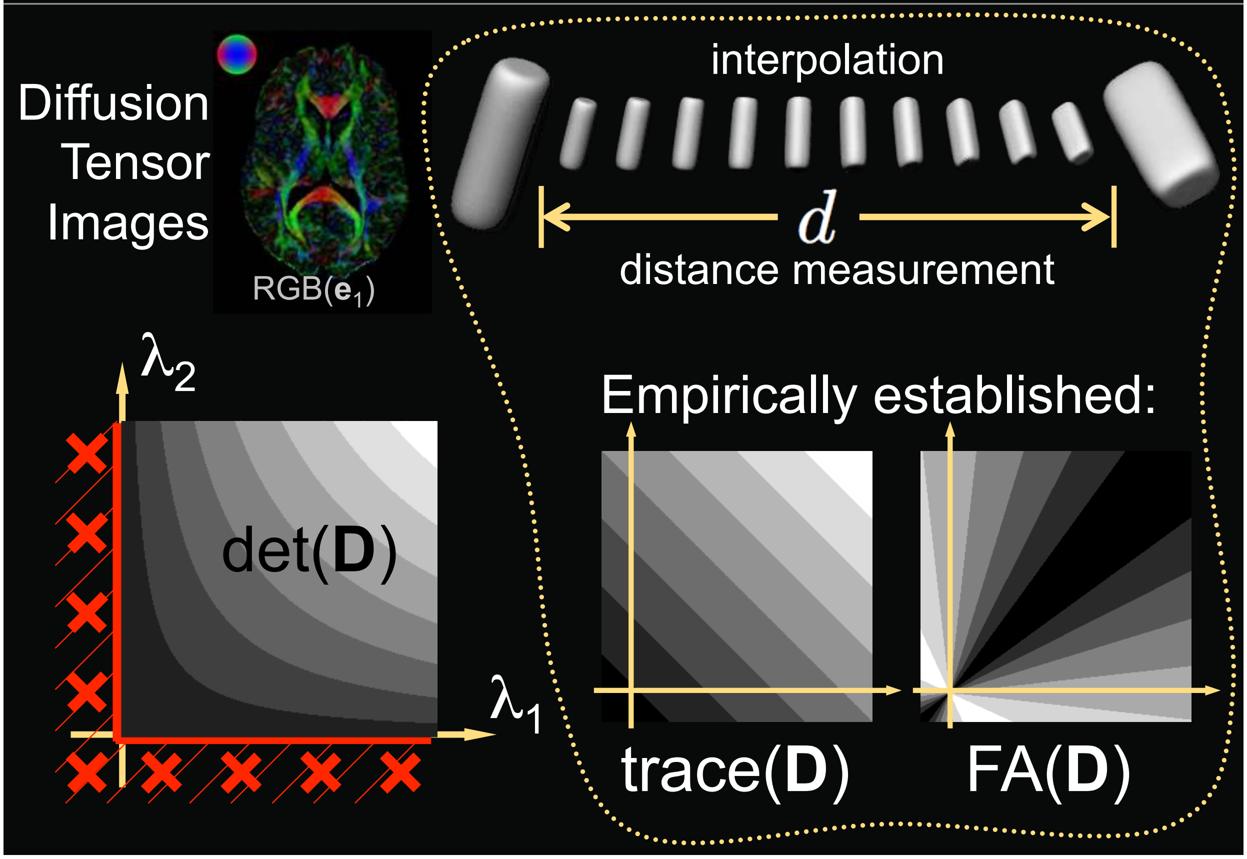
Geodesic-Loxodromes for Diffusion Tensor Interpolation and Difference Measurement

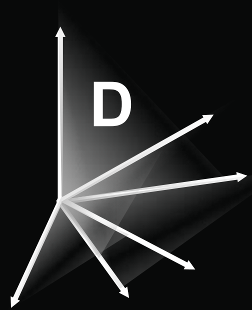
Gordon Kindlmann, Raúl San José Estépar,
Marc Niethammer, Steven Haker,
Carl-Fredrik Westin

Laboratory of Mathematics in Imaging
Department of Radiology
Brigham & Women's Hospital
Harvard Medical School

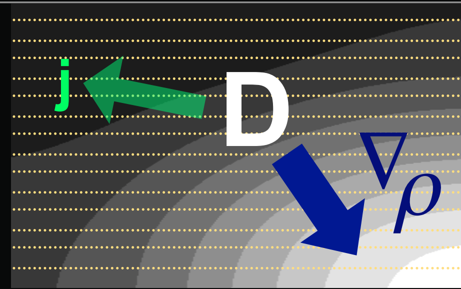
Introduction



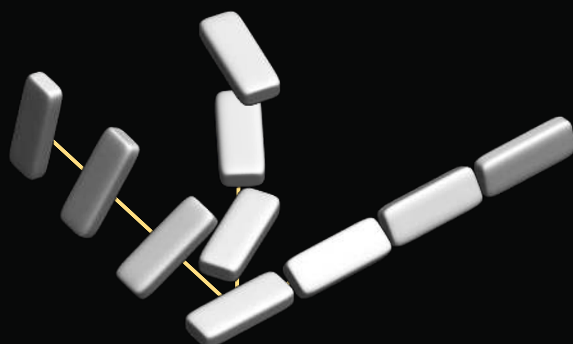
Introduction

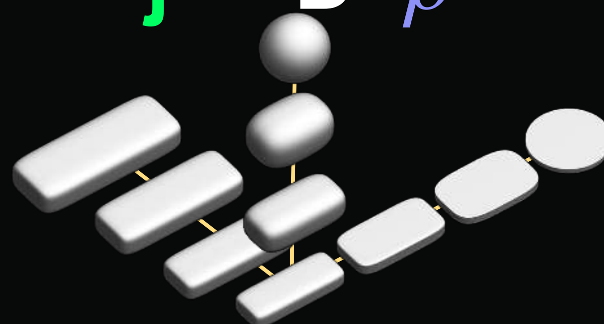


Diffusion tensors as elements of vector space



$$\mathbf{j} = -\mathbf{D}\nabla\rho$$





Orientation: Geodesic

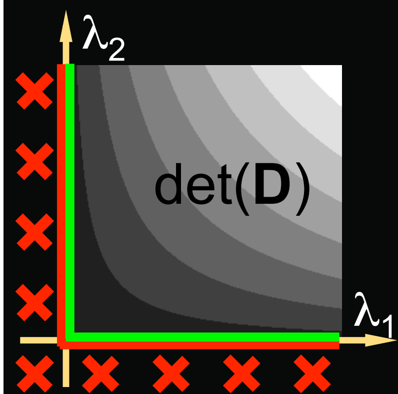
Shape: Loxodrome

Distance measures from path integrals

Rest of talk

- Previous work
- Tensor shape
- Loxodromes
- Geodesic-Loxodromes, Results
- Distance measures, Results
- Conclusions

Riemannian, Log-Euclidean approaches



Riemannian: Batchelor et al. 2005, Pennec et al. 2006, Fletcher et al. 2007, Lenglet et al. 2007

$$F(t,\mathbf{A},\mathbf{B}) = \mathbf{A}^{1/2}\,(\mathbf{A}^{-1/2}\,\mathbf{B}\,\mathbf{A}^{-1/2})^{t}\,\mathbf{A}^{1/2}$$

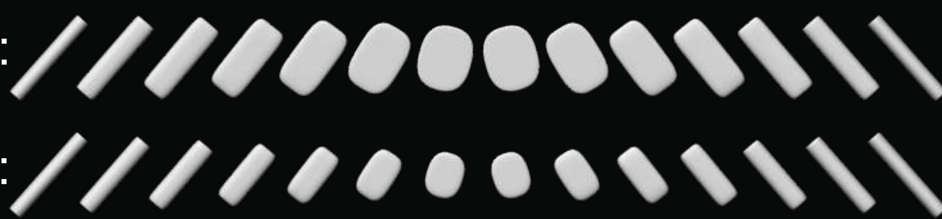
Log-Euclidean: Arsigny et al. 2006

$$F(t,\mathbf{A},\mathbf{B}) = \exp((1-t)\log(\mathbf{A}) + t\log(\mathbf{B}))$$

Both monotonically interpolate det(**D**); no “swelling”:

Linear:

Riemannian:

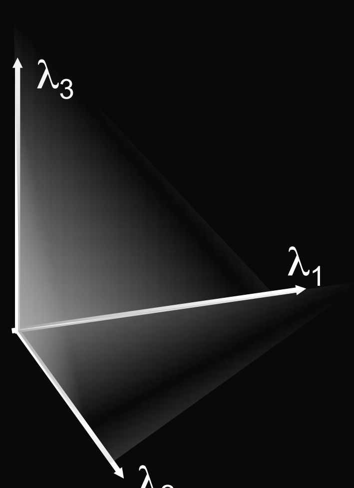


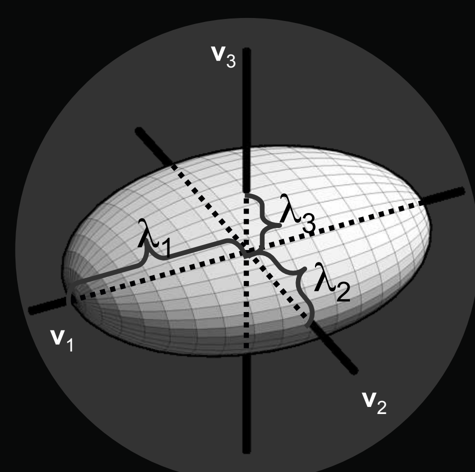
Determinant is only one possible size measure…

Space of Tensor Shape

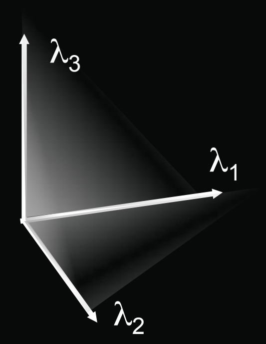
$$\mathbf{D} = \mathbf{R}\Lambda\mathbf{R}^{-1}$$

$$= \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \mathbf{v}_3 \end{bmatrix}$$





Parameterizing Shape with Invariants



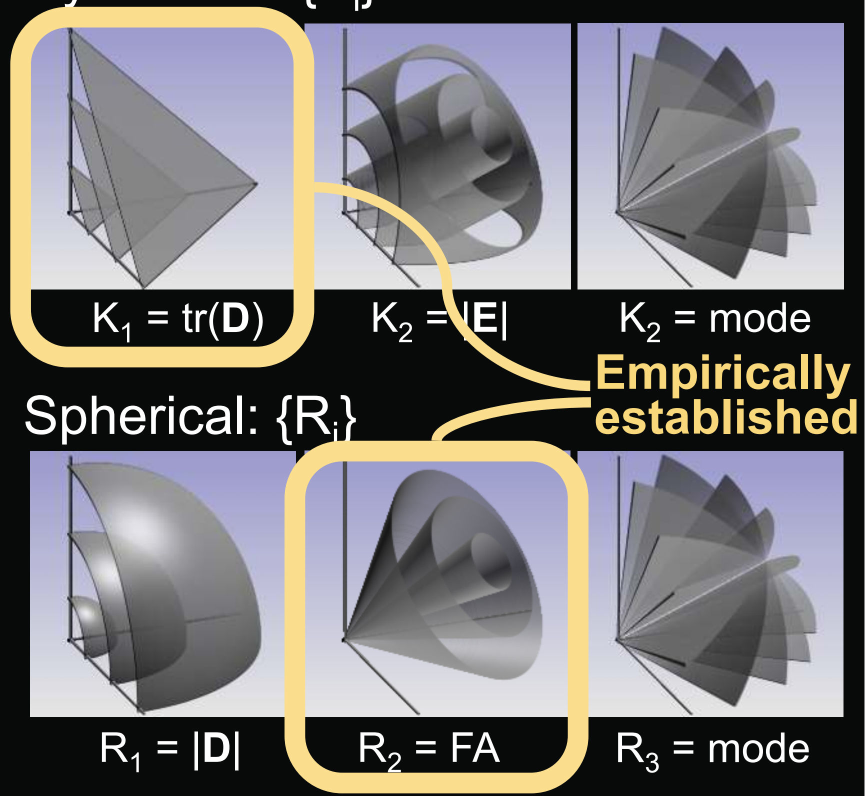
tr(**D**) = Dxx+Dyy+Dzz

|**D**| = sqrt(tr($\mathbf{D}^T\mathbf{D}$))

E = deviatoric(**D**)
 = **D** - tr(**D**)*$\mathbf{I}$/3

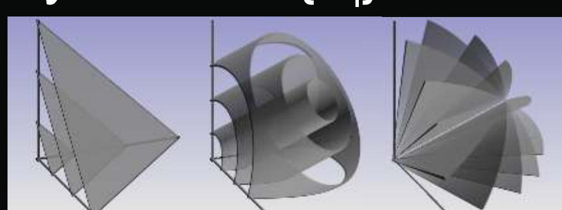
mode = det(**E**/|**E**|)

Cylindrical: {K_i}



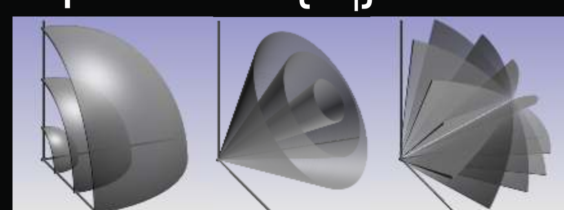
Orthogonal invariant sets

Cylindrical: {K_i}



$$\nabla K_i : \nabla K_j = 0 \text{ if } i \neq j$$

Spherical: {R_i}



$$\nabla R_i : \nabla R_j = 0 \text{ if } i \neq j$$

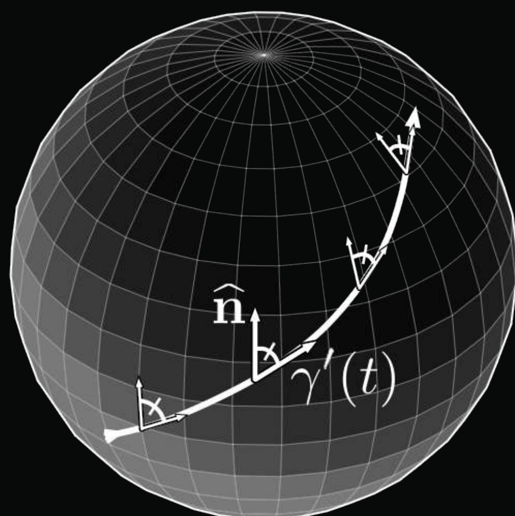
- {K_i} and {R_i} are **orthogonal** sets of shape parameters: their gradients are mutually orthogonal (Ennis & Kindlmann MRM 2006)
- {det(**D**),**?**,**?**} third invariant set?

$\Rightarrow$ Geodesic-Loxodromes monotonically interpolate {K_i} or {R_i}

Loxodrome defined

Path $\gamma(t)$ of constant speed and bearing $\Leftrightarrow$
Fixed angle with respect to North $\widehat{\mathbf{n}}(\mathbf{x})$

$$|\gamma'(t)| = 1 \ \text{ and } \ \gamma'(t) \cdot \widehat{\mathbf{n}}(\gamma(t)) = \alpha \ \text{ for all } t$$



Loxodromes monotonically interpolate **latitude**

Geodesic-Loxodrome defined, intuitively

Minimal-length path between two tensors that monotonically interpolates tensor shape

Geodesic-Loxodrome defined, mathematically

Geodesic-loxodrome $\gamma(t)$ between **A** and **B** is the *shortest* path satisfying:

$$\gamma(0) = \mathbf{A}, \quad \gamma(l) = \mathbf{B}, \ |\gamma'(t)| = 1, \quad \text{and}$$
$$\gamma'(t) \mathbin{:} \widehat{\boldsymbol{\nabla}} J_i(\gamma(t)) = \alpha_i \ \text{for all} \ t \in [0, l], i \in \{1, 2, 3\}$$

where $J_i = K_i$ or $J_i = R_i$ and $\widehat{\boldsymbol{\nabla}} J_i = \dfrac{\boldsymbol{\nabla} J_i}{|\boldsymbol{\nabla} J_i|}$

Constants α_i and l are path "bearing" and length

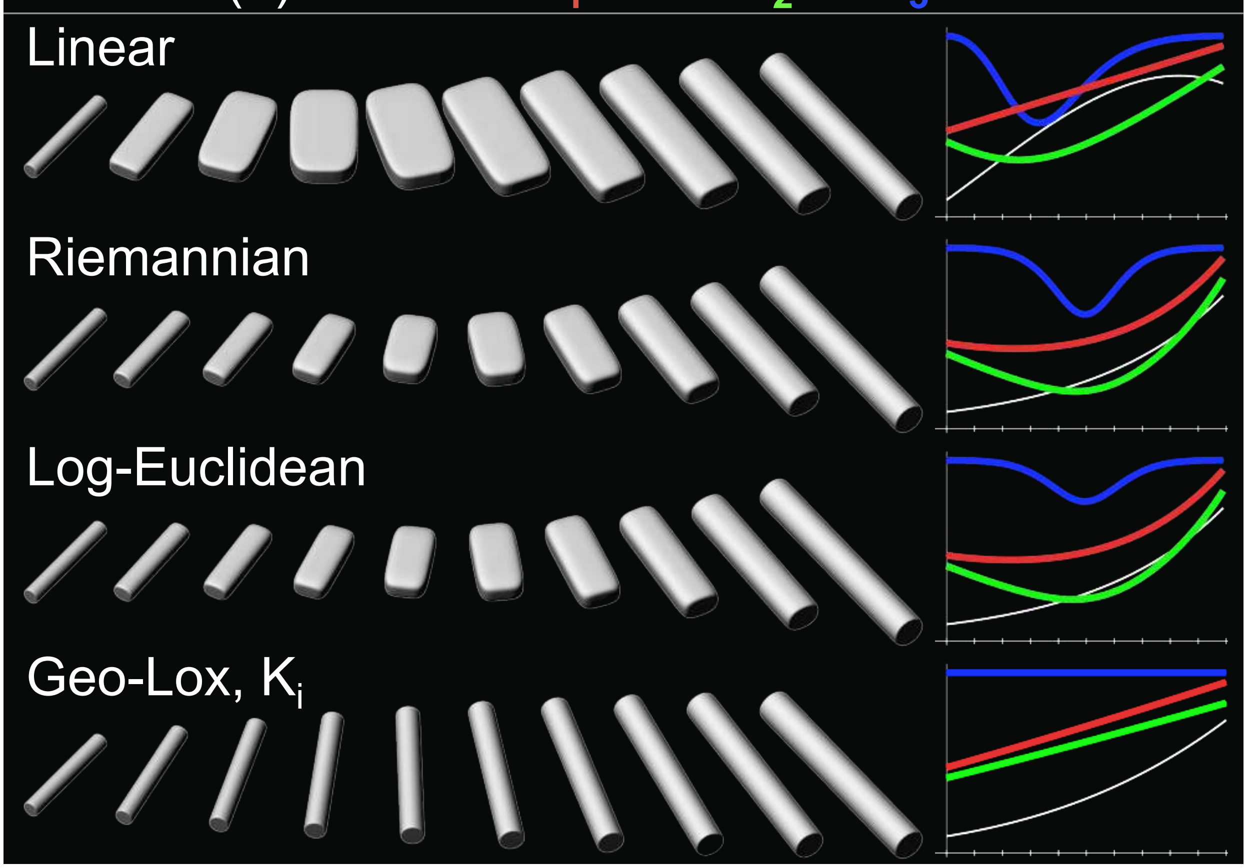
Monotonic: $\dfrac{d}{dt} J_i(\gamma(t)) = \gamma'(t) \mathbin{:} \boldsymbol{\nabla} J_i(\gamma(t)) = \alpha_i |\boldsymbol{\nabla} J_i(\gamma(t))|.$

$$J_i(\mathbf{A}) = J_i(\mathbf{B}) \Rightarrow J_i(\mathbf{A}) = J_i(\gamma(t)) = J_i(\mathbf{B}) \ \text{for all} \ t$$

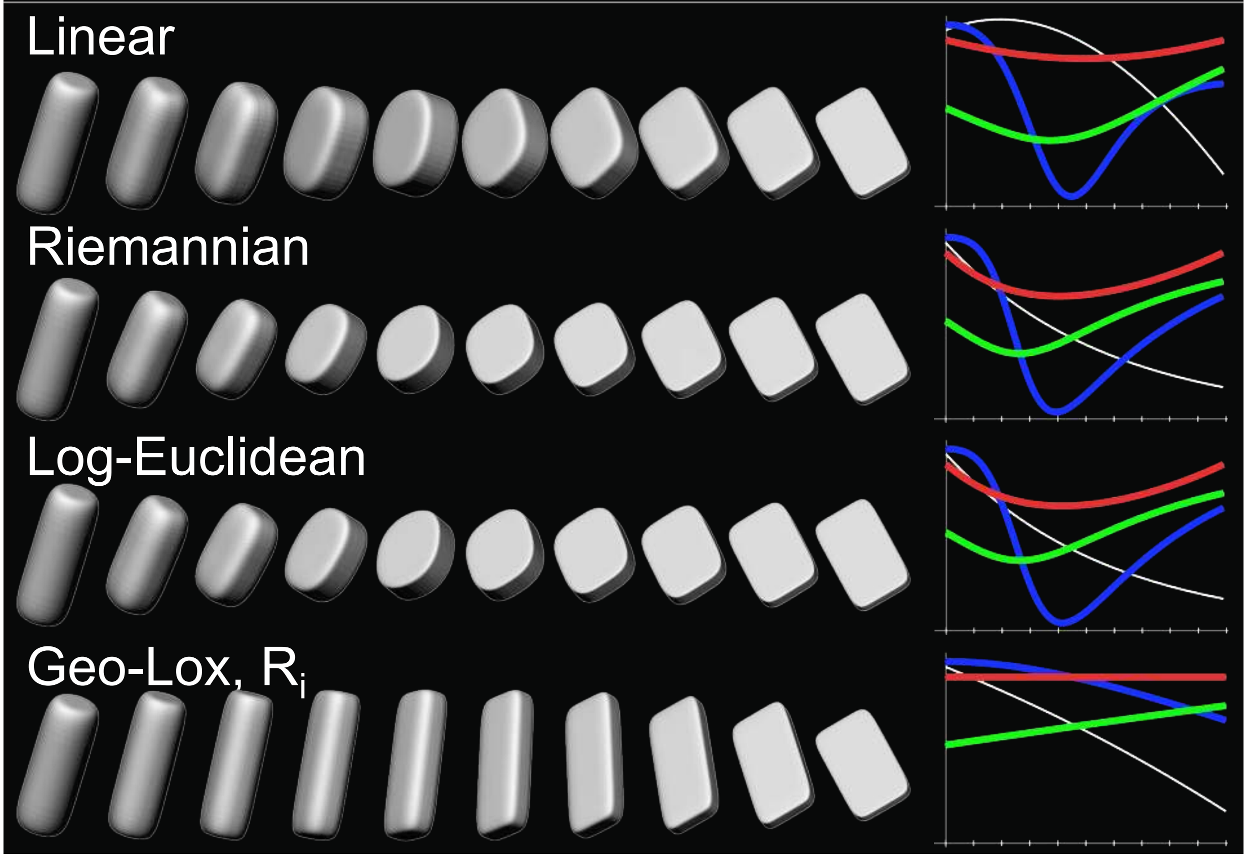
Intersection of isocontours = rotation orbit; geodesic

Results (1)

/ K_1=tr / K_2 / K_3=mode / det



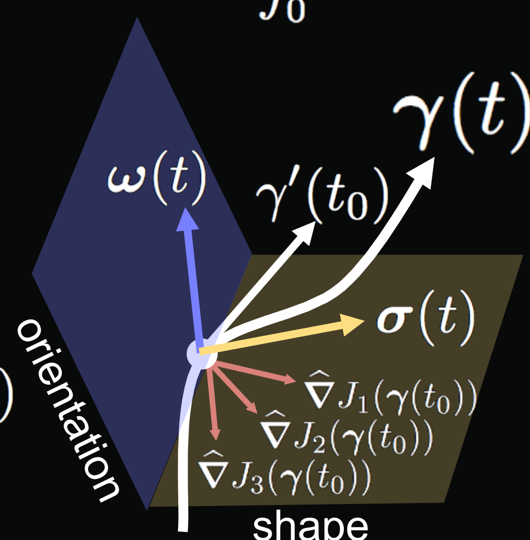
Results (2) /R_1=|D| /R_2=FA /R_3=mode /det



Distance measurement

Integrate tangent norm: $d(\mathbf{A},\mathbf{B}) = \int_0^l |\gamma'(t)|dt = l$

Or, exploit **shape** and **orientation** subspaces:



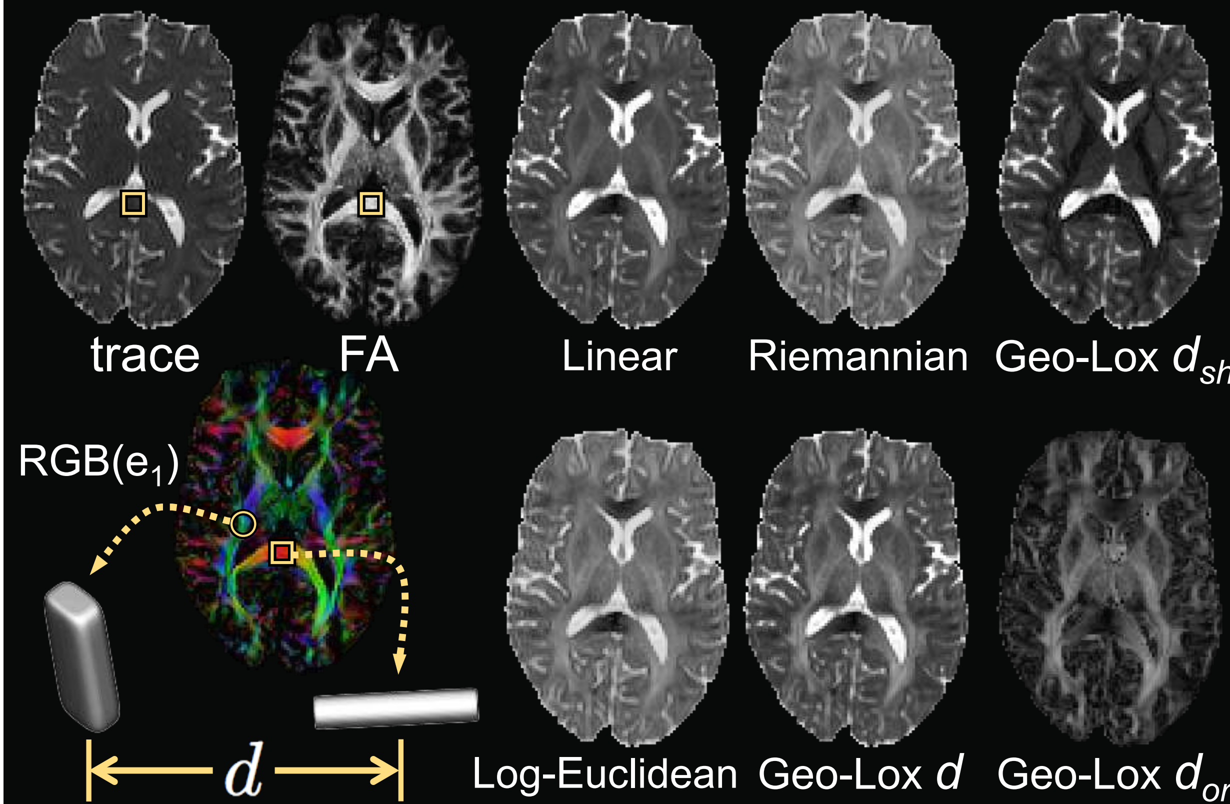
$$\boldsymbol{\omega}(t) = \boldsymbol{\gamma}'(t) - \boldsymbol{\sigma}(t)$$

$$d_{or}(\mathbf{A},\mathbf{B}) = \int_0^l |\boldsymbol{\omega}(t)|dt$$

$$\boldsymbol{\sigma}(t) = \sum_i \boldsymbol{\gamma}'(t) : \hat{\boldsymbol{\nabla}} J_i(\boldsymbol{\gamma}(t)) \hat{\boldsymbol{\nabla}} J_i(\boldsymbol{\gamma}(t))$$

$$d_{sh}(\mathbf{A},\mathbf{B}) = \int_0^l |\boldsymbol{\sigma}(t)|dt$$

Distance measurement, Results



Conclusions, Future Work

- Intuition about orientation and shape mapped to formulation of geodesic and loxodrome properties
- Structure math of interpolation around established tensor **parameters**, not just positive-definiteness
- Tensor Mode ($K_3 = R_3$) important for shape
- Working on fast approximations
- Need log(), exp() map analogs for multi-linear

Acknowledgements

- NIH funding: U41-RR019703, P41-RR13218, R01-MH050740, R01-MH074794.
- Data: Dr. Susumu Mori, Johns Hopkins University, NIH R01-AG-20012-01, P41-RR15241-01A1

gk@bwh.harvard.edu

http://lmi.bwh.harvard.edu/~gk

Thank you