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Geodesics for the Painlevé–Gullstrand Form of Lense–Thirring Spacetime

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Abstract: Recently, the current authors have formulated and extensively explored a rather novel Painlevé–Gullstrand variant of the slow-rotation Lense–Thirring spacetime, a variant which has particularly elegant features—including unit lapse, intrinsically flat spatial 3-slices, and a separable Klein–Gordon equation (wave operator). This spacetime also possesses a non-trivial Killing tensor, implying separability of the Hamilton–Jacobi equation, the existence of a Carter constant, and complete *formal* integrability of the geodesic equations. Herein, we investigate the geodesics in some detail, in the general situation demonstrating the occurrence of "ultra-elliptic" integrals. Only in certain special cases can the complete geodesic integrability be explicitly cast in terms of elementary functions. The model is potentially of astrophysical interest both in the asymptotic large-distance limit and as an example of a "black hole mimic", a controlled deformation of the Kerr spacetime that can be contrasted with ongoing astronomical observations.

Keywords: Painlevé–Gullstrand metrics; Lense–Thirring metric; Killing tensor; Carter constant; integrability; geodesics

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1. Introduction

Recently, the current authors have introduced and extensively explored a specific new variant of the slow-rotation Lense–Thirring spacetime [1,2], described by the explicit line element

$$ds^2 = -dt^2 + \left\{ dr + \sqrt{\frac{2m}{r}}\, dt \right\}^2 + r^2 \left\{ d\theta^2 + \sin^2\theta \left(d\phi - \frac{2J}{r^3} dt \right)^2 \right\}. \quad (1)$$

We shall now extend the physical and mathematical analysis of this spacetime, paying particular attention to the geodesics. We first unavoidably need to provide a brief summary of the key results derived in references [1,2].

In this variant of the Lense–Thirring spacetime the metric possesses both unit lapse [3], and also exhibits a flat spatial 3-metric. That is, the spacetime metric is presented in so-called Painlevé–Gullstrand form [4–7], (sometimes called Gullstrand–Painlevé form [8]), with a relatively simple globally defined tetrad [1,2]. (For a textbook-level physically motivated discussion of the tetrad formalism see, for instance, reference [9].) These purely mathematical observations make this spacetime of particular theoretical interest [10,11]. We point out that, while the nomenclature "lapse function" is borrowed from the ADM foliation formalism [12,13], beyond this purely kinematic adaptation of the terminology, no direct use of the ADM formalism is made in this article.

We emphasize that there is no Birkhoff-like theorem for axisymmetric spacetimes in $(3+1)$ dimensions [14–19]. The Kerr solution need not, (and typically will not), perfectly model rotating horizonless astrophysical sources, such as stars and planets, due to the nontrivial mass multipole moments that such objects typically possess. Instead, the Kerr solution will only model the gravitational field in the asymptotic large-distance regime, a region where the Lense–Thirring spacetime serves as a perfectly valid approximation to Kerr. For a historical background on the Lense–Thirring spacetime, see references [20–22]. For a historical background on the Kerr spacetime, see references [23–26]. For a selection of textbook discussions of the Kerr spacetime, see references [9,27–33]. For a selection of expository articles on the Kerr spacetime, see references [8,34–38]. Given that this variant of the Lense–Thirring metric is a valid approximation for the gravitational fields of rotating stars and planets in the same regime that the Kerr solution is appropriate, there is a compelling physics argument to use the Painlevé–Gullstrand form of Lense–Thirring to model various astrophysically interesting cases [39–45].

From a purely theoretical perspective, the Lense–Thirring metric is algebraically *much* simpler than the Kerr metric, making most calculations significantly easier to conduct, and the Lense–Thirring metric can be recast into Painlevé–Gullstrand form, while the Kerr metric cannot [46–49]. This spacetime exhibits a separable Klein–Gordon equation (wave operator) [2] and also possesses a non-trivial Killing tensor, thereby implying separability and complete (formal) integrability of the Hamilton–Jacobi equations for geodesic motion [2]. Below, we shall discuss two particularly interesting classes of geodesics; the generic case involving ultra-elliptic integrals, and the case of vanishing Carter constant where the analysis can be completely performed in terms of elementary functions. This should be compared to what can and cannot be performed for the usual Kerr spacetime [50–67].

Observationally, apart from its interest in the large-distance asymptotic regime, this Lense–Thirring variant may also be viewed as a "black hole mimic" that can be contrasted with ongoing astronomical observations of various black hole candidates [39,68–72].

We note that a competing slow-rotation model has recently been discussed in references [73,74]. The trade-off made therein was to improve the integrability properties (the "hidden symmetries") at the cost of sacrificing the global Painlevé–Gullstrand form of the metric.

2. Killing Tensor and Carter constant

Based on the algorithm presented in two recent papers by Papadopoulos and Kokkotas [75,76], which are, in turn, based on considerably older results by Benenti and Francaviglia [77], in reference [2] we found the non-trivial Killing tensor:

$$K_{ab}\, \mathrm{d}x^a\, \mathrm{d}x^b = r^4 \left\{ \mathrm{d}\theta^2 + \sin^2\theta \left(\mathrm{d}\phi - \frac{2J}{r^3}\mathrm{d}t \right)^2 \right\}. \tag{2}$$

Explicitly, we wrote the metric as [1,2]:

$$g_{ab} = \left[\begin{array}{c|ccc} -1 + \frac{2m}{r} + \frac{4J^2 \sin^2\theta}{r^4} & \sqrt{\frac{2m}{r}} & 0 & -\frac{2J\sin^2\theta}{r} \\ \hline \sqrt{\frac{2m}{r}} & 1 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ -\frac{2J\sin^2\theta}{r} & 0 & 0 & r^2\sin^2\theta \end{array} \right]_{ab}. \tag{3}$$

Then $\det(g_{ab}) = -r^4 \sin^2\theta$, and for the inverse metric we have [1,2]:

$$g^{ab} = \left[\begin{array}{c|ccc} -1 & \sqrt{\frac{2m}{r}} & 0 & -\frac{2J}{r^3} \\ \hline \sqrt{\frac{2m}{r}} & 1 - \frac{2m}{r} & 0 & \sqrt{\frac{2m}{r}}\,\frac{2J}{r^3} \\ 0 & 0 & \frac{1}{r^2} & 0 \\ -\frac{2J}{r^3} & \sqrt{\frac{2m}{r}}\,\frac{2J}{r^3} & 0 & \frac{1}{r^2\sin^2\theta} - \frac{4J^2}{r^6} \end{array} \right]^{ab}. \tag{4}$$

The (contravariant) non-trivial Killing tensor is [2]:

$$K^{ab} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{\sin^2\theta} \end{bmatrix}^{ab} . \tag{5}$$

Following the analysis of reference [2], the corresponding covariant form of the Killing tensor, $K_{ab} = g_{ac}\, K^{cd}\, g_{db}$, is then explicitly given by:

$$K_{ab} = \begin{bmatrix} \frac{4J^2\sin^2\theta}{r^2} & 0 & 0 & -2Jr\sin^2\theta \\ 0 & 0 & 0 & 0 \\ 0 & 0 & r^4 & 0 \\ -2Jr\sin^2\theta & 0 & 0 & r^4\sin^2\theta \end{bmatrix}_{ab} . \tag{6}$$

One can easily explicitly check (e.g., Maple) that $\nabla_{(c}K_{ab)} = K_{(ab;c)} = 0$.

For any affine parameter λ, the (generalized) Carter constant is now [2]:

$$\mathcal{C} = K_{ab}\,\frac{dx^a}{d\lambda}\,\frac{dx^b}{d\lambda} = r^4\left[\left(\frac{d\theta}{d\lambda}\right)^2 + \sin^2\theta\left(\frac{d\phi}{d\lambda} - \frac{2J}{r^3}\frac{dt}{d\lambda}\right)^2\right] . \tag{7}$$

Without any loss of generality we may choose λ be future-directed, (so $d\lambda/dt > 0$). Note that by construction, since it is a sum of squares, $\mathcal{C} \geq 0$. (For additional recent discussion on general Killing tensors see [78,79].)

3. Conservation Laws

3.1. Four Conserved Quantities

In addition to the Carter constant (7), in this spacetime geometry we have three other conserved quantities [2]. Two of these come from the time-translation and axial Killing vectors, $\xi^a = (1;0,0,0)^a$ and $\psi^a = (0;0,0,1)^a$, respectively: These two conserved quantities are the energy

$$E = -\xi_a\frac{dx^a}{d\lambda} = \left(1 - \frac{2m}{r} - \frac{4J^2\sin^2\theta}{r^4}\right)\frac{dt}{d\lambda} - \sqrt{\frac{2m}{r}}\frac{dr}{d\lambda} + \frac{2J\sin^2\theta}{r}\frac{d\phi}{d\lambda} ; \tag{8}$$

and the azimuthal component of angular momentum

$$L = \psi_a\frac{dx^a}{d\lambda} = r^2\sin^2\theta\frac{d\phi}{d\lambda} - \frac{2J\sin^2\theta}{r}\frac{dt}{d\lambda} . \tag{9}$$

The final conserved quantity, ϵ, is the "mass-shell constraint", with $\epsilon \in \{0, -1\}$ for null and timelike geodesics, respectively. This mass-shell constraint comes from the trivial Killing tensor (the metric g_{ab}):

$$\begin{aligned} \epsilon = g_{ab}\frac{dx^a}{d\lambda}\frac{dx^b}{d\lambda} = &-\left(\frac{dt}{d\lambda}\right)^2 + \left(\frac{dr}{d\lambda} + \sqrt{\frac{2m}{r}}\frac{dt}{d\lambda}\right)^2 \\ &+ r^2\left[\left(\frac{d\theta}{d\lambda}\right)^2 + \sin^2\theta\left(\frac{d\phi}{d\lambda} - \frac{2J}{r^3}\frac{dt}{d\lambda}\right)^2\right] . \end{aligned} \tag{10}$$

3.2. Simplified Conservation Laws

We can greatly simplify these four conserved quantities by rewriting them as [2]:

$$L = r^2\sin^2\theta\left(\frac{d\phi}{d\lambda} - \frac{2J}{r^3}\frac{dt}{d\lambda}\right) ; \tag{11}$$

$$\mathcal{C} = r^4 \left(\frac{\mathrm{d}\theta}{\mathrm{d}\lambda} \right)^2 + \frac{L^2}{\sin^2\theta} ; \tag{12}$$

$$\epsilon = -\left(\frac{\mathrm{d}t}{\mathrm{d}\lambda} \right)^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}\lambda} + \sqrt{\frac{2m}{r}} \frac{\mathrm{d}t}{\mathrm{d}\lambda} \right)^2 + \frac{\mathcal{C}}{r^2} ; \tag{13}$$

$$E = \left(1 - \frac{2m}{r} \right) \frac{\mathrm{d}t}{\mathrm{d}\lambda} - \sqrt{\frac{2m}{r}} \frac{\mathrm{d}r}{\mathrm{d}\lambda} + \frac{2J}{r^3} L . \tag{14}$$

Notice that, by construction, $\mathcal{C} \geq L^2 \geq 0$, and that $(\mathrm{d}t/\mathrm{d}\lambda)^2 + \epsilon \geq 0$.

If $\epsilon = 0$ then, without loss of generality, we can rescale the affine parameter λ to set *one* of the constants $\{\mathcal{C}, E, L\} \to 1$. It is perhaps most intuitive to set $E \to 1$.

In contrast if $\epsilon = -1$ then $\lambda = \tau$ is the proper time and there is no further freedom to rescale the affine parameter. E then has real physical meaning and the qualitative behaviour is governed by the *sign* of $E^2 + \epsilon$. Concretely, at least in the case of Carter constant zero, one asks:

- Is $E < 1$? (Bound orbits);
- Is $E = 1$? (Marginal orbits);
- Or is $E > 1$? (Unbound orbits).

3.3. Forbidden Declination Range

The form of the Carter constant, Equation (12), since it is a positive semi-definite sum of squares, implicitly gives a range of forbidden declination angles for any given, non-zero values of $\mathcal{C}$ and L. We require that $\mathrm{d}\theta/\mathrm{d}\lambda$ be real, and from Equation (12) this implies the following requirement:

$$\left(r^2 \frac{\mathrm{d}\theta}{\mathrm{d}\lambda} \right)^2 = \mathcal{C} - \frac{L^2}{\sin^2\theta} \geq 0 \quad \Longrightarrow \quad \sin^2\theta \geq \frac{L^2}{\mathcal{C}} . \tag{15}$$

Then, provided $\mathcal{C} \geq L^2$, which is automatic in view of (12), we can define a critical angle $\theta_* \in [0, \pi/2]$ by setting

$$\theta_* = \sin^{-1}(|L|/\sqrt{\mathcal{C}}) . \tag{16}$$

Then, the allowed range for θ is the equatorial band:

$$\theta \in \Big[\theta_*, \pi - \theta_* \Big] . \tag{17}$$

- For $L^2 = \mathcal{C}$ we have $\theta = \pi/2$; the motion is restricted to the equatorial plane;
- For $L = 0$ with $\mathcal{C} > 0$ the range of θ is *a priori* unconstrained; $\theta \in [0, \pi]$;
- For $L = 0$ with $\mathcal{C} = 0$ the declination is fixed, $\theta(\lambda) = \theta_0$, and the motion is restricted to a constant declination conical surface.

This completes the *qualitative* assessment of the geodesics, and we can now turn to a more detailed *quantitative* analysis.

4. General Geodesics—Explicit Quantitative Analysis

Using Equations (11)–(14), we can explicitly and analytically solve for the four unknown functions $\mathrm{d}t/\mathrm{d}\lambda$, $\mathrm{d}r/\mathrm{d}\lambda$, $\mathrm{d}\theta/\mathrm{d}\lambda$ and $\mathrm{d}\phi/\mathrm{d}\lambda$ as explicit functions of r and θ, parameterized by the four conserved quantities $\mathcal{C}$, E, L, and ϵ, as well as the quantities m and J characterizing mass and angular momentum of the central object.

In solving for these unknown functions, one encounters three independent sign choices. (Note $\mathrm{d}t/\mathrm{d}\lambda$ is always taken to be positive.) Consequently, it is useful to define the following quantities (the subsequent physical interpretations are chosen from context for each equation):

$$S_r = \begin{cases} +1 & \text{outgoing geodesic} \\ -1 & \text{ingoing geodesic} \end{cases} ; \tag{18}$$

$$S_\theta = \begin{cases} +1 & \text{increasing declination geodesic} \\ -1 & \text{decreasing declination geodesic} \end{cases} ; \tag{19}$$

$$S_\phi = \begin{cases} +1 & \text{prograde geodesic} \\ -1 & \text{retrograde geodesic} \end{cases} . \tag{20}$$

4.1. Trajectories

For the geodesic trajectories we find the four equations:

$$\frac{\mathrm{d}r}{\mathrm{d}\lambda} = S_r \sqrt{X(r)} ; \tag{21}$$

$$\frac{\mathrm{d}t}{\mathrm{d}\lambda} = \frac{E - 2JL/r^3 + S_r\sqrt{(2m/r)X(r)}}{(1-2m/r)} ; \tag{22}$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}\lambda} = S_\theta \frac{\sqrt{\mathcal{C} - L^2/\sin^2\theta}}{r^2} ; \tag{23}$$

$$\frac{\mathrm{d}\phi}{\mathrm{d}\lambda} = \frac{L}{r^2\sin^2\theta} + 2J\,\frac{E - 2JL/r^3 + S_\phi\sqrt{(2m/r)X(r)}}{r^3(1-2m/r)} . \tag{24}$$

Here, $X(r)$ is the sextic Laurent polynomial

$$X(r) = \left(E - \frac{2JL}{r^3}\right)^2 - \left(1 - \frac{2m}{r}\right)\left(-\epsilon + \frac{\mathcal{C}}{r^2}\right) . \tag{25}$$

We note

$$\lim_{r\to\infty} X(r) = E^2 + \epsilon . \tag{26}$$

In terms of the roots of this polynomial in the generic case, we can write

$$X(r) = \frac{E^2+\epsilon}{r^6} \prod_{i=1}^{6}(r - r_i) . \tag{27}$$

In the special case $E^2 + \epsilon = 0$, corresponding to a marginally bound timelike geodesic, the sextic degenerates to a quintic

$$X(r) = \frac{2m}{r^5} \prod_{i=1}^{5}(r - r_i) . \tag{28}$$

Qualitatively, the radial motion can be bounded (if one is trapped between two real roots), or diverge to spatial infinity (if one is trapped above the outermost real root), or be a plunge to $r = 0$ (if one is trapped below the innermost positive real root). In the immediate vicinity of any real root r_i the behaviour will depend on the multiplicity m_i of that root. Approximately, one has

$$\frac{\mathrm{d}r}{\mathrm{d}\lambda} \approx \pm 2K_i\sqrt{|r - r_i|^{m_i}} . \tag{29}$$

If the root is multiplicity 1, (the generic situation) one has a "bounce" at the turning point (perinegricon or aponegricon—periapsis or apoapsis) at some finite value of the affine parameter:

$$|r - r_i| \approx K_i^2(\lambda - \lambda_0)^2 . \tag{30}$$

If the root is multiplicity 2, (a somewhat rarer situation), one has an exponential approach to the root:

$$|r - r_i| \approx C_i \exp(\pm 2K_i\, \lambda). \tag{31}$$

If the root is multiplicity 3 or higher, (an unusual situation; for instance take $\epsilon \to 0$ and $\mathcal{C} \to 0$), one has a very slow polynomial approach to the root as $|\lambda| \to \infty$:

$$|r - r_i| \approx \Big| K_i (m_i - 2)(\lambda - \lambda_0) \Big|^{\frac{2}{2-m_i}}\,. \tag{32}$$

4.2. Integrating the Affine Parameter

From Equation (21), we find

$$\lambda(r) = \lambda_0 + S_r \int_{r_0}^{r} \frac{\mathrm{d}\bar{r}}{\sqrt{X(\bar{r})}} = \lambda_0 + \int_{r_0}^{r} \frac{|\mathrm{d}\bar{r}|}{\sqrt{X(\bar{r})}}\,. \tag{33}$$

This can also be re-expressed as follows:

- Outgoing geodesic $\Longrightarrow S_r = +1 \cap r_0 < r$, hence $\lambda(r) = \lambda_0 + \int_{r_0}^{r} \frac{\mathrm{d}\bar{r}}{\sqrt{X(\bar{r})}}$;
- Ingoing geodesic $\Longrightarrow S_r = -1 \cap r_0 > r$, hence $\lambda(r) = \lambda_0 + \int_{r}^{r_0} \frac{\mathrm{d}\bar{r}}{\sqrt{X(\bar{r})}}$.

Integrals of this type are known as ultra-elliptic integrals [80,81], and date back (at least) to work by Weierstrass and Kovalevskaya in the second half of the 19th century.

If $X(r)$ were to be cubic or quartic, this would be an ordinary elliptic integral [82]. If $X(r)$ is quintic or sextic, as above, this is an ultra-elliptic integral. More generally, for polynomials of arbitrary order, these would be called hyper-elliptic integrals. Even more generally, these integrals are a sub-class of the so-called Abelian integrals [83]. Generically, Equation (33) cannot be explicitly integrated in closed form using only elementary functions, hence we cannot analytically invert this relation to find $r(\lambda)$. However, there is no obstacle, in principle, to numerical integration to explicitly find the affine parameter $\lambda(r)$. (Even for the exact Kerr solution one rapidly finds that use of some level of numerical integration is almost unavoidable [84–86]).

Note that if one is trapped (above or below) any real root of the polynomial $X(r)$ of multiplicity 1, then every time one "bounces" off the root the quantity S_r will flip sign, and $\lambda(r)$ will be double-valued though the inverse function $r(\lambda)$ will always be single-valued. This is as it should be to guarantee that the affine parameter λ is always continuously increasing with time.

Note that if one is trapped between two real roots of the polynomial $X(r)$, say r_{min} and r_{max}, both of multiplicity 1, then each bounce from r_{min} to r_{max}, or from r_{max} to r_{min}, will advance the affine parameter by some finite amount (the appropriate "period" of the ultra-elliptic integral, now typically called a complete ultra-elliptic integral):

$$\Delta\lambda = \int_{r_{min}}^{r_{max}} \frac{\mathrm{d}\bar{r}}{\sqrt{X(\bar{r})}}\,. \tag{34}$$

Then, $\lambda(r)$ will be multi-valued though the inverse function $r(\lambda)$ will always be single-valued. Furthermore, $r(\lambda)$, while analytically intractable, will at least be known to be periodic, with known periodicity $2\,\Delta\lambda$. If, instead, one is approaching a real root of multiplicity 2 or higher, then $\lambda(r)$ will diverge—one will not actually reach the root for any finite amount of affine parameter lapse—that is $r(\lambda)$ will asymptote to that higher multiplicity root.

4.3. Integrating the Epoch

Using Equations (21) and (22), we find

$$\frac{\mathrm{d}t}{\mathrm{d}r} = \frac{\sqrt{2mr}}{r-2m} + S_r \frac{E - 2JL/r^3}{(1-2m/r)\sqrt{X(r)}} \,, \tag{35}$$

so

$$t(r) = t_0 + \int_{r_0}^{r} \left(\frac{\sqrt{2m\bar{r}}}{\bar{r}-2m} + S_r \frac{E - 2JL/\bar{r}^3}{(1-2m/\bar{r})\sqrt{X(\bar{r})}} \right) \mathrm{d}\bar{r} \,, \tag{36}$$

where now $S_r = \mathrm{sign}(r - r_0)$.

It is straightforward to integrate the first term in the integrand, yielding

$$\begin{aligned} t(r) \quad &= \quad t_0 + 2\sqrt{2m}(\sqrt{r} - \sqrt{r_0}) + 2m \ln\left[\frac{(\sqrt{r}-\sqrt{2m})(\sqrt{r_0}+\sqrt{2m})}{(\sqrt{r_0}-\sqrt{2m})(\sqrt{r}+\sqrt{2m})} \right] \\ &\quad + S_r \int_{r_0}^{r} \frac{E - 2JL/\bar{r}^3}{(1-2m/\bar{r})\sqrt{X(\bar{r})}} \, \mathrm{d}\bar{r} \,. \end{aligned} \tag{37}$$

As before, the remaining integral is an ultra-elliptic integral [80,81], now a *different* ultra-elliptic integral, and this equation cannot be explicitly integrated in closed form.

Note that if one is trapped between two real roots of the polynomial $X(r)$, say r_1 and r_2, both of multiplicity 1, and both outside the horizon at $r = 2m$, then each bounce from r_1 to r_2 will advance the Killing time by some finite amount

$$T(r_1, r_2) = \int_{r_1}^{r_2} \left(\frac{\sqrt{2m\bar{r}}}{\bar{r}-2m} + S_r \frac{E - 2JL/\bar{r}^3}{(1-2m/\bar{r})\sqrt{X(\bar{r})}} \right) \mathrm{d}\bar{r} \,, \tag{38}$$

where now $S_r = \mathrm{sign}(r_2 - r_1)$.

Specifically, if $r_{max} = \max\{r_1, r_2\}$ and $r_{min} = \min\{r_1, r_2\}$, then on the upswing from r_{min} to r_{max} one has

$$T_{up} = \int_{r_{min}}^{r_{max}} \left(\frac{\sqrt{2m\bar{r}}}{\bar{r}-2m} + \frac{E - 2JL/\bar{r}^3}{(1-2m/\bar{r})\sqrt{X(\bar{r})}} \right) \mathrm{d}\bar{r} \,. \tag{39}$$

In contrast on the downswing from r_{max} to r_{min} one has

$$T_{down} = \int_{r_{max}}^{r_{min}} \left(\frac{\sqrt{2m\bar{r}}}{\bar{r}-2m} - \frac{E - 2JL/\bar{r}^3}{(1-2m/\bar{r})\sqrt{X(\bar{r})}} \right) \mathrm{d}\bar{r} \,. \tag{40}$$

That is

$$T_{down} = \int_{r_{min}}^{r_{max}} \left(-\frac{\sqrt{2m\bar{r}}}{\bar{r}-2m} + \frac{E - 2JL/\bar{r}^3}{(1-2m/\bar{r})\sqrt{X(\bar{r})}} \right) \mathrm{d}\bar{r} \,. \tag{41}$$

The total period is, therefore,

$$T = T_{up} + T_{down} = 2 \int_{r_{min}}^{r_{max}} \frac{E - 2JL/\bar{r}^3}{(1-2m/\bar{r})\sqrt{X(\bar{r})}} \, \mathrm{d}\bar{r} \,. \tag{42}$$

This is again a complete ultra-elliptic integral, now a *different* complete ultra-elliptic integral.

In this situation $t(r)$ will be multi-valued while the inverse function $r(t)$ will be single valued. Although analytically intractable, $r(t)$ will at least be known to be periodic, with known periodicity $2\,T$. If instead one is approaching a real root of multiplicity 2 or higher, then $t(r)$ will diverge—one will not actually reach the root for any finite amount of Killing time.

4.4. Integrating the Declination

As for our equation involving the declination angle θ, first recall the definition of the critical angle θ_* as $\theta_* = \sin^{-1}\left(|L|/\sqrt{\mathcal{C}}\right)$. Then, from Equation (23), we find

$$\begin{aligned} \frac{\mathrm{d}\cos\theta}{\mathrm{d}\lambda} &= -S_\theta \frac{\sqrt{\mathcal{C}\sin^2\theta - L^2}}{r^2} \\ &= -S_\theta \frac{\sqrt{\mathcal{C}}}{r^2}\sqrt{\sin^2\theta - \sin^2\theta_*} \\ &= -S_\theta \frac{\sqrt{\mathcal{C}}}{r^2}\sqrt{\cos^2\theta_* - \cos^2\theta}\,, \end{aligned} \tag{43}$$

implying

$$\frac{\mathrm{d}\cos\theta}{\sqrt{\cos^2\theta_* - \cos^2\theta}} = -S_\theta \frac{\sqrt{\mathcal{C}}}{r^2}\,\mathrm{d}\lambda = \left(-S_\theta\frac{\sqrt{\mathcal{C}}}{r^2}\right)\left(S_r \frac{\mathrm{d}r}{\sqrt{X(r)}}\right). \tag{44}$$

From this we see

$$\mathrm{d}\cos^{-1}\left(\frac{\cos\theta}{\cos\theta_*}\right) = S_\theta S_r \frac{\sqrt{\mathcal{C}}}{r^2}\frac{\mathrm{d}r}{\sqrt{X(r)}}\,, \tag{45}$$

that is

$$\cos^{-1}\left(\frac{\cos\theta}{\cos\theta_*}\right) = \cos^{-1}\left(\frac{\cos\theta_0}{\cos\theta_*}\right) + S_\theta S_r\sqrt{\mathcal{C}}\int_{r_0}^{r}\frac{\mathrm{d}\bar{r}}{\bar{r}^2\sqrt{X(\bar{r})}}\,. \tag{46}$$

Without loss of generality we may allow the geodesic to reach the critical angle θ_* at some radius r_*, and then use that as our new initial data.

This effectively sets $\theta_0 = \theta_*$, and gives us the following simplified result:

$$\cos^{-1}\left(\frac{\cos\theta}{\cos\theta_*}\right) = S_\theta S_r\sqrt{\mathcal{C}}\int_{r_*}^{r}\frac{\mathrm{d}\bar{r}}{\bar{r}^2\sqrt{X(\bar{r})}}\,. \tag{47}$$

Thence

$$\cos\theta = \cos\theta_*\ \cos\left(S_\theta S_r\sqrt{\mathcal{C}}\int_{r_*}^{r}\frac{\mathrm{d}\bar{r}}{\bar{r}^2\sqrt{X(\bar{r})}}\right) = \cos\theta_*\ \cos\left(\sqrt{\mathcal{C}}\int_{r_*}^{r}\frac{S_r\,\mathrm{d}\bar{r}}{\bar{r}^2\sqrt{X(\bar{r})}}\right),$$

with the last step coming from the fact that $\cos(\cdots)$ is an even function of its argument. That is

$$\cos\theta = \cos\theta_*\ \cos\left(\sqrt{\mathcal{C}}\int_{r_*}^{r}\frac{|\mathrm{d}\bar{r}|}{\bar{r}^2\sqrt{X(\bar{r})}}\right), \tag{48}$$

where the phase

$$(\text{phase}) = \sqrt{\mathcal{C}}\int_{r_*}^{r}\frac{|\mathrm{d}\bar{r}|}{\bar{r}^2\sqrt{X(\bar{r})}} \tag{49}$$

is monotone increasing.

One is, again, reduced to investigating yet another ultra-elliptic integral [80,81], with the declination angle θ oscillating periodically as a function of this phase with period 2π, and with each "bounce" from r_{min} to r_{max} advancing the phase of the cosine by an amount

$$\Delta(\text{phase}) = \sqrt{\mathcal{C}}\int_{r_{min}}^{r_{max}}\frac{\mathrm{d}\bar{r}}{\bar{r}^2\sqrt{X(\bar{r})}}\,. \tag{50}$$

As before, this equation cannot be explicitly integrated in closed form.

4.5. Integrating the Azimuth

We now finally consider the ODE for the evolution of the azimuthal angle: $\mathrm{d}\phi/\mathrm{d}\lambda$. (This particular sub-case is considerably messier than the previous ones.) Using Equations (21) and (24) we find

$$\frac{\mathrm{d}\phi}{\mathrm{d}r} = S_r S_\phi \left[2J \frac{\sqrt{2mr}}{r^3(r-2m)} \right] + S_r \left[\frac{L}{r^2 \sin^2\theta \sqrt{X(r)}} + 2J \frac{E - 2JL/r^3}{r^3(1-2m/r)\sqrt{X(r)}} \right]. \tag{51}$$

Consequently,

$$\begin{aligned} \phi(r) &= \phi_0 + S_r S_\phi \left\{ 2J \int_{r_0}^{r} \frac{\sqrt{2m\bar{r}}}{\bar{r}^3(\bar{r}-2m)} \,\mathrm{d}\bar{r} \right\} \\ &\quad + S_r \int_{r_0}^{r} \left(\frac{L}{\bar{r}^2 \sin^2[\theta(\bar{r})] \sqrt{X(\bar{r})}} + \frac{2J(E - 2JL/\bar{r}^3)}{\bar{r}^3(1-2m/\bar{r})\sqrt{X(\bar{r})}} \right) \mathrm{d}\bar{r}\,, \end{aligned} \tag{52}$$

where as per our previous discussion we have $S_r = \mathrm{sign}(r - r_0)$. The last term appearing here is again an ultra-elliptic integral [80,81]. In contrast, the penultimate term is somewhat worse, because the integrand contains $\theta(\bar{r})$, the overall integral is an *iteration* of an ultra-elliptic integral.

We can explicitly integrate the first integral in closed form:

$$\begin{aligned} 2J \int_{r_0}^{r} \frac{\sqrt{2m\bar{r}}}{\bar{r}^3(\bar{r}-2m)} \,\mathrm{d}\bar{r} &= 2J\sqrt{\frac{2}{m}} \left[\frac{1}{\sqrt{r}} \left(\frac{1}{3r} + \frac{1}{2m} \right) - \frac{1}{\sqrt{r_0}} \left(\frac{1}{3r_0} + \frac{1}{2m} \right) \right] \\ &\quad + \frac{J}{2m^2} \ln \left[\frac{(\sqrt{r} - \sqrt{2m})(\sqrt{r_0} + \sqrt{2m})}{(\sqrt{r_0} - \sqrt{2m})(\sqrt{r} + \sqrt{2m})} \right]. \end{aligned} \tag{53}$$

Thence, in general

$$\begin{aligned} \phi(r) &= \phi_0 + S_r S_\phi \left\{ \frac{J}{2m^2} \ln \left[\frac{(\sqrt{r} - \sqrt{2m})(\sqrt{r_0} + \sqrt{2m})}{(\sqrt{r_0} - \sqrt{2m})(\sqrt{r} + \sqrt{2m})} \right] \right. \\ &\quad \left. + 2J\sqrt{\frac{2}{m}} \left[\frac{1}{\sqrt{r}} \left(\frac{1}{3r} + \frac{1}{2m} \right) - \frac{1}{\sqrt{r_0}} \left(\frac{1}{3r_0} + \frac{1}{2m} \right) \right] \right\} \\ &\quad + S_r \int_{r_0}^{r} \left(\frac{L}{\bar{r}^2 \sin^2[\theta(\bar{r})] \sqrt{X(\bar{r})}} + \frac{2J(E - 2JL/\bar{r}^3)}{\bar{r}^3(1-2m/\bar{r})\sqrt{X(\bar{r})}} \right) \mathrm{d}\bar{r}\,. \end{aligned} \tag{54}$$

As before, this equation being ultra-elliptic means it cannot be explicitly integrated in closed fully analytic form, at least not in terms of elementary functions.

4.6. Summary of Generic Geodesic Evolution

Overall, while these generic geodesic equations cannot be integrated in closed and fully analytic form, they are still integrable in the formal technical sense. In terms of complete ultra-elliptic integrals some of the key quantities of interest (both mathematically and physically) are the "periods":

$$\int_{r_{min}}^{r_{max}} \frac{\mathrm{d}\bar{r}}{\sqrt{X(\bar{r})}}; \qquad \int_{r_{min}}^{r_{max}} \frac{\mathrm{d}\bar{r}}{\bar{r}^2\sqrt{X(\bar{r})}}; \tag{55}$$

and

$$\int_{r_{min}}^{r_{max}} \frac{E - 2JL/\bar{r}^3}{(1-2m/\bar{r})\sqrt{X(\bar{r})}} \,\mathrm{d}\bar{r}; \qquad \int_{r_{min}}^{r_{max}} \frac{E - 2JL/\bar{r}^3}{\bar{r}^3(1-2m/\bar{r})\sqrt{X(\bar{r})}} \,\mathrm{d}\bar{r}; \tag{56}$$

and the iterated integral

$$\int_{r_{min}}^{r_{max}} \frac{L}{\bar{r}^2 \sin^2[\theta(\bar{r})]\sqrt{X(\bar{r})}}\, \mathrm{d}\bar{r} \,. \tag{57}$$

Taylor expanding $(1-2m/r)^{-1}$ focusses attention on the quantities

$$\int_{r_{min}}^{r_{max}} \frac{\mathrm{d}\bar{r}}{\bar{r}^n \sqrt{X(\bar{r})}}; \qquad (n \in \mathbb{N}). \tag{58}$$

Typically, the "periods" of these ultra-elliptic integrals will be incommensurate.

If further specific constraints are now imposed, then these equations can indeed be integrated in closed form. In the next section, we explicate both the null and timelike geodesics for when $\mathcal{C}=0$ in closed fully analytic form. We are able to recover the "rain" geodesics from [1], as well as present a simple derivation of both the "drip" and "hail" geodesics.

5. Geodesics with Carter Constant Zero

If the Carter constant is zero then the geodesic equations simplify radically. Firstly, if $\mathcal{C}=0$ then from the manifest positivity of

$$\mathcal{C} = \left(r^2 \frac{\mathrm{d}\theta}{\mathrm{d}\lambda}\right)^2 + \left(\frac{L}{\sin\theta}\right)^2, \tag{59}$$

we see that we must have *both* $L=0$ *and* $\mathrm{d}\theta/\mathrm{d}\lambda \equiv 0$. The condition that $L=0$ physically constrains the geodesics to the trajectories of zero angular momentum observers (ZAMOs), whilst $\mathrm{d}\theta/\mathrm{d}\lambda = 0 \Longrightarrow \theta(r)=\theta_0$; some constant θ_0.

Furthermore, our expression for $X(r)$ significantly simplifies since now:

$$X(r) = E^2 + \epsilon\left(1 - \frac{2m}{r}\right). \tag{60}$$

We find that the four trajectory equations reduce to:

$$\frac{\mathrm{d}r}{\mathrm{d}\lambda} = S_r\sqrt{X(r)}\,; \tag{61}$$

$$\frac{\mathrm{d}t}{\mathrm{d}\lambda} = \frac{E + S_r\sqrt{(2m/r)X(r)}}{1-2m/r}\,; \tag{62}$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}\lambda} = 0\,; \tag{63}$$

$$\frac{\mathrm{d}\phi}{\mathrm{d}\lambda} = \frac{2J}{r^3}\left(\frac{E + S_\phi\sqrt{(2m/r)X(r)}}{1-2m/r}\right). \tag{64}$$

The form of these equations suggests it would be particularly useful to separate our analysis of null geodesics (photons) from timelike geodesics (massive particles).

5.1. Null Geodesics (Photons) with Carter Constant Zero

For null geodesics (photons) with Carter constant zero, we have the following conditions:

$$\mathcal{C}=0\,; \qquad L=0\,; \qquad \theta(r)=\theta_0\,; \qquad X(r)=E^2\,. \tag{65}$$

Furthermore, without loss of generality, we can rescale the affine parameter to set $E \to 1$, so that $X(r) \to 1$. From Equation (33), we now find

$$\lambda(r) = \lambda_0 + S_r\int_{r_0}^{r} \frac{\mathrm{d}\bar{r}}{\sqrt{X(\bar{r})}} = \lambda_0 + S_r\int_{r_0}^{r} \mathrm{d}\bar{r} = \lambda_0 + S_r(r-r_0)\,. \tag{66}$$

That is, we find the simple linear relation

$$r(\lambda) = S_r(\lambda - \lambda_0) + r_0\,. \tag{67}$$

Thus, in this particular situation, r is an affine parameter. Ingoing geodesics will crash into the central singularity in finite affine time, whereas outgoing geodesics can emerge from the horizon ($r = 2m$) at finite affine time (which we shall soon see corresponds to minus infinity in Killing time). The apparent asymmetry between ingoing and outgoing null geodesics is a side-effect of the initial choices made in setting up the Painlevé–Gullstrand coordinate system. (Did one choose an ingoing or outgoing Painlevé–Gullstrand coordinate system?)

Furthermore, Equation (37) for $t(r)$ now reduces to

$$\begin{aligned} t(r) = t_0 + 2\sqrt{2m}(\sqrt{r} - \sqrt{r_0}) \quad &+ \quad 2m \ln\left[\frac{(\sqrt{r} - \sqrt{2m})(\sqrt{r_0} + \sqrt{2m})}{(\sqrt{r_0} - \sqrt{2m})(\sqrt{r} + \sqrt{2m})}\right] \\ &+ S_r \int_{r_0}^{r} \frac{\bar{r}}{\bar{r} - 2m}\,\mathrm{d}\bar{r}\,. \end{aligned} \tag{68}$$

This integrates explicitly to

$$\begin{aligned} t(r) = t_0 + 2\sqrt{2m}(\sqrt{r} - \sqrt{r_0}) \quad &+ \quad 2m \ln\left[\frac{(\sqrt{r} - \sqrt{2m})(\sqrt{r_0} + \sqrt{2m})}{(\sqrt{r_0} - \sqrt{2m})(\sqrt{r} + \sqrt{2m})}\right] \\ &+ S_r\left[r - r_0 + 2m \ln\left(\frac{r - 2m}{r_0 - 2m}\right)\right]. \end{aligned} \tag{69}$$

This can also be rewritten as

$$\begin{aligned} t(r) \quad &= \quad t_0 + 2\sqrt{2m}(\sqrt{r} - \sqrt{r_0}) + |r - r_0| \\ &\quad + 2m\left\{ S_r \ln\left[\frac{r + 2m - 2\,S_r\sqrt{2mr}}{r_0 + 2m - 2\,S_r\sqrt{2mr_0}}\right]\right\}, \end{aligned} \tag{70}$$

or even

$$t(r) \quad = \quad t_0 + 2\sqrt{2m}(\sqrt{r} - \sqrt{r_0}) + |r - r_0| + 4m\left\{ S_r \ln\left[\frac{\sqrt{r} - S_r\sqrt{2m}}{\sqrt{r_0} - S_r\sqrt{2m}}\right]\right\}.$$

Which form one uses is really a matter of taste.

Finally, from Equation (54) we also have

$$\begin{aligned} \phi(r) \quad &= \quad \phi_0 + S_r S_\phi \left\{ \frac{J}{2m^2} \ln\left[\frac{(\sqrt{r} - \sqrt{2m})(\sqrt{r_0} + \sqrt{2m})}{(\sqrt{r_0} - \sqrt{2m})(\sqrt{r} + \sqrt{2m})}\right] \right. \\ &\quad \left. + 2J\sqrt{\frac{2}{m}}\left[\frac{1}{\sqrt{r}}\left(\frac{1}{3r} + \frac{1}{2m}\right) - \frac{1}{\sqrt{r_0}}\left(\frac{1}{3r_0} + \frac{1}{2m}\right)\right]\right\} \\ &\quad + 2J\,S_r \int_{r_0}^{r} \frac{\mathrm{d}\bar{r}}{\bar{r}^3(1 - 2m/\bar{r})}\,. \end{aligned} \tag{71}$$

This integrates explicitly to

$$\begin{aligned}\phi(r) \quad &= \quad \phi_0 + S_r S_\phi \left\{ \frac{J}{2m^2} \ln\left[\frac{(\sqrt{r}-\sqrt{2m})(\sqrt{r_0}+\sqrt{2m})}{(\sqrt{r_0}-\sqrt{2m})(\sqrt{r}+\sqrt{2m})} \right] \right. \\ &\left. +2J\sqrt{\frac{2}{m}}\left[\frac{1}{\sqrt{r}}\left(\frac{1}{3r}+\frac{1}{2m}\right) - \frac{1}{\sqrt{r_0}}\left(\frac{1}{3r_0}+\frac{1}{2m}\right)\right]\right\} \\ &+S_r \frac{J}{m}\left\{ \frac{1}{r} - \frac{1}{r_0} + \frac{1}{2m}\ln\left[\frac{(1-2m/r)}{(1-2m/r_0)}\right]\right\}. \end{aligned} \tag{72}$$

Notice that while these equations are rather complicated, they are all fully explicit and given in terms of elementary functions. We also note that these equations have sensible limiting behaviour; as $r \to r_0$, we have that $\lambda(r)$, $t(r)$, $\phi(r) \to \lambda_0$, t_0, ϕ_0, respectively.

5.2. Timelike Geodesics (Massive Particles) with Carter Constant Zero

For timelike geodesics (massive particles) with Carter constant zero, we have the following conditions:

$$\mathcal{C} = 0\,; \qquad L = 0\,; \qquad \theta(r) = \theta_0\,; \qquad X(r) = E^2 - 1 + \frac{2m}{r}\,. \tag{73}$$

The form of the polynomial $X(r)$ suggests that we should split our analysis into three cases. For the first case we set $E = 1$ (since in this case $X(r)$ reduces significantly; $X(r) \to 2m/r$), for the second case we set $E > 1$, and for the third case we set $E < 1$. Physically, geodesics with $E = 1$ represent particles that have zero radial velocity at spatial infinity (these are *marginally bound* geodesics). Geodesics with $E > 1$ represent particles that have non-zero velocity at spatial infinity (these are *unbound* geodesics). Geodesics with $E < 1$ represent particles that are in *bound* orbits, and so never escape to spatial infinity. For ingoing geodesics, these correspond to the "rain" ($E = 1$), "hail" ($E > 1$), and "drip" ($E < 1$) geodesics.

5.2.1. Marginal Geodesics $E = 1$

Setting $E = 1$, our conditions reduce to:

$$\mathcal{C} = 0\,; \quad L = 0\,; \quad \theta(r) = \theta_0\,; \quad X(r) = \frac{2m}{r}\,. \tag{74}$$

So, for our expression for $\lambda(r)$ we find

$$\begin{aligned}\lambda(r) = \lambda_0 + S_r \int_{r_0}^{r} \frac{\mathrm{d}\bar{r}}{\sqrt{X(\bar{r})}} \quad &= \quad \lambda_0 + S_r \int_{r_0}^{r} \frac{\mathrm{d}\bar{r}}{\sqrt{2m/\bar{r}}} \\ &= \quad \lambda_0 + S_r\sqrt{\frac{2}{m}}\left[\frac{r^{\frac{3}{2}} - r_0^{\frac{3}{2}}}{3}\right]. \end{aligned} \tag{75}$$

Hence,

$$r(\lambda) = \left\{ \frac{3\sqrt{2m}}{2}[S_r(\lambda-\lambda_0)] + r_0^{\frac{3}{2}} \right\}^{\frac{2}{3}}. \tag{76}$$

Our general expression (37) for $t(r)$ reduces to

$$t(r) = t_0 + 2\sqrt{2m}(\sqrt{r} - \sqrt{r_0}) \quad + \quad 2m \ln\left[\frac{(\sqrt{r} - \sqrt{2m})(\sqrt{r_0} + \sqrt{2m})}{(\sqrt{r_0} - \sqrt{2m})(\sqrt{r} + \sqrt{2m})}\right]$$
$$+ \quad S_r \int_{r_0}^{r} \frac{\mathrm{d}\bar{r}}{(1 - 2m/\bar{r})\sqrt{2m/\bar{r}}} \,. \tag{77}$$

We may now compute the somewhat unwieldy integral

$$\int_{r_0}^{r} \frac{\mathrm{d}\bar{r}}{(1 - 2m/\bar{r})\sqrt{2m/\bar{r}}} \quad = \quad 2\sqrt{2m}(\sqrt{r} - \sqrt{r_0}) + \sqrt{\frac{2}{m}}\left(\frac{r^{\frac{3}{2}} - r_0^{\frac{3}{2}}}{3}\right)$$
$$+2m \ln\left[\frac{(\sqrt{r} - \sqrt{2m})(\sqrt{r_0} + \sqrt{2m})}{(\sqrt{r_0} - \sqrt{2m})(\sqrt{r} + \sqrt{2m})}\right], \tag{78}$$

giving the following explicit form for $t(r)$

$$t(r) = t_0 \quad + \quad 2\sqrt{2m}(\sqrt{r} - \sqrt{r_0}) + 2m \ln\left[\frac{(\sqrt{r} - \sqrt{2m})(\sqrt{r_0} + \sqrt{2m})}{(\sqrt{r_0} - \sqrt{2m})(\sqrt{r} + \sqrt{2m})}\right]$$
$$+S_r \Bigg\{ 2\sqrt{2m}(\sqrt{r} - \sqrt{r_0}) + \sqrt{\frac{2}{m}}\left(\frac{r^{\frac{3}{2}} - r_0^{\frac{3}{2}}}{3}\right)$$
$$+2m \ln\left[\frac{(\sqrt{r} - \sqrt{2m})(\sqrt{r_0} + \sqrt{2m})}{(\sqrt{r_0} - \sqrt{2m})(\sqrt{r} + \sqrt{2m})}\right] \Bigg\} \,. \tag{79}$$

It is worth noting that for $S_r = -1$ (corresponding to ingoing geodesics) we find the particularly simple result

$$t(r) = t_0 - \sqrt{\frac{2}{m}}\left(\frac{r^{\frac{3}{2}} - r_0^{\frac{3}{2}}}{3}\right), \tag{80}$$

whilst for $S_r = +1$ (corresponding to outgoing geodesics), we obtain

$$t(r) \quad = \quad t_0 + \sqrt{\frac{2}{m}}\left(\frac{r^{\frac{3}{2}} - r_0^{\frac{3}{2}}}{3}\right) + 4\sqrt{2m}(\sqrt{r} - \sqrt{r_0})$$
$$+4m \ln\left[\frac{(\sqrt{r} - \sqrt{2m})(\sqrt{r_0} + \sqrt{2m})}{(\sqrt{r_0} - \sqrt{2m})(\sqrt{r} + \sqrt{2m})}\right]. \tag{81}$$

The apparent asymmetry between ingoing and outgoing timelike geodesics is a side-effect of the choices made in setting up the Painlevé–Gullstrand coordinate system. Note that in this situation the formulae for $t(r)$ are J-independent, so this apparent asymmetry is already present for Schwarzschild spacetime in Painlevé–Gullstrand coordinates. The usual choices allow ingoing geodesics to penetrate the horizon and crash into the central singularity in finite Killing time, while outgoing geodesics need an infinite amount of Killing time to escape from the horizon at $r = 2m$ and so become 'stuck'.

Our general expression (54) for $\phi(r)$ reduces to

$$\begin{aligned} \phi(r) &= \phi_0 + S_r S_\phi \left\{ \frac{J}{2m^2} \ln\left[\frac{(\sqrt{r}-\sqrt{2m})(\sqrt{r_0}+\sqrt{2m})}{(\sqrt{r_0}-\sqrt{2m})(\sqrt{r}+\sqrt{2m})} \right] \right. \\ &\left. +2J\sqrt{\frac{2}{m}} \left[\frac{1}{\sqrt{r}} \left(\frac{1}{3r} + \frac{1}{2m} \right) - \frac{1}{\sqrt{r_0}} \left(\frac{1}{3r_0} + \frac{1}{2m} \right) \right] \right\} \\ &+2J\, S_r \int_{r_0}^{r} \frac{\mathrm{d}\bar{r}}{\bar{r}^3 (1-2m/\bar{r})\sqrt{2m/\bar{r}}} \,. \end{aligned} \tag{82}$$

We may integrate the remaining integral in closed form

$$\begin{aligned} \int_{r_0}^{r} \frac{\mathrm{d}\bar{r}}{\bar{r}^3 (1-2m/\bar{r})\sqrt{2m/\bar{r}}} &= \frac{1}{m\sqrt{2m}} \left(\frac{1}{\sqrt{r}} - \frac{1}{\sqrt{r_0}} \right) \\ &+ \frac{1}{4m^2} \ln\left[\frac{(\sqrt{r}-\sqrt{2m})(\sqrt{r_0}+\sqrt{2m})}{(\sqrt{r_0}-\sqrt{2m})(\sqrt{r}+\sqrt{2m})} \right]. \end{aligned} \tag{83}$$

This gives the fully explicit form for $\phi(r)$ as:

$$\begin{aligned} \phi(r) &= \phi_0 + S_r S_\phi \left\{ \frac{J}{2m^2} \ln\left[\frac{(\sqrt{r}-\sqrt{2m})(\sqrt{r_0}+\sqrt{2m})}{(\sqrt{r_0}-\sqrt{2m})(\sqrt{r}+\sqrt{2m})} \right] \right. \\ &\left. +2J\sqrt{\frac{2}{m}} \left[\frac{1}{\sqrt{r}} \left(\frac{1}{3r} + \frac{1}{2m} \right) - \frac{1}{\sqrt{r_0}} \left(\frac{1}{3r_0} + \frac{1}{2m} \right) \right] \right\} \\ &+S_r \left\{ \frac{2J}{m\sqrt{2m}} \left(\frac{1}{\sqrt{r}} - \frac{1}{\sqrt{r_0}} \right) + \frac{J}{2m^2} \ln\left[\frac{(\sqrt{r}-\sqrt{2m})(\sqrt{r_0}+\sqrt{2m})}{(\sqrt{r_0}-\sqrt{2m})(\sqrt{r}+\sqrt{2m})} \right] \right\}. \end{aligned} \tag{84}$$

It is now straightforward to recover the "rain" geodesics explored in [1], which model a ZAMO dropped from spatial infinity with zero initial radial velocity. This physical scenario requires $\mathcal{C} = 0$, $L = 0$, $\epsilon = -1$, and $E = 1$, as above, and, furthermore, is an *ingoing retrograde* geodesic (corresponding mathematically to fixing $S_r = S_\phi = -1$). These geodesics are retrograde since we must give them an initial nonzero angular velocity in the retrograde direction at spatial infinity in order for $L = 0$ to hold along the length of the geodesic. From Equations (80) and (84), we find the explicit "rain" geodesics:

$$t(r) = t_0 - \sqrt{\frac{2}{m}} \left(\frac{r^{\frac{3}{2}} - r_0^{\frac{3}{2}}}{3} \right), \tag{85}$$

which we may re-phrase (defining t_{crash} to be the amount of elapsed Killing time for an ingoing geodesic to crash into the central singularity) as

$$t_{rain}(r) = t_{crash} - \sqrt{\frac{2}{m}}\, \frac{r^{\frac{3}{2}}}{3} \,. \tag{86}$$

Similarly,

$$\phi(r) = \phi_0 + \frac{2J}{3}\sqrt{\frac{2}{m}} \left(\frac{1}{r^{\frac{3}{2}}} - \frac{1}{r_0^{\frac{3}{2}}} \right), \tag{87}$$

which we may rephrase as

$$\phi_{rain}(r) = \phi_\infty + \frac{2J}{3}\sqrt{\frac{2}{m}}\,\frac{1}{r^{\frac{3}{2}}}\,. \tag{88}$$

If we differentiate with respect to r, we find

$$\frac{\mathrm{d}t}{\mathrm{d}r} = -\sqrt{\frac{r}{2m}}\,, \qquad \text{and} \qquad \frac{\mathrm{d}\phi}{\mathrm{d}r} = -\frac{2J}{\sqrt{2m}}\,r^{-5/2}\,, \tag{89}$$

which are exactly the equations defining the rain geodesics as given in reference [1].

5.2.2. Unbound Geodesics $E > 1$

For $E > 1$, let us first restore the full list of conditions from Equation (73):

$$\mathcal{C} = 0\,; \quad L = 0\,; \quad \theta(r) = \theta_0\,; \quad X(r) = E^2 - 1 + \frac{2m}{r}\,. \tag{90}$$

Notice that for $E > 1$, r is *a priori* unconstrained; $r \in (0, +\infty)$.
From our general result (33) for $\lambda(r)$, we have

$$\begin{aligned}
\lambda(r) &= \lambda_0 + S_r \int_{r_0}^{r} \frac{\mathrm{d}\bar{r}}{\sqrt{X(\bar{r})}} = \lambda_0 + S_r \int_{r_0}^{r} \frac{\mathrm{d}\bar{r}}{\sqrt{E^2 - 1 + 2m/\bar{r}}} \\
&= \lambda_0 + \frac{S_r}{E^2 - 1}\Bigg\{ r\sqrt{E^2 - 1 + 2m/r} - r_0\sqrt{E^2 - 1 + 2m/r_0} \\
&\qquad + \frac{m}{\sqrt{E^2 - 1}}\left[\ln\frac{r_0}{r} + 2\ln\left|\frac{\sqrt{E^2 - 1} + \sqrt{E^2 - 1 + 2m/r_0}}{\sqrt{E^2 - 1} + \sqrt{E^2 - 1 + 2m/r}}\right|\right]\Bigg\}\,.
\end{aligned} \tag{91}$$

Conducting a Taylor series expansion around $E = 1$, we find

$$\lambda(r) = \lambda_0 + S_r\sqrt{\frac{2}{m}}\left(\frac{r^{3/2} - r_0^{3/2}}{3}\right) + \mathcal{O}(E - 1)\,. \tag{92}$$

In the limit where $E \to 1$, we see that our expression for $\lambda(r)$ simplifies to (75), as expected.

The unwieldy integral in our general expression (37) for $t(r)$ now becomes

$$\begin{aligned}
&\int_{r_0}^{r} \frac{E}{(1 - 2m/\bar{r})\sqrt{E^2 - 1 + 2m/\bar{r}}}\,\mathrm{d}\bar{r} \\
&\quad = \frac{E}{E^2 - 1}\left(r\sqrt{E^2 - 1 + 2m/r} - r_0\sqrt{E^2 - 1 + 2m/r_0}\right) \\
&\qquad - 2m\ln\left[\frac{(2m - r_0)\left(r - 2m - 2Er\left(E + \sqrt{E^2 - 1 + 2m/r}\right)\right)}{(2m - r)\left(r_0 - 2m - 2Er_0\left(E + \sqrt{E^2 - 1 + 2m/r_0}\right)\right)}\right] \\
&\qquad + \frac{Em(2E^2 - 3)}{(E^2 - 1)^{3/2}}\ln\left[\frac{m - r + E^2 r + r\sqrt{(E^2 - 1)(E^2 - 1 + 2m/r)}}{m - r_0 + E^2 r_0 + r_0\sqrt{(E^2 - 1)(E^2 - 1 + 2m/r_0)}}\right]\,.
\end{aligned} \tag{93}$$

This yields the following fully explicit result for $t(r)$

$$t(r) = t_0 + 2\sqrt{2m}\left(\sqrt{r} - \sqrt{r_0}\right) + 2m\ln\left[\frac{(\sqrt{r}-\sqrt{2m})(\sqrt{r_0}+\sqrt{2m})}{(\sqrt{r_0}-\sqrt{2m})(\sqrt{r}+\sqrt{2m})}\right]$$
$$+S_r\left\{\frac{E}{E^2-1}\left(r\sqrt{E^2-1+2m/r} - r_0\sqrt{E^2-1+2m/r_0}\right)\right.$$
$$-2m\ln\left[\frac{(2m-r_0)\left(r-2m-2Er\left(E+\sqrt{E^2-1+2m/r}\right)\right)}{(2m-r)\left(r_0-2m-2Er_0\left(E+\sqrt{E^2-1+2m/r_0}\right)\right)}\right]$$
$$\left.+\frac{Em(2E^2-3)}{(E^2-1)^{3/2}}\ln\left[\frac{m-r+E^2r+r\sqrt{(E^2-1)(E^2-1+2m/r)}}{m-r_0+E^2r_0+r_0\sqrt{(E^2-1)(E^2-1+2m/r_0)}}\right]\right\}. \tag{94}$$

Conducting a Taylor series expansion around $E=1$ gives

$$t(r) = t_0 + 2\sqrt{2m}(\sqrt{r}-\sqrt{r_0}) + 2m\ln\left[\frac{(\sqrt{r}-\sqrt{2m})(\sqrt{r_0}+\sqrt{2m})}{(\sqrt{r_0}-\sqrt{2m})(\sqrt{r}+\sqrt{2m})}\right]$$
$$+S_r\left\{2\sqrt{2m}(\sqrt{r}-\sqrt{r_0}) + \sqrt{\frac{2}{m}}\left(\frac{r^{\frac{3}{2}}-r_0^{\frac{3}{2}}}{3}\right)\right.$$
$$\left.+2m\ln\left[\frac{(\sqrt{r}-\sqrt{2m})(\sqrt{r_0}+\sqrt{2m})}{(\sqrt{r_0}-\sqrt{2m})(\sqrt{r}+\sqrt{2m})}\right]\right\} + \mathcal{O}(E-1)\,. \tag{95}$$

Hence, in the limit $E \to 1$, we have

$$t(r) = t_0 + 2\sqrt{2m}(\sqrt{r}-\sqrt{r_0}) + 2m\ln\left[\frac{(\sqrt{r}-\sqrt{2m})(\sqrt{r_0}+\sqrt{2m})}{(\sqrt{r_0}-\sqrt{2m})(\sqrt{r}+\sqrt{2m})}\right]$$
$$+S_r\left\{2\sqrt{2m}(\sqrt{r}-\sqrt{r_0}) + \sqrt{\frac{2}{m}}\left(\frac{r^{\frac{3}{2}}-r_0^{\frac{3}{2}}}{3}\right)\right.$$
$$\left.+2m\ln\left[\frac{(\sqrt{r}-\sqrt{2m})(\sqrt{r_0}+\sqrt{2m})}{(\sqrt{r_0}-\sqrt{2m})(\sqrt{r}+\sqrt{2m})}\right]\right\}, \tag{96}$$

which is just Equation (79), as expected.

Lastly, the somewhat unwieldy integral in our general expression (52) for $\phi(r)$ is now given by

$$\int_{r_0}^{r}\frac{2EJ}{\bar{r}^3(1-2m/\bar{r})\sqrt{E^2-1+2m/\bar{r}}}\,\mathrm{d}\bar{r}$$
$$= \frac{EJ}{m^2}\left(\sqrt{E^2-1+2m/r} - \sqrt{E^2-1+2m/r_0}\right)$$
$$-\frac{J}{2m^2}\left\{\ln\left[\frac{1-2m/r_0}{1-2m/r}\right] + \ln\left[\frac{2E(E+\sqrt{E^2-1+2m/r})-(1-2m/r)}{2E(E+\sqrt{E^2-1+2m/r_0})-(1-2m/r_0)}\right]\right\}, \tag{97}$$

so we find that $\phi(r)$ reduces to

$$\phi(r) = \phi_0 - S_r S_\phi \left\{ \frac{J}{2m^2} \ln\left[\frac{(\sqrt{r}-\sqrt{2m})(\sqrt{r_0}+\sqrt{2m})}{(\sqrt{r_0}-\sqrt{2m})(\sqrt{r}+\sqrt{2m})} \right] \right.$$
$$\left. +2J\sqrt{\frac{2}{m}}\left[\frac{1}{\sqrt{r}}\left(\frac{1}{3r}+\frac{1}{2m}\right) - \frac{1}{\sqrt{r_0}}\left(\frac{1}{3r_0}+\frac{1}{2m}\right) \right] \right\}$$
$$+S_r\left\{ \frac{EJ}{m^2}\left(\sqrt{E^2-1+2m/r} - \sqrt{E^2-1+2m/r_0}\right) \right.$$
$$\left. -\frac{J}{2m^2}\left[\ln\left(\frac{1-2m/r_0}{1-2m/r}\right) + \ln\left(\frac{2E(E+\sqrt{E^2-1+2m/r}) - (1-2m/r)}{2E(E+\sqrt{E^2-1+2m/r_0}) - (1-2m/r_0)} \right) \right] \right\}. \tag{98}$$

If we now make the direct substitution $E = 1$, we find

$$\phi(r) = \phi_0 - S_r S_\phi \left\{ \frac{J}{2m^2} \ln\left[\frac{(\sqrt{r}-\sqrt{2m})(\sqrt{r_0}+\sqrt{2m})}{(\sqrt{r_0}-\sqrt{2m})(\sqrt{r}+\sqrt{2m})} \right] \right.$$
$$\left. +2J\sqrt{\frac{2}{m}}\left[\frac{1}{\sqrt{r}}\left(\frac{1}{3r}+\frac{1}{2m}\right) - \frac{1}{\sqrt{r_0}}\left(\frac{1}{3r_0}+\frac{1}{2m}\right) \right] \right\}$$
$$+S_r\left\{ \frac{2J}{m\sqrt{2m}}\left(\frac{1}{\sqrt{r}} - \frac{1}{\sqrt{r_0}}\right) + \frac{J}{2m^2}\ln\left[\frac{(\sqrt{r}-\sqrt{2m})(\sqrt{r_0}+\sqrt{2m})}{(\sqrt{r_0}-\sqrt{2m})(\sqrt{r}+\sqrt{2m})} \right] \right\}, \tag{99}$$

which is just Equation (84), as expected.

Overall, while these equations of motion are rather long, they are fully explicit and have appropriate limits when $E \to 1$.

In all generality, geodesics with $E > 1$ are unbound geodesics. When (as herein) the Carter constant is zero, the ingoing geodesics are the so-called "hail" geodesics, modelling a ZAMO fired in from spatial infinity with a nonzero initial velocity. These geodesics are ingoing retrograde, and are, hence, given by

$$\frac{\mathrm{d}t}{\mathrm{d}r} = \frac{\sqrt{2mr}}{r-2m} - \frac{E}{(1-2m/r)\sqrt{E^2-1+2m/r}}, \tag{100}$$

and

$$\frac{\mathrm{d}\phi}{\mathrm{d}r} = -\frac{2J\sqrt{2mr}}{r^3(r-2m)} - \frac{2EJ}{r^3(1-2m/r)\sqrt{E^2-1+2m/r}}. \tag{101}$$

5.2.3. Bound Geodesics $E < 1$

Once again we repeat the full list of conditions from Equation (73):

$$\mathcal{C} = 0\,; \quad L = 0\,; \quad \theta(r) = \theta_0\,; \quad X(r) = E^2 - 1 + \frac{2m}{r}. \tag{102}$$

If we let $E < 1$, then $X(r) = 0$ has a unique root at $r = \frac{2m}{1-E^2}$, and in order to keep $\sqrt{X(r)}$ *real*, we see that we have the constraint that $r \in (0, \frac{2m}{1-E^2}]$. In particular, $r \leq 2m/(1-E^2)$. Physically, geodesics with $E < 1$ are gravitationally bound (and, because $\mathcal{C} = L = 0$, must eventually crash into the central singularity). These particular bound geodesics are the so-called "drip" geodesics, corresponding to a ZAMO being dropped from some finite r_* with zero initial velocity.

For $\lambda(r)$ we have

$$\lambda(r) = \lambda_0 + S_r \int_{r_0}^{r} \frac{\mathrm{d}\bar{r}}{\sqrt{E^2 - 1 + 2m/\bar{r}}}\,. \tag{103}$$

Explicitly, a brief computation yields

$$\begin{aligned}\lambda(r) \quad = \quad & \lambda_0 + S_r \left\{ \frac{r\sqrt{E^2-1+2m/r}}{1-E^2} + \frac{m \sin^{-1}\left(1+\frac{E^2 r}{m} - \frac{r}{m}\right)}{(1-E^2)^{3/2}} \right\} \\ & -S_r \left\{ \frac{r_0\sqrt{E^2-1+2m/r_0}}{1-E^2} + \frac{m \sin^{-1}\left(1+\frac{E^2 r_0}{m} - \frac{r_0}{m}\right)}{(1-E^2)^{3/2}} \right\}. \end{aligned} \tag{104}$$

At intermediate stages of the computation it is useful to use the identity

$$\ln(x+iy) = \frac{1}{2}\ln(x^2+y^2) + i\cos^{-1}\left(\frac{x}{\sqrt{x^2+y^2}}\right). \tag{105}$$

For $S_r = +1$ these drip geodesics will crash into the central singularity $r = 0$ in finite affine time

$$\lambda_{crash} = \lambda_0 + \frac{m\pi}{2(1-E^2)^{3/2}} - \left\{ \frac{r_0\sqrt{E^2-1+2m/r_0}}{1-E^2} + \frac{m \sin^{-1}\left(1+\frac{E^2 r_0}{m} - \frac{r_0}{m}\right)}{(1-E^2)^{3/2}} \right\}. \tag{106}$$

These "drip" geodesics are *qualitatively* (not quantitatively) somewhat similar to the "rain" geodesics given by Equations (85) and (87).

When it comes to evaluating $t(r)$ the result (94) still *formally* holds, but with the understanding that for $E < 1$ the trailing term in (94) becomes

$$\begin{aligned} & \frac{1}{(E^2-1)^{3/2}} \ln\left[\frac{m - r + E^2 r + r\sqrt{(E^2-1)(E^2-1+2m/r)}}{m - r_0 + E^2 r_0 + r_0\sqrt{(E^2-1)(E^2-1+2m/r_0)}}\right] \\ & = \frac{1}{i^3(1-E^2)^{3/2}} \ln\left[\frac{m - r + E^2 r + ir\sqrt{(1-E^2)(E^2-1+2m/r)}}{m - r_0 + E^2 r_0 + ir_0\sqrt{(1-E^2)(E^2-1+2m/r_0)}}\right] \\ & = -\frac{1}{(1-E^2)^{3/2}} \left\{ \sin^{-1}\left(\frac{r}{m}\sqrt{(1-E^2)(E^2-1+2m/r)}\right) \right. \\ & \qquad \left. - \sin^{-1}\left(\frac{r_0}{m}\sqrt{(1-E^2)(E^2-1+2m/r_0)}\right) \right\}. \end{aligned} \tag{107}$$

At intermediate stages of the computation it is now useful to use the slightly different identity

$$\ln(x+iy) = \frac{1}{2}\ln(x^2+y^2) + i\sin^{-1}\left(\frac{y}{\sqrt{x^2+y^2}}\right). \tag{108}$$

This now enforces *manifest reality* of $t(r)$ for $E < 1$ and, in all its glory, we have

$$
\begin{aligned}
t(r) \quad = \quad & t_0 + 2\sqrt{2m}(\sqrt{r} - \sqrt{r_0}) + 2m \ln\left[\frac{(\sqrt{r} - \sqrt{2m})(\sqrt{r_0} + \sqrt{2m})}{(\sqrt{r_0} - \sqrt{2m})(\sqrt{r} + \sqrt{2m})}\right] \\
& + S_r \left\{ \frac{E}{E^2 - 1}\left(r\sqrt{E^2 - 1 + 2m/r} - r_0\sqrt{E^2 - 1 + 2m/r_0}\right) \right. \\
& \quad -2m \ln\left[\frac{(2m - r_0)\left(r - 2m - 2Er\left(E + \sqrt{E^2 - 1 + 2m/r}\right)\right)}{(2m - r)\left(r_0 - 2m - 2Er_0\left(E + \sqrt{E^2 - 1 + 2m/r_0}\right)\right)}\right] \\
& \quad + \frac{Em(3 - 2E^2)}{(1 - E^2)^{3/2}} \left[\sin^{-1}\left(\frac{r}{m}\sqrt{(1 - E^2)(E^2 - 1 + 2m/r)}\right)\right. \\
& \qquad \left.\left. - \sin^{-1}\left(\frac{r_0}{m}\sqrt{(1 - E^2)(E^2 - 1 + 2m/r_0)}\right)\right]\right\}. \qquad (109)
\end{aligned}
$$

Ingoing geodesics $S_r = -1$ (in this context the "drip" geodesics) will crash into the central singularity in finite Killing time

$$
\begin{aligned}
t_{crash} \quad = \quad & t_0 + 2\sqrt{2mr_0} + 2m \ln\left[\frac{(\sqrt{r_0} + \sqrt{2m})}{(\sqrt{r_0} - \sqrt{2m})}\right] + \left\{ \frac{-Er_0}{1 - E^2}\left(\sqrt{E^2 - 1 + 2m/r_0}\right) \right. \\
& \left. + 2m \ln\left[\frac{(r_0 - 2m)}{\left(r_0 - 2m - 2Er_0\left(E + \sqrt{E^2 - 1 + 2m/r_0}\right)\right)}\right]\right\}. \qquad (110)
\end{aligned}
$$

There are of course many other ways of rearranging this result.

Finally for $\phi(r)$ the result (98) can be extended to the region $E < 1$ without alteration.

6. Conclusions

From this discussion we have seen that, once given the non-trivial Killing tensor for the Lense–Thirring spacetime, we can extract the Carter constant; the fourth constant of the motion. Then, the geodesic equations become integrable, at least in principle (in terms of ultra-elliptic integrals). This allows us to formally solve for myriads of general geodesics. However, we saw that in full generality, we could not explicitly integrate the equations of motion in closed form, the ultra-elliptic integrals are not "elementary". Only when imposing further conditions, such as Carter constant zero, could we then explicitly integrate the equations of motion in an algebraically closed form.

The explicit geodesics given in this discussion are quite tractable and can be applied to a number of astrophysically interesting cases. For example, the Carter constant zero geodesics with $E < 1$ are the "drip" geodesics of the spacetime, while the Carter constant zero geodesics with $E = 1$ are the "rain" geodesics, and the Carter constant zero geodesics with $E > 1$ are the "hail" geodesics of the spacetime.

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