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GEODESIC MAPPINGS BETWEEN KÄHLERIAN SPACES

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Abstract

Geodesic mappings from a Kählerian space K_n onto a Kählerian space $\bar{K}_n$ will be investigated in this paper. We present a construction of Kählerian space K_n which admits non-trivial geodesic mapping onto Kählerian space $\bar{K}_n$.

1 Introduction

Geodesic mappings of special Riemannian spaces were studied by many authors (see e.g. [9], [13], [15]). Geodesic mappings of Kählerian spaces were investigated namely by N. Coburn [1], K. Yano [17], V.J. Westlake [16], K. Yano, T. Nagano [18] and others. In these works the authors prove that in the case of preserving the structure of Kählerian spaces by geodesic mappings these mappings are trivial (i.e. affine). Koga Mitsuru [5] has found more general conditions for the structure of Kählerian spaces forcing any geodesic mapping to be trivial. Similar questions for geodesic mappings of almost Hermitian spaces were investigated by A. Karmazina and I.N. Kurbatova [3].

The geodesic mappings from a Kählerian space K_n onto a Riemannian space $\bar{V}_n$ were studied by J. Mikeš (see [6], [7], [8], [9]).

Papers of J. Mikeš, G. Starko, M. Shiha were devoted to geodesic mappings of hyperbolical and parabolical Kählerian spaces which are generalizations of classical Kählerian spaces (see [9], [11],[12]).

In the sequel, by Kählerian space we mean both classical (i.e. elliptical) as well as hyperbolical and parabolical Kählerian space.

In this paper, we investigate geodesic mappings from a Kählerian space K_n onto a Kählerian space $\bar{K}_n$. We present a construction of non-trivial Kählerian spaces K_n which are geodesically mapped onto Kählerian spaces $\bar{K}_n$.

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2 Geodesic mappings of Kählerian spaces

A diffeomorphism f from a Riemannian space V_n onto a Riemannian space $\bar{V}_n$ is called a *geodesic mapping* if f maps any geodesic line of V_n into a geodesic line of $\bar{V}_n$ (see [2], [9], [13], [15]).

A mapping from V_n onto $\bar{V}_n$ is geodesic if and only if, in the common coordinate system x with respect to the mapping the conditions

$$\bar{\Gamma}^h_{ij}(x) = \Gamma^h_{ij}(x) + \delta^h_i \psi_j + \delta^h_j \psi_i \tag{1}$$

hold, where $\psi_i(x)$ is a covector, Γ^h_{ij} and $\bar{\Gamma}^h_{ij}$ are the Christoffel's symbols of V_n and $\bar{V}_n$, respectively, δ^h_i is the Kronecker symbol.

Conditions (1) are equivalent to

$$\bar{g}_{ij,k} = 2\psi_k \bar{g}_{ij} + \psi_i \bar{g}_{jk} + \psi_j \bar{g}_{ik}, \tag{2}$$

where $\bar{g}_{ij}$ is the metric tensor of $\bar{V}_n$ and "," denotes the covariant derivative with respect to the connection of the space V_n.

Conditions (1) and (2) are called the *Levi-Civita equations*. The covector ψ_i is gradient-like, i.e. $\psi_i = \psi_{,i}$. If $\psi_i \not\equiv 0$ then a geodesic mapping is called *non-trivial*; otherwise it is said to be *trivial* or *affine*.

If a mapping $f\colon V_n \to \bar{V}_n$ is geodesic then the following conditions hold:

$$\begin{array}{lrcl} \text{a)} & \bar{R}^h_{ijk} & = & R^h_{ijk} + \delta^h_k \psi_{ij} - \delta^h_j \psi_{ik}, \\ \text{b)} & \bar{R}_{ij} & = & R_{ij} + (n+1)\psi_{ij}, \\ \text{c)} & \bar{W}^h_{ijk} & = & W^h_{ijk}, \end{array} \tag{3}$$

where

$$\psi_{ij} \equiv \psi_{i,j} - \psi_i \psi_j, \tag{4}$$

R^h_{ijk} ($\bar{R}^h_{ijk}$) are the Riemannian tensors ofV_n ($\bar{V}_n$), R_{ij} ($\bar{R}_{ij}$) are the Ricci tensors of V_n ($\bar{V}_n$), W^h_{ijk} ($\bar{W}^h_{ijk}$) are the Weyl tensors of the projective curvature of V_n ($\bar{V}_n$). The Weyl tensor of the projective curvature is an invariant object of the geodesic mapping.

In the present paper, by a Kählerian space we mean a wide class of spaces defined as follows [9]: a Riemannian space is called a *Kählerian space* K_n if, together with the metric tensor $g_{ij}(x)$, an affine structure $F^h_i(x)$ is defined on K_n which satisfies the relations

$$F^h_\alpha F^\alpha_i = e\,\delta^h_i; \quad F^\alpha_i g_{\alpha j} + F^\alpha_j g_{\alpha i} = 0; \quad F^h_{i,j} = 0, \tag{5}$$

where $e = \pm 1, 0$. If $e = -1$ then K_n is said to be an *elliptic Kählerian space* K_n^-, if $e = +1$ then K_n is said to be a *hyperbolic Kählerian space* K_n^+, and if $e = 0$ and $\mathrm{Rg}\|F_i^h\| = m \leq 2$ then K_n is said to be an *m-parabolic Kählerian space* $K_n^{o(m)}$. The space $K_n^{o(n/2)}$ is called the *parabolic Kählerian space* K_n^o.

As in K_n the structure F is covariantly constant, from [6] follows that on a Kählerian space K_n admitting a nontrivial geodesic mapping, there exists a nonzero convergent vector field (see [6], [7], [8], [9], [12], [11]).

In every space K_n^- with a covariantly nonconstant convergent vector field there exists a coordinate system x with the following metrics and structure (see [7], [8], [9]):

$$g_{ab} = g_{a+m\,b+m} = \partial_{ab} f + \partial_{a+m\,b+m} f; \quad g_{a\,b+m} = \partial_{a\,b+m} f - \partial_{a+m\,b} f;$$

$$F_b^{a+m} = -F_{b+m}^a = \delta_b^a; \quad F_b^a = F_{b+m}^{a+m} = 0,$$

where $a, b = \overline{1, m}$, $m = n/2$,

$$f = \exp(2x^1)\, G(x^2, x^3, ..., x^m, x^{2+m}, x^{3+m}, ..., x^n).$$

If $G \in C^3$, then these formulas generate (provided $|g_{ij}| \neq 0$) the metric of a Kählerian space K_n^-, where a non-constant convergent vector field exists.

A similar property holds (see [12]) for the hyperbolic Kählerian spaces K_n^+ with metrics and structure of the type

$$g_{ab} = g_{a+m\,b+m} = 0; \quad g_{a\,b+m} = \partial_{a\,b+m} f;$$

$$F_b^{a+m} = F_{b+m}^a = 0; \quad F_b^a = -F_{b+m}^{a+m} = \delta_b^a,$$

where $a, b = \overline{1, m}$, $m = n/2$,

$$f = \exp(x^1 + x^{1+m})\, G(x^2 + x^{2+m}, x^3 + x^{3+m}, ..., x^m + x^n).$$

Metrics of parabolically Kählerian spaces K_n^o, admitting covariantly nonconstant convergent vector fields were found by J. Mikeš and M. Shiha [11].

3 Geodesic mappings onto Kählerian spaces

In this section, we determine conditions which are necessary and sufficient for a Riemannian space V_n to admit a nontrivial geodesic mapping onto a Kählerian space $\bar{K}_n$ satisfying the formulas (5). The following theorem holds:

Theorem 1 *The Riemannian space V_n admits a nontrivial geodesic mapping onto a Kählerian space $\bar{K}_n$ if and only if, in the common coordinate system x with respect to the mapping, the conditions*

$$\begin{array}{lrcl} a) & \bar{g}_{ij,k} & = & 2\psi_k \bar{g}_{ij} + \psi_i \bar{g}_{jk} + \psi_j \bar{g}_{ik}, \\ b) & \bar{F}^h_{i,k} & = & \bar{F}^h_k \psi_i - \delta^h_k \bar{F}^\alpha_i \psi_\alpha \end{array} \tag{6}$$

hold, where $\psi_i \not\equiv 0$ and tensors $\bar{g}_{ij}$ and $\bar{F}^h_i$ satisfy the following conditions:

$$\bar{g}_{ij} = \bar{g}_{ji}, \quad \det\|\bar{g}_{ij}\| \neq 0, \quad \bar{F}^h_\alpha \bar{F}^\alpha_i = \bar{e}\delta^h_i, \quad \bar{F}^\alpha_i \bar{g}_{\alpha j} + \bar{F}^\alpha_j \bar{g}_{\alpha i} = 0. \tag{7}$$

Then $\bar{g}_{ij}$ and $\bar{F}^h_i$ are the metric tensor and the structure of $\bar{K}_n$, respectively.

Proof. The Levi-Civita equation (6a)≡(2) guarantees the existence of geodesic mappings from a Riemannian space V_n onto a Riemannian space $\bar{V}_n$ with metric tensor $\bar{g}_{ij}$.

The formula (6b) implies that the structure $\bar{F}^h_i$ in $\bar{V}_n$ is covariantly constant. Further, the algebraic conditions (7) guarantee that $\bar{g}_{ij}$ and $\bar{F}^h_i$ are the metric tensor and the structure of the same Kählerian space $\bar{K}_n$, respectively.

The system (6) is a system of partial differential equations with respect to the unknown functions $\bar{g}_{ij}(x)$, $\bar{F}^h_i(x)$ and $\psi_i(x)$ which moreover must satisfy algebraic conditions (7).

4 Geodesic mappings between Kählerian spaces

As was said in the introduction, a geodesic mapping between Kählerian spaces K_n and $\bar{K}_n$ which preserves the structure (i.e. in the common coordinate system x with respect to the mapping the conditions $\bar{F}^h_i(x) = F^h_i(x)$ hold, where F^h_i and $\bar{F}^h_i$ are structures of K_n and $\bar{K}_n$, respectively) is trivial (i.e. affine).

Since the structures F^h_i and $\bar{F}^h_i$ are covariantly constant in K_n and $\bar{K}_n$, respectively, we can deduce from the results of [6], [9] that for the tensor ψ_{ij} under a geodesic mapping K_n onto $\bar{K}_n$ the relation $\psi_{ij} = 0$ holds, i.e.

$$\psi_{i,j} = \psi_i \psi_j. \tag{8}$$

It follows from the relations (3) and (8) that the Riemannian tensor for a geodesic mapping of K_n onto $\bar{K}_n$ is invariant.

We shall construct a Kählerian space K_n admitting a nontrivial geodesic mapping; of course, the structure of K_n is not preserved.

Obviously, the existence of a nontrivial geodesic map between (pseudo-) euclidean spaces E_n and $\bar{E}_n$ follows from the Beltrami theorem. On the other hand, under some specific conditions on the dimension and the signature of metrics, the spaces E_n and $\bar{E}_n$ are Kählerian spaces K_n and $\bar{K}_n$ in our sense.

For example, E_{2m} is K^-_{2m} and K^+_{2m}, too, where

$$g=(I);\quad F=\begin{pmatrix}0 & I\\ -I & 0\end{pmatrix}\qquad\text{and}\qquad g=\begin{pmatrix}0 & I\\ I & 0\end{pmatrix};\quad F=\begin{pmatrix}I & 0\\ 0 & -I\end{pmatrix}$$

hold, respectively.

We now construct a nontrivial example of a geodesic mapping between Kählerian spaces.

Let K_n be a product of Riemannian spaces with the metric

$$ds^2 = d\tilde{s}^2 + d\tilde{\tilde{s}}^2, \tag{9}$$

where $d\tilde{s}^2$ is the metric of the euclidean Kählerian space $\tilde{K}_{\tilde{n}}$ with the metric tensor $\tilde{g}_{ab}$ and the structure $\tilde{F}^a_b$, $(a,b,c=1,2,...,\tilde{n})$;

$d\tilde{\tilde{s}}^2$ is the metric of a Kählerian space $\tilde{\tilde{K}}_{\tilde{\tilde{n}}}$ with the metric tensor $\tilde{\tilde{g}}_{AB}$ and the structure $\tilde{\tilde{F}}^A_B$, $(A,B,C=\tilde{n}{+}1,\ \dots,\tilde{n}{+}\tilde{\tilde{n}})$, and such that a noncovariantly constant concircular vector field $\tilde{\tilde{\xi}}^h$ exists on K_n.

This space is a Kählerian space, and

$$g=\begin{pmatrix}\tilde{g} & 0\\ 0 & \tilde{\tilde{g}}\end{pmatrix}\qquad\text{and}\qquad F=\begin{pmatrix}\tilde{F} & 0\\ 0 & \tilde{\tilde{F}}\end{pmatrix}$$

are its metrics and structure, respectively.

The spaces $\tilde{K}_{\tilde{n}}$ and $\tilde{\tilde{K}}_{\tilde{\tilde{n}}}$ must be of the same type, i.e. both of them must be either elliptic or hyperbolic or parabolic.

We prove the following result.

Theorem 2 *The Kählerian space K_n, constructed above, admits a nontrivial geodesic mapping onto a Kählerian space $\bar{K}_n$.*

Proof. In the space $\tilde{K}_{\tilde{n}}$ we shall investigate the equations (analogical to (6)):

$$\begin{array}{rcl}\tilde{q}_{ab\wr c} &=& 2\tilde{\psi}_c\tilde{q}_{ab}+\tilde{\psi}_a\tilde{q}_{\underline{bc}}+\tilde{\psi}_b\tilde{q}_{ac},\\ \tilde{B}^a_{b\wr c} &=& \tilde{B}^a_c\tilde{\psi}_b-\delta^a_c\tilde{B}^d_b\tilde{\psi}_d,\\ \tilde{\psi}_{a\wr b} &=& \tilde{\psi}_a\tilde{\psi}_b,\quad \tilde{\psi}_a=\tilde{\psi}_{\wr a}\neq 0,\end{array}\tag{10}$$

where "$\wr$" is the covariant derivative in $\tilde{K}_{\tilde{n}}$; $\tilde{q}_{ab}$, $\tilde{B}^a_b$, $\tilde{\psi}_a$ are some tensors satisfying the algebraic conditions

$$\tilde{B}^a_c\tilde{B}^c_b = e\delta^a_b, \quad \tilde{B}^c_a\tilde{q}_{cb} + \tilde{B}^c_b\tilde{q}_{ca} = 0, \quad \tilde{q}_{ab} = \tilde{q}_{ba}, \quad |\tilde{q}_{ab}| \neq 0. \tag{11}$$

The solution of the equations (10) satisfying (11) exists, because the equations (10) are completely integrable in the euclidean space $\tilde{K}_{\tilde{n}}$.

On the other hand, since there exists a noncovariantly constant concircular vector field in $\tilde{\tilde{K}}_{\tilde{\tilde{n}}}$, we can find a function $\tilde{\tilde{\xi}}$ satisfying the conditions

$$2\tilde{\tilde{\xi}} = \tilde{\tilde{\xi}}^A\tilde{\tilde{\xi}}_A, \quad \tilde{\tilde{\xi}}^A_{\wr\wr B} = \delta^A_B, \tag{12}$$

where $2\tilde{\tilde{\xi}}^A \equiv \tilde{\tilde{\xi}}_B\tilde{\tilde{g}}^{AB}$, $\tilde{\tilde{\xi}}_A \equiv \tilde{\tilde{\xi}}_{\wr\wr A}$, $\|\tilde{\tilde{g}}^{AB}\| = \|\tilde{\tilde{g}}_{AB}\|^{-1}$ and "$\wr\wr$" denotes the covariant derivative of $\tilde{\tilde{K}}_{\tilde{\tilde{n}}}$.

We put

$$\begin{aligned}
\bar{g}_{ab} &= 2\,k\exp(2\bar{\psi})\,\tilde{\tilde{\xi}}\,\bar{\psi}_a\bar{\psi}_b + \tilde{q}_{ab},\\
\bar{g}_{aB} &= k\exp(2\bar{\psi})\,\tilde{\tilde{\xi}}_B\bar{\psi}_a,\\
\bar{g}_{AB} &= k\exp(2\bar{\psi})\,\tilde{\tilde{g}}_{AB},\\
\bar{F}^a_b &= \tilde{B}^a_b,\\
\bar{F}^a_B &= 0,\\
\bar{F}^A_B &= \tilde{\tilde{F}}^A_B,\\
\bar{F}^A_b &= \tilde{\tilde{F}}^A_B\tilde{\tilde{\xi}}\bar{\psi}_b - \tilde{\tilde{\xi}}^A\tilde{B}^c_b\tilde{\xi}_c,
\end{aligned}$$

where k is a constant such that $|g_{ij}| \neq 0$.

Putting $\psi = \tilde{\psi}$, we can verify the formulas (6) and (7). Hence the tensors $\bar{g}_{ij}$ and $\bar{F}^h_i$ constructed by Theorem 1 are the metric and structure tensors of the Kählerian space $\bar{K}_n$, respectively, and $\bar{K}_n$ is a geodesic image of K_n.

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