Generically split projective homogeneous varieties

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Abstract

Let G be an exceptional simple algebraic group over a field k and X a projective G-homogeneous variety such that G splits over $k(X)$. We classify such varieties X.

This classification allows to relate the Rost invariant of groups of type E_7 and their isotropy and to give a two-line proof of the triviality of the kernel of the Rost invariant for such groups. Apart from this it plays a crucial role in the solution [S08] of a problem posed by J.-P. Serre for groups of type E_8.

Keywords: algebraic groups, Chow rings.

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1 Introduction

Let k denote a field and G an exceptional simple algebraic group over k of inner type. In the present paper we classify projective G-homogeneous varieties X such that G splits over the field of rational functions $k(X)$. Such varieties are called *generically split*.

Observe that for groups G of classical types, different from type D_n, the solution of this problem is known. For groups of type A_n and C_n it is a consequence of the index reduction formula for central simple algebras, and for groups of type B_n and D_n corresponding to quadratic forms it follows from the motivic theory of quadrics developed by Karpenko and Vishik (see [Vi08], Theorem 5.8 and Remark 5.9). Therefore we concentrate ourselves

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on exceptional groups, though our main result (Theorem 5.5) is valid for all groups.

In this paper we reduce the problem of the classification of generically split varieties to a purely combinatorial question related to the divisibility of integral polynomials. The fact that a given variety is not generically split (which is the most difficult part of the classification) follows simply from the non-divisibility of certain rather concrete polynomials. These polynomials can be computed in terms of a motivic invariant of G, called the *J-invariant*, which has been introduced and classified in [PSZ] by Petrov, Semenov, and Zainoulline.

Apart from the fact that our classification is an important structural theorem in the theory of linear algebraic groups, it provides several applications to J.-P. Serre's theory of cohomological invariants (see [GMS]). For example, Corollary 5.10 gives a uniform two-line proof of the triviality of the kernel of the Rost invariant for groups of type E_7. Opposite to the traditional approach this proof is independent on the classification and concrete realizations of algebraic groups and uses only the combinatorics of root systems. Moreover, for such groups G our results can be applied to get a refined statement relating the Rost invariant and isotropy of G (see [PS, Corollary 6.10]).

The most powerful application of our classification is the construction of cohomological invariants for groups of type E_8 given by the second author in [S08] and the solution of a problem posed by Serre concerning compact Lie groups of type E_8 (see [S08, Section 8]). Theorem 5.5 and its corollaries play here an essential role.

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The paper is organized as follows. In Section 2 we describe tools to perform computations in the Chow rings of projective homogeneous varieties and reduce these computations to the combinatorics of root systems. Section 3 introduces the J-invariant of algebraic groups. In Section 4 we relate the J-invariant and the Tits algebras of a group. In the last section we provide a classification of generically split projective homogeneous varieties.

2 Algebraic cycles on projective homogeneous varieties

In this section we briefly describe the main properties of projective homogeneous varieties and their Chow rings (see [Ca72], [De74], [Hi82]).

2.1. Let G be a split semisimple algebraic group of rank n defined over a field k. We fix a split maximal torus T in G and a Borel subgroup B of G containing T and defined over k. We denote by Φ the root system of G, by $\Pi = \{\alpha_1, \dots, \alpha_n\}$ the set of simple roots of Φ with respect to B, by W the Weyl group, and by $S = \{s_1, \dots, s_n\}$ the corresponding set of fundamental reflections.

Let $P = P_\Theta$ be the (standard) parabolic subgroup corresponding to a subset $\Theta \subset \Pi$, i.e., $P = BW_\Theta B$, where $W_\Theta = \langle s_\theta, \theta \in \Theta \rangle$ (Bruhat decomposition). Denote

$$W^\Theta = \{w \in W \mid \forall\, s \in \Theta \quad l(ws) = l(w) + 1\},$$

where l is the length function on W. It is easy to see that W^Θ consists of all representatives in the left cosets W/W_Θ which have minimal length.

As P_i we denote the maximal parabolic subgroup $P_{\Pi\setminus\{\alpha_i\}}$ *of type* i and as w_0 the longest element of W. Enumeration of simple roots follows Bourbaki.

Any projective G-homogeneous variety X is isomorphic to G/P_Θ for some subset Θ of the simple roots.

2.2 (Generators of the Chow ring; see [Ko91, Proposition 1.3])**.** Now consider the Chow ring of the variety $X = G/P_\Theta$. It is known that $\mathrm{CH}^*(G/P_\Theta)$ is a free abelian group with a basis given by varieties $[X_w]$ that correspond to the elements $w \in W^\Theta$. The degree (codimension) of the basis element $[X_w]$ equals $l(w_\theta) - l(w)$, where w_θ is the longest element of W_Θ. We call the generators $[X_w]$ *standard*.

Moreover, there exists a natural injective pull-back homomorphism

$$\mathrm{CH}^*(G/P) \to \mathrm{CH}^*(G/B)$$

$$[X_w] \mapsto [X_{ww_\theta}]$$

2.3 (Pieri formula; see [De74, Corollary 2 of 4.4])**.** In order to multiply two basis elements of $\mathrm{CH}^*(G/B)$ one of which is of codimension 1 we use the following formula (Pieri formula):

$$[X_{w_0 s_\alpha}][X_w] = \sum_{\beta\in\Phi^+,\, l(ws_\beta)=l(w)-1} \langle \beta^\vee, \bar\omega_\alpha\rangle [X_{ws_\beta}], \tag{1}$$

where α is a simple root and the sum runs through the set of positive roots $\beta \in \Phi^+$, s_β denotes the reflection corresponding to β, $\beta^\vee$ the coroot of β, and $\bar\omega_\alpha$ is the fundamental weight corresponding to α. Here $[X_{w_0 s_\alpha}]$ is an element of codimension 1.

2.4 Example. Let G be of type D_n, $\Theta = \Pi \setminus \{\alpha_1\}$ and $Q = G/P_1$. Denote by h the unique standard generator of codimension 1 in $\mathrm{CH}^*(Q)$.

A straightforward application of the Pieri formula shows that for $i = 0, \ldots, n-2$ the elements h^i are the unique standard generators of codimension i, for $i = n, \ldots, 2n-2$ the elements h^i are twice the standard generators of codimension i, and h^{n-1} is the sum of the standard generators of codimension $n-1$.

Apart from this, one can notice (but this is not necessary for computations) that Q is a hyperbolic $(2n-2)$-dimensional projective quadric. The Chow ring of such a quadric is of course well-known. The generator h is the class of its hyperplane section.

2.5 Definition. The *Poincaré polynomial* of a free $\mathbb{Z}^{\geq 0}$-graded finitely generated abelian group A_* (resp. $\mathbb{F}_p$-vector space for some fixed prime number p) is, by definition, the polynomial

$$g(A_*, t) = \sum_{i=0}^{\infty} a_i t^i \in \mathbb{Z}[t]$$

with $a_i = \operatorname{rank} A_i$.

The following formula (the Solomon theorem) allows to compute the Poincaré polynomial of $\mathrm{CH}_*(X) = \mathrm{CH}^{\dim X - *}(X)$:

$$g(\mathrm{CH}_*(X), t) = \frac{r(\Pi)}{r(\Theta)},\ \ r(-) = \prod_{i=1}^{m} \frac{t^{e_i(-)} - 1}{t-1}, \tag{2}$$

where $e_i(\Theta)$ (resp. $e_i(\Pi)$) denote the degrees of the fundamental polynomial invariants of the root subsystem of Φ generated by Θ (resp. Π) and m its rank (see [Ca72, 9.4 A]). The dimension of X equals $\deg g(\mathrm{CH}_*(X), t)$.

The values of degrees of fundamental polynomial invariants are given in the following table (see [Ca72, p. 155]):

Φ	$e_i(\Phi)$
A_m	$2, 3, \ldots, m+1$
$\mathrm{B}_m, \mathrm{C}_m$	$2, 4, \ldots, 2m$
D_m	$2, 4, \ldots, 2m-2, m$
E_6	$2, 5, 6, 8, 9, 12$
E_7	$2, 6, 8, 10, 12, 14, 18$
E_8	$2, 8, 12, 14, 18, 20, 24, 30$
F_4	$2, 6, 8, 12$
G_2	$2, 6$

There exists a Maple package [St] of Stembridge that provides tools to compute the Poincaré polynomials of projective homogeneous varieties.

2.6 Example. Let Q be as in Example 2.4. For a group of type D_n the degrees of fundamental polynomial invariants are: $e_i = 2i$ for $i = 1, \ldots, n-1$, and $e_n = n$. Therefore we have

$$g(\mathrm{CH}_*(Q), t) = \frac{(t^{n-1}+1)(t^n-1)}{t-1}.$$

2.7. Let $\mathrm{P} = \mathrm{P}(\Phi)$ denote the weight lattice. We denote as $\bar{\omega}_1, \ldots \bar{\omega}_n$ the basis of P consisting of the fundamental weights. The symmetric algebra $S^*(\mathrm{P})$ is isomorphic to $\mathbb{Z}[\bar{\omega}_1, \ldots \bar{\omega}_n]$. The Weyl group W acts on P, hence, on $S^*(\mathrm{P})$. Namely, for a simple root α_i

$$s_i(\bar{\omega}_j) = \begin{cases} \bar{\omega}_i - \alpha_i, & i = j; \\ \bar{\omega}_j, & \text{otherwise.} \end{cases}$$

We define a linear map $c\colon S^*(\mathrm{P})^{W_\Theta} \to \mathrm{CH}^*(G/P_\Theta)$ as follows. For a homogeneous W_Θ-invariant $u \in \mathbb{Z}[\bar{\omega}_1, \ldots, \bar{\omega}_n]$

$$c(u) = \sum_{w \in W^\Theta,\, l(w) = \deg(u)} \Delta_w(u)[X_{w_0 w w_\theta}],$$

where for $w = s_{i_1} \ldots s_{i_k}$ we denote by Δ_w the composition of derivations $\Delta_{s_{i_1}} \circ \ldots \circ \Delta_{s_{i_k}}$ and the derivation $\Delta_{s_i}\colon S^*(\mathrm{P}) \to S^{*-1}(\mathrm{P})$ is defined by $\Delta_{s_i}(u) = \dfrac{u - s_i(u)}{\alpha_i}$ (see [Hi82, Ch. IV, §1]).

2.8 (Chern classes; see [BH58, §10]). Let $U = \Sigma_u(P_\Theta)$ denote the set of the (positive) roots lying in the unipotent radical of the parabolic subgroup P_Θ. Then the elementary symmetric polynomials $\sum_{u\in U}\sigma_i(u)$ are W_{P_Θ}-invariant and, in fact, coincide with the Chern classes of the tangent bundle T_X:

$$c(T_X) = c(\prod_{\gamma\in U}(1+\gamma)). \tag{3}$$

The effective procedures to multiply cycles in the Chow rings of projective homogeneous varieties and compute the Chern classes of the tangent bundles are implemented in the Maple package [map][1].

3 J-invariant

In this section we recall the definition and the main properties of a motivic invariant of a semisimple algebraic group introduced in [PSZ] and called the J-invariant. We assume that the reader is familiar with Galois cohomology and twisted forms of algebraic groups as described, for instance, in [Inv, Ch. VII].

3.1. Let G_0 be a split semisimple algebraic group over k with a split maximal torus T and a Borel subgroup B containing T. Let $G = {}_\gamma G_0$ be the twisted form of G_0 given by a 1-cocycle $\gamma \in H^1(k, G_0)$.

Let X be a projective G-homogeneous variety and p a prime integer. To simplify the notation we denote

$$\mathrm{Ch}^*(X) = \mathrm{CH}^*(X) \otimes \mathbb{Z}/p$$

and $\overline{X} = X \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, k_s$, where k_s stands for a separable closure of k. We say that a cycle $J \in \mathrm{CH}^*(\overline{X})$ (resp. $J \in \mathrm{Ch}^*(\overline{X})$) is *rational* if it lies in the image of the natural restriction map $\mathrm{res}\colon \mathrm{CH}^*(X) \to \mathrm{CH}^*(\overline{X})$ (resp. $\mathrm{res}\colon \mathrm{Ch}^*(X) \to \mathrm{Ch}^*(\overline{X})$). We denote as $\overline{\mathrm{CH}}^*(X)$ (resp. as $\overline{\mathrm{Ch}}^*(X)$) the image of this map.

3.2. From now on and till the end of this section we consider the variety $X = {}_\gamma(G_0/B)$ of complete flags. Let $\widehat{T}$ denote the group of characters of T and $S(\widehat{T}) \subset S^*(\mathrm{P})$ be the symmetric algebra (see Section 2). By R^* we denote the image of the characteristic map $c\colon S(\widehat{T}) \to \mathrm{Ch}^*(\overline{X})$ defined above. It is easy to see that $R^* \subseteq \overline{\mathrm{Ch}}^*(X)$ (see [KM05, Theorem 6.4]).

[1] Created in collaboration with S. Nikolenko and K. Zainoulline.

3.3. Let $\mathrm{Ch}^*(\overline{G})$ denote the Chow ring with $\mathbb{F}_p$-coefficients of the group $(G_0)_{k_s}$. An explicit presentation of $\mathrm{Ch}^*(\overline{G})$ in terms of generators and relations is known for all groups and all primes p. Namely, by [Kc85, Theorem 3]

$$\mathrm{Ch}^*(\overline{G}) = \mathbb{F}_p[x_1, \ldots, x_r]/(x_1^{p^{k_1}}, \ldots, x_r^{p^{k_r}}) \tag{4}$$

for certain numbers k_i, $i = 1, \ldots, r$, and $\deg x_i = d_i$ for certain numbers $1 \le d_1 \le \ldots \le d_r$ coprime to p. A complete list of numbers $\{d_i p^{k_i}\}_{i=1,\ldots,r}$, called *$p$-exceptional degrees* of G_0, is provided in [Kc85, Table II] (see Example 3.8(5) for exceptional groups). Taking the p-primary and p-coprimary parts of each p-exceptional degree one immediately restores the respective k_i's and d_i's.

3.4. Now we introduce an order on the set of additive generators of $\mathrm{Ch}^*(\overline{G})$, i.e., on the monomials $x_1^{m_1} \ldots x_r^{m_r}$. To simplify the notation, we denote the monomial $x_1^{m_1} \ldots x_r^{m_r}$ by x^M, where M is an r-tuple of integers $(m_1, \ldots, m_r)$. The codimension (in the Chow ring) of x^M is denoted by $|M|$. Observe that $|M| = \sum_{i=1}^r d_i m_i$.

Given two r-tuples $M = (m_1, \ldots, m_r)$ and $N = (n_1, \ldots, n_r)$ we say $x^M \le x^N$ (or equivalently $M \le N$) if either $|M| < |N|$, or $|M| = |N|$ and $m_i \le n_i$ for the greatest i such that $m_i \ne n_i$. This gives a well-ordering on the set of all monomials (r-tuples) known also as *DegLex order*.

3.5. Consider the pull-back induced by the quotient map

$$\pi\colon\ \mathrm{Ch}^*(\overline{X}) \to \mathrm{Ch}^*(\overline{G})$$

According to [Gr58, Rem. 2°] π is surjective with the kernel generated by the subgroup of the non-constant elements of R^*.

Now we are ready to define the J-invariant of a group G.

3.6 Definition. Let $X = {}_\gamma(G_0/B)$ be the twisted form of the variety of complete flags by means of a 1-cocycle $\gamma \in H^1(k, G_0)$. Denote as $\overline{\mathrm{Ch}}^*(G)$ the image of the composite map

$$\mathrm{Ch}^*(X) \xrightarrow{\mathrm{res}} \mathrm{Ch}^*(\overline{X}) \xrightarrow{\pi} \mathrm{Ch}^*(\overline{G}).$$

Since both maps are ring homomorphisms, $\overline{\mathrm{Ch}}^*(G)$ is a subring of $\mathrm{Ch}^*(\overline{G})$.

For each $1 \leq i \leq r$ set j_i to be the smallest non-negative integer such that the subring $\overline{\mathrm{Ch}}^*(G)$ contains an element a with the greatest monomial $x_i^{p^{j_i}}$ with respect to the DegLex order on $\mathrm{Ch}^*(\overline{G})$, i.e., of the form

$$a = x_i^{p^{j_i}} + \sum_{x^M \lneq x_i^{p^{j_i}}} c_M x^M, \quad c_M \in \mathbb{F}_p.$$

The r-tuple of integers $(j_1, \ldots, j_r)$ is called the *J-invariant of G modulo p* and is denoted by $J_p(G)$. Note that $j_i \leq k_i$ for all i.

3.7. We say that a prime number p is a *torsion prime* of G if $\mathrm{Ch}^*(\overline{G}) \neq \mathbb{F}_p$. The latter occurs if $p|n+1$ for groups of type A_n; $p = 2$ for B_n, C_n, D_n, G_2; $p = 2, 3$ for E_6, E_7, F_4; $p = 2, 3, 5$ for E_8.

A table of possible values of the J-invariants is given in [PSZ, Section 4]. To illustrate Definition 3.6 of the J-invariant we give the following examples. For a prime integer p we denote as v_p the p-adic valuation.

3.8 Examples.

1) For an adjoint group G of type ${}^1\mathrm{A}_n$ or C_n let A be the underlying central simple algebra. Then $J_p(G) = (v_p(\mathrm{ind} A))$ for all torsion primes p, and $d_1 = 1$.

2) Let p be a prime integer and A and B be central simple k-algebras that generate the same subgroup in the Brauer group $\mathrm{Br}(k)$. Set $G = \mathrm{PGL}_1(A) \times \mathrm{PGL}_1(B)$. Then $J_p(G) = (v_p(\mathrm{ind} A), 0)$.

Indeed, the Chow ring

$$\mathrm{Ch}^*(\overline{G}) = \mathbb{F}_p[x_1, x_2]/(x_1^{p^{k_1}}, x_2^{p^{k_2}})$$

with $k_1 = v_p(\deg A)$, $k_2 = v_p(\deg B)$. Therefore r in the definition of the J-invariant equals 2. Denote $J_p(G) = (j_1, j_2)$ and consider the map

$$\mathrm{res}\colon \mathrm{Pic}(X_A \times X_B) \to \mathrm{Pic}(\overline{X}_A \times \overline{X}_B),$$

where X_A (resp. X_B) denote the $\mathrm{PGL}_1(A)$- (resp. $\mathrm{PGL}_1(B)$-) variety of complete flags and Pic stands for the Picard group modulo p. Denote by h_A (resp. h_B) the image of $\bar{\omega}_1 \in S(\mathrm{P})$ in $\mathrm{Pic}(\overline{X}_A)$ (resp. $\mathrm{Pic}(\overline{X}_B)$) by means of the map c defined in Section 2.

Since A and B generate the same subgroup in the Brauer group, the cycle $1 \times h_B + \alpha h_A \times 1 \in \mathrm{Pic}(\overline{X}_A \times \overline{X}_B)$ is rational for some $\alpha \in \mathbb{F}_p^\times$ (see [MT95] or Section 4 below for the description of the Picard groups of projective homogeneous varieties). The image of this cycle in $\mathrm{Ch}^*(\overline{G})$ by means of π equals $x_2 + \alpha x_1$ (at least we can choose the generators x_1 and x_2 in such a way). Therefore, since $x_1 < x_2$ in the DegLex order, $j_2 = 0$. The proof that $j_1 = v_p(\mathrm{ind} A)$ is the same as in [PSZ, Section 7, case A_n] and we omit it.

3) In [KM08, Section 2] Karpenko and Merkurjev define the notion of a *minimal basis* of a finite subgroup of the Brauer group $\mathrm{Br}(k)$. Let $\{a_1, \ldots, a_m\}$ be a minimal basis, and $A_1, \ldots, A_m$ central simple algebras representing $a_1, \ldots, a_m$. Then by [KM08, Proposition 2.5]

$$J_p(\mathrm{PGL}_1(A_1) \times \ldots \times \mathrm{PGL}_1(A_m)) = (v_p(\mathrm{ind} A_1), \ldots, v_p(\mathrm{ind} A_m)).$$

4) If q is an anisotropic n-fold Pfister quadric or its codimension 1 subform, and G the respective group of type $\mathrm{D}_{2^{n-1}}$ or $\mathrm{B}_{2^{n-1}-1}$ resp., then $J_2(G) = (0, 0, \ldots, 0, 1)$ with $d_r = 2^{n-1} - 1$.

5) For exceptional adjoint groups we have:

G	p	r	d_i, $i = 1 \ldots r$	k_i, $i = 1 \ldots r$
E_6, F_4, G_2	2	1	3	1
E_7, F_4	3	1	4	1
E_6	3	2	$1, 4$	$2, 1$
E_7	2	4	$1, 3, 5, 9$	$1, 1, 1, 1$
E_8	2	4	$3, 5, 9, 15$	$3, 2, 1, 1$
E_8	3	2	$4, 10$	$1, 1$
E_8	5	1	6	1

6) If G is a generic group, i.e., the twisted form of a split group by a versal torsor (over some field extension of the base field), then $J_p(G) = (k_1, \ldots, k_r)$ (see [PSZ, Example 4.7]).

7) For any G and any prime p if $J_p(G)$ is trivial, i.e., $J_p(G) = (0, \ldots, 0)$, then G splits over a field extension of degree coprime to p.

Next we describe some useful properties of the J-invariant.

3.9 Proposition. *Let G be a semisimple algebraic group of inner type over k, p a prime integer and $J_p(G) = (j_1, \ldots, j_r)$. Then*

1. *Let K/k be a field extension. Denote $J_p(G_K) = ((j_1)_K, \ldots, (j_r)_K)$. Then $(j_i)_K \leq j_i$, $i = 1, \ldots, r$.*

2. *Fix an $l = 1, \ldots, r$. Let $\mathrm{Ch}^*(G_{\mathrm{an}}) = \mathbb{F}_p[x'_1, \ldots, x'_{r'}]/(x_1^{p^{k'_1}}, \ldots, x_{r'}^{p^{k'_r}})$ with $\deg x'_i = d'_i$ be presentation (4) for the semisimple anisotropic kernel*

G_{an} of G. Assume that in this presentation none of x'_i has degree d_l. Then $j_l = 0$.

3. *Assume that the group G does not have simple components of type* E_8 *and for all primes p the J-invariant $J_p(G)$ is trivial. Then G is split.*

Proof. 1. This is an obvious consequence of the definition of the J-invariant.

2. Since G_{an} is the semisimple anisotropic kernel of G, its Dynkin diagram is a subdiagram of the Dynkin diagram of G. Therefore by [Kc85, Table II] we have $r' \le r$ and $\{d'_i, i = 1, \dots, r'\} \subset \{d_i, i = 1, \dots, r\}$. On the other hand, by [PSZ, Corollary 5.19], the polynomials $\prod_{i=1}^{r} \dfrac{t^{d_i p^{j_i}} - 1}{t^{d_i} - 1}$ and $\prod_{i=1}^{r'} \dfrac{t^{d'_i p^{j'_i}} - 1}{t^{d'_i} - 1}$ are equal. This implies the claim.

3. By [PSZ, Corollary 6.7] $J_p(G)$ is trivial iff G splits over a field extension of k of degree coprime to p. So, by our assumptions G splits over field extensions of coprime degrees. Therefore it is split already over k by [Gi97, Theorem C]. □

4 J-invariant and Tits algebras

Let G be a semisimple algebraic group over k of inner type. With each irreducible representation ρ of G one can associate a unique central simple algebra A such that there is a homomorphism $\mu\colon G \to \mathrm{GL}_1(A)$ having the property that $\mu \otimes k_s$ is isomorphic to the representation $\rho \otimes k_s$ (see [Ti71]). The algebra A is called the *Tits algebra* of G corresponding to ρ.

Let X be a projective G-homogeneous variety of type $\Theta \subset \Pi$. Consider the Picard group of X. By Section 2 the group $\mathrm{Pic}(\overline{X})$ is a free abelian group with basis $\bar\omega_i$, $i \in \Pi \setminus \Theta$. Let $\alpha_X\colon\ \mathrm{Pic}(\overline{X}) \to \mathrm{Br}(k)$ be the map sending $\bar\omega_i$ to the Brauer-class of the Tits algebra corresponding to the fundamental representation with the highest weight $\bar\omega_i$.

By [MT95, §2] the following sequence of groups is exact:

$$0 \to \mathrm{Pic}(X) \overset{\mathrm{res}}{\to} \mathrm{Pic}(\overline{X}) \overset{\alpha_X}{\to} \mathrm{Br}(k). \tag{5}$$

This sequence allows to express the group $\mathrm{Pic}(X)$ in terms of the Tits algebras of G.

4.1 Examples.

1) It is well-known that the Tits algebras of any group G of type G_2, F_4, or E_8 are split. Therefore the Picard group of any projective G-homogeneous variety X is rational.

2) Let X be the Severi-Brauer variety $\mathrm{SB}(A)$ of a central simple algebra A of degree n. Then X is the variety of parabolic subgroups of type 1 of the group $G = \mathrm{SL}_1(A)$. It is well-known that the Tits algebra of G corresponding to $\bar{\omega}_1$ is A (see [MT95, 2.4.1]). Therefore the map

$$\mathrm{res}\colon \operatorname{Pic}(\overline{\mathrm{SB}(A)}) \to \operatorname{Pic}(\mathbb{P}^{n-1}) = \mathbb{Z}$$

is the multiplication by $\exp A$.

3) Let q be a regular quadratic form of dimension ≥ 5 with trivial discriminant and Q the respective projective quadric. The variety Q is the variety of parabolic subgroups of $G = \mathrm{O}^+(q)$ of type 1. Since the Tits algebra of G corresponding to $\bar{\omega}_1$ is split (see [MT95, 2.4.3 and 2.4.5]), the Picard group of Q is rational.

4) Let G be a group of type E_7. There is a central simple algebra A of index $1, 2, 4$, or 8 and exponent 1 or 2 such that A is Brauer-equivalent to the Tits algebras corresponding to $\bar{\omega}_2, \bar{\omega}_5$, and $\bar{\omega}_7$. The Tits algebras for $\bar{\omega}_1, \bar{\omega}_3, \bar{\omega}_4$, and $\bar{\omega}_6$ are all split (see [Ti71, 6.5]).

4.2 Proposition. *Let G be a semisimple algebraic group of inner type over k, X a projective homogeneous variety of type $\Theta \subset \Pi$, p a prime integer, and $J_p(G) = (j_1, \ldots, j_r)$. Set $l_i = v_p(\mathrm{ind} A_i)$, where $\alpha_X(\bar{\omega}_i) = [A_i]$, $i \in \Pi \setminus \Theta$.*

Then the cycle $\bar{\omega}_i^{p^{l_i}} \in \mathrm{Ch}^{p^{l_i}}(\overline{X})$ is rational. The following partial converse holds: If G is simple, the cycle $\bar{\omega}_i \in \mathrm{Ch}^1(\overline{X})$, $i \in \Pi \setminus \Theta$, is rational, and $\bar{\omega}_i$ is not equal to any root (modulo p), then $l_i = 0$.

Moreover, $j_i = 0$ for all i with $d_i = 1$ iff the indices of all Tits algebras of G are coprime to p.

Proof. Consider the projective homogeneous variety $X \times \mathrm{SB}(A_i)$, where $\mathrm{SB}(A_i)$ denotes the Severi-Brauer variety of right ideals of A_i of reduced dimension 1. Let $n = \deg A_i$. Denote by $h_i \in \operatorname{Pic}(\overline{\mathrm{SB}(A_i)}) = \operatorname{Pic}(\mathbb{P}^{n-1})$ the standard generator as in Section 2.

By the results of Merkurjev and Tignol (see formula (5)) the cycle $\alpha = \bar{\omega}_i \times 1 - 1 \times h_i \in \operatorname{Pic}(\overline{X} \times \overline{\mathrm{SB}(A_i)})$ is rational. Since the cycles $\alpha^{p^{l_i}} = \bar{\omega}_i^{p^{l_i}} \times 1 - 1 \times h_i^{p^{l_i}} \in \mathrm{Ch}^*(\overline{X} \times \overline{\mathrm{SB}(A_i)})$ and $h_i^{p^{l_i}} \in \mathrm{Ch}^*(\overline{\mathrm{SB}(A_i)})$ are rational, the cycle $\bar{\omega}_i^{p^{l_i}} \times 1 \in \mathrm{Ch}^*(\overline{X} \times \overline{\mathrm{SB}(A_i)})$ is rational as well.

The projection morphism $\mathrm{pr}\colon X\times\mathrm{SB}(A_i)\to X$ is a projective bundle by [PSZ, Corollary 3.4]. In particular, $\mathrm{CH}^*(X\times\mathrm{SB}(A_i))=\bigoplus_{j=0}^{n-1}\mathrm{CH}^{*-j}(X)$. Therefore the pull-back pr^* has a section δ. By the construction of this section it is compatible with a base change. Passing to the splitting field k_s we obtain that the cycle $\bar\delta(\bar\omega_i^{p^{l_i}}\times 1)=\bar\omega_i^{p^{l_i}}\in\mathrm{Ch}^*(\overline{X})$ is rational.

Conversely, assume that $\bar\omega_i$ is rational modulo p, i.e., $\bar\omega_i+pa_1$ is rational for some $a_1\in\mathrm{CH}^1(\overline{X})$. Denote by C the quotient of the weight lattice of G by its root lattice. If p does not divide the order of C, then, since $|C|[A_i]=0\in\mathrm{Br}(k)$, we have $l_i=0$. So, we may assume that p divides $|C|$. Then either C is a p-torsion group, and hence pa_1 belongs to the root lattice, and hence is rational, or C/pC is a cyclic group of order p. (This occurs in cases A_l and D_{2l+1}). Then by the assumptions on $\bar\omega_i$ we can take $\bar\omega_i$ as a generator of C/pC. So, $a_1=m_1\bar\omega_i+pa_2+b_1$ for some $a_2\in\mathrm{CH}^1(\overline{X})$ and rational b_1. Thus, $(1+pm_1)\bar\omega_i+p^2a_2$ is rational. Continuing this way, we see that $\bar\omega_i$ is rational as an element of $\mathrm{CH}^1(\overline{X})/p^N$ for all N. Let now $N=v_p(|C|)$ and $|C|=mp^N$. Then $m\bar\omega_i+|C|a$ is rational for some a. But $|C|a$ is rational for all a. Therefore $m\bar\omega_i$ is rational. Since m is coprime to p, formula (5) implies that $m[A_i]=0$ in $\mathrm{Br}(k)$, as claimed.

Finally, if the indices of all Tits algebras of G are coprime to p, then all $\bar\omega_i=\bar\omega_i^{p^0}$ are rational, and hence by the very definition of the J-invariant all j_i with $d_i=1$ are zero. Conversely, assume that all such j_i equal zero. Let A be a Tits algebra of G such that $v_p(\exp A)$ is maximal and non-zero. By assumption and since sequence (5) is exact, there is a Tits algebra B of G such that $[A]+p[B]$ is zero in $\mathrm{Br}(k)$. But $v_p(\exp(B^{\otimes p}))$ is strictly less than $v_p(\exp A)$ by the choice of A. Contradiction. □

4.3 Remark. It is not true in general that $l_i\le l$ whenever $\bar\omega_i^{p^l}$ is rational. A counter-example is e.g. a group of type E_7 with a Tits algebra of index more than 2.

It is also not true in general that if G is semisimple, but not simple, and $\bar\omega_i\in\mathrm{Ch}^1(\overline{X})$ is rational, then $l_i=0$. A counter-example is e.g. $G=\mathrm{PGL}_1(A)\times\mathrm{PGL}_1(B)$ with $[A]=p[B]$ in $\mathrm{Br}(k)$ and $\bar\omega_i$ with $\alpha_X(\bar\omega_i)=[A]$.

5 Generically split varieties

5.1 Definition. Let G be a semisimple algebraic group over k and X a projective G-homogeneous variety. We say that X is *generically split*, if the group G splits (i.e., contains a split maximal torus) over $k(X)$.

5.2 Remark. There are no generically split homogeneous varieties of positive dimension for groups of outer type.

Indeed, let G be of outer type and X be a projective G-homogeneous variety of positive dimension. If G comes from a cocycle $\xi \in H^1(k, G_0 \rtimes \mathrm{Aut}(D))$, where G_0 is a split group and D is the Dynkin diagram of G, then there is a natural étale extension L/k corresponding to the image of ξ in $H^1(k, \mathrm{S}_n)$, where n is the rank of G. Let m be the number of $*$-action orbits of the absolute Galois group of k on D (see [Inv]).

The map

$$\mathrm{CH}^0(\operatorname{Spec} L \times_{\operatorname{Spec} k} X) \to \mathrm{CH}^0((\operatorname{Spec} L)_{k(X)})$$

is always surjective. On the other hand, $\mathrm{CH}^0(\operatorname{Spec} L \times_{\operatorname{Spec} k} X) = \mathbb{Z}^m$, since X is geometrically irreducible, but $\mathrm{CH}^0((\operatorname{Spec} L)_{k(X)}) = \mathbb{Z}^n$, since X is generically split, and $n > m$, since G is of outer type.

5.3 Remark. If X is generically split, then the Chow motive of X splits over $k(X)$ as a direct sum of Tate motives. This explains the terminology "generically split". One can also call such varieties *generically cellular*, since over $k(X)$ they are cellular via the Bruhat decomposition.

5.4 Example. Pfister quadrics and Severi-Brauer varieties are generically split.

Before formulating the main results of the present paper we recall the notation from the previous sections. For a variety X and a prime number p we denote by $\mathrm{Ch}^*(\overline{X})$ the Chow group of the variety $X \times_k k_s$ with $\mathbb{F}_p$-coefficients; $\overline{\mathrm{Ch}}^*(X)$ stands for the image of the restriction map $\mathrm{res}\colon \mathrm{Ch}^*(X) \to \mathrm{Ch}^*(\overline{X})$, and $g(-,t) \in \mathbb{Z}[t]$ for the Poincaré polynomial of the graded group $-$. Recall also that for an algebraic group G we denote by $J_p(G) = (j_1, \ldots, j_r) \in \mathbb{Z}^r$ its J-invariant modulo p, and $d_1, \ldots, d_r$ are p-coprimary parts of p-exceptional degrees of G defined in 3.3.

5.5 Theorem. *Let G_0 be a split semisimple algebraic group over k, $G = {}_\gamma G_0$ the twisted form of G_0 given by a 1-cocycle $\gamma \in H^1(k, G_0)$, and X a projective G-homogeneous variety. If X is generically split, then for all primes p the following identity on the Poincaré polynomials holds:*

$$\frac{g(\mathrm{Ch}^*(\overline{X}), t)}{g(\overline{\mathrm{Ch}}^*(X), t)} = \prod_{i=1}^{r} \frac{t^{d_i p^{j_i}} - 1}{t^{d_i} - 1}, \tag{6}$$

where $J_p(G) = (j_1, \dots, j_r)$ and d_i's are the p-coprimary parts of the p-exceptional degrees of G_0.

Proof. In the proof of this theorem we use results established in our paper [PSZ].

Let p be a prime integer and Y the variety of Borel subgroups of G. We fix preimages $e_i \in \mathrm{Ch}^*(\overline{Y})$ of $x_i \in \mathrm{Ch}^*(\overline{G})$ (see definition of the J-invariant). For an r-tuple $M = (m_1, \dots, m_r)$ we set $e^M = \prod_{i=1}^r e_i^{m_i}$.

First we recall the definition of filtrations on $\mathrm{Ch}^*(\overline{Y})$ and $\overline{\mathrm{Ch}}^*(Y)$ (see [PSZ, Definition 5.5]). Given two pairs (L, l) and (M, m), where L and M are r-tuples and l and m are integers, we say that $(L, l) \leq (M, m)$ if either $L < M$, or $L = M$ and $l \leq m$.

The (M, m)-th term of the filtration on $\mathrm{Ch}^*(\overline{Y})$ is the subgroup of $\mathrm{Ch}^*(\overline{Y})$ generated by the elements $e^I\alpha$, $I \leq M$, $\alpha \in R^{\leq m}$. We denote as $A^{*,*}$ the graded ring associated to this filtration. As $A^{*,*}_{\mathrm{rat}}$ we denote the graded subring of $A^{*,*}$ associated to the subring $\overline{\mathrm{Ch}}^*(Y) \subset \mathrm{Ch}^*(\overline{Y})$ of rational cycles with the induced filtration.

Consider the Poincaré polynomial of A_{rat} with respect to the grading induced by the usual grading of $\mathrm{Ch}^*(\overline{Y})$. Proposition 5.10 of [PSZ] which explicitely describes an $\mathbb{F}_p$-basis of $A^{*,*}_{\mathrm{rat}}$ implies that the Poincaré polynomial $g(A_{\mathrm{rat}}, t) =: \sum_{i=0}^{\dim X} a_i t^i$ ($a_i \in \mathbb{Z}$) of A_{rat} equals the right hand side of formula (6).

On the other hand, $\dim \overline{\mathrm{Ch}}^*(Y) = \dim A_{\mathrm{rat}}$ and the coefficients b_i of the Poincaré polynomial $g(\overline{\mathrm{Ch}}^*(Y), t) =: \sum_{i=0}^{\dim Y} b_i t^i$ are obviously bigger than or equal to a_i for all i. Therefore $g(\overline{\mathrm{Ch}}^*(Y), t) = g(A_{\mathrm{rat}}, t)$.

Finally, since X is generically split, we have $\dfrac{g(\overline{\mathrm{Ch}}^*(Y), t)}{g(\overline{\mathrm{Ch}}^*(X), t)} = \dfrac{g(\mathrm{Ch}^*(\overline{Y}), t)}{g(\mathrm{Ch}^*(\overline{X}), t)}$ by [PSZ, Theorem 3.7]. This finishes the proof of the theorem. □

5.6 Example. Let Q be an anisotropic $2n$-dimensional generically split quadric and G the respective group. It is well-known that the left hand side of formula (6) equals $1 + t^n$. Therefore $J_2(G) = (0, \dots, 0, 1)$ and by [PSZ, Last column of Table in Section 4] $n = 2^l - 1$ for some l.

The right hand side of formula (6) depends only on the value of the J-invariant of G. In turn, the left hand side depends on the rationality of cycles on X. Available information on cycles that are rational, allows to establish the following result.

5.7 Theorem. *Let G be a group over a field k of type $\Phi = \mathrm{F}_4$, E_6, E_7 or E_8 given by a 1-cocycle from $H^1(k, G_0)$, where G_0 stands for the split adjoint group of the same type as G, and let X be the variety of the parabolic subgroups of G of type i. Assume that the characteristic of k is different from any torsion prime of G.*

The variety X is not generically split if and only if

	Root system	*Parabolic subgroups*	*J-invariant*
1	$\Phi = \mathrm{F}_4$	$i = 4$	$J_2(G) = (1)$
2	$\Phi = \mathrm{E}_6$	$i = 1, 6$	$J_2(G) = (1)$
3	$\Phi = \mathrm{E}_6$	$i = 2, 4$	$J_3(G) = (j_1, *)$, $j_1 \neq 0$
4	$\Phi = \mathrm{E}_7$	$i = 1, 3, 4, 6$	$J_2(G) = (j_1, *, *, *)$, $j_1 \neq 0$
5	$\Phi = \mathrm{E}_7$	$i = 7$	$J_3(G) = (1)$
6	$\Phi = \mathrm{E}_7$	$i = 1, 6, 7$	$J_2(G) = (*, j_2, *, *)$, $j_2 \neq 0$
7	$\Phi = \mathrm{E}_8$	$i = 1, 6, 7, 8$	$J_2(G) = (j_1, *, *, *)$, $j_1 \neq 0$
8	$\Phi = \mathrm{E}_8$	$i = 7, 8$	$J_3(G) = (1, *)$

("$*$" means any value).

Proof. First we prove using Theorem 5.5 that the cases listed in the table are not generically split. Indeed, assume the contrary.

Case 1. Using Example 3.8(5) one immediately sees that the right hand side of formula (6) equals $t^3 + 1$. On the other hand, a straightforward computation using Pieri formula (1) shows that $\mathrm{Ch}^3(\overline{X}) \subset \langle \mathrm{Ch}^1(\overline{X}) \rangle$. Another straightforward computation using formula (3) shows that $\mathrm{Ch}^1(\overline{X})$ is generated by the first Chern class of the tangent bundle of $\overline{X}$, and therefore is rational. Therefore $\mathrm{Ch}^3(\overline{X})$ is rational as well, and thus the left hand side of formula (6) does not have a term of degree 3. Contradiction with Theorem 5.5.

Case 2. By Example 3.8(5) the right hand side of (6) equals $t^3 + 1$. On the other hand, by (2)

$$g(\mathrm{Ch}^*(\overline{X}), t) = \sum_{i=0}^{16} t^i + \sum_{i=4}^{12} t^i + t^8.$$

In particular, $g(\mathrm{Ch}^*(\overline{X}), t)$ is not divisible by $t^3 + 1$. Contradiction.

Cases 3. and 4. By Example 3.8(5) the right hand side of (6) has a term of degree 1. On the other hand, a computation by formula (3) shows that the Picard group $\mathrm{Ch}^1(\overline{X})$ is generated by the first Chern class of the tangent bundle, and hence is rational. (Alternatively, one can see this from exact sequence (5) and Example 4.1(4)). Therefore the left hand side of formula (6) does not have a term of degree 1. Contradiction.

Case 5. By Example 3.8(5) the right hand side of (6) equals t^8+t^4+1. On the other hand, a computation using formula (2) shows that $g(\mathrm{Ch}^*(\overline{X}),t) = \frac{(t^{10}-1)(t^{14}-1)(t^{18}-1)}{(t-1)(t^5-1)(t^9-1)}$ and is not divisible by t^8+t^4+1. Contradiction.

Cases 6. and 7. Assume first that $(\Phi, i) \neq (\mathrm{E}_7, 7)$. Using Example 3.8(5) one immediately sees that the right hand side of formula (6) has a term of degree 3. On the other hand, a straightforward computation using Pieri formula (1) and formula (3) shows that

$$\mathrm{Ch}^3(\overline{X}) \subset \langle c_1(T_{\overline{X}}), c_2(T_{\overline{X}}), c_3(T_{\overline{X}})\rangle.$$

Therefore $\mathrm{Ch}^3(\overline{X})$ is rational, and thus the left hand side of formula (6) does not have a term of degree 3. Contradiction with Theorem 5.5.

Consider now the case E_7 with $i=7$. A straightforward computation using the Pieri formula shows that $\mathrm{Ch}^3(\overline{X}) = \langle \mathrm{Ch}^1(\overline{X})\rangle$. If $J_2(G) = (0,1,*,*)$, then by Proposition 4.2 and sequence (5) $\mathrm{Ch}^1(\overline{X})$ is rational. Therefore $\mathrm{Ch}^3(\overline{X})$ is rational, and the same considerations as in the previous paragraph lead to a contradiction.

Otherwise $J_2(G) = (1,1,*,*)$. The expression $\dfrac{g(\mathrm{Ch}^*(\overline{X}),t)}{(t+1)(t^3+1)}$ is a polynomial which has negative coefficients. Hence, by formula (6) the polynomial $g(\overline{\mathrm{Ch}}^*(X),t)$ has negative coefficients. Contradiction.

Case 8. Using Example 3.8(5) one immediately sees that the right hand side of formula (6) has a term of degree 4. On the other hand, a straightforward computation using Pieri formula (1) and formula (3) shows that $\mathrm{Ch}^4(\overline{X}) \subset \langle c_i(T_{\overline{X}}), i=1,\dots,4\rangle$. Therefore $\mathrm{Ch}^4(\overline{X})$ is rational, and thus the left hand side of formula (6) does not have a term of degree 4. Contradiction with Theorem 5.5.

Next we show that all other varieties not listed in the table are generically split. Let G and X be an exceptional group and a G-variety of maximal parabolic subgroups of type s not listed in the table. Consider $G_{k(X)}$. The vertex s in the Tits diagram of the $G_{k(X)}$ is circled.

Then using Proposition 3.9(1,2) and the classification of Tits indices [Ti66] one can see case-by-case that for all primes p the J-invariant of the semisimple anisotropic kernel $(G_{k(X)})_{\mathrm{an}}$ is trivial. Therefore this anisotropic kernel is trivial by Proposion 3.9(3) and the group G splits over $k(X)$. $\square$

It turns out that if the group G is given by a cocycle with values in a split simple *simply connected* group, then the converse of Theorem 5.5 holds. More generally, one can show:

5.8 Theorem. *Let G_0 be a split simple algebraic group over k of type different from A_n and C_n. If G_0 has type D_n, we assume that $G_0 = \mathrm{SO}_{2n}$. Let $G = {}_\gamma G_0$ be the twisted form of G_0 given by a 1-cocycle $\gamma \in H^1(k, G_0)$, and X a projective G-homogeneous variety. Assume that the characteristic of k is different from any torsion prime of G.*

Then the following conditions are equivalent:

1. *X is generically split;*
2. *For all primes p*

$$\frac{g(\mathrm{Ch}^*(\overline{X}), t)}{g(\overline{\mathrm{Ch}}^*(X), t)} = \prod_{i=1}^{r} \frac{t^{d_i p^{j_i}} - 1}{t^{d_i} - 1}, \tag{7}$$

where $J_p(G) = (j_1, \ldots, j_r)$ and d_i's are the p-coprimary parts of the p-exceptional degrees of G_0.

Proof. Implication 1) $\Rightarrow$ 2) follows from Theorem 5.5.

Next we prove the opposite implication.

Cases B_n and D_{n+1}. Assume that G_0 has type B_n or D_{n+1} and $p = 2$. Let Θ denote the type of the homogeneous variety X, $s = \max\{i, \alpha_i \in \Theta\}$ (for notation see Section 2), and Y denote the projective G-homogeneous variety of maximal parabolic subgroups of type s. By [CPSZ, Theorem 2.9]

$$\frac{g(\mathrm{Ch}^*(\overline{X}), t)}{g(\mathrm{Ch}^*(\overline{Y}), t)} = \frac{g(\overline{\mathrm{Ch}}^*(X), t)}{g(\overline{\mathrm{Ch}}^*(Y), t)}.$$

(Formally speaking, Theorem 2.9 of [CPSZ] is proved for groups of type B_n, but absolutely the same proof works for groups of type D_{n+1} that are given by a cocycle from $\mathrm{SO}_{2(n+1)}$).

On the other hand, by [Vi08, Corollary 2.12] the group $\mathrm{Ch}^i(\overline{Y})$ is rational for all $i < n - s + 1$. Therefore by formula (7) $j_i = 0$ for all i such that $2i - 1 < n - s + 1$. Denote $J_p(G_{k(Y)}) = (j'_1, j'_2, \ldots, j'_r)$. By Proposition 3.9(1) $j'_l \leq j_l$ for all l. Therefore $j'_i = 0$ if $2i - 1 < n - s + 1$. On the other hand, by Proposition 3.9(2) and [PSZ, Table in Section 4] $j'_i = 0$ for all $i > [\frac{n-s+1}{2}]$.

Therefore $J_p(G_{k(Y)})$ is trivial, hence by Proposition 3.9(3) $G_{k(Y)}$ is split. By the Tits classification [Ti66] $G_{k(X)}$ is split as well.

Exceptional types. For homogeneous varieties of maximal parabolic subgroups the statement immediately follows from the proof of Theorem 5.7. The same considerations as in that proof also show the validity of the statement of the present theorem for general exceptional homogeneous varieties. □

5.9 Remark. Let G_0 and G be groups of type B_n or D_{n+1} as in the previous theorem. Let X be the variety of maximal parabolic subgroups of G of type s and $J_2(G) = (j_1, \ldots, j_r)$. It follows from the proof that X is generically split iff $j_i = 0$ for all $1 \leq i \leq [\frac{n+1-s}{2}]$.

As mentioned in the introduction this classification of generically split varieties has diverse applications. Below we give a two-line proof of the triviality of the kernel of the Rost invariant for groups of type E_7 (cf. [Ga01]).

5.10 Corollary (Kernel of the Rost invariant)**.** *Let G_0 denote a split simply connected group of type* E_7 *over an arbitrary field k. Suppose $\eta \in H^1(k, G_0)$ and the Rost invariant of η is trivial. Then $\eta = 0$.*

Proof. Consider the twisted form G of G_0 given by η and the varieties X_i of maximal parabolic subgroups of G of type i ($i = 1, \ldots, \operatorname{rank} G$).

The Rost invariant of $\eta_{k(X_i)}$ is still zero. On the other hand, the cocycle $\eta_{k(X_i)}$ is equivalent to a cocycle with values in the semisimple anisotropic kernel $(G_{k(X_i)})_{\mathrm{an}}$. Since the rank of $(G_{k(X_i)})_{\mathrm{an}}$ is at most 6, $\eta_{k(X_i)}$ is trivial by [Ga01, Theorem 0.5 a),b)], i.e., $G_{k(X_i)}$ is split.

Thus, for all i the varieties X_i are generically split. Their classification (Theorem 5.7) immediately implies that for all primes p the J-invariant of G modulo p is trivial. Therefore G is split by Corollary 3.9(3), i.e., $\eta = 0$. □

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