GENERIC TRANSFER FOR GENERAL SPIN GROUPS

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Abstract

We prove Langlands functoriality for the generic spectrum of general spin groups (both odd and even). Contrary to other recent instances of functoriality, our resulting automorphic representations on the general linear group are not self-dual. Together with cases of classical groups, this completes the list of cases of split reductive groups whose L-groups have classical derived groups. The important transfer from GSp_4 *to* GL_4 *follows from our result as a special case.*

Contents

1. Introduction

Let $\mathbf{G}$ be a connected reductive group over a number field k. Let $G = \mathbf{G}(\mathbb{A})$, where $\mathbb{A}$ is the ring of adèles of k. Let ${}^L G$ denote the L-group of $\mathbf{G}$, and fix an embedding

$$\iota : {}^L G \hookrightarrow \mathrm{GL}_N(\mathbb{C}) \rtimes W(\bar{k}/k),$$

where $W(\bar{k}/k)$ is the Weil group of k. Without loss of generality, we may assume that N is minimal. Let $\pi = \bigotimes_v \pi_v$ be an automorphic representation of G. Then for almost all v, the local representation π_v is an unramified representation, and its class is determined by a semisimple conjugacy class $[t_v]$ in ${}^L G$. Here v is a finite place of

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k. Let Π_v be the unramified representation of $\mathrm{GL}_N(k_v)$ determined by the conjugacy class $[\iota(t_v)]$ generated by $\iota(t_v)$. *Langlands's functoriality conjecture* then demands the existence of an automorphic representation $\Pi' = \bigotimes_v \Pi'_v$ of $\mathrm{GL}_N(\mathbb{A})$ such that $\Pi'_v \simeq \Pi_v$ for all the unramified places v.

In this article we prove functoriality in the cases where $\mathbf{G}$ is not classical but the derived group ${}^LG^0_D$ of the connected component of its L-group is. (We follow the convention that a classical group is the stabilizer of a symplectic, orthogonal, or Hermitian nondegenerate bilinear form. Hence, e.g., spin groups are not considered classical.) These groups do not have a useful matrix representation. This fact creates a major difficulty in proving stability of the corresponding root numbers, forcing us to use rather complicated abstract structure theory.

We are mainly concerned with quasi-split groups and those automorphic representations that are induced from generic cuspidal ones. The theory of Eisenstein series reduces our problem to establishing functoriality for generic cuspidal representations of $G = \mathbf{G}(\mathbb{A})$.

The cases when $\mathbf{G}$ is a quasi-split classical group were addressed in [7], [8], and [23], unless $\mathbf{G}$ is a quasi-split special orthogonal group, which should be taken up by the authors of [8].

In this article we establish the functorial transfer of generic cuspidal representations when $\mathbf{G} = \mathrm{GSpin}_m$, the split general spin group of semisimple rank $n = [m/2]$. These groups are split reductive linear algebraic groups of type B_n or D_n whose derived groups are double coverings of split special orthogonal groups. Moreover, the connected component of their Langlands dual groups are ${}^LG^0 = \mathrm{GSp}_{2n}(\mathbb{C})$ or $\mathrm{GSO}_{2n}(\mathbb{C})$, respectively. Then ${}^LG = \mathrm{GSO}_{2n}(\mathbb{C}) \times W(\bar{k}/k)$ or $\mathrm{GSp}_{2n}(\mathbb{C}) \times W(\bar{k}/k)$, according to whether m is even or odd. The map ι is the natural embedding. Observe that ${}^LG^0_D$ is now a classical group and that these groups are precisely the ones for which $\mathbf{G}^0_D$ is not classical, but ${}^LG^0_D$ is. The transfer is to the space of automorphic representations of $\mathrm{GL}_{2n}(\mathbb{A})$.

It is predicted by the theory of (twisted) endoscopy of Kottwitz and Shelstad [28] and Langlands and Shelstad [31] that the representations of $\mathrm{GL}_{2n}(\mathbb{A})$ which are in the image of this transfer must satisfy

$$\Pi = \widetilde{\Pi} \otimes \omega \tag{1}$$

for some Hecke character ω, where $\widetilde{\Pi}$ denotes the contragredient of Π. If ω_Π is the central character of Π, this implies that ω_Π/ω^n must be a quadratic character μ of $k^\times\backslash\mathbb{A}^\times$. Each μ then determines a quadratic extension of k via class field theory, and the group $\mathbf{G}$ which has transfers to automorphic representations of the type just mentioned is the quasi-split form GSpin^*_{2n} of GSpin_{2n} associated to the quadratic extension. The split case then corresponds to $\mu \equiv 1$, which is the content of the present article.

If a representation π of $\mathrm{GSpin}_{2n}^*(\mathbb{A})$ with central character ω_π transfers to Π on $\mathrm{GL}_{2n}(\mathbb{A})$ satisfying $\Pi \simeq \widetilde{\Pi} \otimes \omega$ for some Hecke character ω, then $\omega = \omega_\pi$ and $\omega_\Pi = \omega_\pi^n \mu$, where μ is the quadratic Hecke character associated with the quasi-split GSpin_{2n}^*. While we are not able to show that every Π satisfying (1) is the transfer of an automorphic representation π, we do show that our transfers satisfy (1). (In fact, we prove that Π is *nearly equivalent* to $\widetilde{\Pi} \otimes \omega$ for now; see Theorem 1.1.)

We should note here that if Π is an automorphic transfer to $\mathrm{GL}_{2n+1}(\mathbb{A})$ satisfying (1), then $\omega = \theta^2$ for some θ, and $\Pi \otimes \theta^{-1}$ is then self-dual. Therefore, this is already subsumed in the self-dual case, which is a case of standard twisted endoscopy. On the other hand, the case of $\mathrm{GL}_{2n}(\mathbb{A})$ discussed above is an example of the most general form of transfer that twisted endoscopy can handle.

As explained earlier, in this article we prove Langlands's functoriality conjecture in the form discussed for all generic cuspidal representations of split $\mathrm{GSpin}_m(\mathbb{A})$. In other words, we establish generic transfer from $\mathrm{GSpin}_m(\mathbb{A})$ to $\mathrm{GL}_{2n}(\mathbb{A})$. Extension of this transfer to the nongeneric case requires either the use of models other than Whittaker models or that of Arthur's twisted trace formula. As far as we know, new models for these groups have not been developed, and the fact that these groups are not classical may make matters complicated. On the other hand, the use of Arthur's twisted trace formula depends at present on the validity of the fundamental lemmas that are not available for these groups. We refer to [2] for information on the case of GSp_4.

To state our main theorem, fix a Borel subgroup $\mathbf{B}$ in $\mathbf{G}$ with a maximal (split) torus $\mathbf{T}$, and denote the unipotent radical of $\mathbf{B}$ by $\mathbf{U}$. Let ψ be a nontrivial continuous additive character of $k\backslash\mathbb{A}$. As usual, we use ψ and a fixed splitting (i.e., the choice of Borel subgroup above along with a collection of root vectors, one for each simple root of $\mathbf{T}$; e.g., see [28, page 13]) to define a nondegenerate additive character of $\mathbf{U}(k)\backslash\mathbf{U}(\mathbb{A})$, again denoted by ψ (see also [41, Section 2]).

Let (π, V_π) be an irreducible cuspidal automorphic representation of $\mathbf{G}(\mathbb{A})$. The representation π is said to be *globally generic* if there exists a cusp form $\phi \in V_\pi$ such that

$$\int_{\mathbf{U}(k)\backslash\mathbf{U}(\mathbb{A})} \phi(ng)\psi^{-1}(n)\,dn \neq 0. \tag{2}$$

Note that cuspidal automorphic representations of general linear groups are always globally generic. Two irreducible automorphic representations Π and Π' of $\mathrm{GL}_N(\mathbb{A})$ are said to be *nearly equivalent* if there is a finite set of places T of k such that $\Pi_v \simeq \Pi'_v$ for all $v \notin T$. Our main result is the following.

THEOREM 1.1
Let k be a number field, and let $\pi = \bigotimes \pi_v$ be an irreducible globally generic cuspidal automorphic representation of either $\mathrm{GSpin}_{2n+1}(\mathbb{A})$ *or* $\mathrm{GSpin}_{2n}(\mathbb{A})$*. Let S be a nonempty finite set of non-Archimedean places v such that for $v \notin S$, we have that π_v and ψ_v are unramified. Then there exists an automorphic representation $\Pi = \bigotimes \Pi_v$ of* $\mathrm{GL}_{2n}(\mathbb{A})$ *such that for all places $v \notin S$, the homomorphism parameterizing the local representation Π_v is given by*

$$\Phi_v = \iota \circ \phi_v : W_v \longrightarrow \mathrm{GL}_{2n}(\mathbb{C}),$$

where W_v denotes the local Weil group of k_v and $\phi_v : W_v \longrightarrow {}^L G^0$ is the homomorphism parameterizing π_v. Moreover, if ω_Π and ω_π denote the central characters of Π and π, respectively, then $\omega_\Pi = \omega_\pi^n$. Furthermore, if v is an Archimedean place or a non-Archimedean place with $v \notin S$, then $\Pi_v \simeq \widetilde{\Pi}_v \otimes \omega_{\pi_v}$. In particular, the representations Π and $\widetilde{\Pi} \otimes \omega_\pi$ are nearly equivalent.

At the non-Archimedean places v where π_v is unramified with the semisimple conjugacy class $[t_v]$ in ${}^L G^0$ as its Frobenius-Hecke (or Satake) parameter, this amounts to the fact that the local representation Π_v is the unramified irreducible admissible representation determined by the conjugacy class generated by $\iota(t_v)$ in $\mathrm{GL}_{2n}(\mathbb{C})$. At the Archimedean places, the existence of ϕ_v is contained in [30].

Our method of proof is that of applying an appropriate version of the Converse Theorems (see [9], [11]) to a family of L-functions whose required properties, except for one, are proved in [39], [15], [24], and [20]. The exception, that is, the main stumbling block for applying the converse theorem, is that of stability of certain root numbers under highly ramified twists. In [41] the root numbers, or more precisely the inverses of the local coefficients, were expressed as a Mellin transform of certain Bessel functions. Applying this to our case requires a good amount of development and calculations. This is particularly important since the necessary Bruhat decompositions for these groups are more complicated than for the classical groups. For that we have to resort to using the abstract theory of roots which is harder since no reasonable matrix representation is available for these groups. Moreover, the main theorem in [41] is based on certain assumptions whose verification requires our calculations.

The fact that GSpin_{2n} has a disconnected center makes matters even more complicated. This leads us to use an extended group $\mathrm{GSpin}_{2n}^{\sim}$ of GSpin_{2n}, so that our proof of stability proceeds smoothly.

There are two important transfers that are special cases of this theorem. The first is the generic transfer from $\mathrm{GSp}_4 = \mathrm{GSpin}_5$ to GL_4 whose proof, as far as we know, has never been published before. We should point out that even the unpublished proofs of this result are based on methods that are fairly disjoint from ours. We finally remark that our result in this case also gives an immediate proof of the holomorphy of spinor L-functions for generic cusp forms on GSp_4 (see Remark 7.9).

The second special case is when $\mathbf{G} = \mathrm{GSpin}_6$. In this case our transfer gives the exterior square transfer from GL_4 to GL_6 due to H. H. Kim [22] which, when composed with the symmetric cube of a cuspidal representation on $\mathrm{GL}_2(\mathbb{A})$, leads to its symmetric fourth.

The issue of whether the local components of the transfer at places in S are the "correct" ones (*strong transfer*) has been dealt with for classical groups thanks to the existence of the theory of descent from GL_n to classical groups (see [16], [42]). The analogous results for our cases have to wait until the descent or other techniques are established for representations of $\mathrm{GL}_{2n}(\mathbb{A})$ which satisfy (1). We should note here that, contrary to the case of general linear groups, the local Langlands conjecture is not yet available for the groups we deal with in the present article. What we mean by the correct local component is that for places $v \in S$ we do get the local transfer, which can be defined, at least for generic local representations, using γ-factors of representations of these groups twisted by those of general linear groups (see [8, Section 7]).

Further applications such as global estimates toward the Ramanujan conjecture as well as some of the other applications established in [7] and [8] will be addressed in future articles. As mentioned earlier, the cases of quasi-split GSpin groups will be the subject matter of our next article.

Here is an outline of the contents of each section. In Section 2 we review the structure theory of the groups involved in this article. In particular, we give a detailed description of the root data for GSpin groups and their extensions. We then prove the necessary analytic properties of local L-functions in Section 3. In particular, we discuss the Standard Module Conjecture, which is another local ingredient. In Section 4 we go on to prove the most crucial local result, stability of γ-factors under twists by highly ramified characters. This is where we do the calculations with root data mentioned above and use the extended group. We then prove the necessary analytic properties of the global L-functions in Section 5 which prepares us to apply the converse theorem in Section 6. In Section 7 we include the special cases mentioned above along with some other local and global consequences of the main theorem.

2. Structure theory

We review the structure theory for the families of algebraic groups relevant to the current work, namely, GSpin_{2n+1} and GSpin_{2n}, as well as their duals GSp_{2n} and GSO_{2n}. We also introduce the group $\mathrm{GSpin}^{\sim}_{2n}$, which is closely related to GSpin_{2n}. It shares the same derived group with GSpin_{2n}. However, contrary to GSpin_{2n}, which has disconnected center, the center of $\mathrm{GSpin}^{\sim}_{2n}$ is connected. We need this group for our purposes, as we explain later.

2.1. Root data for GSpin *groups*

We first describe the algebraic group $\mathbf{G} = \mathrm{GSpin}_m$, $m = 2n+1$ or $2n$, and its standard Levi subgroups in terms of their root data. We rely heavily on these descriptions in the computations of Section 4.

The group GSpin_m is the quotient of $\mathrm{GL}_1 \times \mathrm{Spin}_m$ by a central subgroup of order 2 (see Proposition 2.2).

PROPOSITION 2.1

The root datum of GSpin_m *can be described as the following. Let*

$$X = \mathbb{Z}e_0 \oplus \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_n,$$
$$X^\vee = \mathbb{Z}e_0^* \oplus \mathbb{Z}e_1^* \oplus \cdots \oplus \mathbb{Z}e_n^*,$$

and let $\langle\, , \rangle$ *be the standard* $\mathbb{Z}$*-pairing on* $X \times X^\vee$*. The root datum for* GSpin_m *is* $(X, R, X^\vee, R^\vee)$ *with* R *and* $R^\vee$ *generated, respectively, by*

$$\Delta = \{\alpha_1 = e_1 - e_2, \alpha_2 = e_2 - e_3, \ldots, \alpha_{n-1} = e_{n-1} - e_n, \alpha_n = e_n\},$$
$$\Delta^\vee = \{\alpha_1^\vee = e_1^* - e_2^*, \alpha_2^\vee = e_2^* - e_3^*, \ldots, \alpha_{n-1}^\vee = e_{n-1}^* - e_n^*, \alpha_n^\vee = 2e_n^* - e_0^*\},$$

if $m = 2n+1$ *and by*

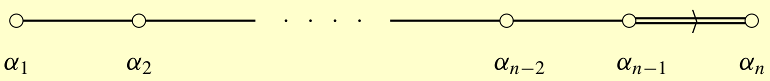
$$\Delta = \{\alpha_1 = e_1 - e_2, \ldots, \alpha_{n-1} = e_{n-1} - e_n, \alpha_n = e_{n-1} + e_n\},$$
$$\Delta^\vee = \{\alpha_1^\vee = e_1^* - e_2^*, \ldots, \alpha_{n-1}^\vee = e_{n-1}^* - e_n^*, \alpha_n^\vee = e_{n-1}^* + e_n^* - e_0^*\},$$

if $m = 2n$.

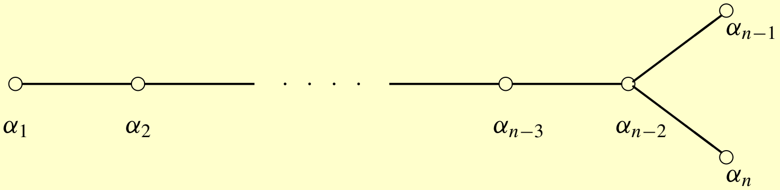
Proof

See [3, Section 2]. □

In the odd case, $\mathbf{G}$ has a Dynkin diagram of type B_n:



In the even case, it has a Dynkin diagram of type D_n:



PROPOSITION 2.2
The derived group of $\mathbf{G}$ *is isomorphic to* Spin_{2n+1} *or* Spin_{2n}*, the double coverings, as algebraic groups, of special orthogonal groups. In fact,* $\mathbf{G}$ *is isomorphic to*

$$(\mathrm{GL}_1 \times \mathrm{Spin}_m)/\big\{(1, 1), (-1, c)\big\},$$

where $c = \alpha_n{}^\vee(-1)$ *if* $m = 2n + 1$ *and* $c = \alpha_{n-1}{}^\vee(-1)\alpha_n{}^\vee(-1)$ *if* $m = 2n$*. The dual of* $\mathbf{G}$ *is* GSp_{2n} *if* $m = 2n + 1$ *and* GSO_{2n} *if* $m = 2n$*.*

Moreover, if $\mathbf{M}$ *is the Levi component of a maximal standard parabolic subgroup of* $\mathbf{G}$*, then* $\mathbf{M}$ *is isomorphic to* $\mathrm{GL}_k \times \mathrm{GSpin}_{m-2k}$ *with* $k = 1, 2, \ldots, n$ *if* $m = 2n + 1$ *and* $k = 1, 2, \ldots, n - 2, n$ *if* $m = 2n$*.*

Proof
See [3, Section 2]. □

We can also describe the Levi subgroup $\mathbf{M}$ in terms of its root datum. Without loss of generality, we may assume that $\mathbf{M}$ is maximal. Obviously, $\mathbf{M}$ has the same character and cocharacter lattices as those of $\mathbf{G}$. Denote the set of roots of $\mathbf{M}$ by $R_{\mathbf{M}}$, and denote its coroots by $R^\vee_{\mathbf{M}}$. They are generated by $\Delta - \{\alpha\}$ and $\Delta^\vee - \{\alpha^\vee\}$, respectively, where $\alpha = \alpha_k$ unless $m = 2n$ and $k = n$, in which case α can be of either α_n or α_{n-1} (resulting in two nonconjugate isomorphic Levi components). In the sequel, the case of $m = 2n$ and $k = n - 1$ is therefore always ruled out, and we do not repeat this again.

PROPOSITION 2.3
(a) *The center of* $\mathbf{G}$ *is given by*

$$\mathbf{Z_G} = \begin{cases} \mathbf{A}_0 & \text{if } m = 2n + 1, \\ \mathbf{A}_0 \cup (\zeta_0 \mathbf{A}_0) & \text{if } m = 2n, \end{cases}$$

where

$$\mathbf{A}_0 = \big\{e_0^*(\lambda) : \lambda \in \mathrm{GL}_1\big\}$$

and $\zeta_0 = e_1^*(-1)e_2^*(-1)\cdots e_n^*(-1)$*.*

(b) *The center of* $\mathbf{M}$ *is given by*

$$\mathbf{Z_M} = \begin{cases} \mathbf{A}_k & \text{if } m = 2n+1, \\ \mathbf{A}_k \cup (\zeta_k \mathbf{A}_k) & \text{if } m = 2n, \end{cases}$$

where

$$\mathbf{A}_k = \left\{ e_0^*(\lambda) e_1^*(\mu) e_2^*(\mu) \cdots e_k^*(\mu) \ : \ \lambda, \mu \in \mathrm{GL}_1 \right\}$$

and $\zeta_k = e_{k+1}^*(-1) e_{k+2}^*(-1) \cdots e_n^*(-1)$.

Proof

The maximal torus $\mathbf{T}$ of $\mathbf{G}$ (or $\mathbf{M}$) consists of elements of the form

$$t = \prod_{j=0}^{n} e_j^*(t_j)$$

with $t_j \in \mathrm{GL}_1$. Now t is in the center of $\mathbf{G}$, respectively, $\mathbf{M}$, if and only if it belongs to the kernel of all simple roots of $\mathbf{G}$, respectively, $\mathbf{M}$. For $\mathbf{G}$, this leads to

$$\frac{t_1}{t_2} = \frac{t_2}{t_3} = \cdots = \frac{t_{n-1}}{t_n} = t_n = 1$$

if $m = 2n+1$ and

$$\frac{t_1}{t_2} = \frac{t_2}{t_3} = \cdots = \frac{t_{n-1}}{t_n} = t_{n-1} t_n = 1$$

if $m = 2n$. For $\mathbf{M}$, we get

$$\frac{t_1}{t_2} = \frac{t_2}{t_3} = \cdots = \frac{t_{k-1}}{t_k}, \qquad \frac{t_{k+1}}{t_{k+2}} = \cdots = \frac{t_{n-1}}{t_n} = t_n = 1$$

if $m = 2n+1$ and

$$\frac{t_1}{t_2} = \frac{t_2}{t_3} = \cdots = \frac{t_{k-1}}{t_k}, \qquad \frac{t_{k+1}}{t_{k+2}} = \cdots = \frac{t_{n-1}}{t_n} = t_{n-1} t_n = 1$$

if $m = 2n$. These relations prove the proposition. □

Remark 2.4

When $m = 2n$, the nonidentity component of $\mathbf{Z_G}$ can also be written as $z' \mathbf{A}_0$, where z' is a nontrivial element in the center of Spin_{2n}, the derived group of $\mathbf{G}$. We now specify this element explicitly in terms of the central element z of [3, Proposition 2.2]. There is a typographical error in the description of z in that article which we correct here:

$$z = \begin{cases} \prod_{j=1}^{n-2} \alpha_j{}^\vee\big((-1)^j\big) \cdot \alpha_{n-1}{}^\vee(-1) & \text{if } n \text{ is even,} \\ \prod_{j=1}^{n-2} \alpha_j{}^\vee\big((-1)^j\big) \cdot \alpha_{n-1}{}^\vee(-\sqrt{-1})\alpha_n{}^\vee(\sqrt{-1}) & \text{if } n \text{ is odd.} \end{cases}$$

To compute z', note that with $m = 2n$ we have

$$e_1^* + \cdots + e_{n-1}^* + e_n^* = \sum_{j=1}^{n-2} j\alpha_j{}^\vee + \Big(\frac{n}{2} - 1\Big)\alpha_{n-1}{}^\vee + \frac{n}{2}\alpha_n{}^\vee + \frac{n}{2}e_0^*,$$

which, when evaluated as a character at (-1), yields

$$\zeta_0 = \begin{cases} z & \text{if } n = 4p, \\ ze_0^*(\sqrt{-1}) & \text{if } n = 4p+1, \\ cze_0^*(-1) & \text{if } n = 4p+2, \\ cze_0^*(-\sqrt{-1}) & \text{if } n = 4p+3. \end{cases}$$

Therefore, $\zeta_0 \mathbf{A}_0 = z'\mathbf{A}_0$, where z' is an element in the center of Spin_{2n} given by

$$z' = \begin{cases} z & \text{if } n \equiv 0, 1 \bmod 4, \\ cz & \text{if } n \equiv 2, 3 \bmod 4. \end{cases}$$

2.2. Root data for $\mathrm{GSpin}^\sim$ *groups*

We describe the structure theory for the group $\mathrm{GSpin}_{2n}^\sim$ as well as its standard Levi subgroups in this section. For our future discussion on stability of γ-factors in Section 4, we need to work with a group with a connected center. The center of GSpin_{2n} is not connected, as we observed in Proposition 2.3. To remedy this, we define a new group that is just GSpin_{2n} extended by a one-dimensional torus. This group has a connected center, while its derived group remains the same as that of GSpin_{2n}, that is, Spin_{2n}. This allows us to work with $\mathrm{GSpin}_{2n}^\sim$, as we explain in Section 4.

Definition 2.5

Let $\mathrm{GSpin}_{2n}^\sim$ be the group

$$(\mathrm{GL}_1 \times \mathrm{GSpin}_{2n})/\big\{(1, 1), (-1, \zeta_0)\big\},$$

where ζ_0 is as in Proposition 2.3. Note that the derived group of $\mathrm{GSpin}_{2n}^\sim$ is again Spin_{2n}.

PROPOSITION 2.6
Let

$$X = \mathbb{Z}E_{-1} \oplus \mathbb{Z}E_0 \oplus \mathbb{Z}E_1 \oplus \cdots \oplus \mathbb{Z}E_n,$$
$$X^\vee = \mathbb{Z}E^*_{-1} \oplus \mathbb{Z}E^*_0 \oplus \mathbb{Z}E^*_1 \oplus \cdots \oplus \mathbb{Z}E^*_n,$$

and let $\langle\, , \,\rangle$ *be the standard* $\mathbb{Z}$*-pairing on* $X \times X^\vee$*. Then* $(X, R, X^\vee, R^\vee)$ *is the root datum for* $\mathrm{GSpin}^{\sim}_{2n}$ *with* R *and* $R^\vee$ *generated, respectively, by*

$$\Delta = \{\alpha_1 = E_1 - E_2, \ldots, \alpha_{n-1} = E_{n-1} - E_n, \alpha_n = E_{n-1} + E_n - E_{-1}\}$$

and

$$\Delta^\vee = \{\alpha_1^\vee = E^*_1 - E^*_2, \ldots, \alpha_{n-1}^\vee = E^*_{n-1} - E^*_n, \alpha_n^\vee = E^*_{n-1} + E^*_n - E^*_0\}.$$

Proof
Our proof is similar to the proof of [3, Proposition 2.4]. We compute the root datum of $\mathrm{GSpin}^{\sim}_{2n}$ using that of GSpin_{2n} described earlier, and we verify that it can be written as above.

Start with the character lattice of $\mathrm{GL}_1 \times \mathrm{GSpin}_{2n}$ which can be written as the $\mathbb{Z}$-span of $e_0, e_1, \ldots, e_n$, and e_{-1}. Here, e_{-1} is a generator for the character lattice of the GL_1 factor. Now, characters of $\mathrm{GSpin}^{\sim}_{2n}$ are those characters of $\mathrm{GL}_1 \times \mathrm{GSpin}_{2n}$ which are trivial when evaluated at the element $(-1, \zeta_0)$. Note that $e_i(\zeta_0) = -1$ for $1 \le i \le n$, $e_0(\zeta_0) = 1$, and $e_{-1}(-1) = -1$. This implies that the character lattice of $\mathrm{GSpin}^{\sim}_{2n}$ can be written as the $\mathbb{Z}$-span of $2e_{-1}, e_0, e_1 + e_{-1}, \ldots, e_n + e_{-1}$. Now, set $E_{-1} = 2e_{-1}$, $E_0 = e_0$, and $E_i = e_{-1} + e_i$ for $1 \le i \le n$. We can compute a basis for the cocharacter lattice using the $\mathbb{Z}$-pairing of the root datum. It turns out to consist of $E^*_{-1} = e^*_{-1}/2 - (e^*_1 + \cdots + e^*_n)/2$, $E^*_0 = e^*_0$, and $E^*_i = e^*_i$ for $1 \le i \le n$. Writing the simple roots and coroots in terms of the new bases finishes the proof. For example,

$$\alpha_n = e_{n-1} + e_n = (e_{n-1} + e_{-1}) + (e_n + e_{-1}) - 2e_{-1} = E_{n-1} + E_n - E_{-1}$$

and

$$\alpha_n{}^\vee = e^*_{n-1} + e^*_n - e^*_0 = E^*_{n-1} + E^*_n - E^*_0. \qquad \square$$

We can also describe the root datum of any standard Levi subgroup $\mathbf{M}$ in $\mathrm{GSpin}^{\sim}_{2n}$. Again, without loss of generality, we may assume that $\mathbf{M}$ is maximal. Similar to the case of GSpin_{2n}, the roots and coroots of $\mathbf{M}$ are, respectively, generated by $\Delta - \{\alpha_k\}$ and $\Delta^\vee - \{\alpha_k{}^\vee\}$ for some k. The character and cocharacter lattices are the same as those of $\mathrm{GSpin}^{\sim}_{2n}$.

PROPOSITION 2.7

(a) *The center of* $\mathrm{GSpin}^{\sim}_{2n}$ *is given by*

$$\left\{E_0^*(\mu)E_1^*(\lambda)E_2^*(\lambda)\cdots E_n^*(\lambda)E_{-1}^*(\lambda^2)\ :\ \lambda,\mu\in \mathrm{GL}_1\right\},$$

and it is hence connected.

(b) *The center of* $\mathbf{M}$ *is given by*

$$\left\{E_0^*(\mu)E_1^*(\nu)\cdots E_k^*(\nu)E_{k+1}^*(\lambda)\cdots E_n^*(\lambda)E_{-1}^*(\lambda^2)\ :\ \lambda,\mu,\nu\in \mathrm{GL}_1\right\},$$

and it is hence connected.

Proof

The maximal torus of $\mathrm{GSpin}^{\sim}_{2n}$ (or $\mathbf{M}$) consists of elements of the form

$$t=\prod_{j=-1}^{n} E_j^*(t_j)$$

with $t_j\in\mathrm{GL}_1$. Now t is in the center of $\mathbf{G}$, respectively, $\mathbf{M}$, if and only if it belongs to the kernel of all simple roots of $\mathbf{G}$, respectively, $\mathbf{M}$. For $\mathbf{G}$, this leads to

$$\frac{t_1}{t_2}=\frac{t_2}{t_3}=\cdots=\frac{t_{n-1}}{t_n}=\frac{t_{n-1}t_n}{t_{-1}}=1.$$

For $\mathbf{M}$, we get

$$\frac{t_1}{t_2}=\frac{t_2}{t_3}=\cdots=\frac{t_{k-1}}{t_k},\qquad \frac{t_{k+1}}{t_{k+2}}=\cdots=\frac{t_{n-1}}{t_n}=\frac{t_{n-1}t_n}{t_{-1}}=1.$$

These relations prove the proposition. □

We also describe the structure of standard Levi subgroups in $\mathrm{GSpin}^{\sim}_{2n}$.

PROPOSITION 2.8

The standard Levi subgroups of $\mathrm{GSpin}^{\sim}_{2n}$ *are isomorphic to*

$$\mathrm{GL}_{k_1}\times\cdots\times\mathrm{GL}_{k_r}\times\mathrm{GSpin}^{\sim}_{2l},$$

where $k_1+\cdots+k_r+l=n$.

Proof

Without loss of generality, we may assume that $\mathbf{M}$ is maximal. The character and cocharacter lattices of $\mathbf{M}$, which are the same as those of $\mathbf{G}$, were described in

Proposition 2.6 and can be written as

$$(\mathbb{Z}E_1 \oplus \cdots \oplus \mathbb{Z}E_k) \oplus (\mathbb{Z}E_{-1} \oplus \mathbb{Z}E_0 \oplus \mathbb{Z}E_{k+1} \oplus \cdots \oplus \mathbb{Z}E_n)$$

and

$$(\mathbb{Z}E_1^* \oplus \cdots \oplus \mathbb{Z}E_k^*) \oplus (\mathbb{Z}E_{-1}^* \oplus \mathbb{Z}E_0^* \oplus \mathbb{Z}E_{k+1}^* \oplus \cdots \oplus \mathbb{Z}E_n^*),$$

respectively. This, along with the description of roots and coroots of **M** given above, implies that the root datum of **M** can be written as a direct sum of two root data. The first one is now the well-known root datum of GL_k, and the second is just our earlier description of the root datum of $\mathrm{GSpin}^{\sim}_{2(n-k)}$. Therefore, **M** is isomorphic to $\mathrm{GL}_k \times \mathrm{GSpin}^{\sim}_{2(n-k)}$. □

2.3. Root data for GSp_{2n} *and* GSO_{2n}

We describe the root data for the two groups GSp_{2n} and GSO_{2n} in detail. Since these two groups are usually introduced as matrix groups, we also describe the root data in terms of their usual matrix representations. It is evident from this description that the two groups GSpin_{2n+1} and GSp_{2n}, as well as GSpin_{2n} and GSO_{2n}, are pairs of connected reductive algebraic groups with dual root data.

Consider the group defined as

$$\left\{g \in \mathrm{GL}_{2n} \,:\, {}^t g J g = \mu(g) J\right\},$$

where the $(2n \times 2n)$-matrix J is defined via

$$J = \begin{pmatrix} & & & & & 1 \\ & & & & \iddots & \\ & & & 1 & & \\ & & -1 & & & \\ & \iddots & & & & \\ -1 & & & & & \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} & & & & & 1 \\ & & & & \iddots & \\ & & & 1 & & \\ & & 1 & & & \\ & \iddots & & & & \\ 1 & & & & & \end{pmatrix},$$

respectively. The former is the connected reductive algebraic group GSp_{2n}. However, the latter is not connected as an algebraic group. This group is sometimes denoted by GO_{2n} (see [34, Section 2]). Its connected component is the group GSO_{2n} (also denoted by SGO_{2n}).

PROPOSITION 2.9

The root datum of the groups GSp_{2n} *and* GSO_{2n} *can be described as follows. Let*

$$X = \mathbb{Z}e_0 \oplus \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_n,$$

$$X^\vee = \mathbb{Z}e_0^* \oplus \mathbb{Z}e_1^* \oplus \cdots \oplus \mathbb{Z}e_n^*,$$

and let $\langle\ ,\ \rangle$ be the standard $\mathbb{Z}$-pairing on $X \times X^\vee$. Then $(X, R, X^\vee, R^\vee)$ is the root datum for the connected reductive algebraic group GSp_{2n} *or* GSO_{2n} *with R and $R^\vee$ generated, respectively, by*

$$\Delta = \{\alpha_1 = e_1 - e_2, \alpha_2 = e_2 - e_3, \ldots, \alpha_{n-1} = e_{n-1} - e_n, \alpha_n = 2e_n - e_0\},$$
$$\Delta^\vee = \{\alpha_1^\vee = e_1^* - e_2^*, \alpha_2^\vee = e_2^* - e_3^*, \ldots, \alpha_{n-1}^\vee = e_{n-1}^* - e_n^*, \alpha_n^\vee = e_n^*\},$$

for GSp_{2n} *and*

$$\Delta = \{\alpha_1 = e_1 - e_2, \ldots, \alpha_{n-1} = e_{n-1} - e_n, \alpha_n = e_{n-1} + e_n - e_0\},$$
$$\Delta^\vee = \{\alpha_1^\vee = e_1^* - e_2^*, \ldots, \alpha_{n-1}^\vee = e_{n-1}^* - e_n^*, \alpha_n^\vee = e_{n-1}^* + e_n^*\},$$

for GSO_{2n}.

Proof
See [45, pages 133 – 136] for GSp_{2n}. Similar computations work for GSO_{2n}. □

The Dynkin diagrams are of type C_n and D_n, respectively. A computation similar to the proof of Proposition 2.3 proves the following.

PROPOSITION 2.10
Let $\mathbf{G}$ *be either* GSp_{2n} *or* GSO_{2n}. *Then the center of* $\mathbf{G}$ *is given by*

$$\mathbf{Z} = \left\{ e_0^*(\lambda^2)\, e_1^*(\lambda) \cdots e_n^*(\lambda) : \lambda \in \mathrm{GL}_1 \right\}.$$

The maximal split torus in both GSp_{2n} and GSO_{2n} can be described as

$$\widehat{T} = \left\{ t(a_1, \ldots, a_n, b_n, \ldots, b_1) = \begin{pmatrix} a_1 & & & & & \\ & \ddots & & & & \\ & & a_n & & & \\ & & & b_n & & \\ & & & & \ddots & \\ & & & & & b_1 \end{pmatrix} : a_i b_i = \mu \right\}. \tag{3}$$

We can now describe e_i and e_i^* in terms of matrices. In either case, we have

$$\begin{aligned} &e_0(t) = \mu, e_0^*(\lambda) = t(1, \ldots, 1, \lambda, \ldots, \lambda), \\ &e_i(t) = a_i, e_i^*(\lambda) = t(1, \ldots, 1, \underset{\substack{\uparrow \\ i}}{\lambda}, 1, \ldots, 1, \underset{\substack{\uparrow \\ 2n+1-i}}{\lambda^{-1}}, 1, \ldots, 1), \quad 1 \le i \le n. \end{aligned} \tag{4}$$

3. Analytic properties of local L-functions

Let F denote a local field of characteristic zero, either Archimedean or non-Archimedean. Let $\mathbf{G}_n$ denote the algebraic group GSpin_{2n+1} (resp., GSpin_{2n}), and let σ be an irreducible admissible generic representation of $M = \mathbf{M}(F)$ in $G = \mathbf{G}_{r+n}(F)$, where $\mathbf{M} \simeq \mathrm{GL}_r \times \mathbf{G}_n$ is the Levi subgroup of a standard parabolic subgroup $\mathbf{P}$ in $\mathbf{G}_{r+n}$. Let $\widehat{M} \simeq \mathrm{GL}_r(\mathbb{C}) \times \mathrm{GSp}_{2n}(\mathbb{C})$ (resp., $\widehat{M} \simeq \mathrm{GL}_r(\mathbb{C}) \times \mathrm{GSO}_{2n}(\mathbb{C})$) be the Levi component of the corresponding standard parabolic subgroup $\widehat{P}$ in the dual group $\widehat{G} = {}^L G^0 = \mathrm{GSp}_{2(n+r)}(\mathbb{C})$ (resp., $\widehat{G} = \mathrm{GSO}_{2(n+r)}(\mathbb{C})$). Let r denote the adjoint action of $\widehat{M}$ on the Lie algebra of the unipotent radical of $\widehat{P}$. Then by [3, Proposition 5.6], $r = r_1 \oplus r_2$ if $n \geq 1$ (resp., $n \geq 2$) with $r_1 = \rho_r \otimes \widetilde{R}$ and $r_2 = \mathrm{Sym}^2 \rho_r \otimes \mu^{-1}$ (resp., $r_2 = \bigwedge^2 \rho_r \otimes \mu^{-1}$). Here, ρ_r denotes the standard representation of $\mathrm{GL}_r(\mathbb{C})$, $\widetilde{R}$ denotes the contragredient of the standard representation of $\mathrm{GSp}_{2n}(\mathbb{C})$ (resp., $\mathrm{GSO}_{2n}(\mathbb{C})$), and μ denotes the similitude character of $\mathrm{GSp}_{2n}(\mathbb{C})$ (resp., $\mathrm{GSO}_{2n}(\mathbb{C})$). If $n = 0$, then $r = r_1$ with $r_1 = \mathrm{Sym}^2 \rho_r \otimes \mu^{-1}$ (resp., $r_1 = \bigwedge^2 \rho_r \otimes \mu^{-1}$). Recall that we have excluded $n = 1$ in the even case. The Langlands-Shahidi method defines the L-functions $L(s, \sigma, r_i)$ and ϵ-factors $\epsilon(s, \sigma, r_i, \psi)$ for $1 \leq i \leq 2$, where ψ is a nontrivial additive character of F. (In the global setting, it is the local component of our fixed global additive character ψ of Section 1.) If π denotes a representation of $\mathbf{G}_n(F)$ and τ denotes one of $\mathrm{GL}_r(F)$, then we sometimes employ the following notation for these L-functions as well as their global analogues:

$$L(s, \pi \times \tau) := L(s, \tau \otimes \widetilde{\pi}, \rho_r \otimes \widetilde{R}) = L(s, \tau \otimes \widetilde{\pi}, r_1), \tag{5}$$

$$\epsilon(s, \pi \times \tau, \psi) := \epsilon(s, \tau \otimes \widetilde{\pi}, \rho_r \otimes \widetilde{R}, \psi) = \epsilon(s, \tau \otimes \widetilde{\pi}, r_1, \psi). \tag{6}$$

PROPOSITION 3.1

Assume that σ is tempered. Then the local L-function $L(s, \sigma, r_i)$ is holomorphic for $\Re(s) > 0$ for $1 \leq i \leq 2$.

Proof

The result is known more generally for Archimedean F (see [36] or [1]). For non-Archimedean F, this is [3, Theorem 5.7]. Here, $i = 1, 2$, and the first L-function gives the Rankin-Selberg product while the second is either the twisted symmetric or the twisted exterior square L-function. When $n = 0$, we get only the second L-function (see [3, Proposition 5.6]). □

The following proposition is due to W. Kim (in his Ph.D. dissertation [25]) when F is non-Archimedean and, in more generality, due to Kostant [27] and Vogan [48] when F is Archimedean.

PROPOSITION 3.2 (The Standard Module Conjecture for $\mathbf{G}_n$)

Let σ be an irreducible admissible generic representation of $\mathbf{M}(F)$, where $\mathbf{M}$ is a standard maximal Levi subgroup as before, and let ν be an element in the positive

Weyl chamber. Let $I(\nu, \sigma)$ be the representation unitarily induced from ν and σ (the standard module), and denote by $J(\nu, \sigma)$ its unique Langlands quotient. Assume that $J(\nu, \sigma)$ is generic. Then $J(\nu, \sigma) = I(\nu, \sigma)$. In particular, $I(\nu, \sigma)$ is irreducible.

A similar result also holds for general linear groups (see [49]).

Remark 3.3
For small values of n, we need not rely on [25]. We can obtain the result for small n from published ones as we now explain. The group GSpin_5 is isomorphic to GSp_4, whose derived group is Sp_4. G. Muić has proved the Standard Module Conjecture for (quasi-split) classical groups (see [33, Theorem 1.1]). The result for GSpin_5 now follows from Corollary 3.5.

Similarly, note that the derived group of GSpin_6 is isomorphic to $\mathrm{Spin}_6 \simeq \mathrm{SL}_4$ and hence equal to the derived group of GL_4. Therefore, again by Corollary 3.5, the result for GSpin_6 follows from the Standard Module Conjecture for GL_6.

PROPOSITION 3.4
Let $\mathbf{G} \subset \widetilde{\mathbf{G}}$ be two connected reductive groups whose derived groups are equal. Let $\widetilde{\mathbf{P}} = \widetilde{\mathbf{M}}\mathbf{N}$ be a maximal standard parabolic subgroup of $\widetilde{\mathbf{G}}$, and let $\mathbf{P} = \mathbf{MN}$ be the corresponding one in $\mathbf{G}$ with $\mathbf{M} = \widetilde{\mathbf{M}} \cap \mathbf{G}$. Also, let $\widetilde{\mathbf{T}} \subset \widetilde{\mathbf{M}}$ and $\mathbf{T} = \widetilde{\mathbf{T}} \cap \mathbf{G} \subset \mathbf{M}$ be maximal tori in $\widetilde{\mathbf{G}}$ and $\mathbf{G}$, respectively. Let $\widetilde{\sigma}$ be a quasi-tempered representation of $\widetilde{M} = \widetilde{\mathbf{M}}(F)$, and denote by σ its restriction to $M = \mathbf{M}(F)$. Write $\sigma = \bigoplus_i \sigma_i$ with σ_i irreducible representations of M. Let $I(\widetilde{\sigma})$ denote the induced representation $\operatorname{Ind}_{\widetilde{M}N \uparrow \widetilde{G}} \widetilde{\sigma} \otimes 1$ of $\widetilde{G} = \widetilde{\mathbf{G}}(F)$, and let $I(\sigma_i)$ denote $\operatorname{Ind}_{MN \uparrow G} \sigma_i \otimes 1$, a representation of $G = \mathbf{G}(F)$. Then the standard module $I(\widetilde{\sigma})$ is irreducible if and only if each standard module $I(\sigma_i)$ is irreducible.

Proof
By the irreducibility of $\widetilde{\sigma}$ and the fact that $\widetilde{M} = \widetilde{T} M$, choose

$$\left\{t_1 = 1, t_2, \ldots, t_k \,:\, t_i \in \widetilde{T} = \widetilde{\mathbf{T}}(F)\right\}$$

such that $\sigma_i(m) = \sigma_1(t_i^{-1} m t_i)$. Observe that

$$I(\widetilde{\sigma})|_G = \bigoplus_i I(\sigma_i). \tag{7}$$

In fact, for $f_1 \in V(\sigma_1)$, define $f_i(g) = f_1(t_i^{-1} g t_i)$. Then $f_i \in V(\sigma_i)$, the space of $I(\sigma_i)$, and the representation $I(\sigma_i)(t_i^{-1} g t_i)$ on $V(\sigma_1)$ is isomorphic to $I(\sigma_i)$ since

$$\left(I(\sigma_1)(t_i^{-1} g t_i) f_1\right)_i = I(\sigma_i)(g) f_i \tag{8}$$

for all $g \in G$. In particular, $I(\sigma_i)$ is irreducible if and only if $I(\sigma_1)$ is. The assumption of the equality of the derived groups implies that $\widetilde{G} = \widetilde{T}G$, which, in turn, implies that $\widetilde{T}$ acts transitively on the set of $I(\sigma_i)$.

If each $I(\sigma_i)$ is irreducible, then $I(\widetilde{\sigma})$ has to be irreducible. In fact, if $(\widetilde{I}_1, \widetilde{V}_1)$ is an irreducible subrepresentation of $I(\widetilde{\sigma})$, then

$$\widetilde{I}_1 \mid G = \bigoplus_j I_j, \quad I_j \neq \{0\}, \tag{9}$$

and given j, there exists i such that $I_j \subset I(\sigma_i)$. Conversely, for each i there exists j such that $0 \neq I_j \subset I(\sigma_i)$. Fix i such that $V(\sigma_i) \cap \widetilde{V}_1 \neq 0$. Since $\widetilde{V}_1$ is invariant under $\widetilde{T}$, applying $I(\widetilde{\sigma})(\widetilde{T})$ to this intersection, one concludes that $0 \neq V(\sigma_i) \cap \widetilde{V}_1 \subsetneq V(\sigma_i)$ for all i. Consequently, $0 \neq I_j \subsetneq I(\sigma_i)$, which is a contradiction.

Conversely, suppose that $I(\widetilde{\sigma})$ is irreducible but $I(\sigma_i)$'s are (all) reducible. Let V_i be an irreducible G-subspace of $V(\sigma_i)$. Then

$$\bigoplus_i I(\widetilde{\sigma})(t_i)V_i \tag{10}$$

is a $\widetilde{G}$-invariant subspace of $V(\widetilde{\sigma})$ which is strictly smaller than $V(\widetilde{\sigma})$, a contradiction. □

COROLLARY 3.5
Suppose that $\mathbf{G}$ *and* $\mathbf{G}'$ *are two connected reductive groups having the same derived group. Then the Standard Module Conjecture is valid for* G *if and only if it is valid for* G'.

Proof
Let $\mathbf{H}$ be the common derived group. Apply Proposition 3.4 once to $\mathbf{H} \subset \mathbf{G}$ and again to $\mathbf{H} \subset \mathbf{G}'$. □

The Langlands-Shahidi method defines the local L-functions via the theory of intertwining operators. With notation as above, let the standard maximal Levi subgroup $\mathbf{M}$ in $\mathbf{G}$ correspond to the subset θ of the set of simple roots Δ of $\mathbf{G}$. Then $\theta = \Delta - \{\alpha\}$ for a simple root $\alpha \in \Delta$. We denote by w the longest element in the Weyl group of $\mathbf{G}$ modulo that of $\mathbf{M}$. Then w is the unique element with $w(\theta) \subset \Delta$ and $w(\alpha) < 0$. Let $A(s, \sigma, w)$ denote the intertwining operator as in [39, (1.1), page 278], and let $N(s, \sigma, w)$ be defined via

$$A(s, \sigma, w) = r(s, \sigma, w)N(s, \sigma, w), \tag{11}$$

$$r(s, \sigma, w) = \frac{L(s, \sigma, \widetilde{r}_1)L(2s, \sigma, \widetilde{r}_2)}{L(1+s, \sigma, \widetilde{r}_1)\epsilon(s, \sigma, \widetilde{r}_1, \psi)L(1+2s, \sigma, \widetilde{r}_2)\epsilon(2s, \sigma, \widetilde{r}_2, \psi)}. \tag{12}$$

In fact, the Langlands-Shahidi method inductively defines the γ-factors using the theory of local intertwining operators out of which the L- and ϵ-factors are defined via the relation

$$\gamma(s, \sigma, r_i, \psi) = \epsilon(s, \sigma, r_i, \psi) \frac{L(1-s, \sigma, \widetilde{r_i})}{L(s, \sigma, r_i)}. \tag{13}$$

The following proposition about analytic properties of local L-functions is the main result of this section. We use it to prove the necessary global analytic properties.

PROPOSITION 3.6

Let σ be a local component of a globally generic cuspidal automorphic representation of $\mathbf{M}(\mathbb{A})$. Then the normalized local intertwining operator $N(s, \sigma, w)$ is holomorphic and nonzero for $\Re(s) \geq 1/2$.

Proof

First, assume that σ is tempered. Then $A(s, \sigma, w)$ is holomorphic for $\Re(s) > 0$ by a result of Harish-Chandra. Moreover, for $\Re(s) > 0$ we have that $r(s, \sigma, w)$ is nonzero by definition and holomorphic by Proposition 3.1. This implies that $N(s, \sigma, w)$ is also holomorphic for $\Re(s) > 0$.

Next, assume that σ is not tempered but still unitary. Write $\sigma = \tau \otimes \widetilde{\pi}$, where τ is a representation of $\mathrm{GL}_r(F)$ and $\widetilde{\pi}$ is one of $\mathbf{G}_n(F)$. (We use $\widetilde{\pi}$ in order to get the usual Rankin-Selberg factors for pairs of general linear groups below.) By Proposition 3.2 and the classification of irreducible unitary representations of general linear groups, we can write τ and $\widetilde{\pi}$ as

$$\tau = \mathrm{Ind}(\nu^{\alpha_1}\tau_1 \otimes \cdots \otimes \nu^{\alpha_p}\tau_p \otimes \tau_{p+1} \otimes \nu^{-\alpha_p}\tau_p \otimes \cdots \otimes \nu^{-\alpha_1}\tau_1)$$

and

$$\widetilde{\pi} = \mathrm{Ind}(\nu^{\beta_1}\pi_1 \otimes \cdots \otimes \nu^{\beta_q}\pi_q \otimes \pi_0)$$

with $0 = \alpha_{p+1} < \alpha_p < \cdots < \alpha_1 < 1/2$ and $0 < \beta_q < \cdots < \beta_1$, where $\tau_1, \ldots, \tau_{p+1}$ and $\pi_1, \ldots, \pi_q$ are tempered representations of the corresponding general linear groups and π_0 is a generic tempered representation of $\mathbf{G}_t(F)$ for some t. Here, $\nu(\cdot)$ denotes $|\det(\cdot)|_F$. Since we are assuming that σ is a component of a global cuspidal representation (see Remark 3.7), it follows exactly as in [21, Lemma 3.3] that $\beta_1 < 1$. However, note that one should use our $\widetilde{\pi}$ in the argument.

Now $N(s, \sigma, w)$ is equal to a product of rank-one operators for either $\mathrm{GL}_k \times \mathrm{GL}_l \subset \mathrm{GL}_{k+l}$ (Rankin-Selberg products) or $\mathrm{GL}_k \times \mathbf{G}_l \subset \mathbf{G}_{k+l}$. Lemma 2.10 of [21] implies that the former rank-one operators are holomorphic since $\Re(s - \alpha_1 - \beta_1) > -1$ for

$\Re(s) \geq 1/2$. The latter rank-one operators are also holomorphic for $\Re(s) \geq 1/2$ by the tempered case at the beginning of this proof since $\alpha_1 < 1/2$.

The fact that $N(s, \sigma, w)$ is a nonvanishing operator now follows from applying a result of Y. Zhang to our case (see [50, pages 393–394]). Note that because of Proposition 3.1, no assumptions are needed in applying [50]. □

Remark 3.7

Note that the proof of [21, Lemma 3.3] does depend on the fact that σ is assumed to be a local component of a global cuspidal representation. To be more precise, the proof uses [21, Theorem 3.2(3)], and it refers to [21, Proposition 1.8], which is a global result.

4. Stability of γ-factors

We continue to denote by $\mathbf{G}_n$ either of the groups GSpin_{2n+1} or GSpin_{2n} in this section. In Sections 4.1 and 4.2 we denote by $\mathbf{G}_n^{\sim}$ the groups GSpin_{2n+1} in the odd case and $\mathrm{GSpin}_{2n}^{\sim}$ in the even case (see Remark 4.2), and $\mathbf{G}$ denotes $\mathbf{G}_{n+1}^{\sim}$ in either case.

In this section we prove a key local fact, called the *stability of γ-factors*, which is what allows us to connect the Langlands-Shahidi L- and ϵ-factors to those in the converse theorem. Similar results have been proved for the group SO_{2n+1} in [10] and [7] and for other classical groups in [8], which we have followed. A more general result appears in [12] and [13].

Let F denote a non-Archimedean local field of characteristic zero. Thus, F could be one of the k_v's, where v is a finite place. Composing a fixed splitting of G_n with ψ as in [41] defines a generic character of U as well as U_M which we still denote by ψ. Let π be an irreducible admissible ψ-generic representation of $\mathrm{GSpin}_{2n+1}(F)$ or $\mathrm{GSpin}_{2n}(F)$, and let η be a continuous character of $F^\times$. The associated γ-factors of the Langlands-Shahidi method defined in [39, Theorem 3.5] are denoted by $\gamma(s, \eta \times \pi, \psi)$. They are associated to the pair $(\mathrm{GSpin}_{m+2}, \mathrm{GL}_1 \times \mathrm{GSpin}_m)$ of the maximal Levi subgroup $\mathbf{M} = \mathrm{GL}_1 \times \mathrm{GSpin}_m$ in the connected reductive group GSpin_{m+2}, where $m = 2n+1$ or $2n$. Recall that the γ-factor is related to the L- and ϵ-factors by

$$\gamma(s, \eta\times, \pi, \psi) = \epsilon(s, \eta \times \pi, \psi)\frac{L(1-s, \eta^{-1} \times \widetilde{\pi})}{L(s, \eta \times \pi)}. \tag{14}$$

The main result of this section is the following.

THEOREM 4.1

Let π_1 and π_2 be irreducible admissible generic representations of $\mathrm{GSpin}_m(F)$ *with equal central characters $\omega_{\pi_1} = \omega_{\pi_2}$. Then for a sufficiently ramified character η of*

$F^\times$ we have

$$\gamma(s, \eta \times \pi_1, \psi) = \gamma(s, \eta \times \pi_2, \psi).$$

The proof of this theorem is the subject matter of this section, including a review of some facts about *partial Bessel functions*.

4.1. $\gamma(s, \eta \times \pi, \psi)$ *as the Mellin transform of a Bessel function*

Recall that $\mathbf{G}_n^\sim$ denotes either GSpin_{2n+1} or $\mathrm{GSpin}_{2n}^\sim$. Then $\mathrm{GL}_1 \times \mathbf{G}_n^\sim$ is a maximal Levi subgroup in $\mathbf{G} = \mathbf{G}_{n+1}^\sim$. We refer to $\mathbf{G} = \mathrm{GSpin}_{2n+3}$ as *odd* and $\mathbf{G} = \mathrm{GSpin}_{2n+2}^\sim$ as *even* in the rest of this section. Therefore, in the odd case, $\mathbf{G}_n^\sim$ and $\mathbf{G}_n$ are the same, but they are not in the even case.

Remark 4.2

We need to assume that the center of our group $\mathbf{G}$ is connected for the proof of Proposition 4.16. This is not true if $\mathbf{G}$ is taken to be GSpin_{2n+2}, as we pointed out in Proposition 2.3. To remedy this, we can alternatively work with the group $\mathrm{GSpin}_{2n+2}^\sim$ of Section 2.2. Since $\mathrm{GSpin}_{2n+2}^\sim$ has the same derived group as GSpin_{2n+2}, its corresponding γ-factors are the same as those of GSpin_{2n+2} since they (and, in fact, the local coefficients via which they are defined) depend only on the derived group of our group. This has no effect on the arguments of the next few sections, as all of our crucial computations take place inside the derived group.

Let $\mathbf{G}$ be as above with a fixed Borel subgroup $\mathbf{B} = \mathbf{TU}$ as before. We continue to denote its root data by $(X, R, X^\vee, R^\vee)$, which we have described in detail in Section 2. Consider the maximal parabolic subgroup $\mathbf{P} = \mathbf{MN}$ in $\mathbf{G}$, where $\mathbf{N} \subset \mathbf{U}$ and the Levi component, $\mathbf{M}$, is isomorphic to $\mathrm{GL}_1 \times \mathbf{G}_n^\sim$. The standard Levi subgroup $\mathbf{M} \supset \mathbf{T}$ corresponds to the subset $\theta = \Delta - \{\alpha_1\}$ of the set of simple roots $\Delta = \{\alpha_1, \alpha_2, \ldots, \alpha_n, \alpha_{n+1}\}$ of $(\mathbf{G}, \mathbf{T})$ with notation as in Section 2. Let $\widetilde{w}_0$ denote the unique element of the Weyl group of $\mathbf{G}$ such that $\widetilde{w}_0(\theta) \subset \Delta$ and $\widetilde{w}_0(\alpha_1) < 0$. Notice that the parabolic subgroup $\mathbf{P}$ is self-associate; that is, $\widetilde{w}_0(\theta) = \theta$. We denote by G, P, M, N, B, and so on, the groups $\mathbf{G}(F), \mathbf{P}(F), \mathbf{M}(F), \mathbf{N}(F), \mathbf{B}(F)$, and so on, in what follows. Also, denote the opposite parabolic subgroup to P by $\overline{P} = M\overline{N}$.

Let $\mathbf{Z} = \mathbf{Z_G}$ and $\mathbf{Z_M}$ be the centers of $\mathbf{G}$ and $\mathbf{M}$, respectively. The following is [41, Assumption 5.1] for our cases. We need this when dealing with Bessel functions.

PROPOSITION 4.3

There exists an injection $e^* : F^\times \longrightarrow Z_G \backslash Z_M$ such that for all $t \in F^\times$ we have $\alpha_1(e^*(t)) = t$.

Proof
We define $e^*(t)$ to be the image in $Z_G \backslash Z_M$ of $e_1^*(t)$ in the odd case and that of $E_1^*(t)$ in the even case. The proposition is now clear from our explicit descriptions in Section 2. □

Denote the image of e^* by Z_M^0 as in [41]. (Note that [41] uses the notation $\alpha^\vee$ for e^*.)

We now review some standard facts about the reductive group G whose proofs could be found in either [43] or [44], for example. For $\alpha \in R$, let $u_\alpha : F \longrightarrow G$ be the root group homomorphism determined by the equation

$$tu_\alpha(x)t^{-1} = u_\alpha\big(\alpha(t)x\big), \quad t \in T, x \in F. \tag{15}$$

Moreover, define $w_\alpha : F^\times \longrightarrow G$ by

$$w_\alpha(\lambda) = u_\alpha(\lambda)u_{-\alpha}(-\lambda^{-1})u_\alpha(\lambda). \tag{16}$$

Also, set $w_\alpha := w_\alpha(1)$. Then

$$w_\alpha(\lambda) = \alpha^\vee(\lambda)w_\alpha = w_\alpha \alpha^\vee(\lambda^{-1}), \tag{17}$$

where $\alpha^\vee$ is the coroot corresponding to α. The element w_α normalizes T, and we denote its image in the Weyl group by $\widetilde{w}_\alpha$.

Remark 4.4
Our choice of w_α is indeed the same as n_α in [43, page 133]. This choice differs up to a sign from those made in [41, (4.43), (4.19), or (4.56)], requiring w_α to be the image of $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ under the homomorphism from SL_2 into $\mathbf{G}$ determined by α. The latter choice introduces an occasional negative sign in some of the equations, for example, (17). Of course, this choice is irrelevant to the final results, and we have chosen Springer's since we are using some detailed information on structure constants from [43] in what follows (see, e.g., Lemma 4.10).

Recall that

$$w_{-\alpha}(\lambda) = w_\alpha\Big(\frac{-1}{\lambda}\Big), \tag{18}$$

$$w_\alpha^2 = \alpha^\vee(-1), \tag{19}$$

$$w_{-\alpha} = w_\alpha^{-1}. \tag{20}$$

For any two linearly independent roots α and β in R and a total order on R, which we now fix, we have

$$u_\alpha(x)u_\beta(y)u_\alpha(-x) = u_\beta(y) \prod_{\substack{i,j>0 \\ i\alpha+j\beta\in R}} u_{i\alpha+j\beta}(c_{ij}x^i y^j) \tag{21}$$

for certain *structure constants* $c_{ij} = c_{\alpha,\beta;i,j}$ (which may depend on the total order R). In particular, if there are no roots of the form $i\alpha + j\beta$ with $i, j > 0$, then $u_\alpha(x)u_\beta(y) = u_\beta(y)u_\alpha(x)$.

We recall the following result.

PROPOSITION 4.5
Let α and β be two arbitrary linearly independent roots, and let $(\beta - c\alpha, \ldots, \beta + b\alpha)$ be the α-string through β. Then

$$w_\alpha w_\beta(x) w_\alpha^{-1} = w_{\widetilde{w}_\alpha(\beta)}(d_{\alpha,\beta}x), \tag{22}$$

where

$$d_{\alpha,\beta} = \sum_{i=\max(0,c-b)}^{c} (-1)^i \, c_{-\alpha,\beta;i,1} \, c_{\alpha,\beta-i\alpha;i+b-c,1}. \tag{23}$$

Moreover,

$$d_{-\alpha,\beta} = (-1)^{\langle\beta,\alpha^\vee\rangle} d_{\alpha,\beta}, \tag{24}$$

and

$$d_{\alpha,\beta} d_{\alpha,\widetilde{w}_\alpha(\beta)} = (-1)^{\langle\beta,\alpha^\vee\rangle}. \tag{25}$$

Proof
This is [43, Lemma 9.2.2]. Note that Springer defines $d_{\alpha,\beta}$ via

$$w_\alpha u_\beta(x) w_\alpha^{-1} = u_{\widetilde{w}_\alpha(\beta)}(d_{\alpha,\beta}x),$$

which, using (16), immediately implies (22). □

Denote the image of u_α in G by U_α. Notice that M is generated by T and U_α's with α ranging over $\Sigma(\theta)$, the set of all (positive and negative) roots spanned by $\alpha_2, \alpha_3, \ldots, \alpha_n, \alpha_{n+1}$, while N is generated by U_α's, where α ranges over $R(N) = R^+ - \Sigma(\theta)$, the set of positive roots of G not in M (i.e., involving a positive coefficient of α_1 when written as a sum of simple roots with nonnegative coefficients), and $\overline{N}$ is generated by U_α's, where α ranges over $R(\overline{N}) = R^- - \Sigma(\theta)$, the set of negative roots of G not in M (i.e., involving a negative coefficient of α_1 when written as a sum of simple roots with nonpositive coefficients). Let $U_M = U \cap M$. Then U_M is generated by U_α's with $\alpha \in \Sigma(\theta)^+ = \Sigma(\theta) \cap R^+$.

The group M acts via the adjoint action on N; in particular, both U_M and Z_M^0 act on N. We are interested in the orbits of N under conjugation by $Z_M^0 U_M$.

LEMMA 4.6

Up to a subset of measure zero of N, the following is a complete set of representatives for the orbits of N under conjugation by U_M:

$$U_M \backslash N \simeq \left\{ u_{\alpha_1}(a) u_\gamma(x) \,:\, a \in F^\times,\ x \in F \right\},$$

where

$$\gamma = \begin{cases} \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{n+1} & \text{if } \mathbf{G} \text{ is odd,} \\ \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{n-1} + \alpha_n + \alpha_{n+1} & \text{if } \mathbf{G} \text{ is even} \end{cases}$$

is the longest positive root in G.

Proof

Using the same Bourbaki notation as in Section 2, $R(N)$ is given by (see [6])

$$\left\{ \begin{array}{l} \alpha_1, \alpha_1 + \alpha_2, \ldots, \alpha_1 + \alpha_2 + \cdots + \alpha_{n+1}, \alpha_1 + \alpha_2 + \cdots + \alpha_n + 2\alpha_{n+1}, \\ \alpha_1 + \alpha_2 + \cdots + \alpha_{n-1} + 2\alpha_n + 2\alpha_{n+1}, \ldots, \gamma = \alpha_1 + 2\alpha_2 + \cdots + \\ 2\alpha_n + 2\alpha_{n+1} \end{array} \right\} \tag{26}$$

for the odd case and by

$$\left\{ \begin{array}{l} \alpha_1, \alpha_1 + \alpha_2, \ldots, \alpha_1 + \alpha_2 + \cdots + \alpha_{n-1} + \alpha_n, \alpha_1 + \alpha_2 + \cdots + \alpha_{n-1} + \\ \alpha_{n+1}, \alpha_1 + \alpha_2 + \cdots + \alpha_{n-1} + \alpha_n + \alpha_{n+1}, \alpha_1 + \alpha_2 + \cdots + \alpha_{n-2} + \\ 2\alpha_{n-1} + \alpha_n + \alpha_{n+1}, \alpha_1 + \alpha_2 + \cdots + 2\alpha_{n-2} + 2\alpha_{n-1} + \alpha_n + \alpha_{n+1}, \\ \ldots, \gamma = \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{n-1} + \alpha_n + \alpha_{n+1} \end{array} \right\} \tag{27}$$

for the even case.

An arbitrary element $n \in N$ is of the form

$$n = \prod_{\alpha \in R(N)} u_\alpha(x_\alpha) \tag{28}$$

with $x_\alpha \in F$. The ordering in the product can be arbitrarily chosen since any linear combination with positive integer coefficients of two roots in $R(N)$ has α_1 with an integer coefficient of at least two which cannot be a root; hence by (21), any two terms in the above product commute. We make use of this fact in the rest of the proof.

Observe that the set $R(N)$ has the property that if α belongs to $R(N)$, then so does $\gamma - \alpha + \alpha_1$. Notice that if $\alpha' \in R(N) - \{\alpha_1, \gamma\}$, then $\beta = \alpha' - \alpha_1 \in \Sigma(\theta)$ and $\beta > 0$; hence $g = u_\beta(x_\beta) \in U_M$ for any $x_\beta \in F$. The observation means that $\gamma - \beta = \gamma - \alpha' + \alpha_1 \in R(N)$. Fix one such β, and consider the adjoint action of g on n:

$$gng^{-1} = \prod_{\alpha \in R(N)} g u_\alpha(x_\alpha) g^{-1} = \prod_{\alpha \in R(N)} u_\beta(x_\beta) u_\alpha(x_\alpha) u_\beta(-x_\beta).$$

We now look at each term in this product. If $i\beta + j\alpha \notin R$ for positive i and j, then by (21), the term is equal to $u_\alpha(x_\alpha)$. This is the case most of the time. The only roots of the form $i\beta + j\alpha$ with positive i and j are $\beta + \alpha_1$ and $\beta + (\gamma - \beta) = \gamma$, except when we are in the odd case with $\beta = \alpha_2 + \cdots + \alpha_{n+1}$ and $\alpha = \alpha_1$, in which case $2\beta + \alpha_1 = \gamma$ is also a root. Therefore,

$$\prod_{\alpha \in R(N)} u_\beta(x_\beta) u_\alpha(x_\alpha) u_\beta(-x_\beta) = \prod_{\alpha \in R(N)} u_\alpha(y_\alpha),$$

where

$$y_\alpha = \begin{cases} x_\alpha & \text{if } \alpha \neq \beta + \alpha_1, \gamma, \\ x_\alpha + C x_{\alpha_1} x_\beta & \text{if } \alpha = \beta + \alpha_1, \\ x_\alpha + C' x_\beta x_{\gamma - \beta} + C'' x_\beta^2 x_{\alpha_1} & \text{if } \alpha = \gamma, \alpha \neq \beta + \alpha_1. \end{cases}$$

Here, $C, C', C'' \in F^\times$ are the appropriate structure constants as in (21). In fact, C'' is nonzero only in the exceptional case mentioned above, that is, the odd case with $\beta = \alpha_2 + \cdots + \alpha_{n+1}$ and $\alpha = \alpha_1$.*

Assuming that $x_{\alpha_1} \neq 0$, which excludes only a subset of $n \in N$ of measure zero, we can choose $x_\beta \in F$ appropriately in order to have $x_\alpha + C x_{\alpha_1} x_\beta = 0$. Applying this process for all the β in $\Sigma(\theta)$ described above, we can make all x_α in (28) equal to zero, except for x_{α_1} and x_γ. In the process, the value of x_{α_1} does not change, but the value of x_γ may change. We let $a = x_{\alpha_1}$, and we let x be the final value of x_γ. This proves the lemma. □

We now consider conjugation by Z_M^0.

LEMMA 4.7
Let $n = u_{\alpha_1}(a) u_\gamma(x) \in N$ with $a \in F^\times$ and $x \in F$. Then n and $u_{\alpha_1}(1) u_\gamma(y)$ are in the same conjugacy class of N under conjugation by Z_M^0 for some $y \in F$.

*We thank the referee who brought the exceptional case to our attention.

Proof
For $z = e_1^*(\lambda) \in Z_M^0$, in the odd case, we have

$$\begin{aligned} znz^{-1} &= e_1^*(\lambda) u_{\alpha_1}(a) e_1^*(\lambda^{-1})\ e_1^*(\lambda) u_\gamma(x) e_1^*(\lambda^{-1}) \\ &= u_{\alpha_1}\big(\alpha_1(e_1^*(\lambda))\, a\big)\ u_\gamma\big(\alpha_1(e_1^*(\lambda))\, x\big) \\ &= u_{\alpha_1}(\lambda^{\langle \alpha_1, e_1^* \rangle}\, a)\ u_\gamma(\lambda^{\langle \gamma, e_1^* \rangle}\, x) \\ &= u_{\alpha_1}(\lambda\, a)\ u_\gamma(\lambda\, x). \end{aligned}$$

In the even case, e_1^* above should be replaced by E_1^*. Take $\lambda = 1/a$ and $y = x/a$ to finish the proof. □

Lemmas 4.6 and 4.7 immediately imply the following.

COROLLARY 4.8
Up to a subset of measure zero of N, the following is a complete set of representatives for the orbits of N under conjugation by $Z_M^0 U_M$:

$$Z_M^0 U_M \backslash N \simeq \big\{ u_{\alpha_1}(1) u_\gamma(x) \ :\ x \in F \big\}.$$

If we set

$$w_0 = \begin{cases} w_{\alpha_1} w_{\alpha_2} \cdots\ w_{\alpha_n} w_{\alpha_{n+1}} w_{\alpha_n} \cdots\ w_{\alpha_2} w_{\alpha_1} & \text{if } \mathbf{G} \text{ is odd,} \\ w_{\alpha_1} w_{\alpha_2} \cdots\ w_{\alpha_{n-1}} w_{\alpha\alpha} w_{\alpha_{n-1}} \cdots\ w_{\alpha_2} w_{\alpha_1} & \text{if } \mathbf{G} \text{ is even,} \end{cases} \tag{29}$$

where $w_{\alpha\alpha}$ is the product of the commuting elements $w_{\alpha_n} w_{\alpha_{n+1}} = w_{\alpha_{n+1}} w_{\alpha_n}$, then w_0 is the representative in G of the unique Weyl group element $\widetilde{w}_0$ introduced at the beginning of Section 4.1.

Remark 4.9
Note that in [41] the analogue of our element $w_{\alpha\alpha}$ is denoted by $w_{\alpha_{n+1}}$ as explained in [41, (4.45)]. However, our current choices, which apply equally well to SO_{2n} or other cases treated in [41, Section 4], replace the two commuting matrices on the left-hand side of [41, (4.45)] with their transposes (see Remark 4.4).

As in [41] we are interested in elements $n \in N$ such that

$$w_0^{-1} n = m n' \overline{n} \in P \overline{N}. \tag{30}$$

The decomposition in (30) is clearly unique, and we compute the m-, n'-, and $\overline{n}$-parts of an element n, as in Corollary 4.8. We do this in Proposition 4.12. First, we prove the following auxiliary lemma.

LEMMA 4.10
We can normalize the u_α's such that the element w_γ satisfies

$$\gamma^\vee(d)w_\gamma = w_\gamma(d) =$$
$$\begin{cases} w_{\alpha_2}\cdots w_{\alpha_n}w_{\alpha_{n+1}}w_{\alpha_n}\cdots w_{\alpha_2}w_{\alpha_1}w_{\alpha_2}^{-1}\cdots w_{\alpha_n}^{-1}w_{\alpha_{n+1}}^{-1}w_{\alpha_n}^{-1}\cdots w_{\alpha_2}^{-1} & \text{if } \mathbf{G} \text{ is odd,} \\ w_{\alpha_2}\cdots w_{\alpha_{n-1}}w_{\alpha\alpha}w_{\alpha_{n-1}}\cdots w_{\alpha_2}w_{\alpha_1}w_{\alpha_2}^{-1}\cdots w_{\alpha_{n-1}}^{-1}w_{\alpha\alpha}^{-1}w_{\alpha_{n-1}}^{-1}\cdots w_{\alpha_2}^{-1} & \text{if } \mathbf{G} \text{ is even,} \end{cases}$$

where

$$d = \begin{cases} (-1)^n & \text{if } \mathbf{G} \text{ is odd,} \\ (-1)^{n-1} & \text{if } \mathbf{G} \text{ is even.} \end{cases} \tag{31}$$

Remark 4.11
The d in the odd case is slightly different from the corresponding value for the group SO_{2n+3} carried out in [8, Section 4.2.1]; that is, it differs by a factor of -1. The reason for this discrepancy is that the representative we have fixed for the longest element of the Weyl group would, in the case of the group SO_3, lead to

$$\begin{pmatrix} & & -1 \\ & -1 & \\ -1 & & \end{pmatrix}.$$

This is the correct element that should have been used in [8, Section 4.2.1] and [41] instead of

$$\begin{pmatrix} & & 1 \\ & -1 & \\ 1 & & \end{pmatrix}.$$

The latter is the corresponding Weyl group representative for the group SL_3. However, since the group SO_3 does not have a Cartan of the same dimension as that of SL_3, there is no natural way (i.e., not requiring a choice of basis) of embedding it in SL_3. Therefore, there is no reason why the representative for SO_3 would be the same as that of SL_3. Of course, both of these two matrices correspond to the same Weyl group element since they differ only by a diagonal matrix in SO_3. In the notation of the present article, we can fix $u_\alpha(\ )$ and $u_{-\alpha}(\ )$ in SO_3 such that

$$w_\alpha(\lambda) = \begin{pmatrix} & & -\lambda^2 \\ & -1 & \\ -1/\lambda^2 & & \end{pmatrix}.$$

The choice of representative is now simply $w_\alpha(1)$.

Proof of Lemma 4.10
We begin by noting that

$$\gamma = \begin{cases} \widetilde{w}_{\alpha_2} \cdots \widetilde{w}_{\alpha_n} \widetilde{w}_{\alpha_{n+1}} \widetilde{w}_{\alpha_n} \cdots \widetilde{w}_{\alpha_2}(\alpha_1) & \text{if } \mathbf{G} \text{ is odd,} \\ \widetilde{w}_{\alpha_2} \cdots \widetilde{w}_{\alpha_{n-1}} \widetilde{w}_{\alpha_n} \widetilde{w}_{\alpha_{n+1}} \widetilde{w}_{\alpha_{n-1}} \cdots \widetilde{w}_{\alpha_2}(\alpha_1) & \text{if } \mathbf{G} \text{ is even.} \end{cases} \tag{32}$$

Let $\beta_1 = \alpha_1$, and denote by β_i the consecutive images of β_1 under the first $i-1$ Weyl group elements above for

$$\begin{cases} 1 \leq i \leq 2n & \text{if } \mathbf{G} \text{ is odd,} \\ 1 \leq i \leq 2n-1 & \text{if } \mathbf{G} \text{ is even.} \end{cases}$$

In fact, the β_i's are precisely the roots listed in (26) and (27) and in the same order.

Now apply (22) repeatedly to conclude that the right-hand side of the expression of the statement of the lemma is equal to $w_\gamma(d)$, where

$$d = \begin{cases} d_{\alpha_2,\beta_1} \cdots d_{\alpha_n,\beta_{n-1}} d_{\alpha_{n+1},\beta_n} d_{\alpha_n,\beta_{n+1}} \cdots d_{\alpha_2,\beta_{2n-1}} & \text{if } \mathbf{G} \text{ is odd,} \\ d_{\alpha_2,\beta_1} \cdots d_{\alpha_{n-1},\beta_{n-2}} d_{\alpha_n,\beta_{n-1}} d_{\alpha_{n+1},\beta_n} \cdots d_{\alpha_2,\beta_{2n-2}} & \text{if } \mathbf{G} \text{ is even.} \end{cases}$$

For $1 \leq i \leq n-1$ in the odd case and for $1 \leq i \leq n$ in the even case, the α_{i+1}-string through β_i is $(\beta_i, \beta_i + \alpha_{i+1})$; that is, $c = 0$ and $b = 1$ in the notation of Proposition 4.5. In the odd case with $i = n$, the α_{i+1}-string through β_i is $(\beta_i, \beta_i + \alpha_{i+1}, \beta_i + 2\alpha_{i+1})$; that is, $c = 0$ and $b = 2$. Similarly, for $1 \leq j \leq n-1$ in the odd case, the α_{n-j-1}-string through β_{n+j} is $(\beta_{n+j}, \beta_{n+j} + \alpha_{n-j-1})$; that is, $c = 0$ and $b = 1$. Also, for $1 \leq j \leq n-2$ in the even case, the α_{n-j}-string through β_{n+j} is $(\beta_{n+j}, \beta_{n+j} + \alpha_{n-j})$; that is, $c = 0$ and $b = 1$. Putting all these together and using (23), we can write

$$\begin{aligned} d_{\alpha_2,\beta_1} &= c_{\alpha_2,\beta_1;1,1} \\ &\vdots \\ d_{\alpha_n,\beta_{n-1}} &= c_{\alpha_n,\beta_{n-1};1,1} \\ d_{\alpha_{n+1},\beta_n} &= c_{\alpha_{n+1},\beta_n;2,1} \\ d_{\alpha_n,\beta_{n+1}} &= c_{\alpha_n,\beta_{n+1};1,1} \\ &\vdots \\ d_{\alpha_2,\beta_{2n-1}} &= c_{\alpha_2,\beta_{2n-1};1,1} \end{aligned} \quad \text{and} \quad \begin{aligned} d_{\alpha_2,\beta_1} &= c_{\alpha_2,\beta_1;1,1} \\ &\vdots \\ d_{\alpha_{n-1},\beta_{n-2}} &= c_{\alpha_{n-1},\beta_{n-2};1,1} \\ d_{\alpha_n,\beta_{n-1}} &= c_{\alpha_n,\beta_{n-1};1,1} \\ d_{\alpha_{n+1},\beta_n} &= c_{\alpha_{n+1},\beta_n;1,1} \\ &\vdots \\ d_{\alpha_2,\beta_{2n-2}} &= c_{\alpha_2,\beta_{2n-2};1,1} \end{aligned}$$

in the odd and even cases, respectively.

We can now normalize the u_α's such that we have $c_{\alpha_i,\alpha_{i+1};1,1} = 1$ and, in the odd case, $c_{\alpha_n,\alpha_{n+1};1,2} = -1$. These normalizations are motivated by the explicit matrix realizations of the related groups SO_{2n+3} and SO_{2n+2}, such as those in [41]. The values of other structure constants are now uniquely determined by these (see [43, Lemma 9.2.3]). We then get

$$\begin{aligned} c_{\alpha_2,\beta_1;1,1} &= -1 \\ &\vdots \\ c_{\alpha_n,\beta_{n-1};1,1} &= -1 \\ c_{\alpha_{n+1},\beta_n;2,1} &= -1 \\ c_{\alpha_n,\beta_{n+1};1,1} &= +1 \\ &\vdots \\ c_{\alpha_2,\beta_{2n-1};1,1} &= +1 \end{aligned} \quad \text{and} \quad \begin{aligned} c_{\alpha_2,\beta_1;1,1} &= -1 \\ &\vdots \\ c_{\alpha_{n-1},\beta_{n-2};1,1} &= -1 \\ c_{\alpha_n,\beta_{n-1};1,1} &= -1 \\ c_{\alpha_{n+1},\beta_n;1,1} &= +1 \\ &\vdots \\ c_{\alpha_2,\beta_{2n-2};1,1} &= +1. \end{aligned}$$

Therefore, in the expression for d, the first n terms in the odd case and the first $n-1$ terms in the even case are equal to -1, and others are equal to $+1$. This implies that $d = (-1)^n$ in the odd case and $d = (-1)^{n-1}$ in the even case. □

PROPOSITION 4.12

Assume that $x \in F$, and assume that $n = u_{\alpha_1}(1)u_\gamma(x)$ satisfies (30). Moreover, assume that x is nonzero (which rules out only a subset of N of measure zero). Then

$$\begin{aligned} m &= w'\gamma^\vee\left(\frac{d}{x}\right), \\ n' &= u_\gamma(-x)u_{\alpha_1}(-1), \\ \overline{n} &= u_{-\gamma}\left(\frac{1}{x}\right)u_{-\alpha_1}(1), \end{aligned}$$

where d is as in (31) and

$$w' = \begin{cases} w_{\alpha_2}^{-1}\cdots w_{\alpha_n}^{-1}w_{\alpha_{n+1}}^{-1}w_{\alpha_n}^{-1}\cdots w_{\alpha_2}^{-1} & \text{if } \mathbf{G} \text{ is odd,} \\ w_{\alpha_2}^{-1}\cdots w_{\alpha_{n-1}}^{-1}w_{\alpha\alpha}^{-1}w_{\alpha_{n-1}}^{-1}\cdots w_{\alpha_2}^{-1} & \text{if } \mathbf{G} \text{ is even,} \end{cases}$$

with $w_{\alpha\alpha}$ again as in (29). Moreover, we could also write m as

$$m = {\alpha_1}^\vee\left(\frac{d}{x}\right)w', \tag{33}$$

which is analogous to [41, Propositions 4.4, 4.8] modulo our Remark 4.4.

Proof

By the uniqueness of the decomposition in (30), it is enough to prove that these values do satisfy (30). This is a straightforward computation utilizing (21) multiple times.

First, observe that if $i,\ j > 0$ are integers, then $i\alpha_1 + j\gamma$ cannot be a root. Hence, $u_\gamma(\cdot)$ and $u_{\alpha_1}(\cdot)$ commute by (21). Also, $i\alpha_1 + j(-\gamma)$ cannot be a root, which again implies by (21) that $u_{-\gamma}(\cdot)$ and $u_{\alpha_1}(\cdot)$ commute. Similarly, $u_\gamma(\cdot)$ and $u_{-\alpha_1}(\cdot)$ commute.

Moreover, by (29), we have

$$w_0 = w_{\alpha_1} w'^{-1} w_{\alpha_1} \tag{34}$$

and, by Lemma 4.10,

$$w_\gamma(d) = w'^{-1} w_{\alpha_1} w'. \tag{35}$$

Now,

$$\begin{aligned}
w_0^{-1} n &= w_{\alpha_1}^{-1} w' w_{\alpha_1}^{-1} u_{\alpha_1}(1) u_\gamma(x) \\
&= w_{\alpha_1}^{-1} w' u_{\alpha_1}(-1) u_{-\alpha_1}(1) u_{\alpha_1}(-1) u_{\alpha_1}(1) u_\gamma(x) \\
&= w_{\alpha_1}^{-1} w' u_\gamma(x) u_{\alpha_1}(-1) u_{-\alpha_1}(1) \\
&= w_{\alpha_1}^{-1} w' w_\gamma(x) u_\gamma(-x) u_{-\gamma}\left(\frac{1}{x}\right) u_{\alpha_1}(-1) u_{-\alpha_1}(1) \\
&= w_{\alpha_1}^{-1} w' w_\gamma(x) \cdot u_\gamma(-x) u_{\alpha_1}(-1) \cdot u_{-\gamma}\left(\frac{1}{x}\right) u_{-\alpha_1}(1) \\
&= w' w'^{-1} w_{\alpha_1}^{-1} w' w_\gamma(x) \cdot n' \cdot \overline{n} \\
&= w' w_\gamma(d)^{-1} w_\gamma(x) \cdot n' \cdot \overline{n} \\
&= w' \left(w_\gamma \gamma^\vee\left(\frac{1}{d}\right)\right)^{-1} w_\gamma \gamma^\vee\left(\frac{1}{x}\right) \cdot n' \cdot \overline{n} \\
&= w' \gamma^\vee\left(\frac{d}{x}\right) \cdot n' \cdot \overline{n} \\
&= m n' \overline{n}.
\end{aligned}$$

To see (33), we use (22) repeatedly to write

$$\begin{aligned}
m &= w_{\alpha_1}^{-1} w' w_\gamma(x) \\
&= w_{\alpha_1}^{-1} \cdot w' w_\gamma(x) w'^{-1} \cdot w' \\
&= w_{\alpha_1}^{-1} w_{\alpha_1}(Dx) w' \\
&= w_{\alpha_1}^{-1} w_{\alpha_1} \alpha_1{}^\vee\left(\frac{1}{Dx}\right) w' \\
&= \alpha_1{}^\vee\left(\frac{1}{Dx}\right) w',
\end{aligned}$$

where

$$D = \begin{cases} d_{-\alpha_2,\beta_2} \cdots d_{-\alpha_n,\beta_n} d_{-\alpha_{n+1},\beta_{n+1}} d_{-\alpha_n,\beta_{n+2}} \cdots d_{-\alpha_2,\beta_{2n}} & \text{if } \mathbf{G} \text{ is odd,} \\ d_{-\alpha_2,\beta_2} \cdots d_{-\alpha_{n-1},\beta_{n-1}} d_{\alpha_n,\beta_n} d_{-\alpha_{n+1},\beta_{n+1}} d_{-\alpha_{n-1},\beta_{n+2}} \cdots d_{-\alpha_2,\beta_{2n-1}} & \text{if } \mathbf{G} \text{ is even.} \end{cases} \tag{36}$$

Notice that conjugation by w'^{-1} sends $\gamma = \beta_{2n}$ in the odd case and $\gamma = \beta_{2n-1}$ in the even case back to α_1.

Finally, we claim that $Dd = 1$ in both even and odd cases. To see this, note that we can write

$$Dd = \begin{cases} \prod_{i=1}^{n} d_{\alpha_{i+1},\beta_i} d_{-\alpha_{i+1},\beta_{i+1}} \prod_{j=1}^{n-1} d_{\alpha_{j+1},\beta_{2n-j}} d_{-\alpha_{j+1},\beta_{2n+1-j}} & \text{if } \mathbf{G} \text{ is odd,} \\ \prod_{i=1}^{n} d_{\alpha_{i+1},\beta_i} d_{-\alpha_{i+1},\beta_{i+1}} \prod_{j=1}^{n-2} d_{\alpha_{j+1},\beta_{2n-1-j}} d_{-\alpha_{j+1},\beta_{2n-j}} & \text{if } \mathbf{G} \text{ is even.} \end{cases}$$

Using (24) followed by (25), we can rewrite this as

$$Dd = \begin{cases} \prod_{i=1}^{n} (-1)^{\langle \beta_i+\beta_{i+1},\alpha_{i+1}^\vee\rangle} \prod_{j=1}^{n-1} (-1)^{\langle \beta_{2n-j}+\beta_{2n+1-j},\alpha_{j+1}^\vee\rangle} & \text{if } \mathbf{G} \text{ is odd,} \\ \prod_{i=1}^{n} (-1)^{\langle \beta_i+\beta_{i+1},\alpha_{i+1}^\vee\rangle} \prod_{j=1}^{n-2} (-1)^{\langle \beta_{2n-1-j}+\beta_{2n-j},\alpha_{j+1}^\vee\rangle} & \text{if } \mathbf{G} \text{ is even.} \end{cases}$$

Using the explicit root data that we described earlier, we can see easily that in every single term of these products the power of (-1) is an even integer. In fact, $\beta_i + \beta_{i+1}$ is equal to α_{i+1} plus twice a root for all i. Similarly, $\beta_{2n-j} + \beta_{2n+1-j}$ is equal to α_{j+1} plus twice a root for all j in the odd case, and $\beta_{2n-1-j} + \beta_{2n-j}$ is equal to α_{j+1} plus twice a root for all j in the even case. This completes the proof. □

The following is [41, Assumption 4.1] for our cases.

PROPOSITION 4.13
Let $n \in N$ satisfy (30). Then, except for a subset of measure zero of N, we have

$$U_{M,n} = U'_{M,m}, \tag{37}$$

where the notation is as in [41, Section 4]; that is,

$$U_{M,n} = \{u \in U_M \ : \ u\,n\,u^{-1} = n\}, \tag{38}$$

and

$$U'_{M,m} = \left\{u \in U_M \ : \ m\,u\,m^{-1} \in U_M \text{ and } \psi(mum^{-1}) = \psi(u)\right\}.$$

Note that the condition $\psi(mum^{-1}) = \psi(u)$ in the definition of $U'_{M,m}$ in our case is just the compatibility of ψ with elements of the Weyl group (see [41, pages 2079–2080]).

Proof
By arguments such as those on [41, page 2085], if the proposition is true for $n \in N$, then it is also true for every member of the intersection of its conjugacy class under M with N, provided that for the m-part we use the twisted conjugacy classes instead (see [41, (4.10)]). Hence, it is enough to verify the proposition for those n, as in Corollary 4.8. Fix one such $n = u_{\alpha_1}(1)u_\gamma(q)$ with $q \neq 0$ for the rest of this proof.

We can explicitly compute both sides of (37) as follows. Any $u \in U_M$ can be written as

$$u = \prod_{\beta \in \Sigma(\theta)^+} u_\beta(x_\beta), \tag{39}$$

where the order of the terms in the product is with respect to the total order of R we have fixed.

We have $unu^{-1} = n$ if and only if $uu_{\alpha_1}(1)u_\gamma(q)u^{-1} = u_{\alpha_1}(1)u_\gamma(q)$. Notice that by (21) we know that $u_\gamma(q)$ commutes with all $u_\beta(x_\beta)$ in the product. Hence, $u \in U_{M,n}$ if and only if $uu_{\alpha_1}(1)u^{-1} = u_{\alpha_1}(1)$. Among the terms $u_\beta(x_\beta)$, the element $u_{\alpha_1}(1)$ commutes with those with $\beta \in \Sigma(\Omega)^+$, where $\Omega = \Delta - \{\alpha_1, \alpha_2\}$. Also, if β belongs to $\Sigma(\theta)^+ - \Sigma(\Omega)^+$, then so does $\gamma - \beta - \alpha_1$. Using (21) several times, we can now write

$$\begin{aligned} uu_{\alpha_1}(1)u^{-1} &= \prod_{\beta \in \Sigma(\theta)^+} u_\beta(x_\beta)u_{\alpha_1}(1)\Big(\prod_{\beta \in \Sigma(\theta)^+} u_\beta(x_\beta)\Big)^{-1} \\ &= u_\gamma\Big(\sum c_{\alpha_1,\beta;1,1}c_{\alpha_1+\beta,\delta;1,1}x_\beta x_\delta\Big)\cdot \prod_{\beta \in \Sigma(\theta)^+ - \Sigma(\Omega)^+} u_{\alpha_1+\beta}(-c_{\alpha_1,\beta}x_\beta)\cdot u_{\alpha_1}(1), \end{aligned}$$

where the sum in the first term is over *unordered* pairs (β, δ) of roots in $\Sigma(\theta)^+ - \Sigma(\Omega)^+$ such that $\beta + \delta = \gamma - \alpha$ and $\beta \neq \delta$. Here the order of terms is prescribed by the order we fixed in (39). This implies that $uu_{\alpha_1}(1)u^{-1} = u_{\alpha_1}(1)$ if and only if $x_\beta = 0$ for all $\beta \in \Sigma(\theta)^+ - \Sigma(\Omega)^+$. Therefore,

$$U_{M,n} = \Big\{\prod_{\beta \in \Sigma(\Omega)^+} u_\beta(x_\beta)\Big\}. \tag{40}$$

To compute $U'_{M,m}$, note that with d as in (31) we have

$$\begin{aligned} m\,u\,m^{-1} &= w'\gamma^\vee\Big(\frac{d}{q}\Big) \prod_{\beta \in \Sigma(\theta)^+} u_\beta(x_\beta)\gamma^\vee\Big(\frac{d}{q}\Big)^{-1} w'^{-1} \\ &= w' \prod_{\beta \in \Sigma(\theta)^+} u_\beta\bigg(\beta\Big(\gamma^\vee\Big(\frac{d}{q}\Big)\Big)\, x_\beta\bigg) w'^{-1} \\ &= \prod_{\beta \in \Sigma(\theta)^+} w' u_\beta\bigg(\beta\Big(\gamma^\vee\Big(\frac{d}{q}\Big)\Big)\, x_\beta\bigg) w'^{-1}. \end{aligned}$$

Conjugation by the element w' sends each positive root group of a root in $\Sigma(\theta)^+ - \Sigma(\Omega)^+$ to a root group corresponding to a negative root and sends those with roots in $\Sigma(\Omega)^+$ to themselves. Therefore, again,

$$U'_{M,m} = \Big\{ \prod_{\beta \in \Sigma(\Omega)^+} u_\beta(x_\beta) \Big\}. \tag{41}$$

Now (37) follows from (40) and (41). □

We wish to have an explicit identification of $\mathrm{GL}_1(F) \times \mathbf{G}_n^\sim(F)$ with M as a Levi subgroup of G. Going back to our descriptions of the groups GSpin_{2n+1} and $\mathrm{GSpin}_{2n}^\sim$ in Section 2, note that if we consider the root datum obtained from that of $\mathbf{G}$ by eliminating e_1 and e_1^* in the odd case and E_1 and E_1^* in the even case as well as the root α_1 and its corresponding coroot, then the remaining root datum corresponds to a subgroup of $\mathbf{G}$ isomorphic to $\mathbf{G}_n^\sim$. Denote the F-points of this subgroup by $G_n^\sim$. Let $k \in G_n^\sim$, and let $a \in F^\times$. We claim that $e_1^*(a)$ in the odd case (or $E_1^*(a)$ in the even case) and k commute. To see this, it is enough to observe that $e_1^*(a)$ (or $E_1^*(a)$) commutes with $u_\beta(x)$ for all $\beta \in \Sigma(\theta)$ since $G_n^\sim$ is generated by the corresponding U_β's along with a subtorus of T. By (15), we have

$$e_1^*(a) u_\beta(x) e_1^*(a)^{-1} = u_\beta\big(\beta(e_1^*(a))x\big) = u_\beta(a^{\langle \beta, e_1^* \rangle} x)$$

and similarly for $E_1^*(a)$. Moreover, $\langle \beta, e_1^* \rangle = 0$ for all $\beta \in \Sigma(\theta)$. Therefore, $e_1^*(a)$ in the odd case (or $E_1^*(a)$ in the even case) and all the $u_\beta(x)$ commute. This implies that the maps $(a, k) \mapsto e_1^*(a)k$ in the odd case and $(a, k) \mapsto E_1^*(a)k$ in the even case give an isomorphism identifying $\mathrm{GL}_1(F) \times \mathbf{G}_n^\sim(F)$ with M. In particular, the element $m = \alpha_1{}^\vee(d/x)w'$ in (33) is identified with

$$\left(\frac{d}{x}, e_2^*\Big(\frac{x}{d}\Big)w'\right) \text{ or } \left(\frac{d}{x}, E_2^*\Big(\frac{x}{d}\Big)w'\right) \in \mathrm{GL}_1(F) \times \mathbf{G}_n^\sim(F) \tag{42}$$

since $\alpha_1{}^\vee = e_1^* - e_2^*$ in the odd case and $\alpha_1{}^\vee = E_1^* - E_2^*$ in the even case (and noting $w' \in G_n^\sim$). Moreover, $e_2^*(x/d)w'$ (or $E_2^*(x/d)w'$) is an element of a maximal Levi subgroup in G just as in the case of classical groups in [41].

We are now prepared to express the γ-factors as Mellin transforms.

PROPOSITION 4.14
Let σ be an irreducible admissible ψ-generic representation of $\mathbf{G}_n^\sim(F)$ (see Remark 4.2). Consider $\mathrm{GL}_1 \times \mathbf{G}_n^\sim$ as a standard Levi subgroup in $\mathbf{G}$ as above. Let η be any nontrivial character of $F^\times$ with η^2 ramified. Then

$$\gamma(s, \eta \times \sigma, \psi)^{-1} = g(s, \eta) \cdot \int_{F^\times} j_{v,\overline{N}_0}\big(a(x)w'\big)\eta(x)|x|^{s-n-\delta}\, dx^\times, \tag{43}$$

where $a(x) = e_2^*(x/d)$ or $E_2^*(x/d)$ is as in (42) (with d as in (31)), $\delta = 1/2$ and $g(s,\eta) = \eta(-1)^n$ in the odd case and $\delta = 1$ and $g(s,\eta) = \eta(-1)^{n-1}\gamma(2s,\eta^2,\psi)^{-1}$ in the even case. Here $v \in V_\sigma$ and $W_v \in \mathscr{W}(\sigma,\psi)$ with $W_v(e) = 1$, where $\mathscr{W}(\sigma,\psi)$ denotes the Whittaker model of σ. Moreover, $\overline{N}_0 \subset \overline{N}$ is a sufficiently large compact open subgroup of the opposite unipotent subgroup $\overline{N}$ to N, where $P = MN \subset G$ is the Levi decomposition of the corresponding standard parabolic subgroup. The function $j_{v,\overline{N}_0}$ denotes the partial Bessel function defined in [41].

Proof
Given that η^2 is ramified, this proposition is the main result of [41, Theorem 6.2, (6.39)] applied to our cases. Notice that our Propositions 4.13 and 4.3 verify the two hypotheses of that theorem (i.e., [41, Assumptions 4.1, 5.1]) for our cases.

To get from [41, (6.39)] to (43), note that we have

$$\omega_{\sigma_s}^{-1}(\dot{x}_\alpha)(w_0\omega_{\sigma_s})(\dot{x}_\alpha) = \eta(x)^2|x|^{2s}.$$

Moreover, as in [41, Section 7], we have

$$q^{\langle s\tilde{\alpha}+\rho, H_M(\dot{m})\rangle}\, d\dot{n} = |x|^{-s-n+\delta}\, dx^\times$$

and

$$j_{\tilde{v},\overline{N}_0}(\dot{m}) = \eta\Big(\frac{d}{x}\Big) j_{v,\overline{N}_0}\big(a(x)w'\big),$$

where $a(x) = e_2^*(x/d)$ in the odd case and $E_2^*(x/d)$ in the even case. □

In Section 4.2 we first rewrite (43) in terms of Bessel functions defined similarly to [10], and then we study their asymptotics.

4.2. Bessel functions and their asymptotics
We now briefly review some basic facts from [10]. Because of [8], and particularly [12] and [13], where these issues are studied more generally, we concentrate only on the cases at hand and leave out the details of the more general situation.

We use the same notation as in [10]. Consider the group $\mathbf{G}_n^\sim$ in both even and odd cases, and consider w' as an element of its Weyl group. Notice that w' supports a Bessel function and (in Bruhat order) is minimal nontrivial with respect to this property. Moreover, $A_{w'} = Z_{M_\Omega}$, where $\Omega = \Delta - \{\alpha_1,\alpha_2\}$, M_Ω is the standard Levi subgroup determined by Ω, and Z_{M_Ω} denotes its center.

Let σ be an irreducible admissible ψ-generic representation of $\mathbf{G}_n^\sim(F)$, and take $v \in V_\sigma$ such that the associated Whittaker function $W_v \in \mathscr{W}(\sigma,\psi)$ satisfies

$W_v(e) = 1$. The associated Bessel function on Z_{M_Ω} is defined via

$$J_{\sigma,w'}(a) = \int_{U_{w'}^-} W_v(aw'u)\psi^{-1}(u)\,du \tag{44}$$

with $a \in Z_{M_\Omega}$ and $U_{w'}^- = \prod_\alpha U_\alpha$, where the product is over all those $\alpha \in \Sigma(\theta)^+$ for which $w'(\alpha) < 0$, and U_α is as before.

Similar to [10] and [8], we have that $J_{\sigma,w'}$ exists and is independent of $v \in V_\sigma$, and for convergence purposes we use a slight modification of it, namely, the partial Bessel function

$$J_{\sigma,w',v,Y}(a) = \int_Y W_v(aw'y)\psi^{-1}(y)\,dy, \tag{45}$$

where $Y \subset U_{w'}^-$ is a compact open subgroup.

4.2.1. Domain of integration

We now show that the partial Bessel functions of [41] are the same as those in [10].

Recall that $M = M_\theta = \mathrm{GL}_1(F) \times \mathbf{G}_n^\sim(F) \subset G = \mathbf{G}(F)$, and consider $m \in M$ as in (42); that is, $m = (d/x, e_2^*(x/d)w')$ in the odd case and $m = (d/x, E_2^*(x/d)w')$ in the even case with d as in (31).

LEMMA 4.15

We can choose Y appropriately such that with $m' = e_2^*(x/d)w'$ or $E_2^*(x/d)w'$ as above, we have

$$j_{v,\overline{N}_0}(m') = \begin{cases} J_{\sigma,w',v,Y}\left(e_2^*\left(\dfrac{x}{d}\right)\right) & \text{in the odd case,} \\ J_{\sigma,w',v,Y}\left(E_2^*\left(\dfrac{x}{d}\right)\right) & \text{in the even case.} \end{cases}$$

Here, m' is an element of the maximal Levi subgroup $M' = \mathrm{GL}_1(F) \times \mathbf{G}_{n-1}^\sim(F)$ in $\mathbf{G}_n^\sim(F)$.

Proof

Let us first recall [41, Theorem 6.2]. In the notation of that article, we have $j_{v,\overline{N}_0}(m') = j_{v,\overline{N}_0}(m', y_0)$ with $y_0 \in F^\times$ satisfying $\mathrm{ord}_F(y_0) = -\mathrm{cond}(\psi) - \mathrm{cond}(\eta^2)$. Here, the function $j_{v,\overline{N}_0}(m', y_0)$ is given by

$$\int_{U_{M',n}\backslash U_{M'}} W_v(m'u^{-1})\,\phi\big(u\, e^*(y_0)^{-1}e^*(x_\alpha)\overline{n}e^*(x_\alpha)^{-1}e^*(y_0)\,u^{-1}\big)\psi(u)\,du, \tag{46}$$

where ϕ is the characteristic function of $\overline{N}_0$, $x_\alpha = 1/x$, $\overline{n}$ is as in Proposition 4.12, and e^* is as in Proposition 4.3. Again as in [10] and [8], it follows from Proposition 4.3

that we can take $U_{M,n}\backslash U_M$ to be $U_{w'}^-$. Notice that this depends only on w'. On the other hand, $u \in U_{w'}^-$ is in the domain of integration if and only if

$$ue^*(y_0)^{-1}e^*(x_\alpha)\overline{n}e^*(x_\alpha)^{-1}e^*(y_0)u^{-1} \in \overline{N}_0.$$

This condition is equivalent to $ue^*(x_\alpha)\overline{n}e^*(x_\alpha)^{-1}u^{-1} \in e^*(y_0)\overline{N}_0e^*(y_0)^{-1}$, and $e^*(y_0)\overline{N}_0e^*(y_0)^{-1}$ is another compact open subgroup of the same type as $\overline{N}_0$, which we may replace it with.

Recall that an arbitrary element of $\overline{N}$ is given by

$$\overline{n}(y) = \prod_{\alpha \in R(\overline{N})} u_\alpha(y_\alpha), \tag{47}$$

where $y = (y_\alpha)_{\alpha \in R(\overline{N})}$ and $y_\alpha \in F$. Also recall that $\overline{n}$ in (46) was given by $\overline{n} = u_{-\gamma}(1/x)u_{-\alpha_1}(1)$. Moreover, note that $x_\alpha = 1/x$. Hence, $e^*(x_\alpha)\overline{n}e^*(x_\alpha)^{-1} = \overline{n}(y')$, where $y'_{-\gamma} = 1$, $y'_{-\alpha_1} = x$, and all other coordinates of y' are zero (see (15)). Of course, throughout we have a fixed ordering of the roots in the products similar to that of (28). Hence, the domain of integration is determined by $u\overline{n}(y')u^{-1} \in \overline{N}_0$.

We may take $\overline{N}_0 = \{\overline{n}(y) : y_\alpha \in \mathfrak{p}^{M_\alpha}\}$ for all $\alpha \in R(\overline{N})$ for a sufficiently large integer vector $M = (M_\alpha)_{\alpha \in R(\overline{N})}$. As the M_α's increase, $\overline{N}_0$ exhausts $\overline{N}$.

On the other hand, any $u \in U_{w'}^-$ is given by

$$u = u(b) = \prod_{\alpha \in \Sigma^+(\Omega)} u_\alpha(b_\alpha) \tag{48}$$

for $b = (b_\alpha)_{\alpha \in \Sigma^+(\Omega)}$ and $b_\alpha \in F$. Now, $u\overline{n}(y')u^{-1} = \overline{n}(y'')$, where y'' depends linearly on b and y'. In other words, y'' depends upon x and b. Of course, we could compute y'' explicitly in terms of x and b using structure constants; however, that has no bearing on what follows and is not needed. Now, choose $Y = \{u = u(b) : y'' \geq M\}$. This defines the domain of integration. Enlarging $\overline{N}_0$ if need be would then imply that the domain Y does not depend on m, and we conclude the lemma. □

Therefore, we can rewrite (43) as

$$\gamma(s, \eta \times \sigma, \psi)^{-1} = g(s, \eta) \cdot \int_{F^\times} J_{\sigma,w',v,Y}\big(a(x)\big)\eta(x)|x|^{s-n-\delta}\, dx^\times \tag{49}$$

with $a(x) = e_2^*(x/d)$ or $E_2^*(x/d)$.

4.2.2. Asymptotics of Bessel functions

We now study the asymptotics of our Bessel functions near zero and infinity. This allows us to prove our result on stability given that the γ-factors are already written as the Mellin transform of Bessel functions.

Our starting point is the following analogue of [10, Proposition 5.1] for our groups. Note that the proposition was proved only for the group SO_{2n+1}. However, as was pointed out in [8], the methods used to prove it are quite general. This was pointed out for classical groups (with finite center) in [8]. But the same also holds for GSpin or $GSpin^{\sim}$ groups (see [12]), which are of interest to us since the only difference is the infinite center that is already contained in the fixed Borel subgroup we are dividing out with.

PROPOSITION 4.16
There exist a vector $v'_\sigma \in V_\sigma$ and a compact neighborhood BK_1 of the identity in $B\backslash G_n^{\sim}$ such that if χ_1 is the characteristic function of BK_1, then for all sufficiently large compact open sets $Y \subset U_w^-$ we have

$$J_{\sigma,w,v,Y}(a) = \int_Y W_v(awy)\chi_1(awy)\psi^{-1}(y)\,dy + W_{v'_\sigma}(a). \tag{50}$$

Notice that w here would be our earlier w' if we wanted to consider the group $G_n^{\sim}$ as part of the Levi subgroup M in G, as in (42).

We now rewrite this in a way that depends only on the central character of σ. To this end, we argue similarly to [8], making some necessary modifications along the way. For any positive integer M, set

$$U(M) = \langle u_\alpha(t) : \alpha \in \Delta, |t| \le q^M \rangle.$$

These are compact open subgroups of U, and as the integer M grows, they exhaust U. For any $v \in V_\sigma$, we define

$$v_M = \frac{1}{\mathrm{Vol}(U(M))} \int_{U(M)} \psi^{-1}(u)\sigma(u)v\,du.$$

The smoothness of σ implies that this is a finite sum and $v_M \in V_\sigma$. Then just as in [10], if Y is sufficiently large relative to M, we may choose v'_σ and K_1 such that $K_1 \subset \mathrm{Stab}(v_M)$, and we have

$$\int_Y W_v(awy)\chi_1(awy)\psi^{-1}(y)\,dy = \int_Y W_{v_M}(awy)\chi_1(awy)\psi^{-1}(y)\,dy. \tag{51}$$

Write $awy = utk_1$ with $u \in U$, $t \in T$, and $k_1 \in K_1$. Then $K_1 \subset \mathrm{Stab}(v_M)$ implies that $W_{v_M}(awy) = \psi(u)W_{v_M}(t)$. As in [10], the support of W_{v_M} is contained in

$$T_M = \{t \in T : \alpha(t) \in 1 + \mathfrak{p}^M \text{ for all simple } \alpha\}. \tag{52}$$

At this point we assume that the center $\mathbf{Z}$ is connected, which we can do (see Remark 4.2). This is why we chose to work with the group GSpin$^{\sim}$ in the even case in this section. By the connectedness of $\mathbf{Z}$, and since the groups are split, we have the following exact sequence of F-points of tori:

$$0 \longrightarrow Z \longrightarrow T \longrightarrow T_{ad} \longrightarrow 0, \tag{53}$$

which splits (see [12], [13]). Recall that $Z = \mathbf{Z}(F)$ and so on. Identify T_{ad} with $(F^{\times})^n$ through values of roots, and let $T_M^1 \subset T$ be the image of

$$(1 + \mathfrak{p}^M)^n \subset T_{ad},$$

under the splitting map. Here, the rank of T_{ad} is n and $T_M = ZT_M^1$.

Now if $t \in T$, then we can write $t = zt^1$ with $z \in Z$ and $t \in T_M^1$. Also, we have $W_v(t) = W_{v_M}(t)$, and if we choose M large enough, so that $T_M^1 \subset T \cap \mathrm{Stab}(v)$, then

$$W_{v_M}(t) = W_v(t) = W_v(zt^1) = \omega_\sigma(z) W_v(t^1) = \omega_\sigma(z).$$

Next, we note that in our integral, $W_{v_M}(awy)\chi_1(awy) \neq 0$ if and only if $awy \in UT_MK_1$ or $y \in (aw)^{-1}UT_MK_1$. Writing $awy = utk_1 = u(awy)z(awy)t^1k_1$ then implies that

$$\int_Y W_v(awy)\chi_1(awy)\psi^{-1}(y)\,dy = \int_{Y\cap(aw)^{-1}UT_MK_1} \psi\big(u(awy)\big)\psi^{-1}(y)\omega_\sigma\big(z(awy)\big)\,dy. \tag{54}$$

Therefore, we can rewrite Proposition 4.16 as follows.

PROPOSITION 4.17
Let $v \in V_\sigma$ with $W_v(e) = 1$, and choose M sufficiently large, so that $T_M^1 \subset T \cap \mathrm{Stab}(v)$. There exist a vector $v'_\sigma \in V_\sigma$ and a compact open subgroup K_1 such that for Y sufficiently large we have

$$J_{\sigma,w,v,Y}(a) = \int_{Y\cap(aw)^{-1}UT_MK_1} \psi\big(u(awy)\big)\psi^{-1}(y)\omega_\sigma\big(z(awy)\big)\,dy + W_{v'_\sigma}(a).$$

4.3. Proof of Theorem 4.1

Let $\sigma_i = \pi_i$, $i = 1, 2$, in the odd case. In the even case, choose a character μ of the center $Z^{\sim}$ of $\mathrm{GSpin}_{2n}^{\sim}(F)$ (which contains the center of $\mathrm{GSpin}_{2n}(F)$) such that μ agrees with the central characters $\omega_{\pi_1} = \omega_{\pi_2}$ on the center of $\mathrm{GSpin}_{2n}(F)$. Consider the representation of $\mathrm{GSpin}_{2n}^{\sim}(F)$ induced from the representation $\mu \otimes \pi_i$ on $Z^{\sim} \cdot \mathrm{GSpin}_{2n}(F)$ (which is of finite index in $\mathrm{GSpin}_{2n}^{\sim}(F)$), and let σ_i be an irreducible constituent of this induced representation (see [45]). Note that the choice of σ_i is irrelevant. Then

$$\gamma(s, \eta \times \sigma_i, \psi) = \gamma(s, \eta \times \pi_i, \psi),$$

by Remark 4.2. Also, the assumption $\omega_{\pi_1} = \omega_{\pi_2}$ implies $\omega_{\sigma_1} = \omega_{\sigma_2}$.

Choose $v_i \in V_{\sigma_i}$, $i = 1, 2$, with $W_{v_i}(e) = 1$, and let M be a large-enough integer, so that $T_M^1 \subset T \cap \mathrm{Stab}(v_i)$. Choose a compact open subgroup $K_0 \subset \mathrm{Stab}(v_1) \cap \mathrm{Stab}(v_2)$. Then in Proposition 4.17 we may take

$$K_1 = \bigcap_{u \in U(M)} u^{-1} K_0 u;$$

that is, we can take the same K_1 for both σ_1 and σ_2. Consequently, by Proposition 4.17, there exist $v'_{\sigma_i} \in V_{\sigma_i}$ such that

$$J_{\sigma_i,w,v,Y}(a) = \int_{Y \cap (aw)^{-1} U T_M K_1} \psi\big(u(awy)\big)\psi^{-1}(y)\omega_{\sigma_i}\big(z(awy)\big)\, dy + W_{v'_{\sigma_i}}(a). \quad (55)$$

Now $\omega_{\sigma_1} = \omega_{\sigma_2}$ implies that

$$J_{\sigma_1,w,v,Y}(a) - J_{\sigma_2,w,v,Y}(a) = W_{v'_{\sigma_1}}(a) - W_{v'_{\sigma_2}}(a). \quad (56)$$

Now taking $a = a(x)$ to be $e_2^*(x/d)$ or $E_2^*(x/d)$ and w to be the w' described before, we apply (49) to conclude that

$$\begin{aligned}
&\gamma(s, \eta \times \sigma_1, \psi)^{-1} - \gamma(s, \eta \times \sigma_2, \psi)^{-1} \\
&\quad = g(s, \eta) \int_{F^\times} \big(J_{\sigma_1,w,v,Y}(a(x)) - J_{\sigma_2,w,v,Y}(a(x))\big)\ \eta(x)\ |x|^{s-n+\delta}\, d^\times x \\
&\quad = g(s, \eta) \int_{F^\times} \big(W_{v'_{\sigma_1}}(a(x)) - W_{v'_{\sigma_2}}(a(x))\big)\ \eta(x)\ |x|^{s-n+\delta}\, d^\times x.
\end{aligned}$$

However, note that Whittaker functions are smooth, and for $\Re(s) \gg 0$ and η sufficiently ramified, we have

$$\int_{F^\times} W_{v'_{\sigma_i}}\big(a(x)\big)\eta(x)|x|^{s-n+\delta}\, d^\times x \equiv 0.$$

Hence, for $\Re(s) \gg 0$, we have $\gamma(s, \eta \times \sigma_1, \psi)^{-1} - \gamma(s, \eta \times \sigma_2, \psi)^{-1} \equiv 0$, which then implies $\gamma(s, \eta \times \sigma_1, \psi) = \gamma(s, \eta \times \sigma_2, \psi)$ for all s by analytic continuation. Therefore, $\gamma(s, \eta \times \pi_1, \psi) = \gamma(s, \eta \times \pi_2, \psi)$. □

4.4. Stable form of $\gamma(s, \eta \times \pi, \psi)$

We now prove some consequences of Theorem 4.1 that are needed later.

First, let us compute the stable form of Theorem 4.1 by taking π_2 to be an appropriate principal series representation and computing its right-hand side explicitly.

PROPOSITION 4.18
Let π be an irreducible generic representation of $\mathbf{G}_n(F)$ with central character $\omega = \omega_\pi$. Let $\mu_1, \ldots, \mu_n$ be n characters of $F^\times$. Then for every sufficiently ramified character η of $F^\times$ we have

$$\gamma(s, \eta \times \pi, \psi) = \prod_{i=1}^{n} \gamma(s, \eta\mu_i, \psi)\gamma(s, \eta\omega\mu_i^{-1}, \psi).$$

Proof
Set $\mu_0 = \omega$, and consider the character

$$\mu = (\mu_0 \circ e_0) \otimes (\mu_1 \circ e_1) \otimes \cdots \otimes (\mu_n \circ e_n)$$

of $\mathbf{T}(F)$ with e_i's as in Section 2.1. Proposition 2.3 implies that the restriction of the character μ to the center of $\mathbf{G}_n(F)$ is $\mu_0 = \omega$. Consider the induced representation $\mathrm{Ind}(\mu)$ from the Borel subgroup to $\mathbf{G}_n(F)$. Reordering the μ_i if necessary, we may assume that it has an irreducible admissible generic subrepresentation π_2 (see Proposition 3.2). Since $\omega_{\pi_2} = \mu_0 = \omega = \omega_\pi$, we can apply Theorem 4.1 to get $\gamma(s, \eta \times \pi, \psi) = \gamma(s, \eta \times \pi_2, \psi)$. The multiplicativity of γ-factors can now be used to compute the right-hand side to get

$$\gamma(s, \eta \times \pi_2, \psi) = \prod_{i=1}^{n} \gamma(s, \eta\mu_i, \psi)\gamma(s, \eta\omega\mu_i^{-1}, \psi). \tag{57}$$

This completes the proof. □

COROLLARY 4.19
Let π be an irreducible generic representation of $\mathbf{G}_n(F)$ with central character $\omega = \omega_\pi$. Let $\mu_1, \ldots, \mu_n$ be n characters of $F^\times$ as in Proposition 4.18. Then for every sufficiently ramified character η of $F^\times$ we have

$$L(s, \eta \times \pi) \equiv 1$$

and

$$\epsilon(s, \eta \times \pi, \psi) = \prod_{i=1}^{n} \epsilon(s, \eta\mu_i, \psi)\epsilon(s, \eta\omega\mu_i^{-1}, \psi).$$

Proof
If η is sufficiently ramified, then by [40] we have

$$L(s, \eta \times \pi) \equiv 1.$$

This implies that $\epsilon(s, \eta\times\pi, \psi) = \gamma(s, \eta\times\pi, \psi)$. Moreover, since η is highly ramified, so is each $\eta\mu_i$ and $\eta\omega\mu_i^{-1}$. This implies that $L(s, \eta\mu_i) \equiv 1$ and $L(s, \eta\omega\mu_i^{-1}) \equiv 1$. Therefore, $\epsilon(s, \eta\mu_i, \psi) = \gamma(s, \eta\mu_i, \psi)$ and $\epsilon(s, \eta\omega\mu_i^{-1}, \psi) = \gamma(s, \eta\omega\mu_i^{-1}, \psi)$. The second statement of the corollary follows from Proposition 4.18. □

5. Analytic properties of global L-functions

In this section we prove the properties of global L-functions which we need in order to apply the Converse Theorems.

We again let $\mathbf{G}_n$ denote either the group GSpin_{2n+1} or GSpin_{2n} as in Section 2.1. Let k be a number field, and let $\mathbb{A}$ be its ring of adèles. Let S be a finite set of finite places of k. Let $\mathcal{T}(S)$ denote the set of irreducible cuspidal automorphic representations τ of $\mathrm{GL}_r(\mathbb{A})$ for $1 \le r \le N-1$ such that τ_v is unramified for all $v \in S$. If η is a continuous complex character of $k^\times\backslash\mathbb{A}^\times$, then we let $\mathcal{T}(S;\eta) = \{\tau = \tau' \otimes \eta \,:\, \tau' \in \mathcal{T}(S)\}$.

If π is a globally generic cuspidal representation of $\mathbf{G}_n(\mathbb{A})$ and τ is a cuspidal representation of $\mathrm{GL}_r(\mathbb{A})$ in $\mathcal{T}(S;\eta)$, then $\sigma = \tau \otimes \tilde{\pi}$ is a (unitary) cuspidal globally generic representation of $\mathbf{M}(\mathbb{A})$, where $\mathbf{M} = \mathrm{GL}_r \times \mathbf{G}_n$ is a Levi subgroup of a standard parabolic subgroup in $\mathbf{G}_{r+n}$. The machinery of the Langlands-Shahidi method as mentioned in Section 3 now applies (see [37], [39]). Recall that

$$L(s, \pi \times \tau) = \prod_v L(s, \pi_v \times \tau_v), \tag{58}$$

$$\epsilon(s, \pi \times \tau) = \prod_v \epsilon(s, \pi_v \times \tau_v, \psi_v), \tag{59}$$

where the local factors are as in (5) and (6).

PROPOSITION 5.1
Let S be a nonempty set of finite places of k, and let η be a character of $k^\times\backslash\mathbb{A}^\times$ such that η_v is highly ramified for $v \in S$. Then for all $\tau \in \mathcal{T}(S;\eta)$, the L-function $L(s, \pi \times \tau)$ is entire.

Proof
These L-functions are defined via the Langlands-Shahidi method, as we outlined in Section 3. The proposition is a special case of a more general result, [24, Theorem 2.1] (see [20] for the original idea). Note that we have proved the necessary assumption of that theorem, [24, Assumption 1.1], for our cases in Proposition 3.6. □

The following lemma is an immediate consequence of Proposition 3.6.

LEMMA 5.2
The global normalized intertwining operator $N(s, \sigma, w)$ is a holomorphic and nonzero operator for $\Re(s) \geq 1/2$.

PROPOSITION 5.3
For any cuspidal automorphic representation τ of $\mathrm{GL}_r(\mathbb{A}_F)$, $1 \leq r \leq 2n-1$, the L-function $L(s, \pi \times \tau)$ is bounded in vertical strips.

Proof
This follows as a consequence of [15, Theorem 4.1] along the lines of [15, Corollary 4.5], given the fact that we have proved [15, Assumption 2.1] in our Lemma 5.2 for our cases. □

PROPOSITION 5.4
For any cuspidal automorphic representation τ of $\mathrm{GL}_r(\mathbb{A}_F)$, $1 \leq r \leq 2n$, we have the functional equation

$$L(s, \pi \times \tau) = \epsilon(s, \pi \times \tau) L(1-s, \widetilde{\pi} \times \widetilde{\tau}).$$

Proof
This is a special case of [39, Theorem 7.7]. □

6. Proof of the main theorem

As mentioned before, we use the following variant of converse theorems of Cogdell and Piatetski-Shapiro. This version of the Converse Theorems appeared in [7, Section 2].

THEOREM 6.1
Let $\Pi = \bigotimes \Pi_v$ be an irreducible admissible representation of $\mathrm{GL}_N(\mathbb{A})$ whose central character ω_Π is invariant under $k^\times$ and whose L-function $L(s, \Pi) = \prod_v L(s, \Pi_v)$ is absolutely convergent in some right half-plane. With notation as in Section 5, suppose that for every $\tau \in \mathscr{T}(S; \eta)$, we have that:
(1) $L(s, \Pi \times \tau)$ *and* $L(s, \widetilde{\Pi} \times \widetilde{\tau})$ *extend to entire functions of* $s \in \mathbb{C}$;
(2) $L(s, \Pi \times \tau)$ *and* $L(s, \widetilde{\Pi} \times \widetilde{\tau})$ *are bounded in vertical strips; and*
(3) $L(s, \Pi \times \tau) = \epsilon(s, \Pi \times \tau) L(1-s, \widetilde{\Pi} \times \widetilde{\tau})$.
Then there exists an automorphic representation Π' of $\mathrm{GL}_N(\mathbb{A})$ such that $\Pi_v \simeq \Pi'_v$ for all $v \notin S$.

Here, the twisted L- and ϵ-factors are defined via

$$L(s, \Pi \times \tau) = \prod_v L(s, \Pi_v \times \tau_v), \qquad \epsilon(s, \Pi \times \tau) = \prod_v \epsilon(s, \Pi_v \times \tau_v, \psi_v)$$

with local factors as in [9].

We can now prove Theorem 1.1.

Proof of Theorem 1.1
We apply Theorem 6.1 with $N = 2n$. We continue to denote by $\mathbf{G}_n$ either GSpin_{2n+1} or GSpin_{2n}. First, we introduce a candidate for the representation Π. Consider $\pi = \bigotimes \pi_v$, and let S be as in the statement of the theorem, that is, a nonempty set of non-Archimedean places v such that for all finite $v \notin S$, both π_v and ψ_v are unramified.

(i): $v < \infty$ and π_v unramified. Choose Π_v as in the statement of the theorem via the Frobenius-Hecke (or Satake) parameter. More precisely, since π_v is unramified, it is given by an unramified character χ of the maximal torus $\mathbf{T}(k_v)$. This means that there are unramified characters $\chi_0, \chi_1, \ldots, \chi_n$ of $k_v^\times$ such that for $t \in \mathbf{T}(k_v)$,

$$\chi(t) = (\chi_0 \circ e_0)(t)(\chi_1 \circ e_1)(t) \cdots (\chi_n \circ e_n)(t), \tag{60}$$

where e_i's form the basis of the rational characters of the maximal torus of $\mathbf{G}$ as in Section 2.1. The character χ corresponds to an element $\hat{t}$ in $\widehat{T}$, the maximal torus of (the connected component) of the Langlands dual group which is $\mathrm{GSp}_{2n}(\mathbb{C})$ or $\mathrm{GSO}_{2n}(\mathbb{C})$, uniquely determined by the equation

$$\chi\left(\phi(\varpi)\right) = \phi(\hat{t}), \tag{61}$$

where ϖ is a uniformizer of our local field k_v and $\phi \in X_*(\mathbf{T}) = X^*(\widehat{T})$ (see [14, (I.2.3.3), page 26]). We make this identification explicit via the correspondence $e_i^* \longleftrightarrow e_i$ for $i = 0, \ldots, n$ as in Section 2.1, which gave the duality of $\mathrm{GSpin}_{2n+1} \longleftrightarrow \mathrm{GSp}_{2n}$ and $\mathrm{GSpin}_{2n} \longleftrightarrow \mathrm{GSO}_{2n}$. Applying (61) with the ϕ on the left-hand side replaced with e_i^* and the one on the right-hand side replaced with e_i for $i = 0, 1, \ldots, n$ yields

$$\chi_i(\varpi) = \chi\left(e_i^*(\varpi)\right) = e_i(\hat{t}), \quad i = 0, 1, \ldots, n. \tag{62}$$

We can now compute the Satake parameter explicitly as an element $\hat{t}$ in the maximal torus $\widehat{T}$ of $\mathrm{GSp}_{2n}(\mathbb{C})$ or $\mathrm{GSO}_{2n}(\mathbb{C})$, as described in (3). If we write our unramified characters as $\chi_i(\ ) = |\ |_v^{s_i}$ for $s_i \in \mathbb{C}$ and $0 \le i \le n$, then we get

$$\hat{t} = \begin{pmatrix} |\varpi|^{s_1} & & & & & \\ & \ddots & & & & \\ & & |\varpi|^{s_n} & & & \\ & & & |\varpi|^{s_0-s_n} & & \\ & & & & \ddots & \\ & & & & & |\varpi|^{s_0-s_1} \end{pmatrix}. \tag{63}$$

Hence, Π_v is the unique unramified constituent of the representation of $\mathrm{GL}_{2n}(k_v)$ induced from the character

$$\chi_1 \otimes \cdots \otimes \chi_n \otimes \chi_0\chi_n^{-1} \otimes \cdots \otimes \chi_0\chi_1^{-1} \tag{64}$$

of the k_v-points of the standard maximal torus in GL_{2n}.

A crucial point here is what the central characters of π_v and Π_v are. It follows from Proposition 2.3 that the central character $\omega_{\pi_v} = \chi_0$. Moreover, the central character ω_{Π_v} of Π_v is χ_0^n; hence, we have $\omega_{\Pi_v} = \omega_{\pi_v}^n$.

Furthermore, note that $\widetilde{\Pi}_v$ is the unique unramified constituent of the representation induced from

$$\chi_1^{-1} \otimes \cdots \otimes \chi_n^{-1} \otimes \chi_0^{-1}\chi_n \otimes \cdots \otimes \chi_0^{-1}\chi_1.$$

Therefore, we have $\widetilde{\Pi}_v \simeq \chi_0^{-1} \otimes \Pi_v$. In other words, $\Pi_v \simeq \widetilde{\Pi}_v \otimes \omega_{\pi_v}$.

(ii): $v|\infty$. Choose Π_v as in the statement of Theorem 1.1 (see [30]). To be more precise, Langlands associates to π_v a homomorphism ϕ_v from the local Weil group $W_v = W_{k_v}$ to the dual group $\widehat{G}$ which is $\mathrm{GSp}_{2n}(\mathbb{C})$ or $\mathrm{GSO}_{2n}(\mathbb{C})$ in our cases. Both of these groups have natural embeddings ι into $\mathrm{GL}_{2n}(\mathbb{C})$, and we take Π_v to be the irreducible admissible representation of $\mathrm{GL}_{2n}(k_v)$ associated to $\Phi_v = \phi_v \circ \iota$ in [30].

Again, we want to show that $\omega_{\Pi_v} = \omega_{\pi_v}^n$ and $\Pi_v \simeq \widetilde{\Pi}_v \otimes \omega_{\pi_v}$. To do this, we use some well-known facts regarding representations of W_v and local Langlands correspondence for $\mathrm{GL}_n(\mathbb{R})$ and $\mathrm{GL}_n(\mathbb{C})$. We refer to [26] for a nice survey of these results.

First, assume that $k_v = \mathbb{C}$. Then $W_v = \mathbb{C}^\times$ and any irreducible representation of W_v is one-dimensional and of the form

$$z \mapsto [z]^\ell |z|_{\mathbb{C}}^t, \quad \ell \in \mathbb{Z},\ t \in \mathbb{C},$$

where $[z] = z/|z|$ and $|z|_{\mathbb{C}} = |z|^2$.

The $2n$-dimensional representation Φ_v of W_v can now be written as a direct sum of $2n$ one-dimensional representations as above. Moreover, $\Phi_v(z) = \phi_v(z)$, considered as a diagonal matrix in $\mathrm{GL}_{2n}(\mathbb{C})$, actually lies, up to conjugation, in $\mathrm{GSp}_{2n}(\mathbb{C})$ or $\mathrm{GSO}_{2n}(\mathbb{C})$ as in (3). Therefore, there exist one-dimensional representations $\phi_0, \phi_1, \ldots, \phi_n$ as above such that Φ_v is the direct sum of $\phi_1, \ldots, \phi_n, \phi_n^{-1}\phi_0, \ldots, \phi_1^{-1}\phi_0$. Now the central characters of Π_v and π_v can be written as $\omega_{\Pi_v}(z) = \det(\Phi_v(z))$ and $\omega_{\pi_v}(z) = e_0(\phi_v(z))$, where $\phi_v(z) = \Phi_v(z)$ is considered as a diagonal $(2n \times 2n)$-matrix as in (3) and e_0 is as in (4). In other words, $\omega_{\pi_v} = \phi_0$ and $\omega_{\Pi_v} = \phi_0^n$ or $\omega_{\Pi_v} = \omega_{\pi_v}^n$.

Moreover, $\widetilde{\Pi}_v$ corresponds to the $2n$-dimensional representation of W_v which is the direct sum of $\phi_1^{-1}, \ldots, \phi_n^{-1}, \phi_n\phi_0^{-1}, \ldots, \phi_1\phi_0^{-1}$, implying that the two representations Π_v and $\widetilde{\Pi}_v \otimes \omega_{\pi_v}$ have the same parameters; that is, $\Pi_v \simeq \widetilde{\Pi}_v \otimes \omega_{\pi_v}$.

Next, assume that $k_v = \mathbb{R}$. Then $W_v = \mathbb{C}^\times \cup j\mathbb{C}^\times$ with $j^2 = -1$ and $jzj^{-1} = \bar{z}$ for $z \in \mathbb{C}^\times$. Here the situation is identical, and the only difference is that W_v also has two-dimensional irreducible representations. The one-dimensional representations of W_v can be described as

$$z \mapsto |z|_{\mathbb{R}}^t, \qquad j \mapsto 1, \quad t \in \mathbb{C},$$

$$z \mapsto |z|_{\mathbb{R}}^t, \qquad j \mapsto -1, \quad t \in \mathbb{C},$$

with $|z|_{\mathbb{R}} = |z|$, and the irreducible two-dimensional representations are of the form

$$z = re^{i\theta} \mapsto \begin{pmatrix} r^{2t}e^{i\ell\theta} & \\ & r^{2t}e^{-i\ell\theta} \end{pmatrix}, \qquad j \mapsto \begin{pmatrix} 0 & (-1)^\ell \\ 1 & 0 \end{pmatrix},$$

where $t \in \mathbb{C}$ and $\ell \geq 1$ is an integer. These correspond, respectively, to representations $1 \otimes |\cdot|_{\mathbb{R}}^t$ and $\mathrm{sgn} \otimes |\cdot|_{\mathbb{R}}^t$ of $\mathrm{GL}_1(\mathbb{R})$ and $D_\ell \otimes |\cdot|_{\mathbb{R}}^t$ of $\mathrm{GL}_2(\mathbb{R})$. Here, D_ℓ is the representation of $\mathrm{SL}_2^{\pm}(\mathbb{R})$ induced from the discrete series (limit of discrete series when $\ell = 1$) representation D_ℓ^+ on the group $\mathrm{SL}_2(\mathbb{R})$, the discrete series of lowest weight $\ell + 1$ (see [26, Section 2]).

Notice that again $\Phi_v(z) = \phi_v(z)$ is a diagonal $(2n \times 2n)$-matrix in $\mathrm{GSp}_{2n}(\mathbb{C})$ or $\mathrm{GSO}_{2n}(\mathbb{C})$, as in the previous case, while $\Phi_v(j)$ may have (2×2)-blocks as well. Therefore, we still have $\omega_{\Pi_v} = \omega_{\pi_v}^n$ and $\Pi_v \simeq \widetilde{\Pi}_v \otimes \omega_{\pi_v}$ as before.

(iii): $v < \infty$ and π_v ramified. Choose Π_v to be an arbitrary irreducible admissible representation of $\mathrm{GL}_{2n}(k_v)$ with $\omega_{\Pi_v} = \omega_{\pi_v}^n$.

Let $\Pi = \bigotimes_v \Pi_v$. Then Π is an irreducible admissible representation of $\mathrm{GL}_{2n}(\mathbb{A})$ whose central character ω_Π is equal to ω_π^n and hence is invariant under $k^\times$. Moreover, for all $v \notin S$, we have that $L(s, \pi_v) = L(s, \Pi_v)$ by construction. Hence, $L^S(s, \Pi) =$

$L^S(s, \pi)$, where

$$L^S(s, \Pi) = \prod_{v \notin S} L(s, \Pi_v), \qquad L^S(s, \pi) = \prod_{v \notin S} L(s, \pi_v).$$

Therefore, $L(s, \Pi) = \prod_v L(s, \Pi_v)$ is absolutely convergent in some right half-plane.

Choose $\eta = \bigotimes_v \eta_v$ to be a unitary character of $k^\times \backslash \mathbb{A}^\times$ such that η_v is sufficiently ramified for $v \in S$ in order for Theorem 4.1 to hold and such that at one place η_v^2 is still ramified. For $\tau \in \mathscr{T}(S; \eta)$, we claim the following equalities (along with their analogous equalities for the contragredients):

$$L(s, \Pi_v \times \tau_v) = L(s, \pi_v \times \tau_v), \tag{65}$$

$$\epsilon(s, \Pi_v \times \tau_v, \psi_v) = \epsilon(s, \pi_v \times \tau_v, \psi_v). \tag{66}$$

Here the L- and ϵ-factors on the left are as in [9], and those on the right are defined via the Langlands-Shahidi method (see [39], [37]).

To see (65) and (66), we again consider different places separately.

(i): $v < \infty$ and π_v unramified. Let π_v be again as in (60) with Satake parameter (63). Then Π_v is as in (64). By [17], we have

$$L(s, \Pi_v \times \tau_v) = \prod_{i=1}^{n} L(s, \tau_v \otimes \chi_i) L(s, \tau_v \otimes \chi_0 \chi_i^{-1}), \tag{67}$$

$$L(s, \widetilde{\Pi}_v \times \widetilde{\tau}_v) = \prod_{i=1}^{n} L(s, \widetilde{\tau}_v \otimes \chi_i^{-1}) L(s, \widetilde{\tau}_v \otimes \chi_0^{-1} \chi_i),$$

and

$$\epsilon(s, \Pi_v \times \tau_v, \psi_v) = \prod_{i=1}^{n} \epsilon(s, \tau_v \otimes \chi_i, \psi_v) \epsilon(s, \tau_v \otimes \chi_0 \chi_i^{-1}, \psi_v). \tag{68}$$

On the other hand, it follows from the inductive property of γ-factors in the Langlands-Shahidi method (see [39, Theorem 3.5] or [38]) that

$$\gamma(s, \pi_v \times \tau_v, \psi_v) = \prod_{i=1}^{n} \gamma(s, \tau_v \otimes \chi_i, \psi_v) \gamma(s, \tau_v \otimes \chi_0 \chi_i^{-1}, \psi_v), \tag{69}$$

just as in (57).

Since τ_v is generic, it is a full-induced representation from generic essentially tempered ones. Thus, we can write

$$\tau_v \simeq \mathrm{Ind}(\nu^{b_1} \tau_{1,v} \otimes \cdots \otimes \nu^{b_p} \tau_{p,v}), \tag{70}$$

where each $\tau_{j,v}$ is a tempered representation of some $\mathrm{GL}_{r_j}(k_v)$, $\nu(\ ) = |\det(\ )|_v$ on $\mathrm{GL}_{r_j}(k_v)$, $r_1 + \cdots + r_p = r$, and the $\tau_{j,v}$ are in the Langlands order. Moreover, recall that π_v is the unique irreducible unramified subrepresentation of the representation of $\mathbf{G}_n(k_v)$ induced from the character χ as in (60) after an appropriate reordering, if necessary.

Now, by the definition of L-functions (see [39, Section 7]) and their multiplicative property (see [38, Theorem 5.2]), we have

$$\begin{aligned} L(s, \pi_v \times \tau_v) &= \prod_{j=1}^{p} L(s + b_j, \pi_v \times \tau_{j,v}) \\ &= \prod_{j=1}^{p} \prod_{i=1}^{n} L(s + b_j, \tau_{j,v} \otimes \chi_i) L(s + b_j, \tau_{j,v} \otimes \chi_0 \chi_i^{-1}) \\ &= \prod_{i=1}^{n} L(s, \tau_v \otimes \chi_i) L(s, \tau_v \otimes \chi_0 \chi_i^{-1}), \end{aligned} \tag{71}$$

and likewise,

$$L(s, \widetilde{\pi}_v \times \widetilde{\tau}_v) = \prod_{i=1}^{n} L(s, \widetilde{\tau}_v \otimes \chi_i^{-1}) L(s, \widetilde{\tau}_v \otimes \chi_0^{-1} \chi_i). \tag{72}$$

Note that [38, Conjecture 5.1], which is a hypothesis of [38, Theorem 5.2], is known in our cases by [3, Theorem 5.7].

Equations (69), (71), and (72) in turn imply

$$\epsilon(s, \pi_v \times \tau_v, \psi_v) = \prod_{i=1}^{n} \epsilon(s, \tau_v \otimes \chi_i, \psi_v) \epsilon(s, \tau_v \otimes \chi_0 \chi_i^{-1}, \psi_v). \tag{73}$$

Note that the product L-functions for $\mathrm{GL}_a \times \mathrm{GL}_b$ of the Langlands-Shahidi method and the L-functions of [17] are known to be equal (see [35]). Hence, to see (65) and (66), all we need is to compare the right-hand sides of (67) and (68) with those of (71), (72), and (73).

(ii): $v \mid \infty$. By the local Langlands correspondence (see [30]), the representations π_v and τ_v are given by admissible homomorphisms

$$\phi : W_v \longrightarrow \begin{cases} \mathrm{GSp}_{2n}(\mathbb{C}) & \text{if } \mathbf{G}_n = \mathrm{GSpin}_{2n+1}, \\ \mathrm{GSO}_{2n}(\mathbb{C}) & \text{if } \mathbf{G}_n = \mathrm{GSpin}_{2n}, \end{cases}$$

and

$$\phi' : W_v \longrightarrow \mathrm{GL}_r(\mathbb{C}),$$

respectively, and the tensor product

$$(\iota \circ \phi) \otimes \phi' \,:\, W_v \longrightarrow \mathrm{GL}_{2nr}(\mathbb{C})$$

is again admissible. Now,

$$L(s, \Pi_v \times \tau_v) = L\big(s, (\iota \circ \phi) \otimes \phi'\big) = L(s, \pi_v \times \tau_v),$$

and

$$\epsilon(s, \Pi_v \times \tau_v, \psi_v) = \epsilon\big(s, (\iota \circ \phi) \otimes \phi', \psi_v\big) = \epsilon(s, \pi_v \times \tau_v, \psi_v),$$

where the middle factors are the local Artin-Weil factors (see [47]) and equalities hold by [36] (see also [5]).

(iii): $v < \infty$ and π_v ramified. This is where we need the stability of γ-factors. Since $v \in S$, the representation τ_v can be written as

$$\tau_v \simeq \mathrm{Ind}(\nu^{b_1} \otimes \cdots \otimes \nu^{b_r}) \otimes \eta_v \simeq \mathrm{Ind}(\eta_v \nu^{b_1} \otimes \cdots \otimes \eta_v \nu^{b_r}), \tag{74}$$

where $\nu(x) = |x|_v$. Then

$$L(s, \pi_v \times \tau_v) = \prod_{i=1}^{r} L(s + b_i, \pi_v \times \eta_v), \tag{75}$$

$$\epsilon(s, \pi_v \times \tau_v, \psi_v) = \prod_{i=1}^{r} \epsilon(s + b_i, \pi_v \times \eta_v, \psi_v). \tag{76}$$

However, since η_v is sufficiently ramified (depending on π_v), Corollary 4.19 implies that

$$L(s, \pi_v \times \eta_v) \equiv 1, \tag{77}$$

$$\epsilon(s, \pi_v \times \eta_v) = \prod_{i=1}^{n} \epsilon(s, \eta_v \chi_i, \psi_v) \epsilon(s, \eta_v \chi_0 \mu_i^{-1}, \psi_v), \tag{78}$$

for n arbitrary characters $\chi_1, \chi_2, \ldots, \chi_n$, and $\chi_0 = \omega_{\pi_v}$. We choose them to be as in (60).

On the other hand, by either [17] or [39], we have

$$L(s, \Pi_v \times \tau_v) = \prod_{i=1}^{r} L(s + b_i, \Pi_v \otimes \eta_v), \tag{79}$$

$$\epsilon(s, \Pi_v \times \tau_v, \psi_v) = \prod_{i=1}^{r} \epsilon(s + b_i, \Pi_v \otimes \eta_v, \psi_v). \tag{80}$$

Again, since η_v is highly ramified (depending on Π_v) and $\omega_{\Pi_v} = \omega_{\pi_v}^n = \chi_0^n$ is equal to the product of the $2n$ characters

$$\chi_1, \dots, \chi_n, \chi_0\chi_n^{-1}, \dots, \chi_0\chi_1^{-1},$$

[19, Proposition 2.2] implies that

$$L(s, \Pi_v \otimes \eta_v) \equiv 1, \tag{81}$$

$$\epsilon(s, \Pi_v \otimes \eta_v) = \prod_{i=1}^{n} \epsilon(s, \eta_v\chi_i, \psi_v)\epsilon(s, \eta_v\chi_0\chi_i^{-1}, \psi_v). \tag{82}$$

Comparing equations (75) – (82) now proves (65) and (66) for a non-Archimedean place v at which π_v is ramified.

Now that we have (65) and (66) for all places v of k, we conclude globally that

$$L(s, \Pi \times \tau) = L(s, \pi \times \tau), \qquad L(s, \widetilde{\Pi} \times \widetilde{\tau}) = L(s, \widetilde{\pi} \times \widetilde{\tau}), \tag{83}$$

$$\epsilon(s, \Pi \times \tau) = \epsilon(s, \pi \times \tau), \qquad \epsilon(s, \widetilde{\Pi} \times \widetilde{\tau}) = \epsilon(s, \widetilde{\pi} \times \widetilde{\tau}), \tag{84}$$

for all $\tau \in \mathscr{T}(S; \eta)$. All that remains is to verify the three conditions of Theorem 6.1, which we can now check for the factors coming from the Langlands-Shahidi method thanks to (83) and (84). Conditions (1) – (3) of Theorem 6.1 are Propositions 5.1, 5.3, and 5.4, respectively.

Therefore, there exists an automorphic representation Π' of $\mathrm{GL}_{2n}(\mathbb{A})$ such that for all $v \notin S$, we have $\Pi_v \simeq \Pi'_v$. In particular, for all $v \notin S$, the local representation Π'_v is related to π_v, as prescribed in Theorem 1.1. Moreover, note that for all $v \notin S$, we have $\omega_{\Pi'_v} = \omega_{\Pi_v} = \omega_{\pi_v}^n$. Since $\omega_{\Pi'}$ is a Hecke character that agrees with the Hecke character ω_π^n at all but possibly finitely many places, we conclude that $\omega_{\Pi'} = \omega_\pi^n$.

On the other hand, if v is an Archimedean place or a non-Archimedean place with $v \notin S$, then we proved earlier that

$$\Pi'_v \simeq \Pi_v \simeq \widetilde{\Pi}_v \otimes \omega_{\pi_v} \simeq \widetilde{\Pi}'_v \otimes \omega_{\pi_v},$$

which means, in particular, that Π' is nearly equivalent to $\widetilde{\Pi}' \otimes \omega_{\pi'}$. □

7. Complements

7.1. Local consequences

Our first local result is to show that the local transfers at the unramified places remain generic. Let us first recall a general result of Jian-Shu Li which we use. The following is a special case of [32, Theorem 2.2].

PROPOSITION 7.1 (J.-S. Li; see [32])
Let $\mathbf{G}$ *be a split connected reductive group over a non-Archimedean local field* F*, and let* $\mathbf{B} = \mathbf{TU}$ *be a fixed Borel subgroup, where* $\mathbf{T}$ *is a maximal torus and* $\mathbf{U}$ *is the unipotent radical of* $\mathbf{B}$*. Let* χ *be an unramified character of* $\mathbf{T}(F)$*, and let* $\pi(\chi)$ *be the unique irreducible unramified subquotient of the corresponding principal series representation. Then* $\pi(\chi)$ *is generic if and only if for all roots* α *of* $(\mathbf{G}, \mathbf{T})$*, we have* $\chi(\alpha^\vee(\varpi)) \neq |\varpi|_F$*. Here* $\alpha^\vee$ *denotes the coroot associated to* α*, and* ϖ *is a uniformizer of* F*.*

PROPOSITION 7.2
Let $\pi = \bigotimes_v \pi_v$ *be an irreducible globally generic cuspidal automorphic representation of* $\mathrm{GSpin}_m(\mathbb{A})$*,* $m = 2n+1$ *or* $2n$*, and let* $\Pi = \bigotimes_v \Pi_v$ *be a transfer of* π *to* $\mathrm{GL}_{2n}(\mathbb{A})$ *(see Theorem 1.1). If* $v < \infty$ *is a place of* k *with* π_v *unramified, then the local representation* Π_v *is irreducible and unramified, and we have* $\Pi_v \simeq \widetilde{\Pi}_v \otimes \omega_{\pi_v}$*. Moreover, if* $m = 2n+1$ *(see Remark 7.3), then* Π_v *is generic (and, hence, a full-induced principal series representation).*

Proof
The representation Π_v is irreducible and unramified by construction (see (i) in the proof of Theorem 1.1). We also proved that Π_v satisfies $\Pi_v \simeq \widetilde{\Pi}_v \otimes \omega_{\pi_v}$ in the course of the proof of Theorem 1.1 in Section 6.

Assume that $m = 2n+1$. We now show that Π_v is generic. Our tool is Proposition 7.1. Let χ and $\chi_0, \ldots, \chi_n$ be as in (60). Since π_v is generic, by Proposition 7.1 we have that $\chi(\alpha^\vee(\varpi)) \neq |\varpi|_{k_v}$ for all roots α. Using the notation of Section 2, the roots in the odd case $m = 2n+1$ are $\alpha = \pm(e_i - e_j), \pm(e_i + e_j)$ with $1 \leq i < j \leq n$, and $\pm(e_i)$ with $1 \leq i \leq n$. The corresponding coroots are $\alpha^\vee = \pm(e_i^* - e_j^*), \pm(e_i^* + e_j^* - e_0^*)$ with $1 \leq i < j \leq n$, and $\pm(2e_i^* - e_0^*)$ with $1 \leq i \leq n$, respectively. This implies that $\chi_i \chi_j^{-1} \neq |\ |^{\pm 1}$ for $i \neq j$ and $\chi_i \chi_j \chi_0^{-1} \neq |\ |^{\pm 1}$ for all i, j.

The representation Π_v was chosen to be the unique irreducible unramified subquotient of the the representation on $\mathrm{GL}_{2n}(F)$ induced from the $2n$ unramified characters $\chi_1, \ldots, \chi_n, \chi_0\chi_n^{-1}, \ldots, \chi_0\chi_1^{-1}$ as in (64). Therefore, these relations imply that Π_v is generic and full-induced. □

Remark 7.3
The above argument does not quite work in the even case, and one can easily construct local examples, where the transferred local representation is the (unique) unramified subquotient of an induced representation on GL_{2n} far from the generic constituent.

For example, consider GSpin_6 with $\chi_0 = \mu^2$, $\chi_1 = \mu(5/2)$, $\chi_2 = \mu(1/2)$, and $\chi_3 = \mu(-3/2)$, where μ is a unitary character of $F^\times$ and $\mu(r)$ means $\mu|\ |^r$. Now Π_v is the unique unramified constituent of the representation on $\mathrm{GL}_6(F)$ induced from

$\mu(5/2)$, $\mu(3/2)$, $\mu(1/2)$, $\mu(-1/2)$, $\mu(-3/2)$, and $\mu(-5/2)$ and, in fact, is far from being generic. In this case, there is another constituent that is square-integrable and hence tempered and generic.

Of course, we do expect Π_v in the case of $m = 2n$ to be generic as well. However, this phenomenon is not a purely local one in the case of $m = 2n$. In fact, it is automatic that the local transfers at the unramified places are generic once we prove that the automorphic representation Π is induced from *unitary* cuspidal representations (see Remark 7.5). As we discuss in Remark 7.5 this will follow from our future work.

7.2. Global consequences

In this section we make some comments about the automorphic representation Π which are almost immediate consequences of our main result, and we leave more detailed information about Π for a future article.

PROPOSITION 7.4

Let π be a globally generic cuspidal automorphic representation of $\mathrm{GSpin}_m(\mathbb{A})$, $m = 2n + 1$ or $2n$, and let $\omega = \omega_\pi$. Then there exist a partition $(n_1, n_2, \ldots, n_t)$ of $2n$ and (not necessarily unitary) cuspidal automorphic representations $\sigma_1, \ldots, \sigma_t$ of $\mathrm{GL}_{n_i}(\mathbb{A})$, $i = 1, \ldots, t$, and a permutation p of $\{1, \ldots, t\}$ with $n_i = n_{p(i)}$ and $\sigma_i \simeq \widetilde{\sigma}_{p(i)} \otimes \omega$ such that any transfer Π of π as in Theorem 1.1 is a constituent of $\Sigma = \mathrm{Ind}(\sigma_1 \otimes \cdots \otimes \sigma_t)$, where the induction is, as usual, from the standard parabolic subgroup of GL_{2n} having Levi subgroup $\mathrm{GL}_{n_1} \times \cdots \times \mathrm{GL}_{n_t}$.

Proof

Let Π be any transfer of the globally generic cuspidal representation π as in Theorem 1.1. By [29, Proposition 2] there exist a partition p and σ_i's, such that Π is a constituent of Σ. Furthermore, for finite places v, where π_v is unramified, we have that Π_v is the unique unramified constituent of $\Sigma_v = \mathrm{Ind}(\sigma_{1,v} \otimes \cdots \otimes \sigma_{t,v})$. As part of Theorem 1.1, we showed that Π and $\widetilde{\Pi} \otimes \omega$ are nearly equivalent (see the definition prior to Theorem 1.1). Now, $\widetilde{\Pi} \otimes \omega$ is a constituent of $\widetilde{\Sigma} \otimes \omega = \mathrm{Ind}((\widetilde{\sigma}_1 \otimes \omega) \otimes \cdots \otimes (\widetilde{\sigma}_t \otimes \omega))$, and by the classification theorem of Jacquet and Shalika (see [18, Theorem 4.4]), we have that there is a permutation p of $\{1, \ldots, t\}$ such that $n_i = n_{p(i)}$ and $\sigma_i \simeq \widetilde{\sigma}_{p(i)} \otimes \omega$.

Now let Π' be another transfer of π as in Theorem 1.1. Then Π' is again a constituent of some $\Sigma' = \mathrm{Ind}(\sigma'_1 \otimes \cdots \otimes \sigma'_{t'})$, where each σ'_i is a cuspidal automorphic representation of $\mathrm{GL}_{n'_i}(\mathbb{A})$ and $(n'_1, \ldots, n'_{t'})$ is a partition of $2n$. Moreover, for almost all finite places v, we have that Π'_v is the unique unramified constituent of Σ'_v. On the other hand, by construction, $\Pi_v \simeq \Pi'_v$ for almost all v, and therefore, the classification theorem of Jacquet and Shalika again implies that $t = t'$ and, up to a permutation, $n_i = n'_i$ and $\sigma_i \simeq \sigma'_i$ for $i = 1, \ldots, t$. Therefore, Π' is also a constituent of Σ. □

Remark 7.5

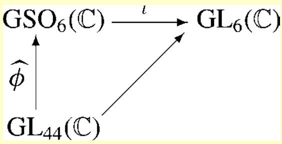
If we write $\sigma_i = \tau_i \otimes |\det(\ )|^{r_i}$ for $i = 1, 2, \ldots, t$, with τ_i unitary cuspidal and $r_i \in \mathbb{R}$, then we expect that all $r_i = 0$; that is, Π is an isobaric sum of unitary cuspidal representations. We will take up this issue, which will have important consequences, in our future work.

7.3. *Exterior square transfer*

In this section we show that exterior square transfer from GL_4 to GL_6 due to H. H. Kim [22] can be deduced as a special case of our main result. However, note that in this article we are proving only the weak transfer. Once we prove the strong version of the transfer from GSpin_{2n} to GL_{2n}, again it will have the full content of the results of [22]. A similar remark also applies to Section 7.4.

PROPOSITION 7.6

Let $\phi : \mathrm{GSpin}_6 \longrightarrow \mathrm{GL}_4$ *be the (double) covering map (see Proposition 2.2), and denote by* $\widehat{\phi}$ *the map induced on the connected components of the* L*-groups:*



Then $\iota \circ \widehat{\phi} = \bigwedge^2$.

Proof

The group GSO_6 is of type D_3, and we denote its simple roots by $\alpha_1, \alpha_2, \alpha_3$ as in Section 2. Also, GL_4 is of type A_3 (or D_3), and we denote its corresponding simple roots by $\overline{\alpha}_2, \overline{\alpha}_1, \overline{\alpha}_3$, respectively, and similarly for other root data (see Section 2). Let $A = \mathrm{diag}(a_1, a_2, a_3, a_4) \in \mathrm{GL}_4(\mathbb{C})$. For a fixed appropriate choice of fourth root of unity and $\delta = (a_1 a_2 a_3 a_4)^{1/4}$, we have

$$\begin{aligned}
\iota \circ \widehat{\phi}(A) &= \iota \circ \widehat{\phi}\left(\delta\, \overline{\alpha}_2{}^{\vee}\left(\frac{a_1}{\delta}\right) \overline{\alpha}_1{}^{\vee}\left(\frac{a_1 a_2}{\delta^2}\right) \overline{\alpha}_3{}^{\vee}\left(\frac{a_1 a_2 a_3}{\delta^3}\right)\right) \\
&= \iota\left(e_0^*(\delta^4) e_1^*(\delta^2) e_2^*(\delta^2) e_3^*(\delta^2) \alpha_2{}^{\vee}\left(\frac{a_1}{\delta}\right) \alpha_1{}^{\vee}\left(\frac{a_1 a_2}{\delta^2}\right) \alpha_3{}^{\vee}\left(\frac{a_1 a_2 a_3}{\delta^3}\right)\right) \\
&= \iota\big(e_0^*(\delta^4) e_1^*(a_1 a_2) e_2^*(a_1 a_3) e_3^*(a_2 a_3)\big) \\
&= \mathrm{diag}(a_1 a_2, a_1 a_3, a_2 a_3, a_2 a_4, a_1 a_4, a_3 a_4) = \bigwedge^2 A.
\end{aligned}$$

Here the third equality follows from Proposition 2.10. □

As a corollary, we see that our Theorem 1.1 in the special case of $m = 2n$ with $n = 3$ gives Kim's exterior square transfer.

PROPOSITION 7.7
If π is an irreducible cuspidal automorphic representation of $\mathrm{GL}_4(\mathbb{A})$ considered as a representation of $\mathrm{GSpin}_6(\mathbb{A})$ via the covering map ϕ, then the automorphic representation Π of Theorem 1.1 is such that $\Pi_v = \bigwedge^2 \pi_v$ for almost all v.

7.4. Transfer from GSp_4 *to* GL_4

The special case of $m = 2n + 1$ with $n = 2$ of our Theorem 1.1 gives the following.

PROPOSITION 7.8
Let π be an irreducible globally generic cuspidal automorphic representation of $\mathrm{GSp}_4(\mathbb{A})$. Then π can be transferred to an automorphic representation Π of $\mathrm{GL}_4(\mathbb{A})$ associated to the embedding $\mathrm{GSp}_4(\mathbb{C}) \hookrightarrow \mathrm{GL}_4(\mathbb{C})$.

Proof
Notice that GSpin_5 is isomorphic, as an algebraic group, to the group GSp_4. Now the corollary is a special case of Theorem 1.1, as mentioned previously. □

In fact, we can prove more in this special case. We refer to our separate work [4] for more details about this as well as its applications to the generalized Ramanujan conjecture for GSp_4.

Remark 7.9
Proposition 7.8, in particular, proves that the spinor L-function of π is entire. R. Takloo-Bighash [46] also has a proof of some cases of this result using an integral representation. His proof differs from our method in that we use the integral representations only through the Converse Theorems.

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