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# GENTZEN SYSTEMS FOR MODAL LOGIC

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Nice Gentzen formulations of the normal modal systems T and S4 have been know for some time; see, for example, Kanger [5] or Curry [1]. A similar formulation of S5 has also been given, but it is not so nice as the Elimination Theorem is not provable for it. I shall present here sets of rules for several of the non-normal modal systems which are akin to T and S4. Each of the L-systems to be defined here has an Elimination Theorem which may be proved by the methods of Gentzen [3]. These systems are useful, in that each of them has a decision procedure, following, for example, Kleene [6], §80.

1 Epistemic Systems In order to provide a decision procedure for Lewis' system S2, Ohnishi and Matsumoto [11] defined a system, Q2, which had the property that a formula, $A$, was provable in S2 if and only if the consecution $\mathrm{N}(p \supset p) \Vdash A$ was provable in Q2. Q2 was formed by adjoining to any appropriate formulation of the classical propositional calculus, such as Gentzen's **LK** [3], the rules:

$$(\mathrm{N}\Vdash)\ \frac{\Gamma, A \Vdash \Delta}{\Gamma, \mathrm{N}A \Vdash \Delta} \quad \text{and} \quad (\Vdash\mathrm{N})\ \frac{\Gamma \Vdash A}{\mathrm{N}\Gamma \Vdash \mathrm{N}A}$$

where for $(\Vdash\mathrm{N})$ $\Gamma$ must be non-empty.

Here, and elsewhere, $A$, $B$, $C$, etc. are well formed formulas formed from atomic formulas by means of propositional connectives, including N for necessity; $\Gamma$, $\Delta$, etc. are any finite sequences of (zero or more) constituent formulas, $A$, $B$, $C$, etc. Consecutions $\alpha$, $\beta$, etc. are expressions $\Gamma \Vdash \Delta$. $\mathrm{N}\Gamma$ is the result of applying the operator N to each member of $\Gamma$.

Q2 is equivalent to the Hilbert style system E2 introduced by Lemmon (see, for example, [8]), in the sense that $A$ is provable in E2 if and only if $\Vdash A$ is provable in Q2.

Besides the axioms and rules for the classical propositional calculus, E2 has only the axioms

A.1 $\mathrm{N}A \supset A$
A.2 $\mathrm{N}(A \supset B) \supset .\, \mathrm{N}A \supset \mathrm{N}B$

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and the rule

$$\text{R.2} \quad \frac{\vdash A \supset B}{\vdash \mathsf{N}A \supset \mathsf{N}B}$$

The proofs of the counterparts of A.1 and A.2 in Q2 are nearly trivial. That *modus ponens* is admissible is guaranteed by the Elimination Theorem, i.e., that the rule

$$\frac{\Gamma \Vdash A, \Delta \qquad \Gamma', A \Vdash \Delta'}{\Gamma, \Gamma' \Vdash \Delta, \Delta'}$$

is admissible in Q2. R2 is clearly admissible in Q2 given ($\Vdash$N). Hence E2 is contained in Q2.

To show that Q2 is likewise contained in E2, we let the E2 counterpart of the consecution $\alpha = A_1, \ldots, A_n \Vdash B_1, \ldots, B_m$ be the formula $\alpha^* = A_1 \supset. \cdots .\supset .A_n \supset .\sim B_1 \supset. \cdots \supset .\sim B_{m-1} \supset B_m$, or, in case $m = 0$, $\alpha^* = A_1 \supset . \cdots \supset .A_{n-1} \supset \sim A_n$. (No counterpart for the case in which $n = m = 0$ need be given since the consecution $\Vdash$ is not provable in Q2.) If $\alpha$ is provable in Q2, then $\alpha^*$ is provable in E2. This is clear in case $\alpha$ is prime or if $\alpha$ is the conclusion of an inference from $\beta$ ($\beta'$) by one of the non-modal rules, supposing, for the induction, that if $\beta$ is provable in Q2 in fewer steps than $\alpha$, then $\beta^*$ is provable in E2. If $\alpha$ is from $\beta$ by ($\Vdash$*N*), then

$$\frac{\beta = \quad A_1, \ldots, A_n \Vdash B}{\alpha = \mathsf{N}A_1, \ldots, \mathsf{N}A_n \Vdash \mathsf{N}B}.$$

By the inductive hypothesis, $\vdash_{\text{E2}} A_1 \supset . \cdots \supset .A_n \supset B$; so $\vdash_{\text{E2}} \mathsf{N}A_1 \supset \mathsf{N}(A_2 \supset . \cdots \supset. A_n \supset B)$ by (R.2), from which it follows that $\vdash_{\text{E2}} \mathsf{N}A_1 \supset. \mathsf{N}A_2 \supset . \cdots \supset. \mathsf{N}A_n \supset \mathsf{N}B$, i.e., $\vdash_{\text{E2}} \alpha^*$, by repeated applications of A.2 and transitivity.

In case $\alpha$ is from $\beta$ by the rule (N$\Vdash$), we have

$$\frac{\beta = C_1, \ldots, C_n, \quad A \Vdash B_1, \ldots, B_m}{\alpha = C_1, \ldots, C_n, \mathsf{N}A \Vdash B_1, \ldots, B_m}.$$

So, by the inductive hypothesis $\vdash_{\text{E2}} C_1 \supset . \cdots \supset. C_n \supset. A \supset .\sim B_1 \supset . \cdots \supset .\sim B_{m-1} \supset B_m$ (or $\vdash_{\text{E2}} C_1 \supset . \cdots \supset . C_n \supset \sim A$). Since $\vdash_{\text{E2}} \mathsf{N}A \supset A$ (A.1), it follows that $\vdash_{\text{E2}} C_1 \supset . \cdots \supset C_n \supset. \mathsf{N}A \supset. \sim B_1 \supset. \cdots \supset. \sim B_{m-1} \supset B_m$ (or $\vdash_{\text{E2}} C_1 \supset . \cdots \supset . C_n \supset \sim \mathsf{N}A$), which is to say, $\vdash_{\text{E2}} \alpha^*$.

Thus, if $\alpha = \Vdash A$ is provable in Q2, then $\alpha^* = A$ is provable in E2, and conversely. The two systems are equivalent, in the sense defined. Hence I shall now call Q2 LE2.

Matsumoto [9] also presented the system Q3, to provide a decision procedure for S3, by modifying the rule ($\Vdash$N) of Q2 to

$$(\Vdash \mathsf{N}) \; \frac{\Gamma \Vdash \mathsf{N}\mathbf{t}, A \qquad \mathsf{N}\Gamma \Vdash \mathsf{N}A}{\mathsf{N}\Gamma \Vdash \mathsf{N}A}$$

where $\mathbf{t}$ is any designated tautology, such as $(p \supset p)$, and $\Gamma$ is again non-empty.

Q3 is equivalent to Lemmon's E3, which results from adding the axiom schema

A.3 $\mathsf{N}A \supset \mathsf{N}(\mathsf{N}B \supset \mathsf{N}A)$

to E2. It is easy to show that E3 is contained in Q3. To establish that Q3 is contained in E3 we use the same representation as given above. The admissibility of the counterpart of $(\mathsf{N}\Vdash)$ follows as with E2. That the counterpart of this $(\Vdash\mathsf{N})$ is admissible in E3 is shown as follows: Suppose that $\beta_1^* = C_1 \supset. \cdots \supset . C_n \supset . \sim\mathsf{N}\mathbf{t} \supset A$ and $\beta_2^* = \mathsf{N}C_1 \supset. \cdots \supset . \mathsf{N}C_n \supset A$, we show that if $\vdash_{E3} \beta_1^*$ and $\vdash_{E3} \beta_2^*$, then $\vdash_{E2} \alpha^* = \mathsf{N}C_1 \supset. \cdots \supset. \mathsf{N}C_n \supset \mathsf{N}A$.

| | | |
|---|---|---|
| 1. | $\vdash \mathsf{N}C_1 \supset. \cdots \supset. \mathsf{N}C_n \supset A$ | given |
| 2. | $\vdash C_1 \supset. \cdots \supset. C_n \supset. \sim\mathsf{N}\mathbf{t} \supset A$ | given |
| 3. | $\vdash \mathsf{N}C_1 \,\&\, \cdots \,\&\, \mathsf{N}C_n \supset A$ | 1, truth functional logic (TFL) |
| 4. | $\vdash \mathsf{N}(C_1 \,\&\, \cdots \,\&\, C_n) \supset A$ | 3, $(\vdash \mathsf{N}(A \,\&\, B) \equiv \mathsf{N}A \,\&\, \mathsf{N}B)$ |
| 5. | $\vdash C_1 \supset. \cdots \supset. C_n \supset. (\mathsf{N}\mathbf{t} \supset \mathsf{N}(C_1 \,\&\, \cdots \,\&\, C_n)) \supset A$ | 2, 4, (TFL) |
| 6. | $\vdash C_1 \,\&\, \cdots \,\&\, C_n \supset. (\mathsf{N}\mathbf{t} \supset \mathsf{N}(C_1 \,\&\, \cdots \,\&\, C_n)) \supset A$ | 5, (TFL) |
| 7. | $\vdash \mathsf{N}(C_1 \,\&\, \cdots \,\&\, C_n) \supset. \mathsf{N}(\mathsf{N}\mathbf{t} \supset \mathsf{N}(C_1 \,\&\, \cdots \,\&\, C_n)) \supset NA$ | 6, R.2, A.2 |
| 8. | $\vdash \mathsf{N}(C_1 \,\&\, \cdots \,\&\, C_n) \supset \mathsf{N}(\mathsf{N}\mathbf{t} \supset \mathsf{N}(C_1 \,\&\, \cdots \,\&\, C_n)) \supset N(C_1 \,\&\, \cdots \,\&\, C_n)$ $\supset \mathsf{N}A$ | 7, (TFL) |
| 9. | $\vdash \mathsf{N}(C_1 \,\&\, \cdots \,\&\, C_n) \supset \mathsf{N}(\mathsf{N}\mathbf{t} \supset \mathsf{N}(C_1 \,\&\, \cdots \,\&\, C_n))$ | A.3 |
| 10. | $\vdash \mathsf{N}(C_1 \,\&\, \cdots \,\&\, C_n) \supset \mathsf{N}A$ | 8, 9, M, P |
| 11. | $\vdash \mathsf{N}C_1 \,\&\, \cdots \,\&\, \mathsf{N}C_n \supset \mathsf{N}A$ | 10, $(\vdash \mathsf{N}(A \,\&\, B) \equiv \mathsf{N}A \,\&\, \mathsf{N}B)$ |
| 12. | $\vdash \mathsf{N}C_1 \supset. \cdots \supset. \mathsf{N}C_n \supset \mathsf{N}A$ | 11, (TFL) |

This suffices to show that Q3 is contained in E3 and that the systems are thus equivalent. I shall call Q3 LE3. (The equivalence of Q2 and E2 and of Q3 and E3 was observed by Ohnishi in [10].)

Lemmon's systems ET and E4 result from adding

A.4 $\mathsf{N}\mathbf{t} \supset \mathsf{N}\mathsf{N}\mathbf{t}$

and

A.5 $\mathsf{N}A \supset \mathsf{N}\mathsf{N}A$

respectively to E.2. Their L formulations are formed by modifying the rule $(\Vdash\mathsf{N})$ again. LET has the rule

$$\frac{\theta,\ \Gamma \Vdash A}{\theta,\ \mathsf{N}\Gamma \Vdash \mathsf{N}A}$$

where $\theta$ is either empty or contains only the constituent $\mathsf{N}\mathbf{t}$, and not both $\theta$ and $\Gamma$ are empty. LE4 has the rule

$$\frac{\mathsf{N}\Gamma \Vdash A}{\mathsf{N}\Gamma \Vdash \mathsf{N}A}$$

provided, again, that $\Gamma$ is not empty. LET and LE4 may easily be shown to be equivalent to ET and E4 in the same manner as before.

If the condition that $\Gamma$ ($\Gamma$ or $\theta$) not be empty is dropped from the rules $(\Vdash\mathsf{N})$ for LE4 and LET, we have the familiar rules for the Gentzen formulations of S4 and T.

**2 Deontic Systems** Deontic modal systems, D$X$, result from the system $X$ (= E2, E3, ET, E4, T, S4) when its axiom A.1 is replaced by

A.1′ $\mathsf{N}A \supset \sim\mathsf{N}\sim A$.

This requires modification in the rule ($\mathsf{N}\Vdash$) in their Gentzen counterparts. Let the new rule be

$$(\mathsf{N}\Vdash)_{\mathsf{D}} \qquad \frac{\Gamma, A \Vdash}{\mathsf{N}\Gamma, \mathsf{N}A \Vdash}$$

where it is required that the succedent in both the premiss and conclusion be void and that every constituent in the antecedent of the conclusion be prefixed by an N. The systems LD$X$ are formed by replacing the rule ($\mathsf{N}\Vdash$) of L$X$ by $(\mathsf{N}\Vdash)_{\mathsf{D}}$. In addition it is necessary to modify the rule ($\Vdash\mathsf{N}$) of LDE4 and LDS4 to

$$\frac{\Gamma' \Vdash A}{\mathsf{N}\Gamma \Vdash \mathsf{N}A}$$

where $\Gamma'$ is a sequence which results by prefixing zero or more members of $\Gamma$ with an N. A similar change is required for the right premiss of ($\Vdash\mathsf{N}$) for LDE3. (These modifications would also be acceptable for LE3, LE4, and LS4.)

Proof of the Elimination Theorem for these systems goes through unimpeded by these changes. Hence, it is easy to show that LDE2, LDE3, LDET, LDE4, LDT, and LDS4 are equivalent to the systems DE2, DE3, DET, DE4, DT, and DS4, as defined, for example, in [4].

**3 Lewis' S-Systems** As observed above, Gentzen formulations of the normal systems T and S4 are obtained by allowing $\Gamma$ in ($\Vdash N$) to be empty. In [10] Ohnishi defines a Gentzen system for Lewis' system S2; it has exactly the same rules as LT but a restriction is placed on its tree proofs to the effect that

(R) *No inference by* ($\Vdash\mathsf{N}$) *may occur below an inference by* ($\Vdash\mathsf{N}$) *in which* $\Gamma$ *is empty.*

Thus $\Vdash \mathsf{N}(p \supset p)$ is provable in LS2, since

$$\cfrac{\cfrac{p \Vdash p}{\Vdash p \supset p}\ (\Vdash\supset)}{\Vdash \mathsf{N}(p \supset p)}\ (\Vdash\mathsf{N})$$

meets the condition, but $\Vdash \mathsf{N}\mathsf{N}(p \supset p)$ is not, since its proof

$$\cfrac{\cfrac{\cfrac{p \Vdash p}{\Vdash p \supset p}\ (\Vdash\supset)}{\Vdash \mathsf{N}(p \supset p)}\ (\Vdash\mathsf{N})}{\Vdash \mathsf{N}\mathsf{N}(p \supset p)}\ (\Vdash\mathsf{N})\text{—invalid}$$

violates the restriction (R).

If $\Gamma$ is allowed to be empty in ($\Vdash\mathsf{N}$) for LE3 but a similar restriction is placed on proofs, the result is a system, LS3, equivalent to Lewis' S3. (*Cf.* Ohnishi [10].)

The restriction (R) may be generalized to read, for $n \geq 1$

$(R)^n$ *No more than* $n-1$ *inferences by* $(\Vdash N)$ *may occur below an inference by* $(\Vdash N)$ *in which* $\Gamma$ *is empty.*

If, for $n \geq 1$, this is applied to proofs by the LT rules for which $\Gamma$ in $(\Vdash N)$ is allowed to be void, systems $LE2^n$ are defined (where the $n$ corresponds to the $n$ of the restriction) which are equivalent to the systems $E2^n$ described by Lemmon [8] and Kripke [7]. These are formed by the addition of the axiom $N^n\mathbf{t}$ (i.e., the result of prefixing $\mathbf{t}$ with $n$ N's) to E2 so formulated that *modus ponens* is its sole rule of inference. Thus $E2^0 = E2$; $E2^1 = E2 + N\mathbf{t} = S2$; $E2^2 = E2 + NN\mathbf{t}$; etc.

The condition $(R)^n$ presents no impediment to the proof of the Elimination Theorem for any of these systems. Thus $LE2^n$ may be shown to be equivalent to $E2^n$. That $E2^n$ is contained in $LE2^n$ is clear, for all its axioms are either provable in E2, and so provable in LE2 which is contained in $LE2^n$, or else the formula $N^n\mathbf{t}$ whose proof in $LE2^n$ is trivial. The admissibility of *modus ponens*, the sole rule of $E2^n$, is guaranteed by the Elimination Theorem. To show that $LE2^n$ is contained in $E2^n$ we use the same representation $\alpha^*$ as before. The result then follows from the fact that

*for every* $n \geq 1$, *if* $\beta$ *occurs in a proof of* $\alpha$ *which contains no more than* $n-1$ *inferences by* $(\Vdash N)$ *which are below an inference by* $(\Vdash N)$ *in which* $\Gamma$ *is empty, then* $\vdash_{E2^n}\beta^*$

which may be proved by induction on $n$, making use of the fact that if $\vdash_{E2^{n-1}} B$, then $\vdash_{E2^n} NB$ $(n \geq 1)$. Hence, $E2^n$ is equivalent to $LE2^n$.

A relaxation of (R) similar to the above immediately produces LS4 from LS3. I.e., if one or more inferences by $(\Vdash N)$ may be below an inference by $(\Vdash N)$ in which $\Gamma$ is void, then the rule $(\Vdash N)$ for LS4 is derivable:

$$\dfrac{(K\Vdash K)\ \dfrac{(\Vdash N)\ \dfrac{(\Vdash K)\ \dfrac{\Vdash \mathbf{t}}{N\mathbf{t}\Vdash \mathbf{t}} \qquad \Vdash \mathbf{t}}{\Vdash N\mathbf{t}}}{\Gamma \Vdash N\mathbf{t}, A} \qquad N\Gamma^* \Vdash A}{N\Gamma \Vdash NA}\ (\Vdash N)$$

This is as it should be, for, it is known, $E3^2 = S4$.

Devices similar to that used in the rule $(\Vdash N)$ for ET may be employed to give L systems for the systems $S4^n$ which result from the extension of T by the axiom: $N^nA \supset N^{n+1}A$ (*cf.* e.g., Feys [2], p. 127). For $LS4^n$ we modify the T rule $(\Vdash N)$ to

$$\frac{\theta, \Gamma \Vdash A}{\theta, N\Gamma \Vdash NA}$$

where $\theta$ is either empty or every constituent in it has the form $N^nB$.

Obviously, $LS4^1 = LS4$. If we required that either $\theta$ or $\Gamma$ not be empty, we would obtain rules for a family of systems $LE4^n$ which would be equivalent to the result of adding $N^nA \supset N^{n+1}A$ to ET. Thus, $E4^n$ stands to ET as $S4^n$ stands to T.

**4** The Ł Modal System A version of Łukasiewicz' modal system (without variable functors), here called Ł, may be obtained by adjoining to E2 either the schema:

$$A \supset B \supset .NA \supset NB$$

or the schema

$$NB \supset .A \supset NA$$

(*cf.* Lemmon [8]; also Kripke [7]). A Gentzen system, LŁ, equivalent to Ł results by adding to rules for the classical propositional calculus the rule ($N\Vdash$) as for E2 and the rule

$$(\Vdash N) \quad \frac{\Gamma, NB \Vdash A}{\Gamma, NB \Vdash NA}$$

where now $\Gamma$ may be any sequence, empty or not.

**5** Intuitionistic Modal Systems If the modal postulates for E2-S4, etc. were added to the intuitionistic propositional calculus for a base rather than to the classical propositional calculus, intuitionistic variants on those systems would be obtained. Equivalent Gentzen systems result simply by adding the appropriate ($N\Vdash$) and ($\Vdash N$) rules to L-rules for the intuitionistic logic.

These intuitionistic modal systems are not, however, the same as those defined in [4] and for which semantics were provided, for in those systems the formula $NA \vee \sim NA$ was provable. Gentzen formulations of these systems $IX$ ($X$ = E2-S4, etc.) having this formula result by adding the modal rules to an L-system for intuitionistic logic which allows multiple constituents in the succedent and modifying the rule ($\Vdash\sim$) to read

$$\frac{\Gamma, A \Vdash \Delta}{\Gamma \Vdash \sim A, \Delta}$$

where $\Delta$ is either empty or every member of it has the form $NB$. (Ordinarily for intuitionistic negation one would require $\Delta$ to be empty.)

The proof that $LIX$ is equivalent to $IX$ is essentially the same as before. The only trick is to show that the counterpart of the new rule ($\Vdash\sim$) is admissible in $IX$. We argue that if $\beta = C_1, \ldots, C_n, A \Vdash NB_1, \ldots, NB_m$ is provable in $LIX$, then by the inductive hypothesis $\beta^* = C_1 \supset. \cdots \supset. C_n \supset. A \supset. \sim NB_1 \supset. \cdots \supset \sim NB_{m-1} \supset NB_m$ is provable in $IX$; hence, by permutation, so is $C_1 \supset. \cdots \supset. C_n \supset. \sim NB_1 \supset. \cdots\cdots \supset. A \supset. NB_m$. From this we have $\vdash C_1 \supset. \cdots \supset. C_n \supset. \sim NB_1 \supset. \cdots\cdots \supset. \sim\sim A \supset \sim\sim NB_m$ by contraposition twice, and so $\vdash C_1 \supset. \cdots \supset. C_n \supset. \sim\sim A \supset. \sim NB_1 \supset. \cdots \supset. \sim NB_{m-1} \supset \sim\sim NB_m$ again by permutation. Given that $NB_m \vee \sim NB_m$, $\sim\sim NB_m \supset NB_m$ is provable in $IX$, so by transitivity it follows that $\alpha^* = C_1 \supset. \cdots \supset. C_n \supset \sim\sim A \supset. \sim NB_1 \supset. \cdots \supset. \sim NB_{m-1} \supset NB$ is also provable in $IX$.

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