapsto (ac, b) + (ac(a - w + w), b(a - w + w + 1)) = \\ &= (ac, b) + (c, b(a + 1)) = (a, b) + (a + 1, c) = [A].\end{aligned}$$

Since the algebra A' maps to A, it follows that $E(\mathrm{SB}(A))/E$ is not excellent. □

Example 5.9. Let $E = \mathbb{Q}(a, b, c)$ be the field of rational function in 3 variables over $\mathbb{Q}$. Let $A = (a, b) \otimes (a + 1, c)$. Then the field extension $E(\mathrm{SB}(A))/E$ is not excellent.

Proof. It is sufficient to set $w = 2$ in Lemma 5.8. □

Proposition 5.10. *Let $E = F(a, b, c, d)$ be the field of rational functions in 4 variables and A be the biquaternion algebra $(a, b) \otimes (c, d)$ over E. The field extension $E(\mathrm{SB}(A))/E$ is not excellent.*

Proof. Let $F' = F(z)$ and $E' = F(x, y, t, z)$ be fields of rational functions in 1 and 4 variables respectively. Let $w_1 = 1 - z$ and $w_2 = 1 + z$. Obviously $w_1, w_2, w_1w_2 \notin (F')^{*2}$. Let $A' = (x, y) \otimes (x(t+1+z), y(t+1-z))$. It follows from Proposition 5.6 that the field extension $E'(\mathrm{SB}(A'))/E'$ is not excellent. Now it is sufficient to identify the fields $E = F(a, b, c, d)$ and $E' = F(x, y, t, z)$ by means of F-isomorphism: $a \mapsto x$, $b \mapsto y$, $c \mapsto x(t+1+z)$, $d \mapsto y(t+1-z)$. □

Appendix A. Surjectivity of $\varepsilon_2 : H^3(F(X)/F) \to \mathrm{Tors}_2\mathrm{CH}^2(X)$ for certain homogeneous varieties

The main purpose of this Appendix is to prove the following theorem.

Theorem A.1. *Let A and B be CS algebras of exponent 2 over a field F of characteristic $\neq 2$. Let $X = \mathrm{SB}(A) \times \mathrm{SB}(B)$. Then the homomorphism $\bar{\varepsilon}_2$*

$$\frac{\ker\left(H^3(F) \to H^3(F(X))\right)}{[A]H^1(F) + [B]H^1(F)} \to \mathrm{Tors}_2\mathrm{CH}^2(X)$$

is an isomorphism. Moreover, $\mathrm{Tors}\mathrm{CH}^2(X) = \mathrm{Tors}_2\mathrm{CH}^2(X)$.

In this Appendix we will use the following notation and agreements.

- We identify the group $H^3(F)$ with the 2-torsion subgroup of the group $H^3(F, \mathbb{Q}/\mathbb{Z}(2))$.
- For any field extension E/F we set
$$H^i(E/F, \mathbb{Q}/\mathbb{Z}(j)) = \ker(H^i(F, \mathbb{Q}/\mathbb{Z}(j)) \to H^i(E, \mathbb{Q}/\mathbb{Z}(j)))$$
and $H^i(E/F) = \ker(H^i(F) \to H^i(E))$.

Lemma A.2. *Let q be a quadratic form over F. Then*
$$2H^3(F(q)/F, \mathbb{Q}/\mathbb{Z}(2)) = 0.$$
In other words, $H^3(F(q)/F, \mathbb{Q}/\mathbb{Z}(2)) = H^3(F(q)/F)$.

Proof. Take a field extension L/F of degree ≤ 2 such that q_L is isotropic. Obviously, $H^3(F(q)/F, \mathbb{Q}/\mathbb{Z}(2)) \subset H^3(L/F, \mathbb{Q}/\mathbb{Z}(2))$. Since
$$[L : F]H^3(L/F, \mathbb{Q}/\mathbb{Z}(2)) = 0,$$
we are done. □

Corollary A.3. *Let q be an Albert form over F. Then*
$$H^3(F(q)/F, \mathbb{Q}/\mathbb{Z}(2)) = 0.$$

Proof. By Arason's Theorem (see [1]), we have $H^3(F(q)/F) = 0$. Applying Lemma A.2, we get $H^3(F(q)/F, \mathbb{Q}/\mathbb{Z}(2)) = 0$. □

We recall that a field F is said to be linked (see [5], [6]) if the following equivalent conditions hold:

(a) all F-algebras of exponent 2 have index ≤ 2,
(b) all Albert forms over F are isotropic.

Lemma A.4. *For any field F there exists a field extension E/F such that E is linked and the homomorphism $H^3(F, \mathbb{Q}/\mathbb{Z}(2)) \to H^3(E, \mathbb{Q}/\mathbb{Z}(2))$ is injective.*

Proof. Let us define the fields $F_0 = F$, F_1, F_2, ... recursively as follows. We set F_i to be the free composite of all the fields of the form $F_{i-1}(q)$ where q runs over all Albert forms over F_{i-1}. Further we set $E = \cup_{i=1}^{\infty} F_i$. By Corollary A.3, the homomorphism $H^3(F, \mathbb{Q}/\mathbb{Z}(2)) \to H^3(E, \mathbb{Q}/\mathbb{Z}(2))$ is injective. By the construction, all Albert forms over E are isotropic. Hence the field E is linked. □

Lemma A.5 (cf. [36, Lemma 5.3])**.** *Let A_1, A_2 be CS F-algebras of index ≤ 2 and let $X = \mathrm{SB}(A_1) \times \mathrm{SB}(A_2)$. Then*
$$H^3(F(X)/F, \mathbb{Q}/\mathbb{Z}(2)) = [A_1]H^1(F, \mathbb{Q}/\mathbb{Z}(1)) + [A_2]H^1(F, \mathbb{Q}/\mathbb{Z}(1)).$$

Proof. We can suppose that A_1 and A_2 are quaternion algebras. By [36, Cor. 3.9], the group $\mathrm{TorsCH}^2(X)$ is trivial. Now it is sufficient to apply Theorem 1.4.1. □

Corollary A.6. *Let A_1, A_2 be F-algebras of index ≤ 2 and let $X = \mathrm{SB}(A_1) \times \mathrm{SB}(A_2)$. Then $2H^3(F(X)/F, \mathbb{Q}/\mathbb{Z}(2)) = 0$.* □

Lemma A.7. *Let A_1 and A_2 be CS F-algebras of exponent 2 and let $X = \mathrm{SB}(A_1) \times \mathrm{SB}(A_2)$. Then $2H^3(F(X)/F, \mathbb{Q}/\mathbb{Z}(2)) = 0$.*

Proof. Let E/F be the field extension constructed in Lemma A.4. Since the homomorphism $H^3(F, \mathbb{Q}/\mathbb{Z}(2)) \to H^3(E, \mathbb{Q}/\mathbb{Z}(2))$ is injective, the homomorphism $H^3(F(X)/F, \mathbb{Q}/\mathbb{Z}(2)) \to H^3(E(X)/E, \mathbb{Q}/\mathbb{Z}(2))$ is injective too. Therefore it is sufficient to prove that $2H^3(E(X)/E, \mathbb{Q}/\mathbb{Z}(2)) = 0$. This assertion follows immediately from Corollary A.6 since any algebra of exponent 2 over a linked field has index ≤ 2. $\square$

Proof of Theorem A.1. By Theorem 1.4.1 it is sufficient to verify surjectivity of the homomorphism $\varepsilon_2 : H^3(F(X)/F) \to \mathrm{TorsCH}^2(X)$. By Lemma A.7, we have $H^3(F(X)/F, \mathbb{Q}/\mathbb{Z}(2)) = \mathrm{Tors}_2 H^3(F(X)/F, \mathbb{Q}/\mathbb{Z}(2)) = H^3(F(X)/F)$. By Theorem 1.4.1, the homomorphism

$$\varepsilon : H^3(F(X)/F, \mathbb{Q}/\mathbb{Z}(2)) \to \mathrm{TorsCH}^2(F)$$

is surjective. Hence the homomorphism ε_2 is surjective as well. $\square$

Corollary A.8. *For any CS F-algebra A of exponent* 2 *the homomorphism* $\bar{\varepsilon}_2$

$$\frac{\ker\left(H^3(F) \to H^3(F(\mathrm{SB}(A)))\right)}{[A]H^1(F)} \to \mathrm{Tors}_2\mathrm{CH}^2(\mathrm{SB}(A))$$

is an isomorphism $\square$

Theorem A.1, Lemma 1.4.2, and equation (1.3.1) imply the following

Corollary A.9. *Let A_1, A_2 and B be CS F-algebras of exponent 2 such that $[A_1] + [A_2] + [B] = 0 \in \mathrm{Br}_2(F)$. Let $X = \mathrm{SB}(A_1) \times \mathrm{SB}(A_2) \times \mathrm{SB}(B)$. Then the homomorphism $\bar{\varepsilon}_2$*

$$\frac{\ker\left(H^3(F) \to H^3(F(X))\right)}{[A_1]H^1(F) + [A_2]H^1(F) + [B]H^1(F)} \to \mathrm{Tors}_2\mathrm{CH}^2(X)$$

is an isomorphism. $\square$

Remark A.10. Let $A_1, \ldots, A_k$ be CS F-algebras of exponent 2. Let

$$X = \mathrm{SB}(A_1) \times \cdots \times \mathrm{SB}(A_k).$$

It is not true (in general) that the homomorphism

$$\frac{\ker\left(H^3(F) \to H^3(F(X))\right)}{[A_1]H^1(F) + \cdots + [A_k]H^1(F)} \xrightarrow{\bar{\varepsilon}_2} \mathrm{Tors}_2\mathrm{CH}^2(X).$$

is bijective. A counterexample for the case $k = 3$ was constructed by E. Peyre (see [36, Rem. 4.1 and Prop. 6.3]).

Appendix B. A criterion of universal excellence for generic splitting fields of quadratic forms

Definition B.1. Let E/F be a finitely generated field extension. We say that E/F is *universally excellent* if for any field extension K/F and any free composite EK of E and K over F the field extension EK/K is excellent.

Remarks. 1) By a free composite of K and E over F we mean the field of fractions of the factor ring $(K \otimes_F E)/\mathcal{P}$, where $\mathcal{P}$ is a minimal prime ideal in $K \otimes_F E$.
2) If X is a geometrically integral variety over F and $E = F(X)$, a free composite EK is uniquely defined and coincides with $K(X)$.

Let ϕ be a non-hyperbolic quadratic form over F. Put $F_0 = F$ and $\phi_0 = \phi_{\mathrm{an}}$. For $i \geq 1$ let $F_i = F_{i-1}(\phi_{i-1})$ and $\phi_i = ((\phi_{i-1})_{F_i})_{\mathrm{an}}$. The smallest h such that $\dim \phi_h \leq 1$ is called the *height* of ϕ. The tower of fields $F_0, F_1, \dots, F_h$ is called the *generic splitting tower* of ϕ. The *degree* of ϕ is defined to be zero if $\dim \phi$ is odd. If $\dim \phi$ is even then there is m such that $\phi_{h-1} \in GP_m(F_{h-1})$. In this case we set $\deg \phi = m$.

The main purpose of Appendix B is to prove the following

Theorem B.2. *Let ϕ be an anisotropic quadratic form over F and let*

$$F_0, F_1, \dots, F_h$$

be the generic splitting tower of ϕ. Let s be a positive integer such that $s \leq h$. Then

1) *if the field extension F_s/F is universally excellent then $s = h$,*
2) *the field extension F_h/F is universally excellent if and only if one of the following conditions holds:*
   (a) *ϕ has the form $\langle\langle a, b\rangle\rangle \gamma$, where γ is an odd-dimensional quadratic form,*
   (b) *$\phi \perp \langle -\det_\pm \phi\rangle$ has the form $\langle\langle a, b\rangle\rangle \gamma$, where γ is an odd-dimensional quadratic form,*
   (c) *ϕ has the form $\langle\langle a\rangle\rangle \gamma$ where γ is an odd-dimensional quadratic form,*
   (d) *there exist $d \notin F^{*2}$, $\pi \in P_2(F)$ and two odd-dimensional quadratic forms γ_1 and γ_2 such that $\pi_{F(\sqrt{d})}$ is anisotropic, the field extension $F(\pi, \sqrt{d})/F$ is universally excellent, and $[\phi] = [\pi\gamma_1] + [\langle\langle d\rangle\rangle \gamma_2]$ (in this case $\dim \phi$ is even and $\det_\pm \phi = d \notin F^{*2}$).*

Remark B.3. We do not know whether there exist d and π (and hence ϕ) as in item (d) of Theorem B.2.

Definition B.4. Let q be a quadratic form and $k \geq 0$. We say that a field extension E/F is *generic in the class of the field extensions over which the Witt index of ϕ is greater or equal to k* (for short (ϕ, k)*-generic*), if the following conditions hold:

1) $i_W(\phi_E) \geq k$,

2) for any field extension K/F with $i_W(\phi_K)_{\mathrm{an}} \geq k$ and for any free composite EK of the fields E and K over F, the field extension KE/K is purely transcendental.

Lemma B.5. *Let q be a quadratic form and k be a positive integer. Let E_1/F and E_2/F be (ϕ, k)-generic field extensions. Then $E_1/F \overset{\mathrm{st}}{\sim} E_2/F$.*

Proof. By Definition B.4, E_1E_2/E_1 and E_1E_2/E_2 are purely transcendental. Hence $E_1/F \overset{\mathrm{st}}{\sim} E_2/F$. □

Proposition B.6 (see [24, Cor. 3.9 and Prop. 5.13])**.** *Let*

$$F_0, F_1, \ldots, F_h$$

be the generic splitting tower of a quadratic form ϕ. Let $k_s = i_W(\phi_{F_s})$ $(0 \leq s \leq h)$. Then the field extension F_s/F is (ϕ, k_s)-generic.

Theorem B.7 (see [16, Th. 1.1])**.** *Let ϕ be an anisotropic form over F. The field extension $F(\phi)/F$ is universally excellent if and only if $\dim \phi \leq 3$ or $\phi \in GP_2(F)$.*

Lemma B.8. *Let ϕ be a non-hyperbolic quadratic form over F and let*

$$F_0, F_1, \ldots, F_h$$

be the generic splitting tower of ϕ. Let r be an integer such that $0 < r \leq h$. Suppose that the field extension F_r/F is universally excellent. Then

1) *for any s with $0 \leq s \leq r$, the field extension F_r/F_s is universally excellent;*
2) *$r = h$ and $\deg \phi \leq 2$.*

Proof. 1). Let $k = i_W(\phi_{F_r})$. By Proposition B.6, both field extensions F_rF_s/F_s and F_r/F_s are (ϕ_{F_s}, k)-generic, where F_rF_s denotes a *free* composite of F_r and F_s over F. By Lemma B.5, we have $F_rF_s/F_s \overset{\mathrm{st}}{\sim} F_r/F_s$.

Since F_r/F is universally excellent it follows that F_rF_s/F_s is universally excellent as well. Since $F_rF_s/F_s \overset{\mathrm{st}}{\sim} F_r/F_s$, it follows that F_r/F_s is universally excellent.

2). Since F_r/F is universally excellent, it follows that F_r/F_{r-1} is universally excellent. Let $\phi_{r-1} = (\phi_{F_{r-1}})_{\mathrm{an}}$. We see that $F_{r-1}(\phi_{r-1})/F_{r-1}$ is universally excellent. It follows from Theorem B.7, that either $\dim \phi_{r-1} \leq 3$ or $\phi_{r-1} \in GP_2(F_{r-1})$. In both cases $\dim \phi_r \leq 1$. Hence, $r = h$. Since $\dim \phi_{h-1} = \dim \phi_{r-1} \leq 4$, it follows that $\deg \phi \leq 2$. □

Definition B.9. Let ϕ be a quadratic form over F and $F_0, F_1, \ldots, F_h$ be the generic splitting tower of ϕ. We denote by F_ϕ the field F_h. For any field extension E/F, we set $E_\phi \overset{\mathrm{def}}{=} E_{\phi_E}$.

Lemma B.10. *Let ϕ be a quadratic form over F and E/F be a field extension. Then $EF_\phi/E \overset{\mathrm{st}}{\sim} E_\phi/E$.*

Proof. Let $k = [\dim \phi/2]$. The field extensions EF_ϕ/E and E_ϕ/E are (ϕ_E, k)-generic. Lemma B.5 completes the proof. □

Corollary B.11. *Let ϕ be a quadratic form over F and E/F be a field extension. Suppose that the field extension F_ϕ/F is universally excellent. Then E_ϕ/E is universally excellent.* □

Corollary B.12. *Let $\phi \in I^3(F)$ be a quadratic form such that the field extension F_ϕ/F is universally excellent. Then ϕ is hyperbolic.*

Proof. Suppose that ϕ is not hyperbolic. Since $\phi \in I^3(F)$, we have $\deg(\phi) \geq 3$. This contradicts to Lemma B.8. □

Corollary B.13. *Let ϕ be a quadratic form over F such that F_ϕ/F is universally excellent. Then for any field extension E/F the condition $\phi_E \in I^3(E)$ implies that ϕ_E is hyperbolic.* □

Lemma B.14. *Let ϕ and ψ be quadratic forms over F. The following conditions are equivalent:* 1) $F_\phi \overset{\mathrm{st}}{\sim} F_\psi$; 2) $\dim(\phi_{F_\psi})_{\mathrm{an}} \leq 1$ *and* $\dim(\psi_{F_\phi})_{\mathrm{an}} \leq 1$.

Proof. 1)⇒2). Obvious.

2)⇒1). It follows from Proposition B.6 and Definition B.4 that the field extensions $F_\phi F_\psi/F_\psi$ and $F_\phi F_\psi/F_\phi$ are purely transcendental. Hence $F_\phi \overset{\mathrm{st}}{\sim} F_\psi$. □

Examples B.15. 1). Let ϕ be an odd-dimensional quadratic form. Let $\psi = \phi \perp \langle -\det_\pm \phi\rangle$. Then $F_\phi/F \overset{\mathrm{st}}{\sim} F_\psi/F$.

2). For $i = 1, \dots, n$, let π_i be anisotropic m_i-fold Pfister forms with $m_1 < m_2 < \cdots < m_n$. Let $\gamma_1, \dots, \gamma_n$ be anisotropic odd-dimensional quadratic forms. Let ϕ be a quadratic form such that $[\phi] = [\pi_1\gamma_1] + \cdots + [\pi_n\gamma_n]$. Then $F_\phi/F \overset{\mathrm{st}}{\sim} F(\pi_1, \dots, \pi_n)/F$.

3). Let $\pi \in GP_n(F)$ and let γ be an odd-dimensional quadratic form. Let $\phi = \pi\gamma$. Then $F_\phi/F \overset{\mathrm{st}}{\sim} F_\pi/F$.

Proof. 1). Since $\psi \in I(F)$, it follows that ψ_{F_ψ} is hyperbolic. Consequently $\dim(\phi_{F_\psi})_{\mathrm{an}} = 1$. Since $\dim(\phi_{F_\phi})_{\mathrm{an}} = 1$, we have $\dim(\psi_{F_\phi})_{\mathrm{an}} \leq 2$. It follows from $\psi \in I^2(F)$ that $\dim(\psi_{F_\phi})_{\mathrm{an}} = 0$. By Lemma B.14, we have $F_\phi/F \overset{\mathrm{st}}{\sim} F_\psi/F$.

2). Obviously $\phi_{F(\pi_1,\dots,\pi_n)}$ is hyperbolic. Let $E = F_\phi$. It is sufficient to verify that $(\pi_1)_E, \dots, (\pi_n)_E$ are hyperbolic. Suppose that there is i such that $[(\pi_i)_E] \neq 0$. Let i be the minimal integer such that $[(\pi_i)_E] \neq 0$. Obviously, $[(\pi_i\gamma_i)_E] \equiv [\phi_E] \equiv 0 \pmod{I^{m_i+1}(F)}$. Since $\dim\gamma$ is odd, we have $[(\pi_i)_E] \equiv [(\pi_i\gamma_i)_E] \equiv 0 \pmod{I^{m_i+1}(F)}$. By APH, we have $[(\pi_i)_E] = 0$, a contradiction.

3). It is sufficient to set $n = 1$ in the previous example. □

The following lemma is a consequence of the index reduction formula [34].

Lemma B.16 (see [15, Th. 1.6] or [13, Prop. 2.1])**.** *Let $\phi \in I^2(F)$ be a quadratic form with* $\mathrm{ind}(C(\phi)) \geq 2^r$. *Then there is s $(0 \leq s \leq h(\phi))$ such that* $\dim\phi_s = 2r + 2$ *and* $\mathrm{ind}C(\phi_s) = 2^r$. □

Lemma B.17. *Let $\phi \in I^2(F)$ be a non-hyperbolic quadratic form such that the field F_ϕ is universally excellent. Then* $\mathrm{ind}C(\phi) = 2$.

Proof. By Corollary B.12, we have $\phi \notin I^3(F)$. Hence $\mathrm{ind} C(\phi) \geq 2$. Suppose that $\mathrm{ind} C(\phi) \geq 4$. By Lemma B.16, there is s such that $\dim \phi_s = 6$. Therefore ϕ_s is an anisotropic Albert form. By Lemma B.8, the field extension F_h/F_s is universally excellent. Replacing F and ϕ by F_s and ϕ_s, we can suppose that ϕ is an anisotropic Albert form. Let $A = C(\phi)$. Clearly $F_\phi/F \overset{\mathrm{st}}{\sim} F(\mathrm{SB}(A))/F$. By Theorem 3.2, the field extension $F(\mathrm{SB}(A))/F$ is not universally excellent, a contradiction. □

Proposition B.18. *Let $\phi \in I^2(F)$ be an anisotropic quadratic form. Then the following conditions are equivalent:*

1) *the field extension F_ϕ/F is universally excellent,*
2) *ϕ has the form $\langle\langle a, b\rangle\rangle \mu$, where μ is an odd-dimensional form.*

Proof. 1)⇒2). Suppose that the field extension F_ϕ/F is universally excellent. By Lemma B.17, we have $\mathrm{ind} C(\phi) = 2$. Therefore there exists an anisotropic 2-fold Pfister form $\pi = \langle\langle a, b\rangle\rangle$ such that $[c(\phi)] = [c(\pi)]$. Let $E = F(\pi)$. Obviously $\phi_E \in I^3(E)$. By Corollary B.13, ϕ_E is hyperbolic. Hence there is γ such that $\phi = \langle\langle a, b\rangle\rangle \gamma$. Since $\phi \notin I^3(F)$, $\dim \gamma$ is odd.

2)⇒1). Suppose that $\phi = \langle\langle a, b\rangle\rangle \gamma$, where γ is an odd-dimensional quadratic form. Let $\pi = \langle\langle a, b\rangle\rangle$. By Example B.15, we have $F_\phi/F \overset{\mathrm{st}}{\sim} F_\pi/F$. By Arason's theorem, the field extension F_π/F is universally excellent. Hence F_ϕ/F is universally excellent. □

Proposition B.19. *Let ϕ be an odd-dimensional anisotropic quadratic form. Then the following conditions are equivalent:*

1) *the field extension F_ϕ/F is universally excellent,*
2) *$\phi \perp \langle -\det_\pm \phi\rangle$ has the form $\langle\langle a, b\rangle\rangle \mu$, where μ is an odd-dimensional form.*

Proof. Obvious by virtue of Proposition B.18 and Example B.15. □

Proposition B.20. *Let ϕ be an even-dimensional anisotropic quadratic form with $d = \det_\pm(\phi) \neq 1 \in F^*/F^{*2}$. Then the following conditions are equivalent:*

1) *the field extension F_ϕ/F is universally excellent,*
2) *there exist $\pi \in GP_2(F)$ and odd-dimensional quadratic forms γ_1, γ_2 such that $[\phi] = [\pi\gamma_1] + [\langle\langle d\rangle\rangle \gamma_2]$ and the field extension $F(\pi, \sqrt{d})/F$ is universally excellent.*

Proof. 1)⇒2). Let $L = F(\sqrt{d})$. Since F_ϕ/F is universally excellent, it follows that L_ϕ/L is universally excellent. If ϕ_L is hyperbolic, we set $\pi = 2\mathbb{H}$, what completes the proof. Suppose now that ϕ_L is not hyperbolic. By Lemma B.17, $\mathrm{ind}(C(\phi_L)) = 2$. Since $C(\phi_L)$ is defined over F, it follows that there is $\pi \in GP_2(F)$ such that $C(\phi_L) = C(\pi_L)$. Let $E = L(\pi) = F(\pi, \sqrt{d})$. Since F_ϕ/F is universally excellent, it follows that E_ϕ/E is universally excellent. We have $C(\phi_E) = C(\pi_E) = 0$. Hence $\phi_E \in I^3(E)$. It follows from Corollary B.13 that ϕ_E is hyperbolic. Therefore, $[\phi] \in W(E/F) = [\pi]W(F) + [\langle\langle d\rangle\rangle]W(F)$. Choose γ_1 and γ_2 such that $[\phi] = [\pi\gamma_1] + [\langle\langle d\rangle\rangle \gamma_2]$. Since $\phi \notin I^2(F)$, the dimension of

γ_2 is odd. Since $\operatorname{ind} C(\phi_L) = 2$, the dimension of γ_1 is odd. By Example B.15, we have $F_\phi/F \overset{\mathrm{st}}{\sim} E/F$. Therefore, the field extension $E/F = F(\pi, \sqrt{d})/F$ is universally excellent.

2)⇒1). Obvious in view of Example B.15. □

Theorem B.2 is now an obvious consequence of Lemma B.8 and Propositions B.18, B.19, and B.20. □

Let ϕ be a non-degenerate quadratic form on an F-vector space V and k be a positive integer such that $k \le \frac{1}{2}\dim V = \frac{1}{2}\dim\phi$. Let $X(\phi, k)$ be the variety of totally isotropic subspaces in V of dimension k. It is well-known that $X(\phi, k)$ is geometrically integral if and only if $k \neq \frac{1}{2}\dim\phi$.

Suppose now that $k < \frac{1}{2}\dim\phi$. Clearly, the field extension $F(X(\phi,k))/F$ is (ϕ,k)-generic. Therefore there exists r $(0 \le r \le h = h(\phi))$ such that the field extension $F(X(\phi,k))/F$ is stably isomorphic to F_r/F. Obviously, $r = 0$ if and only if $k \le i_W(\phi)$. In the case where $k > i_W(\phi)$, the integer r is defined by the condition $\dim\phi_{r-1} - 2 \ge \dim\phi - 2k \ge \dim\phi_r$.

Theorem B.21. *Let ϕ be a quadratic form over F and $X(\phi,k)$ be the variety of totally isotropic subspaces of dimension k $(k < \frac{1}{2}\dim\phi)$. The field extension $F(X(\phi,k))/F$ is universally excellent if and only if at least one of the following conditions holds:*

1) $k \le i_W(\phi)$,
2) $k = \frac{1}{2}\dim\phi - 1$ *and ϕ_{an} has the form $\langle\langle a,b\rangle\rangle\,\gamma$, where γ is an odd-dimensional quadratic form,*
3) $k = \frac{1}{2}(\dim\phi - 1)$ *and $\phi_{\mathrm{an}} \perp \langle -\det_\pm \phi\rangle$ has the form $\langle\langle a,b\rangle\rangle\,\gamma$, where γ is an odd-dimensional quadratic form.*

Proof. Let r be such an integer that $F(X(\phi,k)) \overset{\mathrm{st}}{\sim} F_r/F$. If $r = 0$ then $k \le i_W(\phi)$ and the proof is complete. Suppose now that $r > 0$. By Lemma B.8, we have $r = h = h(\phi)$ and $\deg(\phi) \le 2$. Therefore, $\dim\phi - 2k \le \dim\phi_{h-1} - 2 \le 2^{\deg\phi} - 2 \le 2$. By the assumption of the theorem, we have $\dim\phi - 2k > 0$. Therefore, $k = \frac{1}{2}\dim\phi - 1$ or $k = \frac{1}{2}(\dim\phi - 1)$. Since $\dim\phi_{h-1} \ge 2 + (\dim\phi - 2k) \ge 3$, it follows that either $\phi \in I^2(F)$, or $\dim\phi$ is odd. To complete the proof it is sufficient to apply Theorem B.2. □

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