

GEODESICS AND BOUNDED HARMONIC FUNCTIONS ON INFINITE PLANAR GRAPHS

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(Communicated by William D. Sudderth)

ABSTRACT. It is shown there that an infinite connected planar graph with a uniform upper bound on vertex degree and rapidly decreasing Green's function (relative to the simple random walk) has infinitely many pairwise finitely-intersecting geodesic rays starting at each vertex. We then demonstrate the existence of nonconstant bounded harmonic functions on the graph.

Let \mathfrak{g} be an infinite, simple, connected, planar graph. \mathfrak{g} also denotes the vertex set of the graph. If two vertices x and y are connected by an edge, we write xEy . For a vertex x , the degree of x is $d(x) \equiv |\{y \in \mathfrak{g} : yEx\}|$, and we assume:

$$(1) \quad \delta \equiv \sup_{x \in \mathfrak{g}} d(x) < \infty.$$

A finite [infinite] walk γ is a sequence $(\gamma(0), \dots, \gamma(n))$ $[(\gamma(0), \gamma(1), \dots)]$ of elements of \mathfrak{g} such that $\gamma(k)E\gamma(k+1)$ for all $0 \leq k \leq n-1$ [for all $k \geq 0$]. We say that γ starts at $\gamma(0)$ and, in the first case, ends at $\gamma(n)$ and has length n . Since \mathfrak{g} is connected, we may define a metric:

$$d(x, y) \equiv \inf\{n : n \text{ is the length of a finite walk from } x \text{ to } y\}.$$

A path is a walk whose vertices are distinct. A geodesic γ is a path such that $d(\gamma(m), \gamma(n)) = |m - n|$ for all possible m and n . For $x \in \mathfrak{g}$, $\Gamma(x, n)$ is the set of geodesics that have length n and start at x ; $\Gamma(x)$ is the set of geodesics that have infinite length and start at x .

The following propositions are useful; the first is easy to prove by a diagonal type argument.

Proposition 1. For all $x \in \mathfrak{g}$, $\Gamma(x) \neq \emptyset$.

Proposition 2. Given $x, y \in \mathfrak{g}$ and $\gamma \in \Gamma(x)$, there exists a $\gamma' \in \Gamma(y)$ such that γ and γ' eventually coincide.

Received by the editors October 10, 1989 and, in revised form, September 26, 1990.
1980 *Mathematics Subject Classification* (1985 Revision). Primary 60J45; Secondary 60J15, 05C38.

Key words and phrases. Random walk, planar graphs, geodesic rays, harmonic functions.

During this research, the author was supported by a graduate fellowship at the University of Rochester.

Proof. Let $x, y \in \mathfrak{g}$, $\gamma \in \Gamma(x)$. By the triangle inequality, $|d(y, \gamma(n)) - n| = |d(y, \gamma(n)) - d(x, \gamma(n))| \leq d(x, y)$ and, since $d(y, \gamma(n)) - n$ is nonincreasing, $a \equiv \lim_{n \rightarrow \infty} [d(y, \gamma(n)) - n] = d(y, \gamma(N)) - N$ for some N . Define a path γ' where $(\gamma'(0), \dots, \gamma'(d(y, \gamma(N))))$ is a finite geodesic from y to $\gamma(N)$ and, for $k \geq d(y, \gamma(N))$, $\gamma'(k) = \gamma(k - a)$. Then $\gamma' \in \Gamma(y)$. \square

Consider the transition probabilities for a Markov chain defined by:

$$p(x, y) \equiv \begin{cases} 1/d(x) & \text{if } yEx, \\ 0 & \text{otherwise.} \end{cases}$$

We denote this chain by $X(0), X(1), \dots$. We let $P^x(\cdot) \equiv P(\cdot | X(0) = x)$ and $E^x(\cdot)$ be the associated expectation operator. Hence, $p(x, y) = P(X(1) = y | X(0) = x) = P^x(X(1) = y)$. $X(\cdot)$ is called the simple random walk on \mathfrak{g} .

Let $p^{(n)}(x, y)$ be the n -fold convolution of p with itself, and define Green's function as $G(x, y) = \sum_{n \geq 0} p^{(n)}(x, y)$. Probabilistically, $p^{(n)}(x, y) = P^x(X(n) = y)$ and $G(x, y) = E^x(\sum_{n \geq 0} \chi_{\{y\}}(X(n)))$ = the average number of times that the random walk, starting at x , hits y . It is easy to see that the random walk is transient if and only if G exists (see [2]; his proof for the case when \mathfrak{g} is a tree applies to our case without change). By the strong Markov property,

$$(2) \quad G(x, y) = P^x(\exists n \geq 0: X(n) = y)G(y, y).$$

We assume that Green's function is rapidly decreasing in the sense that

$$(3) \quad \sum_{n \geq 0} n \cdot \sup\{G(x, y): x, y \in \mathfrak{g}, d(x, y) = n\} < \infty.$$

Remark. It is known that the Cheeger condition

$$\exists c > 0: \forall \text{ finite } K \subset \mathfrak{g}: \#\{\text{edges from } K \text{ to } K^c\} / |K| \geq c$$

implies $G(x, y) \leq c\epsilon^{d(x, y)}$ (for some c and ϵ)—see [1] or [4]. Hence the Cheeger condition implies condition (3).

Lemma 1. *For any integer $m \geq 0$, there is an $N(m) \geq 0$ such that if A is the union of m geodesics and $d(x, A) \geq N(m)$, then $P^x(\exists n: X(n) \in A) < 1$.*

Proof. For any $n \geq 0$, $x \in \mathfrak{g}$, let $S(x, n)$ and $B(x, n)$ be the metric sphere and ball respectively with centers x and radii n . If γ is a geodesic, then $|\gamma \cap S(x, n)| \leq |\gamma \cap B(x, n)| \leq 2n + 1$. Hence $|A \cap S(x, n)| \leq (2n + 1)m$ and

we get

$$\begin{aligned}
 P^x(\exists n \geq 0: X(n) \in A) &\leq \sum_{y \in A} P^x(\exists n \geq 0: X(n) = y) \\
 &= \sum_{y \in A} \frac{G(x, y)}{G(y, y)} \quad (\text{by (2)}) \\
 &\leq \sum_{y \in A} G(x, y) \quad (\text{since } G(y, y) \geq 1) \\
 &\leq \sum_{n \geq d(x, A)} |A \cap S(x, n)| \cdot \sup\{G(x, y): d(x, y) = n\} \\
 &\leq m \sum_{n \geq d(x, A)} (2n + 1) \cdot \sup\{G(x, y): d(x, y) = n\}.
 \end{aligned}$$

By (3), choose $N(m)$ so that $m \sum_{n \geq N(m)} (2n + 1) \cdot \sup\{G(x, y): d(x, y) = n\} < 1$. \square

Lemma 2. For any $K \subset \mathfrak{g}$, if $\inf_{x \in \mathfrak{g}} P^x(\limsup_{n \rightarrow \infty} (X(n) \in K)) < 1$, then $\sup_{x \in \mathfrak{g}} d(x, K) = \infty$.

Proof. By condition (1), for any $y \in K$ and $x \in \mathfrak{g}$,

$$P^x(\exists n: X(n) \in K) \geq P^x(X(d(x, y)) = y) \geq (1/\delta)^{d(x, y)}.$$

Thus, if $\sup_{x \in \mathfrak{g}} d(x, K) < \infty$, then $\inf_{x \in \mathfrak{g}} P^x(\exists n: X(n) \in K) > 0$ and, therefore, $\inf_{x \in \mathfrak{g}} P^x(\limsup_{n \rightarrow \infty} (X(n) \in K)) = 1$. \square

Theorem 1. For any $x \in \mathfrak{g}$, there are infinitely many geodesic rays $\gamma_1, \gamma_2, \dots$ starting at x such that if $i \neq j$, then $|\gamma_i \cap \gamma_j| < \infty$.

Proof. We construct such a family inductively. There is always one geodesic ray starting at x (Proposition 1). Suppose $\gamma_1, \gamma_2, \dots, \gamma_m \in \Gamma(x)$ such that if $i \neq j$, then $|\gamma_i \cap \gamma_j| < \infty$. Let $\partial A = \bigcup_{i=1}^m \gamma_i$. By Proposition 2, it is enough to show that there exists a geodesic ray γ such that $\gamma \cap \partial A = \emptyset$. Therefore, by the diagonal method of Proposition 1, it is enough to show that there exists $z \in \mathfrak{g}$ such that for all k , there exists $\gamma_k \in \Gamma(z, k)$ so that $\gamma_k \cap \partial A = \emptyset$.

Let $A = \mathfrak{g} \setminus \partial A$ and $N = N(m + 2)$ where $N(\cdot)$ is as in Lemma 1. As in the proof of Lemma 1, $\sum_{y \in \partial A} G(x, y) < \infty$ and so

$$P^x \left(\limsup_{k \rightarrow \infty} (X(k) \in \partial A) \right) = 0.$$

By Lemma 2, we can choose $z \in A$ such that $d(z, \partial A) \geq N$.

Suppose that there exists n such that for all $\gamma \in \Gamma(z, n)$, $\gamma \cap \partial A \neq \emptyset$. We show that this leads to a contradiction—we show that this implies the existence of two geodesic segments γ_t^* and γ_u^* such that:

- (a) $d(z, \gamma_t^* \cup \gamma_u^*) \geq N$ and
- (b) every infinite path starting at z hits $\gamma_t^* \cup \gamma_u^* \cup \partial A$.

By Lemma 1, condition (a) implies $P^z(\exists j: X(j) \in \gamma_t^* \cup \gamma_u^* \cup \partial A) < 1$ whereas condition (b) implies $P^z(\exists j: X(j) \in \gamma_t^* \cup \gamma_u^* \cup \partial A) = 1$.

For each $y \in S(z, n)$, choose $\gamma_y \in \Gamma(z, n)$. In addition, we choose these geodesics so that $\bigcup \gamma_y$ is a tree. For any $y \in S(z, n)$, let $\gamma_y^* = (\gamma_y(\eta), \dots, \gamma_y(n))$ where $\eta = \max\{j \leq n : \gamma_y(j) \in \partial A\}$. Note that for any $t, u \in S(z, n)$, condition (a) holds. Let $Z = \{y \in S(z, n) : \text{there exists an infinite path in } A, \text{ starting at } z, \text{ which last hits } B(z, n) \text{ at } y\}$. Z is nonempty by choice of N and z . For $Y \subset Z$, let $C(Y)$ be the connected component of $B(z, n) \setminus (\partial A \cup \bigcup_{y \in Y} \gamma_y^*)$ which contains z .

We claim that $C(Z) = C(\{t, u\})$ for some $t, u \in Z$. If so, then condition (b) holds for t and u . To prove this claim, it is enough to show that if t, u, v are distinct elements of Z , then $C(\{t, u, v\}) = C(\{t', u'\})$ for some $t', u' \in \{t, u, v\}$.

Let t, u, v be distinct elements of Z , and let ρ, σ, τ be infinite paths in A starting at z which last hit $B(z, n)$ at t, u, v respectively. Since ∂A is connected, ∂A is in one of the components of $G \setminus (\rho \cup \sigma \cup \tau)$. By planarity, without loss of generality, any path from t to ∂A must hit $\sigma \cup \tau$. Define $\rho^*(j) = \rho(j + M + 1)$ where $M = \max\{k : \rho(k) \in B(z, n)\}$. Then the complement of $\partial A \cup \gamma_t^* \cup \rho^*$ contains two components, say B and C , such that $u \in B, v \in C$, and, without loss of generality, $z \in B$. Then, any path contained in A from z to v must hit either γ_t^* or ρ^* . Since $\rho^* \cap B(z, n) = \emptyset$, $C(\{t, u, v\}) = C(\{t, u\})$. \square

A function $f: \mathfrak{g} \rightarrow \mathbb{R}$ is harmonic if and only if $\sum_{y: yEx} f(y) = d(x)f(x)$ for all x . In particular, since $\liminf_{k \rightarrow \infty} (X(k) \in A)$ is invariant under the Markov shift, $f(x) \equiv P^x(\liminf_{k \rightarrow \infty} (X(k) \in A)) = pf(x)$ and so f is bounded and harmonic. We use an idea similar to one Kendall uses in the case of Brownian motion on manifolds [3] to find a set A so that $P^*(\liminf_{k \rightarrow \infty} (X(k) \in A))$ is nonconstant.

Theorem 2. *There are nonconstant, bounded, harmonic functions on \mathfrak{g} .*

Proof. Let $N = N(2)$ where $N(\cdot)$ is as in Lemma 1. Fix $x \in \mathfrak{g}$ and, by Theorem 1, choose $4N$ rays $\gamma_1, \gamma_2, \dots, \gamma_{4N} \in \Gamma(x)$ whose pairwise intersections are finite. Without loss of generality, these geodesics are numbered in a clockwise fashion (we may do this since \mathfrak{g} is planar). Let M be such that $i \neq j$ implies $(\gamma_i \cap \gamma_j) \setminus B(x, M) = \emptyset$. Let $C = \gamma_1 \cup \gamma_{2N}, u = \gamma_N(M + N), v = \gamma_{3N}(M + N)$, and A and B be the connected components of $\mathfrak{g} \setminus C$ containing u and v respectively. By Lemma 1, since $d(u, C) \geq N$ and $d(v, C) \geq N$,

$$P^u \left(\liminf_{k \rightarrow \infty} (X(k) \in A) \right) \geq P^u(\forall j: X(j) \notin C) > 0$$

and

$$P^v \left(\limsup_{k \rightarrow \infty} (X(k) \in A) \right) \leq P^v(\exists j: X(j) \in C) < 1.$$

By Lemma 2,

$$\sup_{w \in \mathfrak{g}} d(w, A) = \infty.$$

Since, for $w \in B$,

$$P^w \left(\liminf_{k \rightarrow \infty} (X(k) \in A) \right) \leq P^w (\exists j: X(j) \in C) \\ \leq 2c \sum_{n \geq d(w, A)} (2n + 1) \cdot \sup\{G(x, y): d(x, y) = n\}$$

(as in the proof of Lemma 1), and since $d(w, A)$ is unbounded,

$$\inf_w P^w \left(\liminf_{n \rightarrow \infty} (X(n) \in A) \right) = 0$$

and so is not constant. \square

ACKNOWLEDGMENTS

I would like to thank Mike Cranston for suggesting this problem, Carl Mueller for some useful advice, and the referee for suggesting the present form of (3).

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