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Geodesic and metric completeness in infinite dimensions

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Abstract. Some infinite-dimensional Riemannian manifolds are constructed in which the induced metric is incomplete, but all geodesics may be indefinitely extended in both directions and any two points may be joined by a minimizing geodesic. The construction relies on some estimates for truncated Hilbert transforms.

Key words: Riemannian, geodesic.

1. Introduction

Let M be a connected C^{∞} Riemannian manifold without boundary, modelled on a Hilbert space H; the Riemannian structure g on M prescribes on each tangent space to M an inner product which defines the original topology on the tangent space. A metric d on M is determined by g, via path-lengths, in the usual way, and induces the original topology on M. M is described as "complete", or "metrically complete", if (M, d) is a complete metric space.

A "geodesic" in M is a solution of the geodesic equation, defined on an open interval in $\mathbb{R}$. A geodesic is *right-complete* if its domain is unbounded on the right; a complete geodesic has domain $(-\infty, +\infty)$, so that it is either constant or of infinite length in both directions. If every maximal geodesic through the point $x \in M$ is right-complete, say that M is "geodesically complete at x"; it is "geodesically complete" if every maximal geodesic is complete. M is geodesically complete if it is metrically complete (a disguised version of this easy fact is Lemma 2.6 below).

The standard proof (de Rham [17]; see [14], pp. 172–176, or [12], pp. 56–58, or [13], pp. 126–127, etc.) of the Hopf-Rinow theorem shows also that, if M is finite-dimensional and geodesically complete at x for some $x \in M$, then it is metrically complete. The proof evidently uses local compactness, and the result is indeed false in infinite dimensions. If M is complete but infinite-dimensional, there may be a pair of points $a, b \in M$ which cannot be joined by a geodesic (see [2]); then $M_1 = M \backslash \{b\}$ is geodesically complete

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at a, but its metric completion has an ideal point "b". The missing point is, however, clearly the limit of many incomplete geodesics in M_1, obtained from geodesics through b in M. (Similarly, a proper open submanifold of any connected Riemannian manifold must be geodesically incomplete). In fact, Ekeland has proved [10], by a method of great ingenuity, that the set of points of a complete connected Riemannian manifold that cannot be joined to a given point p by a unique minimizing geodesic is of first category. One deduces, taking $p = b$, that the points at which the manifold M_1 is geodesically complete are rare.

It is natural to enquire whether geodesic completeness (at all points) always implies metric completeness. As a conjecture, this has never to my knowledge been publicly proposed or disproved; there are, for instance, a striking silence on p. 126 of [13], and an oddly ambiguous statement in the introduction of [8]. But there may be reasons for this omission. In the first place, the question probably has no interesting consequences, for it is difficult to envisage how a geodesically complete manifold might arise *naturally* without being more obviously metrically complete anyway. Secondly, the conjecture, although it does hold under certain strong but reasonable additional conditions, perhaps does not seem on reflection at all likely in full generality. My principal purpose here is to give a class of explicit counterexamples.

Theorem A *There is a Riemannian manifold M such that*

(i) *M is modelled on separable infinite-dimensional Hilbert space H;*
(ii) *M is of differentiability class C^ω;*
(iii) *M is C^ω-conformally equivalent to the Hilbert space H;*
(iv) *M is geodesically complete;*
(v) *M is metrically incomplete;*
(vi) *The metric completion of M has exactly one ideal point;*
(vii) *M is geodesically convex (that is, any two points of M may be joined by a minimizing geodesic).*

Although the essential idea of the examples, namely to utilize a suitable set of operators, will be obvious to any reader of [2], its realization has not been trivial, and I expect that more natural examples, at least of properties (iv) and (v), may be found in due course. However, the properties (ii), (iii), (vi), and (vii) dispose of a number of further conjectures, of which the most interesting are motivated in §2 (see Remarks 2.11). They are clearly, though distantly, related to the old problem of "prolongation", that is, of embedding an incomplete Riemannian manifold as an open subset of a complete one; but the connection will not be discussed here. See [1], [4], [9], [15], [18] for various aspects of this problem in finite dimensions.

The gist of the example is presented at Proposition 4.4–Theorem 4.10; the proof of geodesic convexity, which depends on a specialized adaptation to the weak topology of an argument of Hilbert (cf. [3], pp. 70–72, and Theorem 1, p. 150, of [16]), is in §7 and §8. Various tedious but intuitively plausible preliminaries to these arguments occupy §3, §5, and §6. Finally, §9 contains the proof of Theorem A and some concluding remarks. Some other addenda will be presented elsewhere.

The arguments and constructions were developed over a long period, and I am particularly grateful for the hospitality and assistance I received at Indiana University, Bloomington, in 1988, and at DPMMS, Cambridge, in 1993.

2. Completions

(2.1) Let $p : [\beta_0, \beta) \to M$ be continuous, where $-\infty < \beta_0 < \beta \leq \infty$. In this case, where the domain is open on the right, p is described as piecewise C^q (here $1 \leq q \leq \infty$) if there is a sequence of parameter values $\beta_0 = t_0 < t_1 < \cdots < t_n \uparrow \beta$ such that, for each $n \geq 0$, p is C^q on $[t_n, t_{n+1}]$. Then p certainly has a length $\ell(p) = \lim_{\alpha \uparrow \beta} \int_{\beta_0}^{\alpha} \|\dot{p}(t)\| \, dt$, possibly with the symbolic value ∞. (Here $\| \ \|$ denotes the norm in $T_{p(t)}M$ induced by the Riemannian inner product). When p is of finite length, $(p(t))_{t \uparrow \beta}$ is a Cauchy net in (M, d), defining a point $p(\beta-)$ of the completion $(\widehat{M}, d)$. (This is certainly the case if p is the restriction to $[\beta_0, \beta)$ of a geodesic, and $\beta < \infty$). Two such paths, both of finite length, $p_i : [0, \beta_i) \to M$ for $i = 1, 2$, will have the same limits $p_i(\beta_i-)$ if and only if there are sequences $u_n^{(i)} \uparrow \beta_i$ such that $d(p_1(u_n^{(1)}), p_2(u_n^{(2)})) \to 0$.

Given $a \in \widehat{M}$, $b \in M$, and $\epsilon > 0$, there is a piecewise C^q path $p : [0, \beta) \to M$ such that $p(0) = b$, $p(\beta-) = a$, and $\ell(p) < d(a, b) + \epsilon$. (Since $d(p(t), b) \leq \ell(p|[0, t])$ for each $t \in [0, \beta)$, it follows in the limit that $d(a, b) \leq \ell(p)$ too). To see this, take a sequence (x_n) in M with $d(a, x_n) < 2^{-n-2}\epsilon$ for $n \geq 1$; then $d(x_n, x_{n+1}) < 3.2^{-n-3}\epsilon$, and join x_n, x_{n+1} by a C^q path $p_n : [n, n+1] \to M$ of length less than $3.2^{-n-3}\epsilon$. Finally let $p_0 : [0, 1] \to M$ be a C^q path with $p_0(1) = x_1$ and $p_0(0) = b$, of length less than $d(x_1, b) + \frac{1}{4}\epsilon$. Concatenate these paths to obtain p, of length less than $d(x_1, b) + \frac{1}{4}\epsilon + \sum_{n=1}^{\infty} 3.2^{-n-3}\epsilon = d(b, x_1) + \frac{5}{8}\epsilon < d(b, a) + \epsilon$, since $d(b, x_1) < d(a, b) + \frac{1}{8}\epsilon$. The limit of p is clearly a.

It follows that $\widehat{M}$ may be constructed as a space of equivalence classes of piecewise C^q paths, rather than of sequences. From now on, all paths are assumed piecewise C^1 in the sense appropriate to their domain. (It would indeed be possible to use rectifiable paths instead). A path $p : J \to M$, where J is any interval in $\mathbb{R}$, is described as "minimizing" if, whenever $\beta_0, \beta \in J$ and $\beta_0 < \beta$, $\ell(p|[\beta_0, \beta]) = d(p(\beta_0), p(\beta))$.

Lemma 2.2 *For each $x \in M$ there is a closed neighbourhood C of x in M such that (C, d) is complete. (In other words, (M, d) is "locally complete").*

Proof. For sufficiently small $\delta > 0$, the exponential map $\exp_x$ at x maps the closed ball $Q(\delta)$ of radius δ about the origin in T_xM diffeomorphically on to a closed neighbourhood $C(\delta)$ of x in M in which any two points may be joined by a minimizing geodesic. (This is the convex neighbourhood theorem, which is still valid in infinite dimensions; see [13], pp. 83–85; and such a neighbourhood $C(\delta)$ is a "convex normal neighbourhood" of x). Thus distances between points of $C(\delta)$ may be computed from paths in $C(\delta)$ alone. But the derivative of $\exp_x$ at 0 is the identity; hence, if δ is small enough, $\exp_x$ will neither increase nor decrease the lengths of paths in $Q(\delta)$ by a factor greater than 2. For such δ, a Cauchy sequence (y_n) in $C(\delta)$ will determine a Cauchy sequence $(\exp_x^{-1} y_n)$ in $Q(\delta)$, which will converge to some point $z \in Q(\delta)$ (as H is complete). Then $y_n \to \exp_x z$ in $C(\delta)$. □

(A proof is also given by Ekeland [10], but the result was known before; a longer but more elementary argument establishes it for Finsler manifolds modelled on Banach spaces. The model must be complete).

Remark 2.3 A metric space (X, d) is locally complete if and only if it is open in its completion $\widehat{X}$. If X is open in $\widehat{X}$ and $x \in X$, take a closed ball about x in $\widehat{X}$ which lies within X. This is a complete neighbourhood of x in X. On the other hand, if $x \in X$ and V is an open neighbourhood of x in X whose closure $\bar{V}$ in X is complete, then the closure $\widehat{V}$ of V in $\widehat{X}$ consists of the equivalence classes of Cauchy sequences in V, and so is identified with $\bar{V}$. However, $\widehat{V}$ is a neighbourhood of x in $\widehat{X}$. So it is a neighbourhood of x in $\widehat{X}$ which is included in X.

Corollary 2.4 *M is open in $\widehat{M}$.*

(2.5) It is natural in the present context to write $\partial M := \widehat{M} \backslash M$, and to call it the "ideal boundary" of M. As M is dense and open in $\widehat{M}$, ∂M is its topological frontier in $\widehat{M}$, and is closed and nowhere dense.

Lemma 2.6 *Let $p : (\gamma, \beta) \to M$ be a maximal geodesic which is right-incomplete, so that $\beta < \infty$. Then $p(\beta-) \in \partial M$.*

Proof. Certainly p is non-constant, so may be reparametrized by arc-length if necessary. Suppose $x = p(\beta-) \in M$. Take a convex normal neighbourhood $C(\delta)$ of x, as in Lemma 2.2. The geodesic through $p(\beta - \frac{1}{4}\delta)$ in the direction $\dot{p}(\beta - \frac{1}{4}\delta)$ may be extended as long as it remains in $C(\delta)$, and so for a distance at least $\frac{1}{2}\delta$. After translation of parameter, this geodesic will extend p to $(\gamma, \beta + \frac{1}{4}\delta)$ at least, which contradicts maximality of p. □

(2.7) If $x \in M$ and $a \in \partial M$, say that a is *optimally accessible* [or *accessible*] from x if there is a minimizing geodesic [or just a geodesic] $p : [0, \beta) \to M$ for which $p(0) = x$ and $p(\beta-) = a$. If a is [optimally] accessible from some point $x \in M$, say that a is [*optimally*] *accessible.* Let $\partial_0 M$ denote the set of accessible points of ∂M, and $\partial_1 M$ the set of optimally accessible points. The equivalence of metric and geodesic completeness can then be formulated as $\partial M = \emptyset \iff \partial_0 M = \emptyset$.

Let me call a "unit geodesic" one parametrized by arc-length. Suppose that M is finite-dimensional and $x \in M$, and that any unit geodesic p with $p(0) = x$ may be defined at least on $[0, k]$, for some $k \geq 0$. Then the argument of de Rham that I referred to in §1 proves incidentally that $C(x; k) := \{y \in M : d(x, y) \leq k\}$ is compact, and that any point of $C(x; k)$ may be joined to x by a minimizing geodesic.

Proposition 2.8 *Let M be finite-dimensional, and $\partial M \neq \emptyset$. Suppose $x \in M$.*

(a) *Let A be a closed subset of ∂M such that $d(x, A) = d(x, \partial M)$. There is at least one point $a \in A$ such that $d(x, a) = d(x, \partial M)$.*

(b) *If $a \in \partial M$ and $d(x, \partial M) = d(x, a)$, then a is optimally accessible from x.*

Proof. Take a sequence (a_n) in A such that $d(x, a_n) < d(x, \partial M) + 1/n$. By (2.1), there is a piecewise C^1 path $p_n : [0, \beta_n) \to M$ such that $p_n(0) = x$,

$p_n(\beta_n-) = a_n$, and $\ell(p_n) < d(x, \partial M) + 1/n$. Let b_n be the last value of p_n in $C(x; d(x, \partial M) - 1/n)$. Then $d(x, b_n) = d(x, \partial M) - 1/n$, and the length of the rest of p_n (after the value b_n) must be less than $2/n$, so that $d(a_n, b_n) < 2/n$. (1)

If $0 < k < d(x, \partial M)$, any unit geodesic starting at x can certainly be defined for a distance k. Set $k_n = d(x, \partial M) - 1/n$. Then there is by (2.7) a unit vector $\xi_n \in T_x M$ such that $\exp_x(k_n \xi_n) = b_n$. Again using finite-dimensionality, extract a subsequence $(\xi_{n(i)})$ for which $n(i)$ increases strictly with i and $\xi_{n(i)} \to \xi$ as $i \to \infty$. Now $q(t) = \exp_x(t\xi)$ is a unit geodesic defined for $0 \le t < d(x, \partial M)$ at least; set $a_0 = q(d(x, \partial M)-)$. Thus $d(q(t), a_0) \le d(x, \partial M) - t$, for $0 \le t < d(x, \partial M)$. (2)

Given $\epsilon > 0$, choose i such that $\frac{1}{2}\epsilon > 3/n(i)$. As $\eta \mapsto \exp_x(k_{n(i)}\eta)$ is continuous on the unit sphere in $T_x M$, and $\xi_{n(j)} \to \xi$ therein as $j \to \infty$, there exists $J \ge i$ such that, whenever $j \ge J$, $d(\exp_x(k_{n(i)}\xi), \exp_x(k_{n(i)}\xi_{n(j)})) < \frac{1}{2}\epsilon$. Hence, if $j \ge J$,

$$
\begin{aligned}
d(a_0, a_{n(j)}) &\le d(a_0, \exp_x(k_{n(i)}\xi)) + d(\exp_x(k_{n(i)}\xi), \exp_x(k_{n(i)}\xi_{n(j)})) \\
&\quad + d(\exp_x(k_{n(i)}\xi_{n(j)}), \exp_x(k_{n(j)}\xi_{n(j)})) \\
&\quad + d(\exp_x(k_{n(j)}\xi_{n(j)}), a_{n(j)}) \\
&\le \frac{1}{n(i)} + \frac{1}{2}\epsilon + |k_{n(i)} - k_{n(j)}| + d(b_{n(j)}, a_{n(j)}), \quad \text{by (2)}, \\
&\le \frac{1}{n(i)} + \frac{1}{2}\epsilon + \frac{1}{n(i)} - \frac{1}{n(j)} + \frac{2}{n(j)}, \quad \text{by (1)}, \\
&\le \frac{3}{n(i)} + \frac{1}{2}\epsilon < \epsilon, \qquad \text{since } n(j) \ge n(J) \ge n(i). \qquad (3)
\end{aligned}
$$

It follows that $a_{n(j)} \to a_0$ as $j \to \infty$. By (2.5), ∂M, and consequently A, are closed in $\widehat{M}$, so this shows that $a_0 \in A$; as $d(x, a_n) \to d(x, \partial M)$ by construction, it also follows that $d(x, a_0) = \lim d(x, a_{n(j)}) = d(x, \partial M)$. This proves (a), if one takes $a = a_0$.

By the construction, a_0 is optimally accessible from x by the path q. Repeat the argument with $A = \{a\}$ to establish (b). □

Corollary 2.9 *If M is finite-dimensional, $\partial_1 M$ is dense in ∂M. In particular, if y is an isolated point of ∂M, there is a neighbourhood U of y in $\widehat{M}$ such that a is optimally accessible from every point of $U \cap M$.*

Proof. Given $y \in \partial M$ and $\epsilon > 0$, take $x \in M$ such that $d(x, y) < \frac{1}{2}\epsilon$.

By Proposition 2.8, there is a point $a \in \partial M$, optimally accessible from x, such that $d(x,a) = d(x,\partial M) < \frac{1}{2}\epsilon$. Thus $a \in \partial_1 M$ and $d(a,y) < \epsilon$. If y is the only point of ∂M in $\{z \in \widehat{M} : d(z,y) < \epsilon\}$, take $U := \{z \in \widehat{M} : d(z,y) < \frac{1}{2}\epsilon\}$, and the argument then shows (since $y = a$ necessarily) that y is optimally accessible from any point of $U \cap M$. □

Corollary 2.10 *If M is finite-dimensional and $\partial M = \{a\}$, then a is optimally accessible from every point of M.* (*From* Proposition 2.8(b)).

Remarks 2.11 It is not always true, even in finite dimensions, that $\partial_1 M = \partial M$ or $\partial_0 M = \partial M$. Let the region M in $\mathbb{R}^2$ be defined in polar coordinates by $\{(r,\theta) : e^{\alpha\theta} < r < ke^{\alpha\theta}\}$, where k, $\alpha > 0$, and $e^{2\pi\alpha} > k$ to prevent overlaps. Then ∂M is identified with the union of the bounding spirals $r = e^{\alpha\theta}$ and $r = ke^{\alpha\theta}$ and the origin; but the origin is not in $\partial_0 M$. More complicated examples of the same type, in which the sets of inaccessible points of the boundary may be dense, are furnished by "Euclidean pseudo-regions with fractal boundary" (see [7] for more details). Such manifolds also show that the completion of a metrically bounded finite-dimensional Riemannian manifold need not be compact. As for Corollary 2.10, the result may fail if the boundary has as few as two points (consider $\mathbb{R}^2 \setminus \{(0,-1),(0,1)\}$).

The significance here of the facts Proposition 2.8–Corollary 2.10 is that they suggest more persuasive versions of the main conjecture. Maybe a Riemannian manifold M which is geodesically complete but metrically incomplete cannot have a "small" ideal boundary (finite or a singleton), or cannot be geodesically convex. One might even conjecture from Ekeland's theorem [10] and from Corollary 2.10 that, if $\partial M = \{a\}$ and M is infinite-dimensional, the points of M from which a is not accessible should still form a set of first category. My example disproves all these conjectures too.

3. The example: metric structure

(3.1) Henceforth, "operator" denotes a bounded linear operator in the real Hilbert space H, and $[\![\,]\!]$ is occasionally used to denote the operator-norm. An operator A is "non-negative", $A \geq 0$, if it is *self-adjoint* and

$$(\forall x \in H) \quad \langle Ax, x\rangle \geq 0; \tag{4}$$

one writes $A \geq 0$. There is a corresponding partial order on the set of self-adjoint operators. Let A, C, and Z be non-zero non-negative operators in H, and let $\alpha := \sqrt{[\![A]\!]}$, $\gamma := \sqrt{[\![C]\!]}$, $\zeta := \sqrt{[\![Z]\!]}$. Suppose further that there is a scalar $c_0 > 0$ such that $c_0 Z \geq C$. (These operators will be further restricted later). Define

$$\left.\begin{aligned} f(x) &:= \frac{\langle Ax, x\rangle + \exp(-\langle Cx, x\rangle)}{\Phi(\langle Zx, x\rangle)}, \qquad \text{where} \\ \Phi(t) &:= (e^e + t)(\log(e^e + t))^2 \quad \text{for any } t \geq 0. \end{aligned}\right\} \tag{5}$$

So f is C^ω on H, and strictly positive because of the exponential term. If $\xi, \eta \in T_x H$, treat them as vectors in H and set

$$g(x) \cdot (\xi, \eta) := f(x)\langle \xi, \eta\rangle. \tag{6}$$

It will be convenient below to have the additional notations

$$\begin{aligned} f_1(x) &:= \frac{\langle Ax, x\rangle + \exp(-\langle Cx, x\rangle)}{e^e + \langle Zx, x\rangle}, \\ f_0(x) &:= \frac{\langle Ax, x\rangle + \exp(-\langle Cx, x\rangle)}{(e^e + \langle Zx, x\rangle)^{1/2}\log(e^e + \langle Zx, x\rangle)}. \end{aligned} \tag{7}$$

Clearly $f(x) \leq e^{-2} f_1(x) \leq 2e^{-3} f_0(x)$. (6)

The Riemannian manifold M is to be H, with its natural C^ω structure and the Riemannian structure g. (Subsequently M_1 will be similarly defined from H_1, A_1, C_1, Z_1). Except in §5, I shall write $|\xi|$ for the norm of ξ as an element of H; when $\xi \in T_x M$, $\|\xi\|$ denotes its Riemannian norm, $\sqrt{g(x) \cdot (\xi, \xi)}$. I shall sometimes need to distinguish the Riemannian length $\ell(p)$ of a path p from its "norm-length" with respect to the norm in H.

The triangle inequalities for $\sqrt{\langle Ax, x\rangle}$, $\sqrt{\langle Cx, x\rangle}$, and $\sqrt{\langle Zx, x\rangle}$ give for $x, y \in H$

$$\left.\begin{aligned} |\sqrt{\langle Ax, x\rangle} - \sqrt{\langle Ay, y\rangle}| &\leq \sqrt{\langle A(x-y, x-y)\rangle} \leq \alpha|x-y|, \\ |\sqrt{\langle Cx, x\rangle} - \sqrt{\langle Cy, y\rangle}| &\leq \sqrt{\langle C(x-y, x-y)\rangle} \leq \gamma|x-y|, \\ |\sqrt{\langle Zx, x\rangle} - \sqrt{\langle Zy, y\rangle}| &\leq \sqrt{\langle Z(x-y, x-y)\rangle} \leq \zeta|x-y|. \end{aligned}\right\} \tag{9}$$

Lemma 3.2 *Suppose that $\partial M \neq \emptyset$. Given $\epsilon > 0$, there exists $\delta > 0$ such that $f_0(x) < \epsilon$ whenever $x \in M$ and $d(x, \partial M) < \delta$.*

Proof. Take $\lambda := \min(\zeta^{-1}e, \gamma^{-1}\alpha)$. Then, if $|x-y| < \lambda$,

$$\left.\begin{aligned} (e^e + \langle Zy,y\rangle)^{1/2} &< (e^e + \langle Zx,x\rangle)^{1/2} + \zeta\lambda \\ &\le 2(e^e + \langle Zx,x\rangle)^{1/2}, \quad \text{by (9), and} \\ \log(e^e + \langle Zy,y\rangle) &< \log 4 + \log(e^e + \langle Zx,x\rangle) \\ &< 2\log(e^e + \langle Zx,x\rangle), \quad \text{and also} \\ \langle Cy,y\rangle &< \langle Cx,x\rangle + \gamma^2\lambda^2 + 2\gamma\lambda\sqrt{\langle Cx,x\rangle} \\ &\le 2(\alpha^2 + \langle Cx,x\rangle) \quad \text{(for instance).} \end{aligned}\right\} \tag{10}$$

Putting these facts (and $\Phi(\langle Zx,x\rangle) \ge 4e^e > 16$) together,

$$f(y) \ge \frac{\exp(-\langle Cy,y\rangle)}{\Phi(\langle Zy,y\rangle)} > K_x := \exp(-2\alpha^2)\left(\frac{\exp(-\langle Cx,x\rangle)}{\Phi(\langle Zx,x\rangle)}\right)^2. \tag{11}$$

If $\Delta_1 := \{y \in H : |x-y| < \lambda\}$, and if the path of finite length $p : [0,\beta) \to M$ joins x to a point of ∂M, there are two possibilities. The whole path may lie in Δ_1; then, by (11), $|p(s)-p(t)| \le K_x^{-1/2}\ell(p|[s,t])$ for $s \le t$ in $[0,\beta)$, so that $p(s)$ must converge in H as $t \uparrow \beta$. This contradicts $p(\beta-) \notin M$. Hence there is in fact a first parameter value t_0 for which $p(t_0) \notin \Delta_1$, and then (11) implies that $\ell(p) > \ell(p|[0,t_0]) > \lambda K_x^{1/2}$.

Since the path p was, however, arbitrary, it follows that $d(x,\partial M) \ge \lambda K_x^{1/2}$, or

$$\frac{\exp(-\langle Cx,x\rangle)}{\Phi(\langle Zx,x\rangle)} \le \lambda^{-1}(\exp\alpha^2)d(x,\partial M). \tag{12}$$

However, since $c_0 Z \ge C$ by (3.1),

$$\begin{aligned} \frac{\exp(-\langle Cx,x\rangle)}{\Phi(\langle Zx,x\rangle)} &\ge \frac{\exp(-c_0\langle Zx,x\rangle)}{e^e(1+e^{-e}\langle Zx,x\rangle)(e+\log(1+e^{-e}\langle Zx,x\rangle))^2} \\ &\ge \frac{\exp(-e-(c_0+e^{-e})\langle Zx,x\rangle)}{(e+e^{-e}\langle Zx,x\rangle)^2} \\ &\ge \exp(-e-2-(c_0+e^{-e}+2e^{-e-1})\langle Zx,x\rangle). \end{aligned} \tag{13}$$

So, by (12),

$$\begin{aligned} -(c_0+e^{-e}+2e^{-e-1})\langle Zx,x\rangle &\le e+2+\log(\lambda^{-1}(\exp\alpha^2)d(x,\partial M)) \\ \text{and} \qquad \langle Zx,x\rangle &\ge -m-n\log d(x,\partial M), \end{aligned} \tag{14}$$

where $n > 0$ and m are constants depending on α, γ, ζ, and c_0. If

$d(x,\partial M) < e^{-m/n}$,

$$\frac{\exp(-\langle Cx,x\rangle)}{(e^e+\langle Zx,x\rangle)^{1/2}} \le \frac{1}{(e^e-m-n\log d(x,\partial M))^{1/2}}. \tag{15}$$

Suppose, if it is possible, that, for given $\tau \in (0,1)$,

$$\langle Ax,x\rangle \ge \tau^2(e^e+\langle Zx,x\rangle)^{1/2}\log(e^e+\langle Zx,x\rangle). \tag{16}$$

If also $|x-y| \le L := \tau\dfrac{(e^e+\langle Zx,x\rangle)^{1/4}\log^{1/2}(e^e+\langle Zx,x\rangle)}{3(\alpha+\zeta)}$, then, by (16),

$$\langle Ay,y\rangle \ge \frac{4}{9}\tau^2(e^e+\langle Zx,x\rangle)^{1/2}\log(e^e+\langle Zx,x\rangle),$$

whilst, similarly,

$$\begin{aligned} e^e+\langle Zy,y\rangle &\le 2(e^e+\langle Zx,x\rangle), \\ \log(e^e+\langle Zy,y\rangle) &< 2\log(e^e+\langle Zx,x\rangle). \end{aligned}$$

Hence

$$\frac{\langle Ay,y\rangle}{\Phi(\langle Zy,y\rangle)} \ge \frac{\tau^2}{18(e^e+\langle Zx,x\rangle)^{1/2}\log(e^e+\langle Zx,x\rangle)}. \tag{17}$$

Next, suppose that $p : [0,\beta) \to M$ is a path of finite length joining $p(0) = x$ to the point $p(\beta-)$ of ∂M. It follows from (14) that $|p(t)| \ge \zeta^{-1}\sqrt{\langle Zp(t),p(t)\rangle} \to \infty$ as $t \uparrow \beta$, since $d(p(t),\partial M) \to 0$, so that there is a first parameter value t_1 for which $|p(t_1)-p(0)| = L$. Then, by (17),

$$\ell(p|[0,t_1]) \ge k(\tau) := \frac{\tau^2}{3(\alpha+\zeta)\sqrt{18}}. \tag{18}$$

Since p was any path from x to ∂M, it follows that $d(x,\partial M) \ge k(\tau)$. Consequently, if $d(x,\partial M) < k(\tau)$, necessarily (16) must be false, and therefore

$$\frac{\langle Ax,x\rangle}{(e^e+\langle Zx,x\rangle)^{1/2}\log(e^e+\langle Zx,x\rangle)} < \tau^2. \tag{19}$$

Given $0 < \epsilon < 2$, take $\tau := \sqrt{\frac{1}{2}\epsilon}$. By (15), there exists $\delta_0 > 0$ such that

$$d(x,\partial M) < \delta_0 \Rightarrow (e^e+\langle Zx,x\rangle)^{-1/2}\exp(-\langle Cx,x\rangle) < \frac{1}{2}\epsilon. \tag{20}$$

Hence, if $\delta :=\min(\delta_0, k(\tau))$ and $d(x,\partial M) < \delta$, necessarily $f_0(x) < \epsilon$. □

Corollary 3.3 (i) *If* $p : [0, \beta) \to M$ *and* $d(p(t), \partial M) \to 0$ *as* $t \uparrow \beta$, *then* $\langle Zp(t), p(t)\rangle \to \infty$ *and* $|p(t)| \to \infty$ *as* $t \uparrow \beta$. *If* p *is of finite length, then* $p(\beta-) \notin M$ *if and only if either* $\langle Zp(t), p(t)\rangle \to \infty$ *or* $|p(t)| \to \infty$ *as* $t \uparrow \beta$.

(ii) *Given* $R > 0$, *there exists* $\epsilon > 0$ *such that* $|y| > R$ *if* $d(y, \partial M) <. \epsilon$.

Proof. Both (ii) and the first two assertions of (i) follow from (14). For the third, notice that, if $|p(t)| \to \infty$, then $p(t)$ cannot converge in M. □

Lemma 3.4 $0 < f(x)\Phi'(\langle Zx, x\rangle) < 3f_1(x)$ *for all* $x \in H$.

Proof. By direct computation:

$$\begin{aligned} f(x)\Phi'(\langle Zx, x\rangle) &= f(x)\{(\log(e^e + \langle Zx, x\rangle))^2 + 2\log(e^e + \langle Zx, x\rangle)\} \\ &< 3f_1(x). \end{aligned}$$

□

(3.5) Let J be an interval in $\mathbb{R}$ which is bounded below and closed on the left, with lower end-point a. Let $\phi : J \to \mathbb{R}$ be a continuous function such that

$$d := \sup \phi(J) \geq b := \phi(a) \tag{21}$$

(d may be ∞). Let $J_1 := [b, d]$ or $J_1 := [b, d)$ according as d is an attained supremum or not; that is, in either case $J_1 := \phi(J) \cap [b, \infty)$. Define

$$\psi : J_1 \to J : x \mapsto \inf \phi^{-1}\{x\}. \tag{22}$$

By the intermediate value theorem, $\phi^{-1}\{x\} \neq \emptyset$ when $b \leq x < d$; if $d \in J_1$, $\phi^{-1}\{d\} \neq \emptyset$. Clearly $\psi(b) = a$. If $c \in \phi^{-1}\{x\}$, $\inf \phi^{-1}\{x\} = \inf(\phi^{-1}\{x\} \cap [a, c])$; since $\phi^{-1}\{x\} \cap [a, c]$ is compact, the infimum is attained, and $\phi(\psi(x)) = x$. If $b \leq x < y \in J_1$ but $u = \psi(x) \geq \psi(y) = v$, then $\phi(v) = y > \phi(u) = x \geq \phi(a) = b$, and, again by the intermediate value theorem, there is a point $w \in [a, v)$ such that $\phi(w) = x$, contradicting (22). Thus ψ is strictly increasing, and consequently Borel-measurable. If $b \leq x_n \uparrow x \in J_1$, then $\psi(x_n) \uparrow \tau \leq \psi(x)$, and $x = \lim \phi(\psi(x_n)) = \phi(\tau)$, so that, by (22), $\tau \geq \psi(x)$. So $\tau = \psi(x)$. This shows that ψ is continuous on the left. In particular, if $d \in J_1$, it is a point of continuity of ψ.

Being strictly increasing, ψ has only countably many points of discontinuity, $(c_i)_{i=1}^{\infty}$. Let $\alpha_i := \psi(c_i)$, $\beta_i := \lim_{x \downarrow c_i} \psi(x)$ (which makes sense as $c_i < d$ necessarily), so that, in view of left-continuity, the "values omitted at c_i" form $(\alpha_i, \beta_i]$. Note that here β_i cannot be the right-hand end-point of J, even if J is closed on the right, for that would force c_i to be a point of continuity of ψ.

Suppose now that $u \in J$ but $u \notin \psi(J_1)$ and $u < \sup \psi(J_1)$. Define $\lambda := \inf L(u)$, where $L(u) := \{x \in J_1 : \psi(x) > u\} \neq \emptyset$. Thus $u \leq \inf \psi(L(u))$. If $b \leq \xi < \lambda$, then $\xi \notin L(u)$ and so $\psi(\xi) \leq u$. However, as u is not a value of ψ, $\psi(\xi) \neq u \neq \psi(\lambda)$. Therefore, if $\psi(\lambda) > u$, $\psi(\xi) < u < \psi(\lambda)$, which is absurd, since $\psi(\xi) \uparrow \psi(\lambda)$ as $\xi \uparrow \lambda$ by left-continuity. Consequently $\psi(\lambda) < u$, and λ is a discontinuity of ψ, $\lambda = c_i$ for some i, $\psi(\lambda) = \alpha_i$, $\inf \psi(L(u)) = \beta_i$, and $u \in (\alpha_i, \beta_i]$. The upshot of this is that

$$\psi(J_1) = J_0 \setminus \left(\bigcup_{i=1}^{\infty} (\alpha_i, \beta_i] \right), \tag{23}$$

where J_0 may be $J \cap (-\infty, \sup \psi(J_1)]$ or $J \cap (-\infty, \sup \psi(J_1))$, depending on whether the supremum of ψ is attained or not.

The right-hand side is a Borel set E in J that I shall call the "core of ϕ". As $\phi \circ \psi$ is the identity, $\phi(E) = J_1$, and ϕ, ψ are mutually inverse Borel bijections between E and J_1. In effect, J_1 is obtained by removing multiple points of ϕ.

If J is enlarged to J', which is still closed at the same left end-point a, and ϕ is extended to $\phi' : J' \to \mathbb{R}$, the foregoing arguments lead to a function $\psi' : J_1' \to J'$ which extends ψ, and $\psi'(J_1') \cap J$ differs from $\psi(J_1)$ by at most one point.

(3.6) Define a C^ω function

$$\begin{aligned} \sigma : M &\to (0, \infty) : \\ x &\mapsto f(x)(e^e + |x|^2)\{\log(e^e + |x|^2) \log\log(e^e + |x|^2)\}^2. \end{aligned} \tag{24}$$

Proposition *For $i = 1, 2, \ldots, k$, let $p_i : [a_i, \beta_i) \to M$ be a path with $|p_i(a_i)| = R_0$ and $\sup\{|p_i(t)| : a_i \leq t < \beta_i\} \geq R_1 > R_0$, and let $\lambda > 0$. Then either there exist $R \in (R_0, R_1)$ and parameter values $u_i \in (a_i, \beta_i)$ such that $|p_i(u_i)| = R$ and $\sigma(p_i(u_i)) < \lambda^2$ for each i; or*

$$\sum_{i=1}^{k} \ell(p_i) \geq \frac{1}{2}\lambda(\log\log\log(e^e + R_1^2) - \log\log\log(e^e + R_0^2)). \tag{25}$$

Proof. Suppose the first alternative is false. Write $r_i(u) := |p_i(u)|$, which is continuous on $[a_i, \beta_i)$, and piecewise C^1 except where it takes the value 0.

The assumption means that, if $v_i \in (a_i, \beta_i)$ for $1 \leq i \leq k$, and $R \in (R_0, R_1)$,

$$\text{if } r_i(v_i) = R \text{ for each } i, \text{ there is an } i \text{ such that } \sigma(p_i(v_i)) \geq \lambda^2. \tag{26}$$

Apply (3.5) to $r_i : [a_i, \beta_i) \to \mathbb{R}$. Let $E(i)$ be the core of r_i, with inverse mapping $q_i : J_1(i) \to E(i)$, where $J_1(i) := r_i([a_i, \beta_i)) \cap [R_0, \infty) \supseteq [R_0, R_1)$. Let

$$F(i) := \{u : a_i \leq u < \beta_i,\ p_i(u) \neq 0,\ \sigma(p_i(u)) \geq \lambda^2\}. \tag{27}$$

Now, $\ell(p_i) = \int_{a_i}^{\beta_i} \sqrt{f(p_i(t))}|\dot{p}_i(t)|dt \geq \int_{E(i)\cap F(i)} \sqrt{f(p_i(t))}|\dot{p}_i(t)|dt$. However, $|p_i(t+h) - p_i(t)| \geq |r_i(t+h) - r_i(t)|$ by the triangle inequality, so that $|\dot{p}_i(t)| \geq \dot{r}_i(t)$ at every point of one-sided differentiability $t \in E(i) \cap F(i)$; by (27), $\dot{r}_i(t)$ is piecewise defined and continuous on $F(i)$. Consequently

$$\ell(p_i) \geq \int_{E(i)\cap F(i)} \sqrt{f(p_i(t))}\dot{r}_i(t)dt = \int_{E(i)\cap F(i)} \sqrt{f(p_i(t))}dr_i(t), \tag{28}$$

where dr_i is of course the Lebesgue-Stieltjes measure on $[a_i, \beta_i)$ induced by r_i. (Note that $\dot{r}_i$ is non-negative on $E(i)$, from (3.5)). By change of variables,

$$\begin{aligned}\int_{E(i)\cap F(i)} &\sqrt{f(p_i(t))}dr_i(t) \\ &= \int_{r_i(E(i)\cap F(i))} \sqrt{f(p_i(q_i(\tau)))}(r_i)_* dr_i)(\tau),\end{aligned} \tag{29}$$

where "$(r_i)_* dr_i$" is the push-forward of the Lebesgue-Stieltjes measure dr_i to a Borel measure on $r_i(E(i) \cap F(i))$. Since r_i is one-one, and piecewise C^1 with non-vanishing derivative, on $E(i) \cap F(i)$, $(r_i)_* dr_i$ is just the restriction of Lebesgue measure. Ergo,

$$\ell(p_i) \geq \int_{r_i(E(i)\cap F(i))} \sqrt{f(p_i(q_i(\tau)))}d\tau. \tag{30}$$

However, if $\tau \in r_i(E(i) \cap F(i)) \cap (R_0, R_1)$, so that $q_i(\tau) \in E(i) \cap F(i)$, then, by (27),

$$\begin{aligned}&\sqrt{f(p_i(q_i(\tau)))} \\ &\geq \frac{\lambda}{(e^e + |p_i(q_i(\tau))|^2)^{1/2} \log(e^e + |p_i(q_i(\tau))|^2) \log\log(e^e + |p_i(q_i(\tau))|^2)}\end{aligned}$$

$$= \frac{\lambda}{(e^e + \tau^2)^{1/2} \log(e^e + \tau^2) \log\log(e^e + \tau^2)}, \tag{31}$$

(since $|p_i(q_i(\tau))| = r_i(q_i(\tau)) = \tau$), and consequently

$$\sum_{i=1}^{k} \ell(p_i) \tag{32}$$
$$\geq \sum_{i=1}^{k} \int_{r_i(E(i)\cap F(i))\cap(R_0,R_1)} \frac{\lambda\, d\tau}{(e^e + \tau^2)^{1/2} \log(e^e + \tau^2) \log\log(e^e + \tau^2)}$$
$$\geq \int_{(R_0,R_1)\cap\{\bigcup r_i(E(i)\cap F(i))\}} \frac{\lambda\, d\tau}{(e^e + \tau^2)^{1/2} \log(e^e + \tau^2) \log\log(e^e + \tau^2)}.$$

Now $\bigcup_{i=1}^{k} r_i(E(i) \cap F(i)) \supseteq (R_0, R_1)$. Indeed, if $\tau \in (R_0, R_1) \subseteq r_i([a_i, \beta_i))$, then $|p_i(q_i(\tau))| = \tau > 0$ for each i, so that, by (26) and (27), there is an i for which $q_i(\tau) \in F(i)$, and $\tau = r_i(q_i(\tau))$ is in $\bigcup_{i=1}^{k} r_i(E(i) \cap F(i))$. Hence

$$\begin{aligned}\sum_{i=1}^{k} \ell(p_i) &\geq \int_{R_0}^{R_1} \frac{\lambda\, d\tau}{(e^e + \tau^2)^{1/2} \log(e^e + \tau^2) \log\log(e^e + \tau^2)} \\ &\geq \frac{1}{2} \int_{R_0}^{R_1} \frac{2\lambda\tau\, d\tau}{(e^e + \tau^2) \log(e^e + \tau^2) \log\log(e^e + \tau^2)} \\ &= \frac{1}{2}\lambda(\log\log\log(e^e + R_1^2) - \log\log\log(e^e + R_0^2)).\end{aligned}$$

□

Note. This argument does not depend on the form of the function of conformality f. Analytically, it is the trick of integrating against a distribution, as on p. 4 of [19]; geometrically, it isolates the radial component of the motion.

Lemma 3.7 *Given $K > 0$, there exists $R_3(K) > 0$ such that, if $x \in M$ and $|x| \geq R_3(K)$ and $\sigma(x) < 1$, necessarily $\langle Zx, x\rangle \geq K$.*

Proof. If $\sigma(x) < 1$, then

$$\begin{aligned}&\exp(-c_0\langle Zx, x\rangle) \\ &\quad\leq \exp(-\langle Cx, x\rangle) \\ &\quad\leq \frac{(e^e + \zeta^2\langle x, x\rangle) \log^2(e^e + \zeta^2\langle x, x\rangle)}{(e^e + \langle x, x\rangle)(\log(e^e + \langle x, x\rangle) \log\log(e^e + \langle x, x\rangle))^2},\end{aligned} \tag{33}$$

which will be less than $\exp(-c_0 K)$ if $\langle x, x\rangle$ is sufficiently large. $\square$

Lemma 3.8 *Suppose that, for $i = 1, 2, 3$, $p_i : [a_i, \beta_i) \to M$ is a path of finite length with $\sup\{|p_i(u)| : a_i \le u < \beta_i\} = \infty$. Then, given $R' \ge 0$ and $\lambda > 0$, there exist $R \ge R'$ and parameter values $u_i \in [a_i, \beta_i)$ such that, for each i, $|p_i(u_i)| = R$ and $\sigma(p_i(u_i)) < \lambda^2$.*

Proof. Take $R_0 := \max\{R', |p_1(a_1)|, |p_2(a_2)|, |p_3(a_3)|\}$; then one may choose $a_i' \in [a_i, \beta_i)$ so that $|p_i(a_i')| = R_0$ for each i. (3.6) applies for any $R_1 > R_0$, and the second alternative is impossible if

$$\log\log\log(e^e + R_1^2) > \log\log\log(e^e + R_0^2) + 2\lambda^{-1}\sum_{i=1}^{3}\ell(p_i). \tag{34}$$

(Thus one even has an estimate on the size of R). $\square$

Lemma 3.9 *Let OAB be a triangle in H in which $\angle AOB = \phi$. Suppose that C divides AB in the ratio $1-t : t$, where $0 \le t \le 1$. Then*

$$(1 - \max(-\cos\phi, 0))(t^2 OA^2 + (1-t)^2 OB^2) \le OC^2. \tag{35}$$

Proof. Let $x := \overrightarrow{OA}$, $y := \overrightarrow{OB}$, $z := \overrightarrow{OC}$. Then

$$\begin{aligned} |z|^2 = |tx + (1-t)y|^2 &= t^2|x|^2 + (1-t)^2|y|^2 + 2t(1-t)|x||y|\cos\phi, \\ 2t(1-t)|x||y|\cos\phi &\ge \min(\cos\phi, 0)(t^2|x|^2 + (1-t)^2|y|^2). \end{aligned} \tag{36}$$

The result follows. $\square$

Lemma 3.10 *Suppose $x, y \in M$, $|x| = |y| = R \ge \exp(\exp(1 + \zeta^{-2} + \zeta^2))$, and $\sigma(x) < 1$, $\sigma(y) < 1$. Then $C^{1/2}x \ne 0 \ne C^{1/2}y$. If $|Z^{1/2}x| \ge e^{e/2} \le |Z^{1/2}y|$, and the angles between $Z^{1/2}x$ and $Z^{1/2}y$, and between $C^{1/2}x$ and $C^{1/2}y$, do not exceed $\psi \in [0, \pi)$, then there are positive numbers $\kappa(\psi)$, $L(\psi)$, not depending on R, such that the path p consisting of the straight-line segments from x to $\kappa(\psi)x$, from $\kappa(\psi)x$ to $\kappa(\psi)y$, and from $\kappa(\psi)y$ to y, satisfies*

$$\begin{aligned} (\forall t \in [0,1])\ |p(t)| &\le \kappa(\psi)R \quad \textit{and} \\ d(x,y) &\le \ell(p) \le L(\psi)(\log\log(e^e + R^2))^{-1}. \end{aligned} \tag{37}$$

Proof. If $\langle Cx, x\rangle = 0$, $(\Phi(\zeta^2\langle x, x\rangle))^{-1} \le f(x)$ (by definition); if also

$\sigma(x) < 1$,

$$(e^e + \zeta^2 R^2)(\log(e^e + \zeta^2 R^2))^2 \\ > (e^e + R^2)\{\log(e^e + R^2)\log\log(e^e + R^2)\}^2, \tag{38}$$

which is impossible if $R \geq \exp(\exp(1+\zeta^2))$. Thus $C^{1/2}x \neq 0$, and similarly for y.

Now assume that $|Z^{1/2}x| \geq e^{e/2} \leq |Z^{1/2}y|$. Then $e^e + \langle Zx, x\rangle \leq 2\langle Zx, x\rangle$, $\log(e^e + \langle Zx, x\rangle) \leq \log 2 + \log\langle Zx, x\rangle < 2\log\langle Zx, x\rangle$, and

$$8f(x) \geq \frac{\langle Ax, x\rangle + \exp(-\langle Cx, x\rangle)}{\langle Zx, x\rangle(\log\langle Zx, x\rangle)^2}, \tag{39}$$

and similarly for y. Let $\tau := 1 - \max(-\cos\psi, 0) > 0$. (40)

Take $\kappa > 1$. If w is any point of the straight-line segment $[x, \kappa x]$, clearly $f(w) \leq \kappa^2 f(x)$; so the $\|\ \|$-length of the segment, and $d(x, \kappa x)$, are not greater than

$$\kappa(\kappa - 1)|x|\sqrt{f(x)} < \frac{\kappa(\kappa-1)R}{R\log(e^e + R^2)\log\log(e^e + R^2)}. \tag{41}$$

Similarly for $d(y, \kappa y)$.

Now, as $R \geq \exp(\exp(1+\zeta^{-2}+\zeta^2)) > \max(\zeta^4, \zeta^{-4})$, one has $4|\log\zeta| < \log R$ and $8(\log R)^2 > (\log(\zeta^2 R^2))^2 \geq (\log\langle Zx, x\rangle)^2$. Hence, from (39), since $\sigma(x) < 1$,

$$\frac{\langle Ax, x\rangle + \exp(-\langle Cx, x\rangle)}{\langle Zx, x\rangle} \leq 8f(x)(\log\langle Zx, x\rangle)^2 \\ < \frac{64}{(R\log\log(e^e + R^2))^2}. \tag{42}$$

The same applies to y.

Let $z := (1-t)\kappa x + t\kappa y$ be a point of $[\kappa x, \kappa y]$. As $\sqrt{\langle A\xi, \xi\rangle}$ is a seminorm on H,

$$\sqrt{\langle Az, z\rangle} \leq \kappa(1-t)\sqrt{\langle Ax, x\rangle} + \kappa t\sqrt{\langle Ay, y\rangle}, \tag{43}$$

whilst, applying Lemma 3.9 to the triangle 0, $\kappa Z^{1/2}x$, $\kappa Z^{1/2}y$, and recalling (40),

$$\sqrt{\langle Zz, z\rangle} \geq \kappa\tau^{1/2}\sqrt{(1-t)^2\langle Zx, x\rangle + t^2\langle Zy, y\rangle}.$$

Similarly

$$\sqrt{\langle Cz,z\rangle} \geq \kappa\tau^{1/2}\sqrt{(1-t)^2\langle Cx,x\rangle + t^2\langle Cy,y\rangle}.$$

Hence

$$\begin{aligned}\frac{\sqrt{\langle Az,z\rangle}}{\sqrt{e^e+\langle Zz,z\rangle}} &\leq \frac{(1-t)\sqrt{\langle Ax,x\rangle}+t\sqrt{\langle Ay,y\rangle}}{\tau^{1/2}\sqrt{(1-t)^2\langle Zx,x\rangle+t^2\langle Zy,y\rangle}} \\ &\leq \tau^{-1/2}\left(\sqrt{\frac{\langle Ax,x\rangle}{\langle Zx,x\rangle}}+\sqrt{\frac{\langle Ay,y\rangle}{\langle Zy,y\rangle}}\right) \\ &\leq 8\tau^{-1/2}(R\log\log(e^e+R^2))^{-1}, \qquad (44)\end{aligned}$$

from (42). Also

$$\sqrt{\frac{\exp(-\langle Cz,z\rangle)}{e^e+\langle Zz,z\rangle}} \leq \frac{\exp(-\frac{1}{2}\kappa^2\tau(1-t)^2\langle Cx,x\rangle)\exp(-\frac{1}{2}\kappa^2\tau t^2\langle Cy,y\rangle)}{\kappa\tau^{1/2}\sqrt{(1-t)^2\langle Zx,x\rangle+t^2\langle Zy,y\rangle}}. \qquad (45)$$

Let $s := t^2+(1-t)^2 \geq \frac{1}{2}$, and $q := s^{-1}t^2$. The right-hand side becomes

$$\frac{\exp(-\frac{1}{2}\kappa^2\tau(1-q)s\langle Cx,x\rangle)\exp(-\frac{1}{2}\kappa^2\tau qs\langle Cy,y\rangle)}{\kappa\tau^{1/2}\sqrt{s(1-q)\langle Zx,x\rangle+sq\langle Zy,y\rangle}}$$

$$\leq \left(\frac{2}{\kappa^2\tau}\right)^{1/2}\left(\frac{\exp(-\frac{1}{2}\kappa^2\tau\langle Cx,x\rangle)}{\langle Zx,x\rangle}\right)^{(1-q)/2}\left(\frac{\exp(-\frac{1}{2}\kappa^2\tau\langle Cy,y\rangle)}{\langle Zy,y\rangle}\right)^{q/2}$$

(since $s \geq \frac{1}{2}$, and the inequality of the means applies to the denominators)

$$\leq 8(R\log\log(e^e+R^2))^{-1} \quad \text{if } \kappa := (\tfrac{1}{2}\tau)^{-1/2}, \quad \text{by (42).}$$

Then, recalling (41),

$$\sqrt{f(z)} \leq \frac{\sqrt{\langle Az,z\rangle}+\exp(-\frac{1}{2}\langle Cz,z\rangle)}{\sqrt{e^e+\langle Zz,z\rangle}} \leq \frac{8(1+\kappa\sqrt{2})}{R\log\log(e^e+R^2)} \qquad (46)$$

at every point z of the segment $[\kappa x,\kappa y]$, and the Riemannian length of the segment does not exceed $16\kappa(1+\kappa\sqrt{2})(\log\log(e^e+R^2))^{-1}$. So (using (41))

$$\begin{aligned}d(x,y) &\leq d(x,\kappa x)+d(\kappa x,\kappa y)+d(\kappa y,y) \\ &\leq (2\kappa(\kappa-1)+16\kappa(1+\kappa\sqrt{2}))(\log\log(e^e+R^2))^{-1}. \qquad (47)\end{aligned}$$

Since $\kappa = (\frac{1}{2}\tau)^{-1/2}$, where τ depends only on ψ, this proves the result; for one may take $\kappa(\psi) := \kappa$ and $L(\psi) := 2\kappa(\kappa-1)+16\kappa(1+\kappa\sqrt{2})$. □

(3.11) Suppose that, for $k = 1,2,3$, $p_k : [0, \beta_k) \to M$ is a path of finite length which does not converge in M. By Corollary 3.3(i) and an inductive application of Lemma 3.8, there exist sequences $(u_i^{(k)})_{i=1}^{\infty}$ of parameter values for $k = 1,2,3$, such that, for each k, $u_i^{(k)} \uparrow \beta_k$; for each i, $|p_k(u_i^{(k)})|$ is independent of k and exceeds $\exp(\exp(1+\zeta^2+\zeta^{-2}))$; $e^e \leq \langle Zp_k(u_i^{(k)}), p_k(u_i^{(k)})\rangle$ for all i and k; $|p_k(u_i^{(k)})| \to \infty$ as $i \to \infty$ for each k; and $\sigma(p_i(u_i^{(k)})) < 1$ for all i and k. For brevity, let me call such a triple of sequences a *triple parameter sequence* for p_1, p_2, p_3. Let $\theta_i^{(kj)}$ be the angle between $C^{1/2}p_k(u_i^{(k)})$ and $C^{1/2}p_j(u_i^{(j)})$, and $\phi_i^{(kj)}$ the angle between $Z^{1/2}p_k(u_i^{(k)})$ and $Z^{1/2}p_j(u_i^{(j)})$; the sequence of pairs $(\theta_i^{(kj)}, \phi_i^{(kj)})$ may be called the (k,j)-*angle sequence* of the triple parameter sequence.

(3.12) The involution $J : x \to -x$ is a Riemannian isometry of M, and therefore extends to an involution $J : \partial M \to \partial M$.

Proposition *∂M has at most four points and two J-conjugacy classes.*

Proof. Suppose x_1, x_2, x_3 are distinct points of ∂M, with $x_1 \neq Jx_2 \neq x_3$. Take paths of finite length p_1, p_2, p_3 with limits x_1, x_2, x_3 respectively. Construct a triple parameter sequence $(u_i^{(k)})_{i=1}^{\infty}$ for p_1, p_2, p_3, with (k,j)-angle sequences $(\theta_i^{(kj)}, \phi_i^{(kj)})$.

If $(\theta_i^{(12)}, \phi_i^{(12)})_{i=1}^{\infty}$ had a cluster point in $[0,\pi) \times [0,\pi)$, there would be a subsequence $(\theta_{i(r)}^{(12)}, \phi_{i(r)}^{(12)})_{r=1}^{\infty}$ for which both components are bounded above by some $\psi < \pi$, and Lemma 3.10 would show that $d(p_1(u_{i(r)}^{(1)}), p_2(u_{i(r)}^{(2)})) \to 0$ as $r \to \infty$. So p_1, p_2 would have the same limit (see (2.1)): $x_1 = x_2$. If $(\theta_i^{(12)}, \phi_i^{(12)})_{i=1}^{\infty}$ had a cluster point in $((0,\pi] \times \{\pi\}) \cup (\{\pi\} \times (0,\pi])$, then substituting Jp_2 for p_2 would change the angles to their supplements, and the resulting sequence would have a cluster point in $[0,\pi) \times [0,\pi)$; thus $x_1 = Jx_2$. Both these possibilities were denied at the outset, so the only admissible cluster points are the ordered pairs $(0,\pi)$ and $(\pi,0)$. If only one is indeed a cluster point, it must be the limit of $(\theta_i^{(12)}, \phi_i^{(12)})_{i=1}^{\infty}$. If both are cluster points, omit (in all three sequences) those indices i for which $\theta_i^{(12)} > \phi_i^{(12)}$, and renumber; after this, necessarily $(\theta_i^{(12)}, \phi_i^{(12)}) \to (0,\pi)$.

The same analysis of $(\theta_i^{(23)}, \phi_i^{(23)})_{i=1}^{\infty}$ now shows that either $x_3 = x_2$ or $x_3 = Jx_2$ (both denied), or else the only cluster points are the pairs $(0,\pi)$ or $(\pi,0)$. In this last case, one may again pass to a further subsequence and assume $(\theta_i^{(23)}, \phi_i^{(23)})$ has a limit, either $(0, \pi)$ or $(\pi, 0)$. But now, it is clear geometrically that $(\theta_i^{(13)}, \phi_i^{(13)})$ also converges — to $(0,0)$ if the limits of $(\theta_i^{(12)}, \phi_i^{(12)})$ and $(\theta_i^{(23)}, \phi_i^{(23)})$ are the same, and to (π, π) otherwise. From Lemma 3.10, this entails that either $x_1 = x_3$ (denied) or $x_3 = Jx_1$.

Therefore, of any three distinct ideal points, two must be J-conjugate. This evidently proves the Proposition. □

(3.13) I expect that (3.12) is the best such result available without more information. A convenient condition is that M have the *tetrapod property*: there exist paths of finite length $\wp_i : [a_i, \beta_i') \to M$, for $i = 1, 2$, and an angle $\phi \in (0, \frac{1}{4}\pi)$, such that $|\wp_i(t)| \to \infty$ as $t \uparrow \beta_i'$ for $i = 1, 2$ (that is, both paths have limits in $\partial M \neq \emptyset$) and, for each choice of $u_1 \in [a_1, \beta_1')$ and $u_2 \in [a_2, \beta_2')$ such that $|\wp_1(u_1)| = |\wp_2(u_2)| > 0$, the angles between $C^{1/2}\wp_1(u_1)$ and $C^{1/2}\wp_2(u_2)$, and between $Z^{1/2}\wp_1(u_1)$ and $Z^{1/2}\wp_2(u_2)$, if defined, both lie in $(2\phi, \pi - 2\phi)$. Here ϕ is a *tetrapod parameter.* (The paths $\wp_i, J\wp_i$ are the four *feet* of the tetrapod; it will be convenient to set $\wp_{i+2} := J\wp_i$). Since ray segments may be added at the start of the paths, and one may reparametrize, there is no loss of generality in assuming henceforth that $a_1 = a_2 = 0$, $\beta_1' = \beta_2' = \beta'$, and $\wp_1(0) = 0 = \wp_2(0)$.

(3.14) Suppose that $H := H_1 \times H_1$ (the Hilbert direct sum of a Hilbert space H_1 with itself), $A := A_1 \times A_1$, $C := C_1 \times C_1$, $Z := Z_1 \times Z_1$. Let M_1 be constructed as at (3.1) from H_1, A_1, C_1, Z_1, whilst M, as before, is defined from H, A, C, Z. In M there is a second isometric involution T, commuting with J: the coordinate transposition $T(x, y) := (y, x)$. Certainly $M_1 \times \{0\}$ is a closed Riemannian submanifold of M, Riemannian-isometric with M_1 in the obvious way. In this case

Lemma *If $\partial M_1 \neq \emptyset$, then M has the tetrapod property of* (3.13).

Proof. Let $\wp$ be a path of finite length in M_1, converging to a point of ∂M_1. It gives rise to paths $\wp_1$, $\wp_2$ in M, where $\wp_1(t) := (\wp(t), 0)$ and $\wp_2 := T \circ \wp_1$; both are of finite length, and cannot converge in M (if $\wp_1$ converges in M, its limit is in $M_1 \times \{0\}$, and is also its limit in the topology of $M_1 \times \{0\}$, which is denied. The metric of M_1 need not agree with the restriction to $M_1 \times \{0\}$ of the metric of M). But the angles between $Z^{1/2}\wp_1(u_1)$ and $Z^{1/2}\wp_2(u_2)$ and between $C^{1/2}\wp_1(u_1)$ and $C^{1/2}\wp_2(u_2)$, if defined, are both $\pi/2$, for any u_1, u_2. Take $\phi := \pi/5$, say. □

Proposition 3.15 *If M, defined as in (3.1), has the tetrapod property of (3.13), then ∂M is a singleton.*

Proof. Suppose $\wp_1$, $\wp_2$, and ϕ are as in (3.13). So $\wp_i$ defines a point x_i of ∂M, for $i = 1, 2$. Let $\wp_0$ be any other path of finite length in M that converges to a point x_0 of ∂M. Take a triple parameter sequence $(u_i^{(k)})_{i=1}^{\infty}$ for $\wp_0$, $\wp_1$, $\wp_2$, with (k, j)-angle sequences $(\theta_i^{(kj)}, \phi_i^{(kj)})$. Then, evidently, $2\phi < \theta_i^{(12)} < \pi - 2\phi$ and $2\phi < \phi_i^{(12)} < \pi - 2\phi$, so that, by Lemma 3.10, $x_1 = x_2$. If $\wp_1$ is replaced by $J \circ \wp_1$, the angles are substituted by their supplements, and Lemma 3.10 gives $Jx_1 = x_2 = x_1$.

Now, if $((\theta_i^{(01)}, \phi_i^{(01)}))_{i=1}^{\infty}$ has a cluster point in $[0, \pi) \times [0, \pi)$, Lemma 3.10 shows that $x_0 = x_1$. If its only cluster points are in $((0, \pi] \times \{\pi\}) \cup (\{\pi\} \times (0, \pi])$, then, as in (3.12), $x_0 = Jx_1$, since taking $J \circ \wp_1$ instead of $\wp_1$ supplements the angles. If its only cluster points are the pairs $(0, \pi)$ or $(\pi, 0)$, then by spherical geometry $\theta_i^{(02)}$, $\phi_i^{(02)}$ can have cluster points only between 2ϕ and $\pi - 2\phi$. Thus Lemma 3.10 shows $x_0 = x_2$. But I have already shown that $Jx_1 = x_2 = x_1$. □

(3.16) For Lemma 3.17–Lemma 3.19, again suppose M, defined as in (3.1), has the tetrapod property of (3.13), with paths $\wp_1, \wp_2$ and parameter ϕ; thus ∂M is a singleton, $\{*\}$. By Corollary 3.3, $|\wp_i(t)| \to \infty$ as $t \uparrow \beta'$, and $\ell(\wp_i|[t, \beta')) \to 0$ too. For any $\delta > 0$ there is an $R_2(\delta) > 0$ such that, for either i, $d(\wp_i(t), *) \leq \ell(\wp_i|[t, \beta')) < \delta$ whenever $|\wp_i(t)| \geq R_2(\delta)$. The same will be true for $\wp_{i+2} = J\wp_i$.

Lemma 3.17 *Suppose $\mu \geq 0$ and $R' \geq 0$. There are numbers $R_1(\mu, R') > R_0(\mu, R') \geq R'$ with the following property. Given $x \in M$ for which $|x| \leq R'$, and a path $p : [a, \beta) \to M$ for which $p(a) = x$, $\ell(p) \leq \mu$, and $\sup\{|p(t)| : t \in [a, \beta)\} \geq R_1$, then there are numbers $R'' \in (R_0, R_1)$, $u \in [a, \beta)$, $v_j \in [0, \beta')$ for $j = 1, 2$, such that $R'' = |p(u)| = |\wp_j(v_j)|$ and $\sigma(p(u)) < 1 > \sigma(\wp_j(v_j))$ for each j. For any such R'', u, and v_j, necessarily $\langle Zp(u), p(u)\rangle \geq e^e \leq \langle Z\wp_j(v_j), \wp_j(v_j)\rangle$ and $C^{1/2}p(u) \neq 0 \neq C^{1/2}\wp_j(v_j)$ for each j.*

Proof. Let $R_3(e^e)$ be as in Lemma 3.7. Set

$$R_0 := \max(R', R_3(e^e), \exp(\exp(1 + \zeta^{-2} + \zeta^2)) \quad \text{and}$$

$$R_1(\mu, R') \tag{48}$$
$$:= \sqrt{\exp(\exp(\exp\{\ell(\wp_1) + \ell(\wp_2) + 1 + \mu + \log\log\log(e^e + R_0^2)\})) - e^e}.$$

Therefore $R_1 > R_0$. Suppose $\sup\{|p(t)| : a \leq t < \beta\} \geq R_1$. Take parameter values $a' \in [a, \beta)$, $b'_j \in [0, \beta')$ which satisfy the equalities $|p(a')| = |\wp_j(b'_j)| = R_0$. Then

$$\begin{aligned} &\frac{1}{2}(\log\log\log(e^e + R_1^2) - \log\log\log(e^e + R_0^2)) \\ &\qquad > \ell(p) + \ell(\wp_j) + \ell(\wp_2), \quad \text{as } \mu \geq \ell(p), \end{aligned} \tag{49}$$

and, setting $k = 3$ in (3.6), there are parameter values $u \in (a', \beta)$, $v_j \in (b'_j, \beta')$ such that $R'' := |p(u)| = |\wp_j(v_j)| \in (R_0, R_1)$ and $\sigma(p(u)) < 1$, $\sigma(\wp_j(v_j)) < 1$. From this, Lemma 3.7, Lemma 3.10, and the choice of R_0 ensure that $\langle Zp(u), p(u)\rangle \geq e^e$, $C^{1/2}p(u) \neq 0$, and similarly for the other paths. □

Lemma 3.18 *Let* $|x| = |\wp_1(v_1)| = |\wp_2(v_2)|$, *where* $v_1, v_2 \in [0, \beta')$. *Set* $v_{i+2} := v_i$ *for* $i = 1, 2$. *Then there exists* $j \in \{1, 2, 3, 4\}$ *such that the angles between* $Z^{1/2}x$ *and* $Z^{1/2}\wp_j(v_j)$, *and between* $C^{1/2}x$ *and* $C^{1/2}\wp_j(v_j)$, *do not exceed* $\pi - \phi$.

Proof. Try first $j = 1$. If the angles are not both less than $\pi - \phi$, but are both greater than ϕ, change to $j = 3$; this supplements both angles. If, however, $j = 1$ makes one angle not less than $\pi - \phi$ and the other not greater than ϕ, change to $j = 2$. (3.13) applies, since $|\wp_1(v_1)| = |\wp_2(v_2)| = |x|$, and the triangle inequality for spherical triangles shows that both angles now lie in $(\phi, \pi - \phi)$). □

Lemma 3.19 *On the hypotheses of* (3.16), *suppose* $x \in M$ *and* $0 \leq \mu < d(x, *)$. *There exists* $R(x, \mu) > |x|$ *such that, if* $y \in M$ *and* $d(x, y) \leq \mu$, *then* $|y| \leq R(x, \mu)$.

Proof. Suppose $d(x, y) < \mu$, and take a path $p : [a, \beta] \to M$ for which $p(a) = x$, $p(\beta) = y$, and $\ell(p) < \mu$. Let $\delta := \frac{1}{2}(d(x, *) - \mu)$, $L(\psi)$ be as in Lemma 3.10, and $R_2(\delta)$ as in (3.16). Set in the statement of Lemma 3.17

$$R' := \max\left(|x|, R_2(\delta), \exp\left(\frac{1}{2}\exp(L(\pi - \phi)/\delta)\right)\right), \tag{50}$$

and $R(x, \mu) := R_1(\mu, R')$ in the conclusion.

Suppose, if possible, that $\sup\{|p(t)| : t \in [a, \beta)\} > R(x, \mu)$. Apply Lemma 3.17, with the same notation R''. By Lemma 3.18, there is a j such that the angles between $Z^{1/2}\wp_j(v_j)$ and $Z^{1/2}p(u)$, and between $C^{1/2}\wp_j(v_j)$ and $C^{1/2}p(u)$, do not exceed $\pi - \phi$. Thus, by the last statement of Lemma 3.17, Lemma 3.10 applies, and

$$\begin{aligned} d(p(u), \wp_j(v_j)) &\leq L(\pi - \phi)/\log\log(e^e + R''^2) \\ &\leq L(\pi - \phi)/\log\log(e^e + R'^2) \leq \delta. \end{aligned} \tag{51}$$

Also, from (3.16), as $|\wp_j(v_j)| = R'' > R' \geq R_2(\delta)$ by (50), $d(\wp_j(v_j), *) < \delta$. So

$$\begin{aligned} d(x, *) &\leq d(x, p(u)) + d(p(u), \wp_j(v_j)) + d(\wp_j(v_j), *) \\ &< \ell(p) + 2\delta < d(x, *), \quad \text{by the definition of } \delta. \end{aligned} \tag{52}$$

This is a contradiction, so that $|\wp_j(v_j)| \leq \sup\{|p(t)| : t \in [a, \beta)\} \leq R(x, \mu)$.

If $d(x, z) = \mu$, and $\epsilon > 0$, take a path p from x to z of length not exceeding $\mu + \frac{1}{2}\epsilon$, and let y be the point on the path such that its length between x and y is $\mu - \frac{1}{2}\epsilon$. Then $d(x, y) < \mu$ and $d(z, y) < \epsilon$. Hence z is in the closure of $\{y : d(x, y) < \mu\}$, and the Lemma follows by a limiting argument. □

4. The example: geodesic structure

(4.1) Let ∇ denote the gradient with respect to the flat Riemannian structure $\langle\, ,\, \rangle$ in H. The geodesic equation for a path $p(t)$ in M, most easily obtained as the Euler-Lagrange equation for the energy integral $\int f(p(t))\langle \dot{p}, \dot{p}\rangle dt$, takes the form

$$f\ddot{p} + \langle \nabla f, \dot{p}\rangle \dot{p} = \frac{1}{2}\langle \dot{p}, \dot{p}\rangle \nabla f. \tag{53}$$

Taking inner products with $\dot{p}$, one has the expected special integral

$$f(p)\langle \dot{p}, \dot{p}\rangle = \mu \quad \text{(a constant on the geodesic)}. \tag{54}$$

Substituting this back into (53), one finds

$$f(p)\ddot{p} + \langle \dot{p}, \nabla f\rangle \dot{p} = \frac{d}{dt}(f\dot{p}) = \frac{1}{2}\,\mu \nabla f/f. \tag{55}$$

Introduce a new parameter s, with respect to which differentiation will be denoted by primes, by $ds := dt/f(p(t))$. Thus (55) becomes

$$p'' = \frac{1}{2}\,\mu \nabla f, \quad \text{and (54) gives} \tag{56}$$

$$\langle p', p'\rangle = \mu f(p). \tag{57}$$

(The constant μ could be normalized to 1, but it is a useful check on the later calculations to retain it). On its own, (56) clearly implies only that

$$\langle p', p'\rangle - \mu f(p) = c', \quad \text{constant on the path.} \tag{58}$$

(56) and (57) are together equivalent to (53) and (54), in the sense that a maximal solution of (56) and (57) corresponds to a maximal solution of (53) and (54), and vice versa, by the appropriate changes of parameter.

Let $X(x) := \frac{1}{2}\nabla f(x)$. Now, taking into account the definition of f given at (3) of (3.1), the equation (56) and the normalization (57) take the forms

$$\begin{aligned} p'' = \mu X(p) = \mu \Bigg(& \frac{Ap - \exp(-\langle Cp, p\rangle)Cp}{\Phi(\langle Zp, p\rangle)} \\ & - \frac{\langle Ap, p\rangle + \exp(-\langle Cp, p\rangle)}{(\Phi(\langle Zp, p\rangle))^2}\Phi'(\langle Zp, p\rangle)Zp\Bigg), \end{aligned} \tag{59}$$

$$\langle p', p'\rangle = \mu \frac{\langle Ap, p\rangle + \exp(-\langle Cp, p\rangle)}{\Phi(\langle Zp, p\rangle)}. \tag{60}$$

Notice that $\Phi(t) \geq e^{2+e} \geq 1$ and $|\Phi(t)^{-1}\Phi'(t)| \leq 2e^{-e} \leq 1$ for all $t \geq 0$. (61)

Lemma 4.2 *Let $\mu > 0$, and suppose $p : [\eta, \beta) \to H$ is a path satisfying (58) at each point of differentiability. Then, for $s \in [\eta, \beta)$,*

$$|p(s)| \leq \max(|p(\eta)|, \mu^{-1/2}\alpha^{-1}(|c'| + \mu)^{1/2}) \exp(\alpha\sqrt{2\mu}\,(\beta - \eta)). \tag{62}$$

Proof. Write $r(s) := \langle p(s), p(s)\rangle$. Then, for each $s \in [\eta, \beta)$,

$$\begin{aligned} (r'(s))^2 &= 4\langle p'(s), p(s)\rangle^2 \leq 4(c' + \mu f(p(s)))r(s) \\ &\leq 4(|c'| + \mu + \mu\alpha^2 r(s))r(s) \end{aligned} \tag{63}$$

by the definition of f, (3.1). The derivative may be one-sided. When $\mu\alpha^2 r(s) \leq |c'| + \mu$, there is nothing to prove. If $\mu\alpha^2 r(s) > |c'| + \mu$, let ξ be the greatest parameter value, if there is one, such that $\eta \leq \xi < s$ and $\mu\alpha^2 r(\xi) = |c'| + \mu$; if no such value exists, set $\xi := \eta$. In either case, for one-sided derivatives

$$r'(s_1) \leq 2\alpha\sqrt{2\mu}\, r(s_1) \quad \text{for } \xi \leq s_1 \leq s. \tag{64}$$

Hence $r(s) \leq r(\xi)\exp(2\alpha\sqrt{2\mu}(s - \xi))$, and the result follows. □

Lemma 4.3 *The maximal solutions of $p'' = \mu X(p)$, for any non-negative constant μ, are defined for all s.*

Proof. If $\mu = 0$, this is clear. If $\mu > 0$, take a maximal solution $y(s)$ whose non-empty domain J has supremum $\beta < \infty$. Choose $\eta \in J$; (58) holds for suitable c'. By Lemma 4.2, there exists a constant $Q > 0$ such that $|y(s)| \leq Q$ for $\eta \leq s < \beta$. For these values of s, (59), (61), and (58) give the feeble estimates

$$\left.\begin{aligned} |y''(s)| &\leq \mu\{(\alpha^2+\gamma^2)Q + (\alpha^2 Q^2 + 1)\zeta^2 Q\}, \\ |y'(s)|^2 &\leq c' + \mu(1+\alpha^2 Q^2). \end{aligned}\right\} \tag{65}$$

By the mean value theorem, $(y'(s))_{s\uparrow\beta}$ and $(y(s))_{s\uparrow\beta}$ are Cauchy nets, and converge to limits v, u. A standard argument proves that the solution of the equation with initial conditions $p(\beta) := u$, $p'(\beta) := v$ extends y beyond β, which contradicts the hypothesis. To show that J is not bounded below, reverse the parameter. □

Proposition 4.4 *Let $p : [a, \beta) \to M$ be differentiable and satisfy* (57), *with $\mu > 0$, and suppose that*

$$\infty > d := \sup\{|p(s)|^2 : a \leq s < \beta\} \geq b := |p(a)|^2 > 0. \tag{66}$$

Define $Q := \{b \log^2 b\}/\{(e^e + b)\log^2(e^e + b)\}$. Then (67)

$$\int_a^\beta \frac{ds}{\Phi(\langle p(s), p(s)\rangle)} \geq \frac{1}{4}\mu^{-1/2} Q \frac{(\log\log d - \log\log b)^2}{\ell(p)}. \tag{68}$$

Proof. Set $r(s) := \langle p(s), p(s)\rangle$. Let E be the core (see (3.5)) of $r : [a, \beta) \to \mathbb{R}$, with inverse $q : J_1 \to E$, where J_1 is $[b, d]$ or $[b, d)$. For $s \in E$, $r(s) \in J_1$, and

$$\int_a^\beta \Phi(r(s))^{-1}\, ds \geq Q \int_E \{r(s) \log^2 r(s)\}^{-1}\, ds. \tag{69}$$

Now, by the Cauchy-Schwarz inequality,

$$\begin{aligned} &\left(\int_a^\beta f(p(s))ds\right)^{1/2} \left(\int_E \{r(s)\log^2 r(s)\}^{-1}\, ds\right)^{1/2} \\ &\qquad \geq \int_E \sqrt{\frac{f(p(s))}{r(s)\log^2 r(s)}}\, ds = \int_{r(E)} \sqrt{\frac{f(p(q(\tau)))}{\tau \log^2 \tau}}\, (r_* ds)(\tau). \end{aligned} \tag{70}$$

Here $r_* ds$ denotes the push-forward of ds to a Borel measure on $r(E) = J_1$.

Next, for $s \in E$ (so that $r(s) \neq 0$ and r is differentiable at s), from (57)

$$r'(s) = 2\langle p'(s), p(s)\rangle \leq 2|p'(s)||p(s)| = 2\sqrt{\mu r(s) f(p(s))}. \tag{71}$$

Hence, as measures on J_1, $r_* ds \geq \frac{d\tau}{2\sqrt{\mu\tau f(p(q(\tau)))}}$ (as in (3.6), I omit the details) and so

$$\int_{r(E)} \sqrt{\frac{f(p(q(\tau)))}{\tau \log^2 \tau}} (r_* ds)(\tau) \\ \geq \int_b^d \frac{\mu^{-1/2} d\tau}{2\tau \log \tau} = \frac{1}{2}\mu^{-1/2}(\log\log d - \log\log b). \tag{72}$$

Recall that $\ell(p) = \int_a^\beta \mu^{1/2} f(p(s)) ds$. Putting this fact together with (69), (70), and (72), one has the result. □

Note. The argument again involves the radial component only.

Corollary 4.4 *Suppose that* $p : [a, \beta) \to M$ *satisfies* (57), *that* $\ell(p) < \infty$, *and that* $\sup\{|p(s)| : a \leq s < \beta\} = \infty$. *Then the integral* $\int_a^\beta \frac{ds}{\Phi(\langle p(s), p(s)\rangle)}$ *diverges.*

Proof. Proposition 4.4 applies to $[a, c)$ for any $c \in [a, \beta)$; let $c \uparrow \beta$. □

(4.5) The conditions (4.7)(i)–(iv) given below are those which are actually used in the subsequent argument, and not necessarily the simplest or most natural hypotheses available. In particular, the requirement in (iii) that $\eta_1 < 1$ is rather asymmetrical. If one had instead

$$[C, S] \leq \eta_1 [A, S] + \nu_1 A, \quad \text{for some } \eta_1, \nu_1 > 0, \tag{73}$$

then (i)–(iv) would hold for the new system obtained by multiplying C by a sufficiently small positive scalar and changing η_1, ν_1 correspondingly.

(4.6) Suppose that, in addition to the non-negative operators A, C, and Z of (3.1), there is a skew-adjoint operator S satisfying the following further conditions:

(i) $[A, S] := AS - SA \geq 0$,
(ii) $\eta_0 C \leq [A, S]$, for some positive constant η_0,
(iii) $[C, S] \leq \eta_1 [A, S] + \nu_1 A$, for some $\eta_1 \in (0, 1)$ and $\nu_1 > 0$,
(iv) $[Z, S] \leq \eta_2 [A, S] + \nu_2 A$, for some $\eta_2, \nu_2 > 0$.

Notes. The commutators $[A,S]$, $[C,S]$, $[Z,S]$ are self-adjoint, so the order relation $\leq$ makes sense. There is no need to assume $[C,S]\geq 0$ or $[Z,S]\geq 0$.

Proposition 4.7 *Let A,C,Z,S satisfy the conditions of* (4.7). *There are constants $\delta_1,\delta_2,\delta_3>0$, depending on $\eta_0,\eta_1,\eta_2,\nu_1,\nu_2$, such that, if $p:[a,c]\to M$ is a path satisfying* (59) *and* (60) *for which $f_1(p(s))\leq\delta_1$ for $a\leq s\leq c$, then*

$$\begin{aligned}\langle p'(c),Sp(c)\rangle-\langle p'(a),Sp(a)\rangle+\mu^{1/2}\delta_2\ell(p)\\ \geq\mu\delta_3\int_a^c\frac{1}{\Phi(\langle Zp(s),p(s)\rangle)}\,ds.\end{aligned}\tag{74}$$

Proof. From (59), since H is real and $S^*=-S$,

$$\begin{aligned}\frac{d}{ds}\langle p',Sp\rangle&=\langle p'',Sp\rangle+\langle p',Sp'\rangle=\langle p'',Sp\rangle\\&=\mu\frac{\langle Ap,Sp\rangle-\exp(-\langle Cp,p\rangle)\langle Cp,Sp\rangle-f(p)\Phi'(\langle Zp,p\rangle)\langle Zp,Sp\rangle}{\Phi(\langle Zp,p\rangle)}.\end{aligned}\tag{75}$$

Also

$$\langle Ap,Sp\rangle=\langle p,ASp\rangle=\langle ASp,p\rangle=\langle -SAp,p\rangle=\frac{1}{2}\langle [A,S]p,p\rangle\tag{76}$$

(and similarly for $\langle Cp,Sp\rangle$ and $\langle Zp,Sp\rangle$). Ergo, (75) may be written

$$\begin{aligned}&\frac{d}{ds}\langle p',Sp\rangle\\&=\mu\frac{\langle [A,S]p,p\rangle-\exp(-\langle Cp,p\rangle)\langle [C,S]p,p\rangle-f(p)\Phi'(\langle Zp,p\rangle)\langle [Z,S]p,p\rangle}{2\Phi(\langle Zp,p\rangle)}\geq\\&\mu\frac{\{1-\eta_1-\eta_2 f(p)\Phi'(\langle Zp,p\rangle)\}\langle [A,S]p,p\rangle-\{\nu_1+\nu_2 f(p)\Phi'(\langle Zp,p\rangle)\}\langle Ap,p\rangle}{2\Phi(\langle Zp,p\rangle)},\end{aligned}\tag{77}$$

by (4.7)(iii), (iv). Take $\delta_1:=\frac{1}{3}\min(1,\frac{1}{2}(1-\eta_1)/\eta_2)$ and $\delta_0:=1-\eta_1-3\eta_2\delta_1$. Thus $\delta_0>0$, and, if $f_1(p(s))\leq\delta_1$ for any $s\in[a,c]$, then, by Lemma 3.4, for such s

$$\begin{aligned}1-\eta_1-\eta_2 f(p(s))\Phi'(\langle Zp(s),p(s)\rangle)&\geq\delta_0,\\ f(p(s))\Phi'(\langle Zp(s),p(s)\rangle&\leq 1.\end{aligned}\tag{78}$$

Integrate (77) between a and c, recalling that A and $[A,S]$ are non-negative:

$$\begin{aligned}\langle p'(c), Sp(c)\rangle - \langle p'(a), Sp(a)\rangle + \frac{1}{2}(\nu_1+\nu_2)\mu \int_a^c \frac{\langle Ap(s), p(s)\rangle}{\Phi(\langle Zp(s), p(s)\rangle)}\,ds \\ \geq \frac{1}{2}\delta_0\mu \int_a^c \frac{\langle [A,S]p(s), p(s)\rangle}{\Phi(\langle Zp(s), p(s)\rangle)}ds \\ \geq \frac{1}{2}\delta_0\mu\eta_0 \int_a^c \frac{\langle Cp(s), p(s)\rangle}{\Phi(\langle Zp(s), p(s)\rangle)}ds \qquad (79)\end{aligned}$$

by (4.7)(ii). Let $\delta_2 := \max(\frac{1}{2}(\nu_1+\nu_2), \frac{1}{2}\delta_0\eta_0)$, $\delta_3 := \frac{1}{2}\delta_0\eta_0$. If necessary (that is, if $\nu_1+\nu_2 < \delta_0\eta_0$) one may increase the left-hand side of (79); in any case, $\delta_2 \geq \delta_3$ and

$$\begin{aligned}&\langle p'(c), Sp(c)\rangle - \langle p'(a), Sp(a)\rangle \\ &\quad + \delta_2\mu \int_a^c \frac{\langle Ap(s), p(s)\rangle + \exp(-\langle Cp(s), p(s)\rangle}{\Phi(\langle Zp(s), p(s)\rangle)}\,ds \\ &\geq \delta_3\mu \int_a^c \frac{\langle Cp(s), p(s)\rangle + \exp(-\langle Cp(s), p(s)\rangle}{\Phi(\langle Zp(s), p(s)\rangle)}ds \qquad (80) \\ &\geq \delta_3\mu \int_a^c \frac{1}{\Phi(\langle Zp(s), p(s)\rangle)}ds \quad (\text{as } \xi + e^{-\xi} \geq 1 \text{ for all } \xi \geq 0).\end{aligned}$$

This is the result stated, as $\ell(p) = \int_a^c \mu^{1/2} f(p(s))ds$. □

Corollary 4.8 *Assume* (4.7). *Suppose that the path* $p : [a, \beta) \to M$ *(where* β *may be* ∞*) satisfies* (59) *and* (60), *that* $\ell(p) < \infty$, *that* $f_1(p(s)) \leq \delta_1$ *for* $a \leq s < \beta$, *and that* $\sup\{|p(s)| : a \leq s < \beta\} = \infty$. *Then* $\langle p'(s), Sp(s)\rangle \to \infty$ *as* $s \uparrow \beta$.

Proof. For each $x \in H$

$$\begin{aligned}e^e + \langle Zx, x\rangle &\leq (1+\zeta^2)(e^e + \langle x, x\rangle) \quad \text{and so} \\ \log(e^e + \langle Zx, x\rangle) &\leq \log(1+\zeta^2) + \log(e^e + \langle x, x\rangle) \\ &\leq (1+\zeta^2)\log(e^e + \langle x, x\rangle). \qquad (81)\end{aligned}$$

Now, for each s, $\dfrac{1}{\Phi(\langle Zp(s), p(s)\rangle)} \geq \dfrac{1}{(1+\zeta^2)^3} \dfrac{1}{\Phi(\langle p(s), p(s)\rangle)}$.

Thus, by Corollary 4.5, the integral in (74) diverges as $c \uparrow \beta$, whilst the length remains bounded. The result follows. □

Theorem 4.9 *If* M *is defined as in* (3.1) *and the conditions* (4.7) *are* *satisfied, then M is geodesically complete.*

Proof. Let $p(s)$ be a right-incomplete maximal geodesic, reparametrized to satisfy (59) and (60). It is defined for all s, by Lemma 4.3, and $\mu \neq 0$ since $p(s)$ must be non-constant; and $\ell(p) < \infty$. Next, as $s \to \infty$, $p(s)$ must tend to a point of ∂M by Lemma 2.6, so that, by Lemma 3.2, $f_1(p(s)) \to 0$ as $s \to \infty$, and by Corollary 3.3(i) $|p(s)| \to \infty$. Hence there exists a such that $f_1(p(s)) \leq \delta_1$ whenever $s \geq a$. Consequently Corollary 4.9 applies, and $\langle p'(s), Sp(s)\rangle \to \infty$ as $s \to \infty$.

Now

$$\begin{aligned} |\langle p'(s), Sp(s)\rangle| &\leq |p'(s)|[\![S]\!]|p(s)| = \mu^{1/2}\,[\![S]\!]|p(s)|\sqrt{f(p(s))} \qquad (82)\\ &< \mu^{1/2}\,[\![S]\!]\,\{\log(e^e + |p(s)|^2)\log\log(e^e + |p(s)|^2)\}^{-1}\sqrt{\sigma(p(s))}, \end{aligned}$$

whilst Lemma 3.8 shows that $\liminf_{s\to\infty} \sigma(p(s)) = 0$. Thus

$$\liminf_{s\to\infty} |\langle p'(s), Sp(s)\rangle| = 0.$$

This contradiction establishes the theorem. □

Remarks 4.10 The proof of Theorem 4.10 could be presented somewhat more directly, since it relies only on a contradiction between convergence of $\int_a^\infty f(p(s))ds$ and divergence of $\int_a^\infty \Phi(\langle p(s), p(s)\rangle)^{-1}\,ds$. The numerical estimates of Proposition 4.8, Proposition 4.4, and (3.6) are unnecessarily detailed for this limited purpose.

5. Construction of suitable operators

In this section, and only here, $\|\ \|$ will denote the norm in the Hilbert spaces H_0 and H_1 (to avoid needless confusion with the absolute value in $\mathbb{R}$).

(5.1) The condition (4.7)(i) says that $A+S$ is a hyponormal operator. In [2], where a similar condition was needed, it was sufficient to choose a strictly hyponormal operator at random, but here the further conditions (4.7)(ii) and (iii) complicate matters. There are definite obstacles to basing the construction on a weighted shift or a modification of one. However, there is another standard class of hyponormal operators, derived from "truncated Hilbert transforms", from which it is possible, with some effort, to construct operators satisfying (4.7). The general information on Hilbert transforms that I shall use is summarized below, (5.2). For hyponormal operators, see [5] — from which I took the idea — and [6]. For the Hilbert transform on the line, see [11]; for higher dimensions and generalizations, [19]; and [20] and [21] (where the transform on the line appears in vol. 2, p. 243).

(5.2) Let $L^p(\mathbb{R})$, $L^p([-1,1])$ denote the Lebesgue spaces of complex-valued functions integrable in pth power with respect to Lebesgue measure on $\mathbb{R}$ or on $[-1,1]$. The second space embeds in the first (extension of functions by zero).

Given $f \in L^p(\mathbb{R})$, where $1 < p < \infty$, one defines for each $\delta > 0$

$$(\mathcal{H}_\delta\{f\})(s) := -\frac{1}{\pi}\int_{|t|\geq\delta} \frac{f(t+s)}{t}\,dt = \frac{1}{\pi}\int_{|s-t|\geq\delta} \frac{f(t)}{s-t}\,dt, \tag{83}$$

(which is clearly in $L^\infty(\mathbb{R})$, by the Hölder inequality). Then

(i) for each $\delta > 0$, $\mathcal{H}_\delta\{f\} \in L^p(\mathbb{R})$;

(ii) as $\delta \downarrow 0$, $\mathcal{H}_\delta\{f\}$ converges in $L^p(\mathbb{R})$ to a limit $\mathcal{H}\{f\}$, the Hilbert transform of f (which may therefore be symbolically represented by the formula

$$\mathcal{H}\{f\}(s) = \frac{1}{\pi}\int \frac{f(t)}{s-t}\,dt,$$

the integral being understood as a "Cauchy principal value in the L^p sense");

(iii) when $p = 2$, $\mathcal{H}$ is a skew-adjoint isometry of $L^2(\mathbb{R})$ (more generally, $\mathcal{H}$ is a bounded linear operator in $L^p(\mathbb{R})$, satisfying $\mathcal{H}^2 = -I$);

(iv) if $\phi(s) = as + b$, where $a \neq 0$, is an affine automorphism of $\mathbb{R}$, then, for any $f \in L^2(\mathbb{R})$, $\mathcal{H}\{f \circ \phi\} = \mathcal{H}\{f\} \circ \phi$;

(v) if f is C^∞ of compact support, $\mathcal{H}\{f\}$ is a.e. equal to a C^∞ function, and $\mathcal{H}\{f'\} = (\mathcal{H}\{f\})'$ a.e. (This is all that is used below, but, with (iii), it evidently implies a more general and elegant statement for Sobolov spaces).

(5.3) The argument will mostly concern the order of magnitude of various non-negative functions, on a domain D which will vary with the context. To avoid introducing unnecessary constants, I shall write interchangeably $\phi = O(\psi)$, $\phi \prec \psi$, or $\psi \succ \phi$ when there is a positive constant Λ such that $\phi(\xi) \leq \Lambda\psi(\xi)$ for all $\xi \in D$. If both $\phi \prec \psi$ and $\psi \prec \phi$, I shall write $\phi \asymp \psi$. (Thus the Landau O is always to be understood uniformly in

D; but the other symbols, which sometimes have notational advantages, do not have their usual asymptotic meanings). The relations $\prec$ and $\asymp$, thus defined, enjoy obvious algebraic and analytical properties.

(5.4) Fix a C^∞ function $\psi : \mathbb{R} \to [0,1]$ such that $\psi(t) = 0$ for $t \leq 0$ and $\psi(t) = 1$ for $t \geq 1$. Suppose that $0 < \theta < \frac{1}{3}$, and set

$$\left.\begin{aligned} h[\theta](t) &:= \begin{cases} 0 & \text{for } t \leq \theta \text{ and for } t \geq 2, \\ \psi\left(\dfrac{t-\theta}{\theta}\right) & \text{for } \theta \leq t \leq 2\theta, \\ 1 & \text{for } 2\theta \leq t \leq 1, \\ \psi(2-t) & \text{for } 1 \leq t \leq 2, \end{cases} \\ g[\theta](t) &:= \exp(\pi i \theta^{-2} t) h[\theta](t). \end{aligned}\right\} \tag{84}$$

Thus $g[\theta]$ and $g[\theta]^2$ are C^∞ of compact support $[\theta, 2]$, and bounded in modulus by 1; their kth derivatives, for $k \geq 1$, are uniformly $O(\theta^{-2k})$.

(5.5) If q is a piecewise C^1 function supported on $[0,2]$ and $\delta \in (0,1)$, then

$$\left|\left(\int_{t \leq s-\delta} + \int_{t \geq s+\delta}\right) \frac{q(t)dt}{s-t}\right| \leq 2\int_\delta^2 \frac{Qdt}{t} = 2Q(\log 2 - \log \delta), \tag{85}$$

where Q is the L^∞-norm of q. Also the principal value

$$\begin{aligned} \left|\int_{s-\delta}^{s+\delta} \frac{q(t)dt}{s-t}\right| &= \lim_{\epsilon \downarrow 0} \left|\int_\epsilon^\delta \frac{q(s-t) - q(s+t)}{t} dt\right| \\ &= \lim_{\epsilon \downarrow 0} \left|\int_\epsilon^\delta \left(\int_{-t}^t (-q'(s+\tau))d\tau\right) \frac{dt}{t}\right| \\ &\leq \lim_{\epsilon \downarrow 0} \int_\epsilon^\delta 2Q'dt = 2Q'\delta, \end{aligned} \tag{86}$$

if $|q'(t)| \leq Q'$ where q' is defined, for $s - \delta \leq t \leq s + \delta$. So in this case, from (83),

$$\pi |\mathcal{H}\{q\}(s)| \leq 2Q(\log 2 - \log \delta) + 2Q'\delta. \tag{87}$$

When $q := g[\theta]^2$, take $Q := 1$ and $Q' = O(\theta^{-2})$, by (5.4); then (87) shows that

$$|\mathcal{H}\{g[\theta]^2\}(s)| \prec |\log \theta| \quad \text{uniformly for all } s. \tag{88}$$

(It is only necessary to take $\delta := \theta^2$). One may proceed similarly when $q := (g[\theta]^2)'$, with $Q = O(\theta^{-2})$ and $Q' = O(\theta^{-4})$, and $\delta := \theta^2$ (or θ^4). With

(5.2)(v), this gives

$$|(\mathcal{H}\{g[\theta]^2\})'(s) \prec \theta^{-2}|\log\theta| \quad \text{uniformly for all } s. \tag{89}$$

(5.6) Let $r[\theta](s) := (1-|g[\theta](s)|^2)\mathcal{H}\{g[\theta]^2\}(s)$. Then, by (88),

$$|r[\theta](s)| \prec |\log\theta| \quad \text{uniformly in } s, \tag{90}$$

whilst, from (89) and the product rule,

$$|r[\theta]'(s)| \prec \theta^{-2}|\log\theta|. \tag{91}$$

Applying (87) with $Q = O(|\log\theta|)$ and $Q' = O(\theta^{-2}|\log\theta|)$, and with $\delta := \theta^2$,

$$|\mathcal{H}\{r[\theta]\}(s)| \prec |\log\theta|^2 \quad \text{uniformly for all } s. \tag{92}$$

So far, the oscillatory factor in $g[\theta]$ has not improved the estimates I have given.

(5.7) Next, consider what happens when $s \notin (0,3)$. Then, writing $\gamma_\theta(t) := (h[\theta](t))^2$,

$$\begin{aligned}\int_\theta^2 \frac{(g[\theta](t))^2}{s-t}dt &= \int_\theta^2 \frac{\exp(2\pi i\theta^{-2}t)\gamma_\theta(t)}{s-t}dt\\ &= \left[\frac{\theta^2\exp(2\pi i\theta^{-2}t)\gamma_\theta(t)}{2\pi i(s-t)}\right]_{t=\theta}^2\\ &\quad - \frac{\theta^2}{2\pi i}\int_\theta^2 \frac{((s-t)\gamma_\theta'(t)+\gamma_\theta(t))\exp(2\pi i\theta^{-2}t)}{(s-t)^2}\,dt.\end{aligned} \tag{93}$$

Since $|s-\theta|$ and $|s-2|$ are not less than θ, and γ_θ takes values in $[0,1]$, the modulus of the first term is uniformly $O(\theta)$; moreover, γ_θ' vanishes on $[2\theta,1]$, and is uniformly $O(1)$ on $[1,2]$ and $O(\theta^{-1})$ on $[\theta,2\theta]$. Hence

$$\left.\begin{aligned}&\left|\int_\theta^2 \frac{\gamma_\theta'(t)\exp(2\pi i\theta^{-2}t)}{s-t}\,dt\right|\\ &\qquad = \frac{\theta.O(\theta^{-1})}{\theta-s} + \frac{1.O(1)}{1-s} \quad \text{for all } s\le 0, \text{ and also}\\ &\left|\int_\theta^2 \frac{\gamma_\theta(t)\exp(2\pi i\theta^{-2}t)}{(s-t)^2}\,dt\right|\\ &\qquad \le \int_\theta^2 \frac{1}{(s-t)^2}\,dt = \frac{1}{2-s} - \frac{1}{\theta-s} = \frac{O(1)}{\theta-s}.\end{aligned}\right\} \tag{94}$$

Similar estimates hold for $s \geq 3$, and my (rather weak) conclusion from (93) is that

$$\begin{aligned} |\mathcal{H}\{g[\theta]^2\}(s)| &= \frac{1}{\pi}\int_\theta^2 \frac{(g[\theta](t))^2}{s-t}\,dt \\ &\prec \frac{\theta}{\min(|\theta - s|, |s-2|)} \quad \text{for } s \notin (0,3). \end{aligned} \tag{95}$$

(5.8) Let $3\theta < s < \frac{1}{2}$. Then, with $r[\theta]$ as in (5.6),

$$\begin{aligned} \left|\int_{-\infty}^0 \frac{r[\theta](t)dt}{s-t}\right| &= \left|\int_{-\infty}^0 \frac{\mathcal{H}\{g[\theta]^2\}(t)dt}{s-t}\right| \quad \text{as } g[\theta](s) = 0 \text{ for } s \leq 0 \\ &\prec \theta \int_{-\infty}^0 \frac{dt}{(s-t)(\theta - t)} \quad \text{by (95)} \\ &\leq \theta \int_{-\infty}^0 \frac{dt}{(\theta-t)^2} = \theta.\frac{1}{\theta} = 1 \quad \text{and similarly} \\ \left|\int_3^\infty \frac{r[\theta](t)dt}{s-t}\right| &\prec \theta \int_3^\infty \frac{dt}{(t-2)^2} = \theta. \quad \text{But (88) implies that} \\ \left|\int_0^{3\theta} \frac{r[\theta](t)dt}{s-t}\right| &\leq \left|\int_0^{2\theta} \frac{|\mathcal{H}\{g[\theta]^2\}(t)|dt}{s-t}\right| \prec \frac{2\theta|\log\theta|}{s-2\theta} \prec |\log\theta|, \end{aligned} \tag{96}$$

as $|g[\theta](t)| = 1$ for $2\theta \leq t \leq 1$, and in the same way

$$\left|\int_{3\theta}^3 \frac{r[\theta](t)dt}{s-t}\right| \leq \left|\int_1^3 \frac{\mathcal{H}\{g[\theta]^2\}(t)dt}{s-t}\right| \prec \frac{2|\log\theta|}{3-s} \prec |\log\theta|. \tag{97}$$

Putting these estimates together,

$$|\mathcal{H}\{r[\theta]\}(s)| \prec |\log\theta| \quad \text{for } 3\theta < s < \frac{1}{2}. \tag{98}$$

However,

$$\begin{aligned} \mathcal{H}\{r[\theta]\} &= \mathcal{H}\{\mathcal{H}\{g[\theta]^2\}\} - \mathcal{H}\{|g[\theta]|^2\,\mathcal{H}\{g[\theta]^2\}\} \\ &= -g[\theta]^2 - \mathcal{H}\{|g[\theta]|^2\mathcal{H}\{g[\theta]^2\}\}, \quad \text{by (5.2)(iii)}, \end{aligned} \tag{99}$$

$$
\left.
\begin{aligned}
&\text{whence}\\
&\quad |\mathcal{H}\{|g[\theta]|^2\mathcal{H}\{g[\theta]^2\}\}(s)| \prec |\log\theta|^2\\
&\qquad\qquad\qquad\qquad \text{for } s \le 3\theta \text{ and } \tfrac{1}{2} \le s, \quad \text{by (92)},\\
&\text{and}\\
&\quad |\mathcal{H}\{|g[\theta]|^2\mathcal{H}\{g[\theta]^2\}(s)| \prec |\log\theta|\\
&\qquad\qquad\qquad\qquad \text{for } 3\theta < s < \tfrac{1}{2}, \quad \text{by (98)}.
\end{aligned}
\right\} \tag{100}
$$

Lemma 5.9 $\int s^{-1}|\overline{g[\theta]}\,\mathcal{H}\{g[\theta]^2\}|^2|\mathcal{H}\{|g[\theta]|^2\,\mathcal{H}\{g[\theta]^2\}\}|^2\,ds \prec |\log\theta|^6.$

Proof. $|g[\theta](s)| \le 1$, and is 0 unless $\theta \le s \le 2$. Hence, by (88) and (100),

$$
\begin{aligned}
&\int s^{-1}|\overline{g[\theta]}\,\mathcal{H}\{g[\theta]^2\}|^2|\mathcal{H}\{|g[\theta]|^2\,\mathcal{H}\{g[\theta]^2\}\}|^2 ds\\
&\prec |\log\theta|^2\int_\theta^2 s^{-1}|\mathcal{H}\{|g[\theta]|^2\,\mathcal{H}\{g[\theta]^2\}\}|^2 ds\\
&\prec |\log\theta|^2\Big\{|\log\theta|^4\Big(\int_\theta^{3\theta} s^{-1}\,ds + \int_{1/2}^2 s^{-1}\,ds\Big)\\
&\qquad\qquad\qquad\qquad + |\log\theta|^2\int_{3\theta}^{1/2} s^{-1}\,ds\Big\}\\
&= |\log\theta|^4\Big(|\log\theta|^2(\log 3 + \log 4) + \log\Big(\frac{1}{2}\Big) - \log 3 - \log\theta\Big)\\
&\prec |\log\theta|^6.
\end{aligned} \tag{101}
$$

□

Note. Because of the oscillatory term in the definition of $g[\theta]$ the estimate is $|\log\theta|^6$, whereas the elementary arguments of (5.6) would lead to $|\log\theta|^7$.

(5.10) Define H_0 to be the complex Hilbert space $L^2([-1,1])$, and set for $f \in H_0$

$$
(A_1 f)(s) := s\chi_{[0,1]}(s)f(s). \tag{102}
$$

Then A_1 is a bounded non-negative operator in H_0. Fix $b > 0$, $c > 1$, and define ${}^n g$ and g_n for $n = 1, 2, 3, \ldots$ by

$$
{}^n g := g[2^{n^c+1-(n+1)^c}], \qquad g_n(t) := n^{-b}({}^n g \circ \phi_n)(t), \tag{103}
$$

where $g[\theta]$ is as in (5.4) and ϕ_n is multiplication by 2^{n^c+1}. Thus g_n is zero off $E_n := (2^{-(n+1)^c}, 2^{-n^c})$, and, by (5.2)(iv), (88) of (5.5), and obviously,

$$\left.\begin{aligned} \sup\{|g_n(t)|^2 : t \in \mathbb{R}\} &= n^{-2b}, \quad \text{and, as } c > 1, \\ \sup\{|\overline{g_n(t)}\mathcal{H}\{g_n^2\}(t)| : t \in \mathbb{R}\} &\prec n^{-3b}((n+1)^c - n^c - 1) \asymp n^{c-1-3b}. \end{aligned}\right\} \quad (104)$$

Define $h_n := \bar{g}_n\mathcal{H}\{g_n^2\}$, which is also zero off E_n. If I suppose that $3b \geq c-1$, then, by (104), there exists a constant K such that $\operatorname{essup}|g_n| \leq K \geq \operatorname{essup}|h_n|$ for all n. Let χ_n denote the characteristic function of E_n. Define $Q_n : H_0 \to H_0$ by

$$(\forall f \in H_0) \quad Q_n(f) := \bar{g}_n\mathcal{H}\{g_nf\} + \bar{h}_n\mathcal{H}\{h_nf\}. \tag{105}$$

As $g_nf, h_nf \in H_0$ and $\|g_nf\| \leq K\,\|\chi_nf\| \geq \|h_nf\|$, (5.2)(iii) shows that $Q_n(f) \in H_0$ and $\|Q_n(f)\| \leq 2K^2\,\|\chi_nf\|$. Moreover, Q_n is skew-adjoint (because $\mathcal{H}$ is). Also,

$$(\forall f \in H_0) \quad \sum_{n=1}^{\infty} \|Q_n(f)\|^2 \leq \sum_{n=1}^{\infty} 4K^4\|\chi_nf\|^2 \leq 4K^4\,\|f\|^2 \tag{106}$$

since the (E_n) are disjoint. As $Q_n(f)$ clearly vanishes off E_n, this proves that $\sum \frac{1}{2}Q_n$ converges strongly to an operator S_1, where $\|S_1\| \leq K^2$. Clearly S_1 is skew-adjoint. I recall that this presupposes $3b \geq c-1 > 0$.

(5.11) A simple computation from (5.2)(ii) and (102) shows that, for each $f \in H_0$,

$$\left.\begin{aligned} [A_1, S_1]f &= \frac{1}{2}\pi^{-1}\sum_{n=1}^{\infty}(\langle f, g_n\rangle g_n + \langle f, h_n\rangle h_n), \\ \text{so that} \quad \langle [A_1, S_1]f, f\rangle &= \frac{1}{2}\pi^{-1}\sum_{n=1}^{\infty}(|\langle f, g_n\rangle|^2 + |\langle f, h_n\rangle|^2). \end{aligned}\right\} \quad (107)$$

Therefore $[A_1, S_1] \geq 0$. Note that g_k and g_n are orthogonal for $k \neq n$, as are h_k and h_n, because E_n and E_k are disjoint. Furthermore, g_n and h_n are orthogonal, because $\mathcal{H}$ is skew-adjoint.

Since g_n and h_n are uniformly essentially bounded and the measure of their support E_n does not exceed 2^{-n^c}, $\sum(\|g_n\|^2 + \|h_n\|^2) < \infty$, and (107) now shows that $[A_1, S_1]$ is in the trace-class.

Take (μ_n) such that $0 \le \mu_n < \mu$ for all n, and define C_0 by

$$(\forall f \in H_0) \quad C_0 f := \frac{1}{2}\sum_{n=1}^{\infty} \mu_n \langle f, g_n\rangle g_n. \tag{108}$$

Hence $0 \le (\pi\mu)^{-1} C_0 \le [A_1, S_1]$, and C_0 is in the trace-class.

Proposition 5.12 *Suppose that* $\mu_n \| s^{-1/2}\,\bar{h}_n \mathcal{H}\{h_n g_n\}\| \le L$ *for all* n. *Then*

$$[C_0, S_1] \le \pi\Big(1 + \frac{1}{2}\mu^2\Big)[A_1, S_1] + \frac{1}{8} L^2 A_1. \tag{109}$$

Proof. Given n,

$$\begin{aligned} S_1 g_n &= \frac{1}{2}\sum_{k=1}^{\infty} Q_k(g_n) = \frac{1}{2} Q_n(g_n) \\ &= \frac{1}{2}\bar{g}_n \mathcal{H}\{g_n^2\} + \frac{1}{2}\bar{h}_n \mathcal{H}\{h_n g_n\} = \frac{1}{2} h_n + \frac{1}{2}\bar{h}_n \mathcal{H}\{h_n g_n\} \end{aligned} \tag{110}$$

(as $g_k g_n = h_k g_n = 0$ for $k \ne n$), and consequently $S_1 g_n - \frac{1}{2} h_n = \frac{1}{2}\bar{h}_n \mathcal{H}\{h_n g_n\}$, which is supported on E_n. Let Γ be the set of indices n for which $S_1 g_n - \frac{1}{2} h_n$ is not zero a.e. (in fact Γ contains all n, but that need not be proved); for each $n \in \Gamma$, let

$$\omega_n := \| s^{-1/2}\,\bar{h}_n \mathcal{H}\{h_n g_n\}\|^{-1} s^{-1/2}\,\bar{h}_n \mathcal{H}\{h_n g_n\}. \tag{111}$$

Then $\{\omega_n : n \in \Gamma\}$ is orthonormal. Now, for any $f \in H_0$, from (108)

$$\langle [C_0, S_1] f, f\rangle = \langle S_1 f, C_0 f\rangle + \langle C_0 f, S_1 f\rangle \tag{112}$$

(compare the real case (76))

$$\begin{aligned} &= \sum_{n=1}^{\infty} \mu_n \Re(\langle f, g_n\rangle\langle g_n, S_1 f\rangle) \le \sum_{n=1}^{\infty} \mu_n |\langle f, g_n\rangle||\langle f, S_1 g_n\rangle| \\ &\le \sum_{n=1}^{\infty} \mu_n |\langle f, g_n\rangle||\langle f, S_1 g_n - \frac{1}{2} h_n\rangle| + \frac{1}{2}\sum_{n=1}^{\infty} \mu_n |\langle f, g_n\rangle||\langle f, h_n\rangle| \\ &\le \frac{1}{2}\Big(1 + \frac{1}{2}\mu^2\Big)\sum_{n=1}^{\infty} |\langle f, g_n\rangle|^2 \\ &\qquad + \frac{1}{4}\sum_{n=1}^{\infty} |\langle f, h_n\rangle|^2 + \frac{1}{2}\sum_{n=1}^{\infty} \mu_n^2 |\langle f, S_1 g_n - \frac{1}{2} h_n\rangle|^2 \end{aligned}$$

$$
\begin{aligned}
&\le \pi\Big(1+\frac{1}{2}\mu^2\Big)\langle [A_1,S_1]f,f\rangle \\
&\qquad + \frac{1}{2}\sum_{n\in\Gamma}\mu_n^2|\langle s^{1/2}f, s^{-1/2}(S_1 g_n - \frac{1}{2}h_n)\rangle|^2 \quad \text{by (107)} \\
&\le \pi\Big(1+\frac{1}{2}\mu^2\Big)\langle [A_1,S_1]f,f\rangle \\
&\qquad + \frac{1}{8}\sum_{n\in\Gamma}\mu_n^2\|s^{-1/2}\bar{h}_n\mathcal{H}\{h_n g_n\}\|^2|\langle s^{1/2}f,\omega_n\rangle|^2 \\
&\le \pi\Big(1+\frac{1}{2}\mu^2\Big)\langle [A_1,S_1]f,f\rangle + \frac{1}{8}L^2\langle s^{1/2}\chi_{[0,1]}f, s^{1/2}\chi_{[0,1]}f\rangle \\
&\qquad\qquad \text{(Bessel's inequality)} \\
&= \pi\Big(1+\frac{1}{2}\mu^2\Big)\langle [A_1,S_1]f,f\rangle + \frac{1}{8}L^2\langle A_1 f,f\rangle,
\end{aligned}
$$

which is the result. □

Lemma 5.13 $\|s^{-1/2}\bar{h}_n\mathcal{H}\{h_n g_n\}\| \prec n^{3(c-1)-7b}$.

Proof. For each index n,

$$
\begin{aligned}
\|s^{-1/2}\,\bar{h}_n\mathcal{H}\{h_n g_n\}\|^2 &= \int s^{-1}|\bar{g}_n\mathcal{H}\{g_n^2\}|^2|\mathcal{H}\{|g_n|^2\mathcal{H}\{g_n^2\}|^2\,ds \\
&= n^{-14b}\int s^{-1}\Big(|(\overline{{}^n g})\,\mathcal{H}\{({}^n g)^2\}|^2|\mathcal{H}\{|{}^n g|^2\,\mathcal{H}\{({}^n g)^2\}|^2\Big)\circ\phi_n\,ds \\
&\qquad \text{by (5.2)(iv)} \\
&= n^{-14b}\int t^{-1}|(\overline{{}^n g})\,\mathcal{H}\{({}^n g)^2\}|^2|\mathcal{H}\{|{}^n g|^2\,\mathcal{H}\{({}^n g)^2\}|^2\,dt, \\
&\qquad \text{where } t := \phi_n(s), \\
&\prec n^{-14b}|\log(2^{n^c+1-(n+1)^c})|^6 \asymp n^{6(c-1)-14b}, \\
&\qquad \text{by Lemma 5.9.} \qquad (113)
\end{aligned}
$$

The result follows. □

(5.14) Let (m_k) be any orthonormal sequence in $L^2([-1,0)) \subseteq L^2(\mathbb{R})$. Thus $g_n m_k = 0$ a.e. for all j and k. Take a bounded sequence (λ_k) of non-negative real numbers, and define the bounded non-negative self-adjoint operator Z_0 by

$$
Z_0 f := \sum_{k=1}^{\infty}\lambda_k\langle f,m_k\rangle m_k. \tag{114}
$$

Lemma 5.15 $[Z_0, S_1] = 0$.

Proof. For any $f \in H_0$,

$$\begin{aligned}\langle [Z_0, S_1]f, f\rangle &= 2\Re(\langle Z_0 f, S_1 f\rangle)\\ &= 2\Re\left(\sum_{k=1}^{\infty} \lambda_k \langle f, m_k\rangle \langle m_k, S_1 f\rangle\right). \qquad (115)\end{aligned}$$

However, for each k, $\langle m_k, S_1 f\rangle = \frac{1}{2}\sum_{n=1}^{\infty}\langle m_k, \bar{g}_n \mathcal{H}\{g_n f\} + \bar{h}_n \mathcal{H}\{h_n f\}\rangle = 0$, since the support of m_k is disjoint from the supports of g_n and of h_n. □

(5.16) Let $\mu_n := \frac{1}{2}n^{-d}$, $\mu := 1$, in (108), where $d \geq 0$. S_1 is defined if $3b \geq c - 1 > 0$, as in (5.10); then $0 \leq \pi^{-1}C_0 \leq [A_1, S_1]$, as in (5.11); and, if $d + 7b \geq 3(c-1)$, Lemma 5.13 ensures that Proposition 5.12 applies and $[C_0, S_1] \leq \frac{3}{2}\pi\,[A_1, S_1] + \frac{1}{8}L^2 A_1$ for some L. Thus the conditions (4.7)(i), (ii), (iii) are satisfied (*mutatis mutandis*, i.e. with subscript 1) if $C_1 := \frac{1}{3}\pi^{-1}C_0$. For (4.7)(iv), it then suffices, by Lemma 5.15, to take $Z_1 := Z_0 + C_1$, so that $C_1 \leq Z_1 \geq Z_0$, and the requirements of (3.1) are met with $c_0 = 1$. Since C_1 is trace-class (see (5.11)), Z_1 will be trace-class if and only if Z_0 is, that is, if $\sum \lambda_n$ converges.

Let H_1 be the real Hilbert space obtained by restriction of the scalar field of H_0 to $\mathbb{R}$, with real inner product given by the real part of the inner product in H_1; the operators act in H_1, and the properties referred to above (non-negative self-adjoint, skew-adjoint, trace-class) hold in H_1 as they did in H_0. The norms in H_1 and in H_0 agree, so the "complex" calculations below give valid results in H_1 too.

Define the manifold M_1 from the operators A_1, C_1, Z_1 in H_1, as at (3.1).

(5.17) Let $e_n := n^a s^{-1} g_n + \sigma_n m_n$, where $a \in \mathbb{R}$, and $\sigma_n > 0$ is defined by

$$\begin{aligned}\sigma_n^2 = \|n^a s^{-1} g_n\|^2 &= n^{2(a-b)} \int s^{-2} |{}^n g|^2 \circ \phi_n ds,\\ &\qquad\qquad \text{which, with } t = \phi_n(s),\\ &= n^{2(a-b)} 2^{n^c+1} \int t^{-2} |g[2^{n^c+1-(n+1)^c}](t)|^2 dt\\ &\asymp n^{2(a-b)}\, 2^{(n+1)^c}, \qquad (116)\end{aligned}$$

as $\chi_{[2\theta,1]} \leq g[\theta] \leq \chi_{[\theta,2]}$, where χ_E denotes the characteristic function of E. Since $c > 1$, the sequence (σ_n) is strictly increasing for sufficiently large n.

But also

$$\|e_n\| = \sigma_n\sqrt{2} \to \infty \quad \text{as} \quad n \to \infty. \tag{117}$$

Moreover,

$$\begin{aligned}\|s^{-1/2}g_n\|^2 &= n^{-2b}\int t^{-1}|g[2^{n^c+1-(n+1)^c}](t)|^2\,dt \\ &\asymp n^{-2b}\log(2^{(n+1)^c-n^c-1}) \asymp n^{c-1-2b},\end{aligned} \tag{118}$$

by an argument similar to that justifying (116).

(5.18) From (116), $\sigma_{n+1} > 2\sigma_n$ for all sufficiently large n (how large depends on a and b); restrict attention to such n. Define $y_n(t) := (1-t)e_n + te_{n+1}$ for $0 \le t \le 1$. Writing y for $y_n(t)$ and ν_n for $\sqrt{\min(\lambda_n, \lambda_{n+1})}$, one then has from (114)

$$\begin{aligned}\langle Z_1y, y\rangle \ge \langle Z_0y, y\rangle &= \lambda_n(1-t)^2\sigma_n^2 + \lambda_{n+1}t^2\sigma_{n+1}^2 \\ &\ge \nu_n^2\{(1-t)^2\sigma_n^2 + t^2\sigma_{n+1}^2\}, \\ \text{and} \qquad \langle A_1y, y\rangle &= \int s\chi_{[0,1]}(s)|(1-t)e_n(s) + te_{n+1}(s)|^2 ds \\ &= \int s^{-1}\{n^{2a}(1-t)^2|g_n|^2 + (n+1)^{2a}t^2|g_{n+1}|^2\}ds \\ &\asymp n^{2a+c-2b-1},\end{aligned} \tag{119}$$

from (118), since m_n, m_{n+1} are supported on $[-1, 0)$, g_n and g_{n+1} have disjoint supports, and $\frac{1}{2} \le (1-t)^2 + t^2 \le 1$. Next,

$$\begin{aligned}\langle C_1e_n, e_n\rangle &= \frac{1}{6}\pi^{-1}n^{2a-d}|\langle s^{-1}g_n, g_n\rangle|^2, \\ &\qquad\qquad \text{by the definitions of } e_n \text{ and } C_1, \\ &\asymp n^{2a-d+2c-4b-2} \asymp \langle C_1e_{n+1}, e_{n+1}\rangle, \quad \text{by (118)}.\end{aligned} \tag{120}$$

Since C_1e_n (a multiple of g_n) and e_{n+1} are orthogonal, it follows that

$$\begin{aligned}\langle C_1y, y\rangle &= (1-t)^2\langle C_1e_n, e_n\rangle \\ &\quad + t^2\langle C_1e_{n+1}, e_{n+1}\rangle \asymp n^{2a-d+2c-4b-2}.\end{aligned} \tag{121}$$

So, for suitable positive constants U and V, large enough n, and $0 \le t \le 1$,

$$f(y_n(t)) \le \frac{Un^{2a+c-2b-1} + \exp(-2Vn^{2a-d+2c-4b-2})}{\nu_n^2\{(1-t)^2\sigma_n^2 + t^2\sigma_{n+1}^2\}\,\{\log(\nu_n^2\{(1-t)^2\sigma_n^2 + t^2\sigma_{n+1}^2\})\}^2}. \tag{122}$$

(5.19) If $2a - d + 2c - 4b - 2 > 0$, then $\exp(-2Vn^{2a-d+2c-4b-2}) \prec n^{2a+c-2b-1}$; so the Riemannian length ℓ_n of the segment $y_n(t)$ may be estimated for large n:

$$\begin{aligned} \ell_n &= \int_0^1 \sqrt{f(y_n(t))}\,\|y_n'(t)\|\,dt = \int_0^1 \sqrt{f(y_n(t))}\,\|e_{n+1} - e_n\|\,dt \\ &\prec \int_0^1 \frac{n^{(2a+c-2b-1)/2}\sigma_{n+1}}{\nu_n\{(1-t)^2\sigma_n^2 + t^2\sigma_{n+1}^2\}^{1/2}\log(\nu_n^2\{(1-t)^2\sigma_n^2 + t^2\sigma_{n+1}^2\})}\,dt, \end{aligned} \tag{123}$$

since $\sigma_{n+1} \ge 2\sigma_n$. Define $\tau := \sigma_{n+1}^{-1}\sigma_n \le \frac{1}{2}$, and then, as $t^2 + (1-t)^2 \ge \frac{1}{4}$,

$$\begin{aligned} &\int_0^\tau \frac{n^{(2a+c-2b-1)/2}\sigma_{n+1}}{\nu_n\{(1-t)^2\sigma_n^2 + t^2\sigma_{n+1}^2\}^{1/2}\log(\nu_n^2\{(1-t)^2\sigma_n^2 + t^2\sigma_{n+1}^2\})}\,dt \\ &\le \frac{\tau n^{(2a+c-2b-1)/2}\,\sigma_{n+1}}{\frac{1}{2}\nu_n\sigma_n.2\log(\frac{1}{2}\nu_n\sigma_n)} = \frac{n^{(2a+c-2b-1)/2}}{\nu_n\log(\frac{1}{2}\nu_n\sigma_n)}, \end{aligned} \tag{124}$$

if $\nu_n\sigma_n > 2$. Also,

$$\begin{aligned} &\int_\tau^1 \frac{n^{(2a+c-2b-1)/2}\sigma_{n+1}}{\nu_n\{(1-t)^2\sigma_n^2 + t^2\sigma_{n+1}^2\}^{1/2}\log(\nu_n^2\{(1-t)^2\sigma_n^2 + t^2\sigma_{n+1}^2\})}\,dt \\ &\le \int_\tau^1 \frac{n^{(2a+c-2b-1)/2}\,\sigma_{n+1}}{2\nu_n t\sigma_{n+1}\log(\nu_n t\sigma_{n+1})}\,dt \\ &= \frac{1}{2}n^{(2a+c-2b-1)/2}\nu_n^{-1}(\log\log(\nu_n\sigma_{n+1}) - \log\log(\nu_n\tau\sigma_{n+1})) \\ &= \frac{1}{2}n^{(2a+c-2b-1)/2}\,\nu_n^{-1}(\log\log(\nu_n\sigma_{n+1}) - \log\log(\nu_n\sigma_n)). \end{aligned} \tag{125}$$

Putting together (124) and (125), I have the estimate, for $\nu_n\sigma_n > 2$,

$$\begin{aligned} \ell_n \prec n^{(2a+c-2b-1)/2}\Big\{&\frac{1}{\nu_n\log(\frac{1}{2}\nu_n\sigma_n)} \\ &+ \frac{\log\log(\nu_n\sigma_{n+1}) - \log\log(\nu_n\sigma_n))}{2\nu_n}\Big\}. \end{aligned} \tag{126}$$

Proposition 5.20 *Suppose that there is a positive number ϵ such that*

$\nu_n \succ n^{-\epsilon}$, *that* $c > 1$, *that* $2a-d+2c-4b-2 > 0$, *and that* $2a+c-2b-1+2\epsilon < 0$. *Then* M_1 *is metrically incomplete.*

Proof. Let $2\delta := -(2a+c-2b-1+2\epsilon) > 0$. Then $\nu_n^{-1} n^{(2a+c-2b-1)/2} \prec n^{-\delta}$. But (ν_n) is bounded above (by (5.14) and (5.18)), and, as $\nu_n \succ n^{-\epsilon}$, from (116)

$$n^{a-b}\, 2^{(n+1)^c/2} \succ \sigma_n \succ \nu_n \sigma_n \succ n^{a-b-\epsilon}\, 2^{(n+1)^c/2}. \tag{127}$$

Thus $\nu_n \sigma_n > 2$ for large n,

$$\left.\begin{aligned} (n+1)^{-c}\log(\nu_n\sigma_n) &= \frac{1}{2}\log 2 + O(n^{-c}\log n), \quad \text{and likewise} \\ (n+2)^{-c}\log(\nu_n\sigma_{n+1}) &= \frac{1}{2}\log 2 + O(n^{-c}\log n), \quad \text{as } n\to\infty. \end{aligned}\right\} \tag{128}$$

As $\log(\frac{1}{2}\nu_n\sigma_n) \asymp n^c$, $\sum \frac{n^{(2a+c-2b-1)/2}}{\nu_n \log(\frac{1}{2}\nu_n\sigma_n)}$ is dominated by $\sum n^{-c-\delta}$, and converges.

Next, from (128),

$$\begin{aligned} &\log\log(\nu_n\sigma_{n+1}) - \log\log(\nu_n\sigma_n) \\ &\quad = \log\left\{\frac{\log(\nu_n\sigma_{n+1})}{\log(\nu_n\sigma_n)}\right\} = \log\left\{\frac{(n+2)^c}{(n+1)^c}(1+O(n^{-c}\log n))\right\} \\ &\quad = c\log\left(1+\frac{1}{n+1}\right) + O(n^{-c}\log n) \asymp n^{-1}. \end{aligned} \tag{129}$$

Hence $\sum n^{(2a+c-2b-1)/2}\left\{\frac{\log\log(\nu_n\sigma_{n+1}) - \log\log(\nu_n\sigma_n)}{\nu_n}\right\}$ is dominated by $\sum n^{-1-\delta}$, and also converges. Now, by (126), $\sum \ell_n$ converges. Thus the path formed by the rectilinear segments $y_n(t)$, for $n = 1, 2, 3, \ldots$, is of finite length in M_1. As it passes through all the points e_n, it has by (117) no limit in H_1, or in M_1. □

Proposition 5.21 *Operators* A_1, C_1, S_1, Z_1 *in the separable Hilbert space* H_1 *may be constructed as above to satisfy* (mutatis mutandis) *all the conditions of* (3.1) *and* (4.7), *and to make the Riemannian manifold* M_1 *of* (3.1) *metrically incomplete. Moreover, both* C_1 *and* Z_1 *may be trace-class operators.*

Proof. This follows from Proposition 5.20 and (5.16) if I can satisfy the assumptions $3b \geq c-1 > 0$ of (5.10), $d > 1$ (to ensure $\sum \lambda_n < \infty$), and $d + 7b \geq 3(c-1)$ from (5.16), $2a - d + 2c - 4b - 2 > 0$ from (5.19), $\epsilon > 0$

and $2a + c - 2b - 1 + 2\epsilon < 0$ from Proposition 5.20. A possible solution is $a := -8$, $b := 30$, $c := 71$, $d := 2$, $\epsilon := 2$. □

Note. Clearly one needs $c - 2b - 1 > 0$ and $a < 0$. The estimate $|\log\theta|^7$ mentioned in the note to Lemma 5.9 would lead, via Lemma 5.13, to the condition $2d + 14b \geq 7(c-1)$, and no solution would be possible; for then $2(c-1-2b) \leq \frac{4}{7}d$, and cannot outweigh $-d$. As for d, it plays no essential part.

6. Normal neighbourhoods

In this section, M and f are defined from operators A, C, and Z as at (3.1), $|\ |$ once more denotes the norm in H, and $\|\ \|$ the Riemannian norm in the tangent spaces of M, whilst $[\![\ ]\!]$, as before, is the operator norm in $L(H)$ (or in other spaces of operators). Recall that $X(x) = \frac{1}{2}\nabla f(x)$.

Lemma 6.1 $|X(x)| = O(1 + \langle x, x\rangle)$ *and* $[\![DX(x)]\!] = O(1 + \langle x, x\rangle)$.

Proof. Note that $|Zx| \leq (\zeta^2\langle Zx, x\rangle)^{1/2}$, for instance by the spectral theorem. So

$$\frac{|Zx|}{1 + \langle Zx, x\rangle} \leq \frac{1}{2}\zeta \quad \text{for all } x\text{. In like manner}$$

$$\exp(-\langle Cx, x\rangle)|Cx| \leq \gamma\exp(-|C^{1/2}x|^2)|C^{1/2}x| \tag{130}$$

is also bounded for all x. The first statement therefore follows from (59) and Lemma 3.4, since $\Phi(\langle Zx, x\rangle) > 1 + \langle Zx, x\rangle$. On differentiating to find $DX(x)\cdot h$, one obtains a sum of many terms all of which are $O(|h|(1+|x|^2))$. (It should be noted that Φ'' and Φ'/Φ are both bounded; compare (52) of (4.1)). □

(6.2) Fix $\Xi \geq 1$ such that $\max(|X(x)|, [\![X'(x)]\!]) \leq \Xi(1 + |x|^2)$ for all $x \in H$. (131)

The desideratum is an estimate for the size of a normal neighbourhood in terms of the norm in H. In the s-parametrization of (4.1) the geodesic equation is $x'' = \mu X(x)$ (see (56)) with initial condition $|x'|^2 = \mu f(x)$ as at (57). It will help to study its modification

$$x'' = \frac{\langle x', x'\rangle}{f(x)} X(x), \tag{132}$$

which, like the geodesic equation itself, is invariant under affine changes of parameter. On "dotting" with $2x'$, (132) has the special integral

$$f(x)^{-1}\langle x', x'\rangle = \mu, \tag{133}$$

μ being a constant of the solution. Hence a solution of (132) is a solution of (56), (57), for a suitable μ, whilst any solution of (56), (57) is a solution of (132). It follows from Lemma 4.3 that the maximal solutions of (132) are defined for all s.

Now fix a point $x \in M$. Denote by $p(s,x,\xi)$, or (when ambiguity does not arise) by $p(s;\xi)$, for $-\infty < s < \infty$, the solution of (132) which satisfies the initial conditions $p(0,x,\xi) = x$, $p'(0,x,\xi) = \xi$ (the prime denotes differentiation with respect to the first variable "s"); and write $E_x(\xi) = E(x,\xi) := p(1,x,\xi)$, so that, by the affine-invariance of (132), $E(x,s\xi) = p(s,x,\xi)$ for all s and ξ. Given $r \geq 0$, let

$$R_1(r) := \Xi^{-1}\exp(-(\alpha^2+1)(\gamma^2+1)(\zeta^2+1)(e^e+r^2)). \tag{134}$$

(Recall that $\Xi \geq 1$. This is just a convenient value, not the "best possible").

Lemma 6.3 $|p(s,x,\xi) - x| < \frac{1}{3}(1+|x|)^{-1}$ *if* $|s\xi| \leq R_1(|x|)$.

Proof. Take τ as the least positive number, if one exists, for which $|p(\tau,x,\xi)-x| = \frac{1}{3}(1+|x|)^{-1}$ (if no such τ exists, there is nothing to prove); and $\mu := f(x)^{-1}\langle\xi,\xi\rangle$. Thus, for $0 \leq s \leq \tau$,

$$\begin{aligned}
|X(p(s;\xi))| &\leq K := \Xi\Big(1+\Big(\frac{1}{3}(1+|x|)^{-1}+|x|\Big)^2\Big) \\
&\qquad \text{by (131), so that, by (132),} \\
|p''(s;\xi)| &\leq \mu K, \quad \text{and, by the mean-value inequality,} \\
|p'(s;\xi)-\xi| &\leq \mu K s; \quad \text{in turn, this leads to} \\
|p(s;\xi)-s\xi-x| &\leq \frac{1}{2}\mu K s^2, \quad \text{so that, recalling the meaning of } \mu, \\
\frac{1}{3}(1+|x|)^{-1} &= |p(\tau;\xi)-x| \leq \frac{1}{2}Kf(x)^{-1}|\tau\xi|^2+|\tau\xi|.
\end{aligned}$$

Hence trivially
$$|\tau\xi| \geq \min\Big(\frac{1}{6}(1+|x|)^{-1}, 2f(x)K^{-1}\Big). \tag{135}$$

But (134) implies $\min(\frac{1}{6}(1+|x|)^{-1}, 2f(x)K^{-1}) > R_1(|x|)$. □

Lemma 6.4 *Suppose that* $\xi, \eta \in H$, $\tau > 0$, *and* $|\tau\xi|, |\tau\eta| \leq R_1(x)$; *then*

$$|p(\tau,x,\xi)-\tau\xi-p(\tau,x,\eta)+\tau\eta| \leq \frac{1}{5}|\tau\xi-\tau\eta|. \tag{136}$$

Proof. Let $\rho(\tau) := \max\{|p(s;\xi) - p(s;\eta)| : 0 \le s \le \tau\}$. By Lemma 6.3, $p(s;\xi)$ and $p(s;\eta)$ (and the straight-line segment joining them) lie for $0 \le s \le \tau$ in the $|\ |$-ball of radius $\frac{1}{3}(1+|x|)^{-1}$ about x, on which $|X(y)|$ and $[\![X'(y)]\!]$ are both bounded by $\Xi(1 + (|x| + \frac{1}{3}(1+|x|)^{-1})^2) < \Xi[x] := \Xi(2 + |x|^2)$. Thus for each such s

$$\begin{aligned}
&|p''(s;\xi) - p''(s;\eta)| \\
&\quad = \left| \frac{\langle p'(s;\xi), p'(s;\xi)\rangle}{f(p(s;\xi))} X(p(s;\xi)) - \frac{\langle p'(s;\eta), p'(s;\eta)\rangle}{f(p(s;\eta))} X(p(s;\eta)) \right| \\
&\quad = \left| \frac{\langle \xi, \xi\rangle}{f(x)} X(p(s;\xi)) - \frac{\langle \eta, \eta\rangle}{f(x)} X(p(s;\eta)) \right| \qquad \text{by (133)} \\
&\quad \le \frac{\langle \xi, \xi\rangle}{f(x)} |X(p(s;\xi)) - X(p(s;\eta))| + \frac{\Xi[x]}{f(x)} |\langle \xi, \xi\rangle - \langle \eta, \eta\rangle| \\
&\quad \le \frac{\langle \xi, \xi\rangle}{f(x)} \Xi[x]\rho(\tau) + \frac{\Xi[x]}{f(x)} |\xi - \eta|(|\xi| + |\eta|), \qquad (137)
\end{aligned}$$

by the mean-value inequality; in turn

$$\begin{aligned}
&|p'(s;\xi) - \xi - p'(s;\eta) + \eta| \\
&\quad \le s\Xi[x]f(x)^{-1}\{\langle \xi, \xi\rangle \rho(\tau) + |\xi - \eta|(|\xi| + |\eta|)\} \quad \text{and} \\
&|p(s;\xi) - s\xi - p(s;\eta) + s\eta| \\
&\quad \le \frac{1}{2} s^2 \Xi[x] f(x)^{-1}\{\langle \xi, \xi\rangle \rho(\tau) + |\xi - \eta|(|\xi| + |\eta|)\}. \qquad (138)
\end{aligned}$$

Therefore

$$\begin{aligned}
\rho(\tau) &\le \frac{1}{2}\tau^2 \Xi[x] f(x)^{-1}\{|\xi|^2 \rho(\tau) + |\xi - \eta|(|\xi| + |\eta|)\} + \tau|\xi - \eta| \\
&\le \frac{1}{240}\rho(\tau) + \frac{7}{6}\tau|\xi - \eta|, \qquad (139)
\end{aligned}$$

since, by (134), $\Xi[x]f(x)^{-1}(R_1(|x|))^2 < \frac{1}{120}$ and $\Xi[x]f(x)^{-1}R_1(|x|) < \frac{1}{6}$; consequently $\rho(\tau) \le \frac{280}{239}\tau|\xi - \eta|$. Substitute back in (138), and note $\frac{1}{240}\,\frac{280}{239} + \frac{1}{6} \le \frac{1}{5}$. □

Lemma 6.5 *For any $w \in H$ such that $|w - x| \le \frac{4}{5}R_1(|x|)$, there exists a unique $\xi \in H$ such that $|\xi| \le R_1(|x|)$ and $E(x,\xi) = w$. This ξ satisfies the further conditions that $|\xi| \le \frac{5}{4}|w - x|$ and $|E(x, s\xi) - x| \le \frac{3}{2}|w - x|$ for $0 \le s \le 1$. Lastly, E_x is a C^ω diffeomorphism between the open sets $V := \{w \in H : |w - x| < \frac{4}{5}R_1(|x|)\}$ and $U := \{\xi \in H : |\xi| < R_1(|x|)$ & $|E(x,\xi) - x| < \frac{4}{5}R_1(|x|)\}$.*

Proof. If $|\xi| \le \frac{5}{4}|w-x| \le R_1(|x|)$, define $T\xi := \xi + w - E(x,\xi)$. Then

$$|T\xi| \le |w-x| + |E(x,\xi) - \xi - x| \le |w-x| + \frac{1}{4}|w-x| \tag{140}$$

also (take $\eta = 0$, $\tau = 1$ in Lemma 6.4, so that $E(x,\eta) = x$). If $|\xi|, |\eta| \le \frac{5}{4}|w-x|$, from Lemma 6.4

$$|T\xi - T\eta| = |E(x,\eta) - \eta - E(x,\xi) + \xi| \le \frac{1}{5}|\xi - \eta|. \tag{141}$$

Since $T\xi = \xi$ if and only if $E(x,\xi) = w$, the existence and uniqueness of ξ follows by the contraction principle. Again from Lemma 6.4, $|E(x,s\xi) - s\xi - x| \le \frac{1}{5}|s\xi|$ for each $s \in [0,1]$, so that $|E(x,s\xi) - x| \le \frac{6}{5}|s\xi|$; the second sentence of the Lemma follows. As for the third, it has just been shown that E_x is a bijective correspondence between the stated sets, and the general theory of differential equations ensures that it is C^ω. It only remains to show that its derivative is invertible at all points of U. For sufficiently small h, Lemma 6.4 gives $|E(x,\xi+h) - E(x,\xi) - h| \le \frac{1}{5}|h|$, so that in the limit $[\![DE_x(\xi) - I]\!] \le \frac{1}{5}$. This proves the result. □

Lemma 6.6 *If $y, x \in H$ and $|y-x| \le R_1(|x|)$, then*

$$\frac{20}{21} f(x) \le f(y) \le \frac{21}{20} f(x). \tag{142}$$

Proof. Since $R_1(|x|) \le \exp(-e^e(1+\zeta^2)) < (1+e^e\zeta^2)^{-1}\exp(-e^e) < \frac{1}{600}\zeta^{-1}$,

$$\begin{aligned} \sqrt{e^e + \langle Zy, y\rangle} &\le \sqrt{e^e + \langle Zx, x\rangle} + \zeta|y-x| \\ &\le \frac{601}{600}\sqrt{e^e + \langle Zx, x\rangle} \end{aligned} \tag{143}$$

and

$$\begin{aligned} \sqrt{e^e + \langle Zy, y\rangle} &\ge \sqrt{e^e + \langle Zx, x\rangle} - \zeta|y-x| \\ &\ge \frac{599}{600}\sqrt{e^e + \langle Zx, x\rangle}. \end{aligned} \tag{144}$$

Secondly,

$$\langle Cy, y\rangle \le (\sqrt{\langle Cx, x\rangle} + \gamma|y-x|)^2 \le \langle Cx, x\rangle + \frac{1}{1800}, \tag{145}$$

since

$$2\gamma|y-x|\sqrt{\langle Cx,x\rangle}\ \le\ 2\gamma^2|x|\exp(-e^e-e^e\gamma^2-|x|^2)<\frac{1}{3600}$$

and

$$\gamma^2|y-x|^2\le\gamma^2\exp(-2e^e-2e^e\gamma^2)<\frac{1}{3600}.$$

Similarly,

$$\langle Cy,y\rangle\ge(\sqrt{\langle Cx,x\rangle}-\gamma|y-x|)^2\ge\langle Cx,x\rangle-\frac{1}{1800};\tag{146}$$

and also

$$\begin{aligned}\langle Ay,y\rangle\ &\le\ (\sqrt{\langle Ax,x\rangle}+\alpha|y-x|)^2\\ &\le\ \langle Ax,x\rangle+\frac{1}{1800}\exp(-\langle Cx,x\rangle),\end{aligned}\tag{147}$$

since

$$\begin{aligned}\alpha^2|y-x|^2\ &\le\ \alpha^2\exp(-2e^e-2e^e\alpha^2-2\gamma^2\langle x,x\rangle)\\ &<\ \frac{1}{3600}\exp(-\langle Cx,x\rangle)\end{aligned}$$

and

$$\begin{aligned}2\alpha|y-x|\sqrt{\langle Ax,x\rangle}\ &\le\ 2\alpha^2|x|\exp(-e^e-e^e\alpha^2-|x|^2-\gamma^2|x|^2)\\ &\le\ \frac{1}{3600}\exp(-\langle Cx,x\rangle).\end{aligned}$$

Likewise

$$\langle Ay,y\rangle\ge\langle Ax,x\rangle-\frac{1}{1800}\exp(-\langle Cx,x\rangle).\tag{148}$$

Now, applying (144), (146), and (147),

$$\begin{aligned}&\frac{\langle Ay,y\rangle+\exp(-\langle Cy,y\rangle)}{\Phi(\langle Zy,y\rangle)}\\ &\le\left(\frac{600}{599}\right)^2\frac{\langle Ax,x\rangle+(e^{1/1800}+\frac{1}{1800})\exp(-\langle Cx,x\rangle)}{(e^e+\langle Zx,x\rangle)(\log(e^e+\langle Zx,x\rangle)+2\log(\frac{599}{600}))^2}\\ &<\frac{21}{20}f(x).\end{aligned}\tag{149}$$

For the other inequality, use (148), (145), and (143):

$$\begin{aligned}\frac{\langle Ay,y\rangle+\exp(-\langle Cy,y\rangle)}{\Phi(\langle Zy,y\rangle)}&\\ \geq\left(\frac{600}{601}\right)^2&\frac{\langle Ax,x\rangle+(e^{-1/1800}-\frac{1}{1800})\exp(-\langle Cx,x\rangle)}{(e^e+\langle Zx,x\rangle)(\log(e^e+\langle Zx,x\rangle)+2\log(\frac{601}{600}))^2}\\ >\frac{20}{21}f(x).&\qquad(150)\end{aligned}$$

For example, $\log(e^e+\langle Zx,x\rangle)+2\log(\frac{601}{600})<\frac{601}{600}\log(e^e+\langle Zx,x\rangle)$. □

Corollary 6.7 *Whenever $|y-x|<\frac{1}{2}R_1(|x|)$, there is a unit geodesic from x to y in M, unique both as a minimizing unit geodesic from x to y and as the only unit geodesic from x to y within the ball $\{w\in H:|w-x|<\frac{4}{5}R_1(|x|)\}$; it is obtained by reparametrizing $E(x,s\xi)$, $0\leq s\leq 1$, for a suitable ξ, where $|\xi|\leq\frac{5}{4}|y-x|$, and it lies within the ball $\{w\in H:|w-x|<\frac{3}{2}|y-x|\}$.*

Proof. For any y with $|y-x|<\frac{1}{2}R_1(|x|)$ there is by Lemma 6.5 a unique ξ such that $|\xi|\leq\frac{5}{4}|y-x|\leq\frac{5}{8}R_1(|x|)$ and $E(x,\xi)=y$. Moreover, $|E(x,s\xi)-x|<\frac{3}{4}R_1(|x|)$ for $0\leq s\leq 1$. It follows from "Gauss's lemma" (see 1.9.2 on p. 80 of [12]) that, if $|y-x|<\frac{1}{2}R_1(|x|)$, the reparametrized geodesic $E(x,s\xi)$, $0\leq s\leq 1$, is the shortest amongst those paths from x to y which remain in the ball $\{z\in H:|z-x|<\frac{4}{5}R_1(|x|)\}$ (and are therefore images under the exponential of paths in T_xM, by Lemma 6.5). Its Riemannian length consequently cannot exceed that of the straight-line segment in H from x to y, which is not greater than $\sqrt{\frac{21}{20}f(x)}\cdot\frac{1}{2}R_1(|x|)$, by Lemma 6.6. On the other hand, Lemma 6.6 also shows that any path from y to x which leaves the ball of radius $\frac{4}{5}R_1(|x|)$ about x must have Riemannian length at least $\sqrt{\frac{20}{21}f(x)}\cdot\frac{11}{10}R_1(|x|)$ (since the segment "from x to outside" has $|\ |$-length at least $\frac{4}{5}R_1(|x|)$, and the segment "from outside to y" at least $\frac{3}{10}R_1(|x|)$), and this is larger than the previous estimate. Hence the reparametrized geodesic $E(x,s\xi)$ is unconditionally the shortest path from x to y. The concluding assertion is taken from Lemma 6.5. □

Proposition 6.8 *There is a decreasing function $\chi:[0,\infty)\to(0,1]$ such that, whenever $x,y,z\in H$ and $|x-y|<\chi(|x|)>|x-z|$, there is a unique minimizing geodesic segment in M which joins y to z. Furthermore, $|w-x|<4\chi(|x|)$ for any point w on this geodesic segment.*

Proof. Take $\chi(r) := \frac{1}{4}R_1(r+1) < \frac{1}{4}$ by (134), and use Corollary 6.7. □

7. Sequential compactness in sets of paths

(7.1) I shall use an analogue for the weak topology of the "sufficiency" part of the Ascoli-Arzelà theorem; it can be modified in many ways.

Let E be a normed space with norm $|\ |$, and let I temporarily denote the interval $[0, a]$, where $a > 0$. Endow the space $C^1(I, E)$ of C^1 (not just piecewise C^1) paths $I \to E$ with the C^1-weak topology; that is, the topology of uniform weak convergence of values and of first derivatives. (This space is convenient for dealing with lengths). A neighbourhood base for $p \in C^1(I, E)$ is furnished by sets of the form

$$\{q \in C^1(I, E) : q(0) - p(0) \in V \ \&\ (\forall t \in I)\dot{q}(t) - \dot{p}(t) \in V\} \quad (151)$$

as V varies over weak neighbourhoods of 0 in E.

(7.2) Given $G_0, G_1 \subseteq E$ and a modulus of continuity $\kappa : (0, \infty) \to (0, \infty)$, let $\mathcal{P}_I(G_0, G_1, \kappa) \subseteq C^1(I, E)$ consist of the differentiable paths p such that $p(0) \in G_0$, $\dot{p}(0) \in G_1$, and, for any $t', t \in I$ and $\epsilon > 0$, $|\dot{p}(t') - \dot{p}(t)| \leq \epsilon$ whenever $|t' - t| < \kappa(\epsilon)$. Then there exists a number $k := k(a, \kappa)$ such that, for all $p \in \mathcal{P}_I(G_0, G_1, \kappa)$ and $t \in I$, $|\dot{p}(t) - \dot{p}(0)| \leq k$. If G_1 is bounded, it follows that $\mathcal{P}_I(G_0, G_1, \kappa)$ is equicontinuous with respect to the norm topology.

Proposition 7.3 *Let E be reflexive, and G_0 and G_1 weakly compact in E. Then $\mathcal{P}_I(G_0, G_1, \kappa)$ is sequentially compact in $C^1(I, E)$ in the C^1-weak topology.*

Proof. Let (p_n) be a sequence in $\mathcal{P}_I(G_0, G_1, \kappa)$. Now G_0 and G_1 are norm-bounded in E, with bounds λ_0, λ_1, and therefore, for any $\xi \in I$, and all n,

$$|\dot{p}_n(\xi)| \leq k + |\dot{p}_n(0)| \leq k + \lambda_1 \quad \text{(see (7.2))}; \quad (152)$$

since the closed balls of radii λ_0, $k + \lambda_1$ about 0 are weakly sequentially compact, a subsequence $(p_{n(i)})$ may be selected by a diagonal process so that, for each $\xi \in \mathbb{Q} \cap I$, $\dot{p}_{n(i)}(\xi)$ converges weakly, say to $q(\xi)$, and also $p_{n(i)}(0)$ converges weakly to v. As G_0 and G_1 are weakly closed, $q(0) \in G_1$ and $v \in G_0$. (153)

Take $\xi, \eta \in \mathbb{Q} \cap I$. Thus $\dot{p}_{n(i)}(\xi) - \dot{p}_{n(i)}(\eta) \to q(\xi) - q(\eta)$ weakly. If $\epsilon > 0$ and $|\xi - \eta| < \kappa(\epsilon)$, then $|\dot{p}_{n(i)}(\xi) - \dot{p}_{n(i)}(\eta)| \leq \epsilon$ for each i, and, by Mazur's lemma,

$$|q(\xi) - q(\eta)| \leq \liminf |\dot{p}_{n(i)}(\xi) - \dot{p}_{n(i)}(\eta)| \leq \epsilon. \tag{154}$$

Consequently q extends to a mapping of I, which I still call q and which still admits the modulus of continuity κ. (155)

Let $\phi \in E'$, with norm 1, and let $\epsilon > 0$. Cover I by open intervals $U_1, U_2, \ldots, U_u$, each of length less than $\kappa(\frac{1}{3}\epsilon)$, and choose $\xi_j \in \mathbb{Q} \cap I \cap U_j$ for each j, $1 \leq j \leq u$. Then there exists N such that $|\phi(\dot{p}_{n(i)}(\xi_j) - q(\xi_j))| \leq \frac{1}{3}\epsilon$ for $1 \leq j \leq u$ whenever $i \geq N$. Given any $t \in I$, take j so that $t \in U_j$, and then, for $i \geq N$,

$$\begin{aligned} &|\phi(\dot{p}_{n(i)}(t) - q(t))| \\ &\leq |\phi(\dot{p}_{n(i)}(t) - \dot{p}_{n(i)}(\xi_j))| + |\phi(\dot{p}_{n(i)}(\xi_j) - q(\xi_j))| + |\phi(q(\xi_j) - q(t))| \\ &\leq \frac{1}{3}\epsilon + \frac{1}{3}\epsilon + \frac{1}{3}\epsilon = \epsilon, \quad \text{by (155).} \end{aligned} \tag{156}$$

Hence $\dot{p}_{n(i)}$ converges uniformly to q in the weak topology.

As q is continuous in norm, by (155), define p_0 by $p_0(s) := v + \int_0^s q(\tau)d\tau$, for each $s \in I$. Then $\dot{p}_0 = q$, and, since $p_{n(i)}(0) \to v$ weakly, $p_{n(i)} \to p_0$ C^1-weakly by the characterization (151) of the C^1-weak topology. By (153) and (155), $p_0 \in \mathcal{P}_I(G_0, G_1, \kappa)$. □

(7.4) Let the Riemannian manifold M again be constructed as at (3.1). As before, $|\ |$ is the norm in H, $[\![\ ]\!]$ the operator-norm, $\|\ \|$ the Riemannian norm in tangent spaces to M, and d the Riemannian distance. Recall from (4.1) that $X(x) = \frac{1}{2}\nabla f(x)$.

Lemma 7.5 *Suppose G_0, G are bounded sets in H and $\Lambda \geq 0$. Then there exist a (Lipschitz) modulus of continuity κ and a weakly compact subset G_1 of H such that all paths $p : I \to H$ which are geodesics in M of Riemannian length not exceeding Λ, start at a point of G_0, and take values in G, belong to $\mathcal{P}_I(G_0, G_1, \kappa)$.*

Proof. As before, let λ_0 be a norm-bound for G_0. Certainly, for any geodesic $p : I \to H$ of length not exceeding Λ, $\|\dot{p}(t)\|$ is constant for all $t \in I$, as at (54) of (4.1), and cannot exceed Λa^{-1}; hence
$\langle \dot{p}(t), \dot{p}(t)\rangle \leq \Lambda^2 a^{-2} f(p(t))^{-1}$. (157)

Also $1/f$ is bounded above on G_0, for instance by

$$\sigma := \Phi(\zeta^2\lambda_0^2)\exp(\gamma^2\lambda_0^2), \tag{158}$$

so that, in particular, $\|\dot{p}(0)\| \le \lambda_1 := \Lambda a^{-1}\sqrt{\sigma}$. Thus one may take G_1 to be the closed ball of radius $\Lambda a^{-1}\sqrt{\sigma}$ about 0.

Let λ be a norm-bound for G. For $x \in G$, $X(x) = \frac{1}{2}\nabla f(x)$ is bounded, by (6.2):

$$\left|\frac{1}{2}\nabla f(x)\right| \le K_1 := \Xi(1+\lambda^2), \tag{159}$$

$$\text{and} \qquad K_2 = K_2(\lambda) := \frac{\exp(-\gamma^2\lambda^2)}{\Phi(\zeta^2\lambda^2)} \le f(x). \tag{160}$$

From the geodesic equation (53) of (4.1), for $0 \le t \le a$

$$\begin{aligned} |\ddot{p}(t)| &= f(p(t))^{-1}\left|\frac{1}{2}\langle\dot{p},\dot{p}\rangle\nabla f - \langle\nabla f,\dot{p}\rangle\dot{p}\right| \le f(p(t))^{-1}\left\{\frac{3}{2}\langle\dot{p},\dot{p}\rangle|\nabla f|\right\} \\ &\le 3\Lambda^2 a^{-2} f(p(t))^{-2}K_1 \le 3\Lambda^2 a^{-2}K_2^{-2}K_1, \\ &\qquad\qquad \text{by (157), (160), (159).} \end{aligned} \tag{161}$$

So take $\kappa(\epsilon) := (3\Lambda^2 a^{-2}K_2^{-2}K_1)^{-1}\epsilon$, which depends only on $\lambda_0, \lambda, \Lambda$ and a. □

(7.6) Besides the assumptions of (3.1) and (4.7), let now C and Z be compact. As before, κ is a modulus of continuity.

Proposition *Let G_0, G_1 be weakly compact sets in H. If (p_n) is a sequence in $\mathcal{P}_I(G_0, G_1, \kappa)$ which converges in the C^1-weak topology to p_0, then p_0 satisfies the inequality $\ell(p_0) \le \limsup_{n\to\infty}\ell(p_n)$.*

Proof. By Proposition 7.3, $p_0 \in \mathcal{P}_I(G_0, G_1, \kappa)$ too. Let λ_0, λ_1 be norm-bounds for G_0, G_1, and let k be as in (7.2). The $\dot{p}_n$ are norm-equicontinuous by the definition of $\mathcal{P}_I(G_0, G_1, \kappa)$, and uniformly bounded by $k + \lambda_1$ (see (152) of Proposition 7.3); so the (p_n) are Lipschitz with common Lipschitz constant $k + \lambda_1$, and uniformly bounded with bound $\lambda := (k + \lambda_1)a + \lambda_0$. On the closed ball $B(\lambda)$ of radius λ, $\nabla f, f$, and $1/f$ are bounded (see (159), (144)); therefore $\sqrt{f}$ is uniformly continuous (and bounded) on $B(\lambda)$.

Define $h_n : I \to \mathbb{R}$ by $h_n(t) := \sqrt{f(p_n(t))}|\dot{p}_n(t)|$, for $n := 1, 2, 3, \ldots$. So (h_n), $(|\dot{p}_n(t)|)$, and $(|A^{1/2}p_n(t)|)$ are equicontinuous and uniformly bounded.

By the classical Ascoli-Arzelà theorem, there is a subsequence of indices $(n(i))$ such that both $(h_{n(i)})$ and $(|A^{1/2}p_{n(i)}|)$ converge uniformly.

Recall that $\ell(p_n) = \int_0^a \sqrt{f(p_n(\tau))}|\dot{p}_n(\tau)|d\tau = \int_0^a h_n(\tau)d\tau$ for $n = 0, 1, 2, \ldots$. Thus

$$\limsup_{n\to\infty} \ell(p_n) \geq \lim_{i\to\infty} \ell(p_{n(i)}) = \int_0^a \lim_{i\to\infty} h_{n(i)}(\tau)d\tau, \tag{162}$$

by uniform convergence. Now $p_{n(i)}(t) \to p_0(t)$ weakly for each t, and $C^{1/2}$ and $Z^{1/2}$ are compact; hence $|C^{1/2}p_{n(i)}(t)| \to |C^{1/2}p_0(t)|$ and $|Z^{1/2}p_{n(i)}(t)| \to |Z^{1/2}p_0(t)|$, so

$$\begin{aligned} \langle Cp_{n(i)}(t), p_{n(i)}(t)\rangle &\to \langle Cp_0(t), p_0(t)\rangle, \\ \Phi(\langle Zp_{n(i)}(t), p_{n(i)}(t)\rangle) &\to \Phi(\langle Zp_0(t), p_0(t)\rangle). \end{aligned} \tag{163}$$

By Mazur's lemma,

$$\begin{aligned} |A^{1/2}p_0(t)| &\leq \lim |A^{1/2}p_{n(i)}(t)|, \\ |\dot{p}_0(t)| &\leq \liminf_{i\to\infty} |\dot{p}_{n(i)}(t)|. \end{aligned}$$

So

$$\begin{aligned} f(p_0(t)) &= \frac{\langle Ap_0(t), p_0(t)\rangle + \exp(-\langle Cp_0(t), p_0(t)\rangle)}{\Phi(\langle Zp_0(t), p_0(t)\rangle)} \\ &\leq \lim_{i\to\infty} f(p_{n(i)}(t)) \end{aligned} \tag{164}$$

(which exists), and in turn

$$\begin{aligned} \sqrt{f(p_0(t))}|\dot{p}_0(t)| &\leq \lim_{i\to\infty} \sqrt{f(p_{n(i)}(t))} \liminf_{i\to\infty} |\dot{p}_{n(i)}(t)| \\ &= \liminf_{i\to\infty} h_{n(i)}(t), \end{aligned} \tag{165}$$

which, by construction, is the limit $\lim_{i\to\infty} h_{n(i)}(t)$. The result follows from integrating this inequality and comparing with (162). □

Note. The proof can be set out in various ways, but subsequences cannot be used to avoid the need for compactness of C and Z. The limits of convergent subsequences $\langle Cp_{n(i)}(s_j), p_{n(i)}(s_j)\rangle$ and $\langle Zp_{n(i)}(s_j), p_{n(i)}(s_j)\rangle$ may not be $\langle Cp_0(s_j), p_0(s_j)\rangle$ and $\langle Zp_0(s_j), p_0(s_j)\rangle$, and Mazur's inequality goes in the wrong direction.

8. Geodesic convexity

In this section, let M be defined from A, C, Z as in (3.1), where C and Z are compact; fix $x \in M$. To construct a minimizing geodesic between x and y, one might try to extract a C^1-weakly convergent subsequence from a sequence of paths between x and y whose lengths approximate $d(x, y)$. Unfortunately, one cannot a priori choose paths with norm-equicontinuous derivatives, as required in Proposition 7.3. A chosen sequence of paths might be unavoidably unbounded in H.

Suppose that M has the tetrapod property of (3.13), with parameter $\phi \in (0, \frac{1}{4}\pi)$ and feet $\wp_1, \wp_2, \wp_3 := J\wp_1, \wp_4 := J\wp_2$. Then ∂M is a singleton $\{*\}$, by Proposition 3.15.

Lemma 8.1 *Suppose $K \geq 0$, $\nu \in (0, 1]$, and let $p : [a, b] \to M$ be a path such that $|p(a)| \leq K$ and $|p(b)| \geq K + \nu$. Then $\ell(p) \geq \nu\sqrt{\omega}$, where*

$$\omega := \Phi(\zeta^2(K+1)^2)^{-1} \exp(-\gamma^2(K+1)^2). \tag{166}$$

Proof. There is a last point $p(s_1)$ such that $|p(s_1)| = K$ and an earliest subsequent point $p(s_2)$ for which $|p(s_2)| = K + \nu$. Then, for $s \in (s_1, s_2]$, $K < |p(s)| \leq K + \nu$. As $\nu \leq 1$, therefore $f(p(s)) \geq \omega$. But the norm-length of $p|[s_1, s_2]$ is at least ν, and so $\ell(p) \geq \ell(p|[s_1, s_2]) \geq \nu\sqrt{\omega}$. □

Lemma 8.2 *Suppose $d(x, y) < d(x, *)$. Then x and y may be joined by a minimizing geodesic.*

Proof. Given $x \in M$, the numbers $r \geq 0$ such that, whenever $d(x, y) < r$, there is a minimizing geodesic from x to y, form an interval containing 0. Suppose its supremum $\mu < d(x, *)$. By Lemma 3.19, all points within Riemannian distance μ of x lie within a ball of radius $K := R(x, \mu) > |x|$ about the origin. Define ω by (166). Note $\mu > \xi(|x|)$ by Proposition 6.8.

Now suppose that $d(x, y) < \mu + \nu\chi(K + 1)\sqrt{\omega}$ (see Proposition 6.8), where $\nu \in (0, 1)$.

Firstly, $|y| \leq K + \nu$. If not, any path p joining x to y will have a last point $p(s_1)$ such that $|p(s_1)| = K$, and a first subsequent point $p(s_2)$ with $|p(s_2)| = K + \nu$. Clearly $\ell(p|[0, s_1]) \geq \mu$ (otherwise s_1 could be increased), and, by Lemma 8.1, $\ell(p|[s_1, s_2]) \geq \nu\sqrt{\omega}$, so that $\ell(p) \geq \mu + \nu\sqrt{\omega}$; since $\chi(K + 1) \leq 1$ by construction (see Proposition 6.8), this contradicts the hypothesis.

Suppose that $y \in M$ satisfies $\mu \le d(x,y) < \mu + \frac{1}{6}\chi(K+1)\sqrt{\omega}$. For $i = 1, 2, 3, \ldots$, take a path p_i between x and y such that $d(x,y) \le \ell(p_i) \le d(x,y) + \frac{1}{6}i^{-1}\chi(K+1)\sqrt{\omega}$. Then, for all relevant values of s, $d(x, p_i(s)) \le \ell(p_i) < \mu + \frac{1}{3}\chi(K+1)\sqrt{\omega}$, and by the last paragraph (with $\nu = \frac{1}{3}$) $|p_i(s)| \le K + \frac{1}{3}$; so, by (166), $f(p_i(s)) \ge \omega$. Let z_i be the last point on p_i for which $d(x, z_i) = \mu - \frac{1}{6}\chi(K+1)\sqrt{\omega}$; by hypothesis, there is a minimizing geodesic q_i from x to z_i, $\ell(q_i) = d(x, z_i)$. The length of the remainder of p_i, between z_i and y, cannot exceed $\ell(p_i) - d(x, z_i) \le \frac{1}{6}(2 + i^{-1})\chi(K+1)\sqrt{\omega}$, and so its length in H (taking $\nu = \frac{1}{2}$ above) is not greater than $\frac{1}{2}\chi(K+1)$. By Proposition 6.8, there is a (unique) minimizing geodesic r_i from z_i to y, and

$$\begin{aligned} \ell(r_i) = d(z_i, y) &\le \ell(p_i) - d(x, z_i) \\ &\le d(x,y) - \mu + \frac{1}{6}(1 + i^{-1})\chi(K+1)\sqrt{\omega}. \end{aligned} \tag{167}$$

The Riemannian distance from x of any point on q_i or on r_i does not exceed

$$\begin{aligned} \mu - \frac{1}{6}\chi(K+1)\sqrt{\omega} + \frac{1}{6}(2 + i^{-1})\chi(K+1)\sqrt{\omega} \\ < \mu + \frac{1}{3}\chi(K+1)\sqrt{\omega}, \end{aligned} \tag{168}$$

so that (taking $\nu = \frac{1}{3}$ again) q_i and r_i are norm-bounded by $K + \frac{1}{3}$ for all i.

Reparametrize affinely so that $q_i, r_i : [0,1] \to M$. Their Riemannian lengths are bounded by $\Lambda := \mu + \frac{1}{3}\chi(K+1)\sqrt{\omega}$. So Lemma 7.5 and Proposition 7.3 may be applied; passing (twice) to a subsequence, I may assume both q_i and r_i converge C^1-weakly to C^1 paths q_0, r_0. Since $q_i(0) = x$, $r_i(1) = y$, and $q_i(1) = z_i = r_i(0)$, for each i, in the weak limit

$$q_0(0) = x, r_0(1) = y, \quad \text{and} \quad q_0(1) = r_0(0). \tag{169}$$

By (7.6),

$$\left.\begin{aligned} &\ell(q_0) \le \limsup_{i\to\infty} \ell(q_i) = \mu - \frac{1}{6}\chi(K+1)\sqrt{\omega} \quad \text{and, from (167),} \\ &\ell(r_0) \le \limsup_{i\to\infty} \ell(r_i) \le d(x,y) - \mu + \frac{1}{6}\chi(K+1)\sqrt{\omega}. \end{aligned}\right\} \tag{170}$$

The concatenation of q_0 and r_0 is defined and has length not exceeding $d(x,y)$; ergo, it is a minimizing path from x to y, and may be reparametrized as a geodesic.

Therefore, if $d(x,y) < \mu + \frac{1}{6}\chi(K+1)\sqrt{\omega}$, there is a minimizing geodesic from x to y — by hypothesis, if $d(x, y) < \mu$, and by the above argument if $d(x, y) \geq \mu$. This contradicts the definition of μ at the beginning of the proof; consequently the assumption that $\mu < d(x, *)$ is untenable, which proves the result. □

Note. q_0 and r_0 are in fact already geodesics, for (7.6) applies to segments of the paths. Their concatenation reparametrizes piecewise-linearly to a geodesic.

Lemma 8.3 *If $x, y \in M$ and $\delta := d(x, *) + d(*, y) - d(x, y) > 0$, then there is a minimizing geodesic from x to y. Furthermore, for any $\epsilon \in (0, \delta)$ there exists $K(x, y, \epsilon) > 0$ such that any path p from x to y for which $\ell(p) < d(x, y) + \epsilon$ lies within the ball of norm-radius $K(x, y, \epsilon)$ about the origin in H.*

Proof. Let

$$\begin{aligned} K(\epsilon) &= K(x, y, \epsilon) \\ &:= \max(R(x, d(x, *) - \frac{1}{2}(\delta - \epsilon)), R(y, d(y, *) - \frac{1}{2}(\delta - \epsilon))) \end{aligned} \tag{171}$$

(recall Lemma 3.19). Take a path p from x to y such that $\ell(p) < d(x, y) + \epsilon$, and split it into two segments: the first, of length $d(x, *) - \frac{1}{2}(\delta - \epsilon)$, starts at x and ends at $z(p, \epsilon)$, and the second, from $z(p, \epsilon)$ to y, is then necessarily of length

$$\begin{aligned} &\ell(p) - d(x, *) + \frac{1}{2}(\delta - \epsilon) \\ &\quad < d(x, y) - d(x, *) + \frac{1}{2}(\delta + \epsilon) = d(y, *) - \frac{1}{2}(\delta - \epsilon). \end{aligned} \tag{172}$$

By Lemma 3.19, these segments lie in the balls of norm-radius $R(x, d(x, *) - \frac{1}{2}(\delta - \epsilon))$ and of radius $R(y, d(y, *) - \frac{1}{2}(\delta - \epsilon))$. So p lies in the ball of radius $K(\epsilon)$.

Now choose a sequence $(p_n)_{n=1}^{\infty}$ of paths from x to y such that $\ell(p_n) < d(x, y) + \frac{1}{2}\delta/n$. If $z_n := z(p_n, \frac{1}{2}\delta/n)$, then $d(x, z_n) \leq d(x, *) - \frac{1}{2}(\delta - \frac{1}{2}\delta/n)$ and, from (172), $d(y, z_n) < d(y, *) - \frac{1}{2}(\delta - \frac{1}{2}\delta/n)$. So Lemma 8.2 yields minimizing geodesics $q_n : [0, 1] \to M$ from x to z_n and $r_n : [0, 1] \to M$ from z_n to y. Since they are included in the ball of radius $K(\frac{1}{2}\delta)$, Lemma 7.5 and Proposition 7.3 and 7.6 apply; there is a sequence $(n(i))$ of indices such that $q_{n(i)}, r_{n(i)}$ have C^1-weak limits q, r, where

$$\left.\begin{aligned} \ell(q) &\le \limsup_{i\to\infty} \ell(q_{n(i)}) \le d(x,*) - \frac{1}{2}\delta, \\ \ell(r) &\le \limsup_{i\to\infty} \ell(r_{n(i)}) \le d(y,*) - \frac{1}{2}\delta. \end{aligned}\right\} \tag{173}$$

Furthermore, $q(1) = r(0)$, both being weak limits of $(z_{n(i)})$, and similarly $q(0) = x$, $r(1) = y$. Thus the concatenation of q and r is defined, and is a path from x to y of length not exceeding $d(x,*) - \frac{1}{2}\delta + d(y,*) - \frac{1}{2}\delta = d(x,y)$ by hypothesis. It may therefore be reparametrized as a minimizing geodesic between x and y. □

Corollary 8.4 *There is a minimizing geodesic from x to y if $d(x,y) \le d(x,*)$.*

Proposition 8.5 *Suppose that M is geodesically complete. Then, for any $x \in M$, the set $\{y \in M : d(x,y) = d(x,*)\}$ is bounded in H.*

Proof. Write $\Delta := d(x,*)$, and suppose there is a sequence (y_n) in M such that $d(x,y_n) = \Delta$ for all n and $|y_n| \to \infty$. By Corollary 8.4, there is a minimizing geodesic p_n from x to y_n, which may be parametrized by arc-length. For each integer $k > \Delta^{-1}$, the sequence $(p_n|[0,\Delta-k^{-1}])$ is uniformly bounded in H, by Lemma 3.19. Thus Lemma 7.5, Proposition 7.3 and a diagonal process may be applied, to select a subsequence $(p_{n(i)})$ such that $(p_{n(i)}|[0,\Delta-k^{-1}])$ converges C^1-weakly as $i \to \infty$ for every choice of k. Renumber it as (p_n). The limits for different k agree (as pointwise weak limits), so they define a C^1 map $p_0 : [0,\Delta) \to M$. By (7.6), p_0 must be distance-nonincreasing:

$$(\forall s,t \in [0,\Delta)) \quad d(p_0(s), p_0(t)) \le |s-t|. \tag{174}$$

Fix $\epsilon \in (0,\Delta)$, $Q > 0$. By Lemma 3.19, $p_n|[0,\Delta-\epsilon]$ is bounded in norm for all n by $\Omega' = \Omega'(Q,\epsilon) := \max(Q, R(x,\Delta-\epsilon))$. Set

$$\Omega_0 = \Omega_0(Q,\epsilon) := R_0(\Delta,\Omega'), \quad \Omega_1 = \Omega_1(Q,\epsilon) := R_1(\Delta,\Omega') \tag{175}$$

as in Lemma 3.17. There exists N such that, whenever $n \ge N$, $|p_n(\Delta)| = |y_n| > \Omega_1 + 1$. For each such n, let s_n be the first value of the parameter such that $p_n(s_n) = \Omega_1$. If

$$\omega = \omega(Q,\epsilon) := \Phi(\zeta^2(\Omega_1+1)^2)^{-1} \exp(-\gamma^2(\Omega_1+1)^2), \tag{176}$$

as at (166), then, by Lemma 8.1, necessarily $\Delta - s_n \geq \sqrt{\omega}$ for $n \geq N$. Hence, if $k > \omega^{-1/2}$, $[\Delta - \epsilon, s_n] \subseteq [0, \Delta - k^{-1}]$.

In Lemma 3.17, I may take, for each $n \geq N$, $a := \Delta - \epsilon$, $\beta := s_n$ and $R' := \Omega'$ (as above). Thus there are numbers $R^{(n)} \in (\Omega_0, \Omega_1)$, $u^{(n)} \in (\Delta - \epsilon, s_n)$, $v_j^{(n)} \in [0, \beta')$ for $j = 1, 2, 3, 4$, such that, for each j,

$$R^{(n)} = |p_n(u^{(n)})| = |\wp_j(v_j^{(n)})| \quad \text{and}$$
$$\max_j(\sigma(p_n(u^{(n)}), \sigma(\wp_j(v_j^{(n)}))) < 1. \tag{177}$$

Reparametrize $p_n|[\Delta - \epsilon, u^{(n)}]$ as $\widetilde{p}_n : [0, 1] \to M$, which remains a geodesic, of length not exceeding $\epsilon - \sqrt{\omega}$, and norm-bounded by Ω_1. Now, by Lemmas 3.17, 3.18, and 3.10, for some index $j(n) \in \{1, 2, 3, 4\}$ there is a path, consisting of straight-line segments $r_i^{(n)} : [0, 1] \to M$ (for $1 \leq i \leq 3$), between $\widetilde{p}_n(1)$ and $\wp_{j(n)}(v_{j(n)}^{(n)})$, of total length not greater than $L(\pi - \phi)/\log\log(e^e + (R^{(n)})^2) \leq L(\pi - \phi)/\log\log(e^e + Q^2)$; and each segment is norm-bounded by $\kappa(\pi - \phi)\Omega_1$. Certainly the Riemannian length of each individual segment does not exceed $L(\pi - \phi)/\log\log(e^e + Q^2)$. (In Lemma 3.10 there are better estimates). By passing to a subsequence and renumbering, I may assume that $j(n)$ is constant, with value j, and that $|p_n(\Delta)| = |y_n| > \Omega_1 + 1$, for all n. After this, $\Omega_0 < |\wp_j(v_j^{(n)})| < \Omega_1$ for all n. Let $\lambda = \lambda(Q, \epsilon)$ be the least parameter for which $|\wp_j(\lambda)| = \Omega_0$ and $\mu = \mu(Q, \epsilon)$ the greatest parameter for which $|\wp_j(\mu)| = \Omega_1$. Then $\lambda < v_j^{(n)} < \mu$ for all n, and, by extracting a further subsequence, I may assume that $v_j^{(n)} \to v_j \in [\lambda, \mu]$ as $n \to \infty$. Consequently $\wp_j(v_j^{(n)})$ converges strongly to $\wp_j(v_j)$.

By Lemma 7.5 and Proposition 7.3 (for the geodesics $\widetilde{p}_n$ in particular — for the rectilinear segments, C^1-weak convergence is merely weak convergence of the end-points), there is a further subsequence $(n(k))$ of the indices such that $(\widetilde{p}_{n(k)})$ and $(r_i^{n(k)})$ for $1 \leq i \leq 3$ all converge C^1-weakly. So does $(p_{n(k)}|[0, \Delta - \epsilon])$. As previously, for instance in Lemma 8.3, the C^1-weak limits may be concatenated, since the appropriate end-points coincide (as weak limits of the same sequences of points). Thus, in the C^1-weak limit, one obtains a path r_0 from $p_0(\Delta - \epsilon)$ to $\wp_j(v_j)$. By Proposition 7.6, the C^1-weak limit of $\widetilde{p}_{n(k)}$ has Riemannian length not greater than $\epsilon - \sqrt{\omega}$, whilst, for $1 \leq i \leq 3$,

$$\ell(\lim r_i^{n(k)}) \le L(\pi - \phi)/\log\log(e^e + Q^2). \tag{178}$$

Hence $\quad \ell(r_0) \le \epsilon + 3L(\pi - \phi)/\log\log(e^e + Q^2)$, and

$$\begin{aligned} d(p_0(\Delta - \epsilon), *) &\le \ell(r_0) + d(\wp_j(v_j), *) \\ &\le \epsilon + 3L(\pi - \phi)/\log\log(e^e + Q^2) + \ell(\wp_j|[\lambda, \beta')). \end{aligned} \tag{179}$$

Here Q is arbitrary, $\lambda \uparrow \beta'$ as $Q \uparrow \infty$, so that $d(p_0(\Delta - \epsilon), *) \le \epsilon$. (180)
But

$$\begin{aligned} \Delta = d(x, *) &\le d(x, p_0(\Delta - \epsilon)) + d(p_0(\Delta - \epsilon), *) \\ &\le d(x, p_0(\Delta - \epsilon)) + \epsilon, \end{aligned} \tag{181}$$

whence $d(x, p_0(\Delta - \epsilon)) \ge \Delta - \epsilon$. This is true for any $\epsilon \in (0, \Delta)$; comparing (174), one sees that $p_0 : [0, \Delta) \to M$ is a minimizing geodesic. By (180), it converges to $*$. This is contrary to the hypothesis that all geodesics are complete. □

Note. The last paragraph of the proof of Lemma 3.19 may be adapted to show that $*$ can be approximated arbitrarily closely by points at distance less than Δ from x, and (see Lemma 8.2) such points may be joined to x by minimizing geodesics. Each such geodesic, on extension to length Δ, must move away from $*$.

(8.6) The case remaining to be considered is when $d(x, y) = d(x, *) + d(*, y)$. Let $\Delta := d(x, *)$, $\Delta' := d(*, y)$, and suppose that (p_n) is a sequence of paths, parametrized by arc-length, between x and y, of lengths decreasing monotonically to $d(x, y)$. Let $v_n = p_n(t_n)$ be the last point on p_n such that $d(x, v_n) = \Delta$, and $w_n = p_n(t'_n)$ the first point such that $d(w_n, y) = \Delta'$. Then $\Delta \le t_n$, $t'_n \le \ell(p_n) - \Delta'$, and $t_n \le t'_n$, for otherwise $\ell(p_n) < \Delta + \Delta'$; and $t'_n - t_n \le \ell(p_n) - \Delta - \Delta' \to 0$. (182)

Lemma 8.7 *The paths p_n of (8.6) are uniformly bounded in H if and only if either of the sequences (v_n), (w_n) is bounded in H.*

Proof. Suppose that $|v_n| \le K$ for all n. Define ω by (166), and choose N so that $\ell(p_n) < \Delta + \Delta' + \frac{1}{4}\sqrt{\omega}$ for $n \ge N$. In that case, by Lemma 8.1, $|p_n(s)| \le K + 1$ for $t_n - \frac{1}{2}\sqrt{\omega} \le s \le t_n + \frac{1}{2}\sqrt{\omega}$. But, if $0 \le s \le t'_n$,

$$\begin{aligned} d(x, y) &\le d(x, p_n(s)) + d(p_n(s), p_n(t'_n)) + d(p_n(t'_n), y) \\ &\le d(x, p_n(s)) + t'_n - s + \Delta' \end{aligned}$$

$$\leq \ell(p_n) < \Delta + \Delta' + \frac{1}{4}\sqrt{\omega}, \tag{183}$$

so that $d(x, p_n(s)) \leq \Delta - \frac{1}{4}\sqrt{\omega}$ when $0 \leq s \leq t'_n - \frac{1}{2}\sqrt{\omega}$. (This shows that $t'_n - \frac{1}{2}\sqrt{\omega} < t_n$ necessarily). Similarly, if $s \geq t_n$,

$$\begin{aligned} d(x,y) &\leq d(x, p_n(t_n)) + d(p_n(t_n), p_n(s)) + d(p_n(s), y) \\ &\leq \Delta + s - t_n + d(p_n(s), y) \\ &\leq \ell(p_n) < \Delta + \Delta' + \frac{1}{4}\sqrt{\omega}, \end{aligned} \tag{184}$$

so that $d(p_n(s), y) \leq \Delta' - \frac{1}{4}\sqrt{\omega}$ when $t_n + \frac{1}{2}\sqrt{\omega} \leq s \leq \ell(p_n)$. For $0 \leq s \leq \ell(p_n)$, either $0 \leq s \leq t_n - \frac{1}{2}\sqrt{\omega} \leq t'_n - \frac{1}{2}\sqrt{\omega}$ or $t_n - \frac{1}{2}\sqrt{\omega} \leq s \leq t_n + \frac{1}{2}\sqrt{\omega}$ or $t_n + \frac{1}{2}\sqrt{\omega} \leq s \leq \ell(p_n)$, so that

$$|p_n(s)| \leq \max\Big(R\Big(x, \Delta - \frac{1}{4}\sqrt{\omega}\Big), K+1, R\Big(y, \Delta' - \frac{1}{4}\sqrt{\omega}\Big)\Big). \tag{185}$$

This holds for all $n \geq N$ and all s, which clearly suffices. There is a symmetrical argument, with t_n and t'_n interchanged, if (w_n) is bounded. □

Proposition 8.8 *If M is geodesically complete, it is geodesically convex. Moreover, for any $x, y \in M$, there exist K, $\epsilon > 0$ such that any path between x and y of length less than $d(x,y) + \epsilon$ is norm-bounded by K.*

Proof. After Lemma 8.3, the only case of interest is when $d(x,y) = d(x,*) + d(*,y)$. Given any sequence (p_n) of paths from x to y with lengths decreasing to $d(x,y)$, reparametrize by arc-length and use the conventions of (8.6). By Proposition 8.5, the sequence (v_n) is bounded; by Lemma 8.7, the paths (p_n) are uniformly bounded. This clearly proves the second sentence of the Proposition. Now choose a specific sequence (p_n).

Define $\omega := \Phi(\zeta^2(K+1)^2)^{-1}\exp(-\gamma^2(K+1)^2)$, as at (166), and restrict attention to indices n which are so large that $\ell(p_n) < d(x,y) + \frac{1}{2}\sqrt{\omega}$. Since

$$\begin{aligned} \Delta + \Delta' + \frac{1}{2}\sqrt{\omega} > \ell(p_n) &\geq d(x, v_n) + d(v_n, w_n) + d(w_n, y) \\ &= \Delta + d(v_n, w_n) + \Delta', \end{aligned} \tag{186}$$

then $d(v_n, w_n) < \frac{1}{2}\sqrt{\omega}$. But, as $|v_n| \leq K$, Lemma 8.1 and Corollary 3.3(i) show $d(v_n, *) \geq \sqrt{\omega}$. From Lemma 8.2, there is a minimizing geodesic $\gamma_n : [0,1] \to M$ joining v_n to w_n, and Corollary 8.4 gives minimizing geodesics

$q_n, r_n : [0,1] \to M$ between x and v_n, w_n and y in turn. Together they form a path of length not greater than $\ell(p_n)$, which is therefore norm-bounded by K for all the n under consideration. By Lemma 7.5 and Proposition 7.3, there is a subsequence $(n(k))$ of the indices such that $(q_{n(k)})$, $(\gamma_{n(k)})$, and $(r_{n(k)})$ all converge C^1-weakly to limits q_0, γ_0 and r_0. As previously (in Lemma 8.2, 8.3, and Proposition 8.5), these paths may be concatenated to give a path between x and y. From Proposition 7.6, and (182) of (8.6),

$$\ell(\gamma_0) \le \limsup \ell(\gamma_{n(k)}) \le \limsup(t'_n - t_n) = 0, \tag{187}$$

since $\ell(\gamma_{n(k)}) = d(v_n, w_n) \le t'_n - t_n \to 0$. Thus γ_0 is constant, and q_0, r_0 may be immediately concatenated. The length of the resulting path is

$$\begin{aligned} \ell(q_0) + \ell(r_0) &\le \limsup_{k\to\infty} \ell(q_{n(k)}) + \limsup_{k\to\infty} \ell(r_{n(k)}) \\ &= \Delta + \Delta' = d(x,y), \end{aligned} \tag{188}$$

by (7.6), so that it is minimizing, and may be reparametrized, in fact piecewise-linearly, to become a minimizing geodesic from x to y. □

9. Concluding remarks

(9.1) *Proof of Theorem* A. Construct operators A_1, S_1, C_1, Z_1 in a separable Hilbert space H_1 to satisfy the conditions of (3.1) and of (4.7), and so that C_1 and Z_1 are compact and the manifold M_1 (defined as at (3.1), but with subscripts) is metrically incomplete. This is possible, by Proposition 5.21. Take $A := A_1 \times A_1$ in $H := H_1 \times H_1$, and so on, as at (3.14), and define M exactly as at (3.1). By Proposition 3.15, M is metrically incomplete and ∂M is a singleton. But A, $S := S_1 \times S_1$, C, Z also satisfy the requirements of (3.1) and (4.7), so that M is geodesically complete by Theorem 4.10, and C and Z are compact, so that M is geodesically convex by Proposition 8.8.

(9.2) *Notes.* Separability of H is quite inessential; one may take orthogonal sums of the operators of §5 with 0 to obtain examples in any infinite-dimensional Hilbert space. Also, the minimizing geodesics between x and y are not in general unique up to parametrization. Choose $a \in M$ such that $d(a,*) < d(a,0)$, and suppose there is a unique minimizing geodesic $p(t)$, $-1 \le t \le 1$, from a to $Ja = -a$. Then $p(t)$ agrees with $-p(-t)$, and therefore $p(0) = 0$. But this means the length of p cannot be less than

$d(a,0)+d(0,Ja)=2d(a,0)>d(a,*)+d(*,Ja)\geq d(a,Ja)$, so that p is not after all minimizing.

(9.3) The choice of A, C, Z may obviously be varied, and both the definition of M at (3.1) and the conditions (4.7) may be modified. Nevertheless, there is a sense in which my construction of M is perhaps the simplest of its type.

One must prove geodesic completeness without metric completeness. For this, conformal modifications of the flat structure on H are convenient, because the geodesic equation can then be put in the transparent form (56), (57) (which is familiar in the geometrical optics of inhomogeneous media). The easiest non-trivial conformal multiplier is perhaps $f(x):=\langle Ax,x\rangle+\exp(-\langle Cx,x\rangle)$, for a suitable choice of A and C, and assumptions like (4.7) (specifically, that $\eta_0 C\leq[A,S]$) yield a proof of geodesic completeness. But M is then forced to be metrically complete.

Let p be of finite length, and set $r(s):=|p(s)|$. Then there exists $\kappa>0$ with

$$\left.\begin{aligned}&\langle Cp(s),p(s)\rangle\leq 2\eta_0^{-1}\langle Ap(s),Sp(s)\rangle\leq 2\kappa|A^{1/2}p(s)|r(s),\\&\text{and so}\\&2(f(s))^{1/2}\geq\sqrt{\langle Ap(s),p(s)\rangle}+\exp(-\kappa r(s)\sqrt{\langle Ap(s),p(s)\rangle})\\&\qquad\qquad\geq(\kappa r(s))^{-1}.\end{aligned}\right\}\tag{189}$$

If p is unbounded in H, its finite length entails (as in (3.5)) the convergence of $\int_1^\infty f^{1/2}\,dr$ and therefore of $\int_1^\infty \frac{dr}{r}$. This contradiction means that p is bounded, and, as f is bounded away from 0 on bounded sets, p must therefore converge in H.

This argument involves only the radial component of the motion of p, and may be overcome by introducing into f a denominator $(1+\langle x,x\rangle)^\epsilon$ for some $\epsilon>0$. For $\epsilon>1$, metric incompleteness is trivial, but geodesic incompleteness is also to be expected (the ray tx, for $t\geq 0$, is a geodesic if $Ax=0$, which implies $Cx=0$ by (189)). Therefore the obvious choice of ϵ is 1. But metric incompleteness still presents difficulties, although they may be due only to my unskilled constructions in §5.

For a path p which tends to infinity to have finite length, one needs that "on the whole" $\langle Ap(s),p(s)\rangle\to 0$ and $\langle Cp(s),p(s)\rangle\to\infty$. To arrange this in §5, $|p(s)|$ must increase so rapidly in comparison that the transverse component of the motion may make an infinite contribution to its length. Therefore I have multiplied the denominator of f by the further term $\log^2(e^e + \langle x, x\rangle)$, which is crucial in (5.19).

Finally, $\langle Zx, x\rangle$ is used in the denominator instead of $\langle x, x\rangle$ for the technical reasons given in the note to (7.6). For a geodesically complete but metrically incomplete Riemannian manifold, it suffices to take the identity for Z.

(9.4) It is noteworthy that the metric structure of M is more troublesome than the geodesic structure. It is less surprising, since the example must be essentially infinite-dimensional, that it is not robust. The conditions (4.7) do allow perturbations, but only very restricted ones. One might vaguely conjecture that there is some *interesting* sense (uninteresting statements are easily found) in which geodesic completeness "generically" implies metric completeness, but it is not clear how such a statement can be credibly and non-trivially formulated.

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