GEOMETRIC CATEGORIES AND O-MINIMAL STRUCTURES*

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INTRODUCTION

The theory of subanalytic sets is an excellent tool in various analytic-geometric contexts; see, for example, Bierstone and Milman [1]. Regrettably, certain "nice" sets—like $\{ (x, x^r) : x > 0 \}$ for positive irrational r, and $\{ (x, e^{-1/x}) : x > 0 \}$—are not subanalytic (at the origin) in $\mathbb{R}^2$. Here we make available an extension of the category of subanalytic sets that has these sets among its objects, and that behaves much like the category of subanalytic sets. The possibility of doing this emerged in 1991 when Wilkie [27] proved that the real exponential field is "model complete", followed soon by work of Ressayre, Macintyre, Marker and the authors; see [21], [5], [7] and [19]. However, there are two obstructions to the use by geometers of this development: (i) while the *proofs* in these articles make essential use of model theory, many *results* are also stated there (efficiently, but unnecessarily) in model-theoretic terms; (ii) the results of these papers apply directly only to the cartesian spaces $\mathbb{R}^n$, and not to arbitrary real analytic manifolds. Consequently, in order to carry out our goal we recast here some results in those papers—as well as many of their consequences—in more familiar terms, with emphasis on results of a geometric nature, and allowing arbitrary (real analytic) manifolds as ambient spaces. We thank W. Schmid and K. Vilonen for their suggestion that this would be a useful undertaking; indeed, they gave us a "wish list" (inspired by Chapters 8 and 9 of Kashiwara and Schapira [12]; see also §10 of [22]) which strongly influenced the form and content of this paper.

We axiomatize in Section 1 the notion of "behaving like the category of subanalytic sets" by introducing the notion of "analytic-geometric category". (The category $\mathcal{C}_{\text{an}}$ of subanalytic sets is the "smallest" analytic-geometric category.) We also state in Section 1 a number of properties shared by all analytic-geometric categories. Proofs of the more difficult results of this nature, like the Whitney-stratifiability of sets and maps in such a category, often involve the use of charts to reduce to the case of subsets of $\mathbb{R}^n$. For subsets of $\mathbb{R}^n$, there already exists the theory of "o-minimal structures on the real field" (defined in Section 2); this subject is developed in detail in [4] and is an abstraction of

*This is a revised version of the paper of the same name appearing in Duke Math. J. **84** (1996), 497–540. References have been updated, and the proof of the "C^p Zero Set" theorem has been corrected and improved. Unfortunately, a gap has been discovered in the proof of C^p Whitney stratification of maps—see D.16—but this gap can be repaired (a corrigendum will be available one of these days). –CLM the theory of semialgebraic sets (see e.g. Bochnak et al. [2]). Each analytic-geometric category arises in a natural way from an o-minimal structure on the real field. On the other hand, every o-minimal structure on the real field gives rise to a "geometric category", a notion more general than "analytic-geometric category". In Section 3 we explain this correspondence between geometric categories and o-minimal structures, and introduce two analytic-geometric categories $\mathcal{C}_{\text{an}}^{\mathbb{R}}$ and $\mathcal{C}_{\text{an,exp}}$, with $\mathcal{C}_{\text{an}}^{\mathbb{R}}$ strictly larger than $\mathcal{C}_{\text{an}}$, and $\mathcal{C}_{\text{an,exp}}$ strictly larger than $\mathcal{C}_{\text{an}}^{\mathbb{R}}$. The subsets $\{\,(x,x^r) : x>0\,\}$ of $\mathbb{R}^2$, $r\in\mathbb{R}$, are objects of $\mathcal{C}_{\text{an}}^{\mathbb{R}}$, while the subset $\{\,(x,e^{-1/x}) : x>0\,\}$ of $\mathbb{R}^2$ is an object of $\mathcal{C}_{\text{an,exp}}$ but not of $\mathcal{C}_{\text{an}}^{\mathbb{R}}$. (Work is underway to construct still larger analytic-geometric categories.)

In Section 4, we list many of the nice properties of o-minimal structures on the real field (some of which are described here for the first time in print). Using the correspondence established in Section 3, it is usually a routine matter to transfer these properties to corresponding properties of geometric categories.

The categories $\mathcal{C}_{\text{an}}^{\mathbb{R}}$ and $\mathcal{C}_{\text{an,exp}}$ have certain special properties, some of which we discuss in Section 5.

We stress that much of the substance of this article derives from the partly model-theoretic papers mentioned above, as well as from the (almost model-theory-free) book [4]; to avoid distraction, we defer to appendices the proofs of assertions made without reference to these (or other) sources. The reader may find it useful to first read Appendix A before consulting other appendices.

Shiota's announcement [24]* lists several results that seem closely related to some of the material presented here. The different axiomatic setting of [24] makes detailed comparisons cumbersome We do consider our setting—where we make a clear distinction between analytic-geometric categories and o-minimal structures—as more convenient.

To make this article accessible to a wider audience we explicitly define some geometric notions like "Whitney stratification".

1. Analytic-geometric categories

We use the following notation: Given a topological space X and $A\subseteq X$, we let $\operatorname{cl}(A)$, $\operatorname{int}(A)$, $\operatorname{bd}(A)$ $(=\operatorname{cl}(A)\setminus\operatorname{int}(A))$ and $\operatorname{fr}(A)$ $(=\operatorname{cl}(A)\setminus A)$ denote respectively the closure, interior, boundary and frontier of A in X.

Throughout this paper, each manifold is assumed to be Hausdorff, with a countable basis for its topology, and of the same (finite) dimension at all of its points. Also, "manifold" will mean "real analytic manifold" unless otherwise specified.

We say that an *analytic-geometric category* $\mathcal{C}$ is given if each manifold M is equipped with a collection $\mathcal{C}(M)$ of subsets of M such that the following five conditions are satisfied for all manifolds M and N:

AG1. $\mathcal{C}(M)$ is a boolean algebra of subsets of M, with $M\in\mathcal{C}(M)$.
AG2. If $A\in\mathcal{C}(M)$, then $A\times\mathbb{R}\in\mathcal{C}(M\times\mathbb{R})$.
AG3. If $f: M\to N$ is a proper analytic map and $A\in\mathcal{C}(M)$, then $f(A)\in\mathcal{C}(N)$.
AG4. If $A\subseteq M$ and (U_i) is an open covering of M (i in some index set I), then $A\in\mathcal{C}(M)$ if and only if $A\cap U_i\in\mathcal{C}(U_i)$ for all $i\in I$.

*Shiota's book "Geometry of Subanalytic and Semialgebraic Sets" (Birkhäuser, 1997) is more current.

AG5. Every bounded set in $\mathcal{C}(\mathbb{R})$ has finite boundary.

This indeed gives rise to a category $\mathcal{C}$, as we show in detail in Appendix D. An object of $\mathcal{C}$ is a pair (A, M) with M a manifold and $A \in \mathcal{C}(M)$. A morphism $(A, M) \to (B, N)$ is a continuous map $f : A \to B$ whose graph

$$\Gamma(f) := \{ (a, f(a)) : a \in A \} \subseteq A \times B$$

belongs to $\mathcal{C}(M \times N)$. Composition of morphisms is given by composition of maps, and $1_{(A,M)}$ is the identity map on A for each object (A, M). We usually refer to an object (A, M) of $\mathcal{C}$ as the *$\mathcal{C}$-set A in M*, or even just the *$\mathcal{C}$-set A* if its ambient manifold is clear from context. Similarly, a morphism $f : (A, M) \to (B, N)$ is called a *$\mathcal{C}$-map* $f : A \to B$ if M and N are clear from context.

All subanalytic subsets of a manifold are $\mathcal{C}$-sets in that manifold; in particular, each finite subset of a manifold is a $\mathcal{C}$-set. (See Appendix D.) Since the subanalytic sets (in manifolds) satisfy axioms AG1 through AG5, it follows that the category $\mathcal{C}_{\mathrm{an}}$ of subanalytic sets and continuous subanalytic maps is the "smallest" analytic-geometric category.

We now record some basic properties (proved in Appendix D).

For the rest of this section we fix an analytic-geometric category $\mathcal{C}$. We let M, N be manifolds of dimension m, n respectively, and let $A \in \mathcal{C}(M)$, $B \in \mathcal{C}(N)$.

1.1. *Every analytic map $f : M \to N$ is a $\mathcal{C}$-map.*

1.2. *Given an open covering (U_i) of M, a map $f : A \to N$ is a $\mathcal{C}$-map if and only if each restriction $f|U_i \cap A : U_i \cap A \to N$ is a $\mathcal{C}$-map.*

1.3. *$A \times B \in \mathcal{C}(M \times N)$, and the projections $A \times B \to A$ and $A \times B \to B$ are $\mathcal{C}$-maps.*

1.4. *If $f : A \to N$ is a proper $\mathcal{C}$-map and $X \subseteq A$ is a $\mathcal{C}$-set, then $f(X) \in \mathcal{C}(N)$.*

1.5. *If A is closed in M and $f : A \to N$ is a $\mathcal{C}$-map, then $f^{-1}(B) \in \mathcal{C}(M)$.*

1.6. *If $B_1, \ldots, B_k$ are $\mathcal{C}$-sets (in possibly different manifolds), then a map*

$$f = (f_1, \ldots, f_k) : A \to B_1 \times \cdots \times B_k$$

is a $\mathcal{C}$-map if and only if each $f_i : A \to B_i$ is a $\mathcal{C}$-map.

1.7. $\mathrm{cl}(A), \mathrm{int}(A) \in \mathcal{C}(M)$.

Convention. Throughout this paper, we let p range over $\{1, 2, 3, \ldots, \infty, \omega\}$.

To state the next few properties, we define for each p the set $\mathrm{Reg}^p(A)$ of C^p smooth points of A (where "C^ω" means "analytic"). More precisely, $\mathrm{Reg}^p(A)$ is the set of all $x \in A$ such that there is an open neighborhood U of x with $U \cap A$ a C^p submanifold of M. (Here and throughout this paper, submanifolds are embedded—not just immersed—submanifolds; in particular, they are locally closed in their ambient manifolds.) Also for each $k \in \mathbb{N}$ we let $\mathrm{Reg}^p_k(A)$ be the set of all $x \in A$ such that there is an open neighborhood U of x with $U \cap A$ a C^p submanifold of M of dimension k; so we have the disjoint union

$$\mathrm{Reg}^p(A) = \mathrm{Reg}^p_0(A) \cup \cdots \cup \mathrm{Reg}^p_m(A).$$

1.8. *For each $k \in \mathbb{N}$ and positive $p \in \mathbb{N}$,* $\operatorname{Reg}_k^p(A) \in \mathcal{C}(M)$.

Remark. 1.8 holds with $p = \omega$ for the analytic-geometric categories $\mathcal{C}_{\mathrm{an}}$ and $\mathcal{C}_{\mathrm{an}}^{\mathbb{R}}$. We don't know if there are analytic-geometric categories for which 1.8 fails with $p = \omega$. (See Section 5 for further information.)

Recall that the tangent bundle TM and the cotangent bundle T^*M of M are again manifolds (of dimension $2m$). If A is a C^1 submanifold of M we consider its tangent bundle

$$TA = \bigcup_{x \in A} T_x A$$

as a subset of the tangent bundle TM, and we define the conormal bundle T_A^*M of A in M by

$$T_A^*M = \bigcup_{x \in A} \{\, \xi \in T_x^*M : \xi | T_x A = 0 \,\},$$

a subbundle of $T^*M|A$.

1.9. *If A is a C^1 submanifold of M, then $TA \in \mathcal{C}(TM)$ and $T_A^*M \in \mathcal{C}(T^*M)$. If in addition $f : A \to N$ is a $\mathcal{C}$-map of class C^1, then $Tf : TA \to TN$ is a $\mathcal{C}$-map.*

The properties listed so far are elementary consequences of axioms AG1 through AG4, unlike the following somewhat deeper results, which also depend on AG5.

1.10. *A has locally only a finite number of components* (*that is, each point $x \in M$ has an open neighborhood U such that $U \cap A$ has finitely many connected components*).

1.11. *If C is a connected component of A, then $C \in \mathcal{C}(M)$.*

1.12. *A is locally connected* (*hence every component of A is open in A*).

1.13. *If A is connected, then A is path connected.*

1.14. *If A is relatively compact and $f : A \to N$ is a $\mathcal{C}$-map, then for each compact set $Y \subseteq N$ there exists $K_Y \in \mathbb{N}$ such that each fiber $f^{-1}(y)$ with $y \in Y$ has at most K_Y connected components.*

Recall that a collection $\mathcal{F}$ of subsets of M is said to be *locally finite* if each point in M has a neighborhood that intersects only finitely many sets in $\mathcal{F}$, or equivalently, if every compact subset of M intersects only finitely many sets in $\mathcal{F}$. Note that if $\mathcal{F} \subseteq \mathcal{C}(M)$ is locally finite, then $\bigcap \mathcal{F} \in \mathcal{C}(M)$ and $\bigcup \mathcal{F} \in \mathcal{C}(M)$.

1.15. *The set of connected components of A is a locally finite subcollection of $\mathcal{C}(M)$.*

We define the *dimension* $\dim A$ of a nonempty set $A \in \mathcal{C}(M)$ to be the maximum of all $d \in \mathbb{N}$ such that A contains a d-dimensional C^1 submanifold of M (so $0 \le \dim A \le m$). We also put $\dim \emptyset := -\infty$. (For C^1 submanifolds of M, this agrees with the usual manifold dimension.)

1.16.

(1) *If* $\mathcal{A} \subseteq \mathcal{C}(M)$ *is locally finite,* $\mathcal{A} \neq \emptyset$, *then* $\dim \bigcup \mathcal{A} = \max\{\dim A : A \in \mathcal{A}\}$.
(2) *If* $f : A \to N$ *is a proper* $\mathcal{C}$*-map, then* $\dim C \geq \dim f(C)$ *for all* $\mathcal{C}$*-sets* $C \subseteq A$.
(3) *If* $A \neq \emptyset$, *then* $\dim \mathrm{fr}(A) < \dim A$.

In the subanalytic category one often reduces problems to the one-dimensional case, where extra tools are available. This works also in our setting, as the next two results show.

1.17. Curve selection. *If* $x \in \mathrm{fr}(A)$, *then there is a* $\mathcal{C}$*-map* $\gamma : [0,1) \to M$ *such that* $\gamma(0,1) \subseteq A$ *and* $\gamma(0) = x$.

Here we consider $[0,1)$ as a $\mathcal{C}$-set in $\mathbb{R}$. Given any positive integer p we can always choose γ to be injective and C^p. If $\mathcal{C} = \mathcal{C}_{\mathrm{an}}^{\mathbb{R}}$ or $\mathcal{C} = \mathcal{C}_{\mathrm{an,exp}}$ we can choose γ to be injective, and analytic on $(0,1)$. If $\mathcal{C} = \mathcal{C}_{\mathrm{an}}$ we can choose γ such that it extends to an analytic function from $(-1,1)$ into M, but this fails in general for $\mathcal{C} = \mathcal{C}_{\mathrm{an}}^{\mathbb{R}}$ or $\mathcal{C} = \mathcal{C}_{\mathrm{an,exp}}$.

1.18. Local parametrization of 1-dimensional $\mathcal{C}$-sets. *Let* $x \in M$, p *be a positive integer and* $\dim A = 1$. *Then there is a relatively compact open* $U \in \mathcal{C}(M)$ *with* $x \in U$ *and there are injective* $\mathcal{C}$*-maps* $\gamma_1, \dots, \gamma_k : [0,1) \to M$ *of class* C^p *such that* $\gamma_i(0) = x$ *for* $i = 1, \dots, k$ *and* $U \cap A \setminus \{x\}$ *is the disjoint union of* $\gamma_1(0,1), \dots, \gamma_k(0,1)$. (*Of course,* $k = 0$ *if* $x \notin \mathrm{cl}(A)$.)

Remark. In fact, the γ_i's can be chosen such that in addition each γ_i maps the interval $(0,1)$ C^p diffeomorphically onto a C^p submanifold of M.

Before we can state the next result, we need several definitions and some notation.

For $k, m \in \mathbb{N}$, let $G_k(\mathbb{R}^m)$ denote the Grassmannian of the k-dimensional vector subspaces of $\mathbb{R}^m$; in particular, $G_1(\mathbb{R}^m) = \mathbb{P}^{m-1}(\mathbb{R})$.

Let X, Y be C^1 submanifolds of $\mathbb{R}^m$ with $\dim X = k$ and let $y \in Y$. We say that the triple (X, Y, y) has the *Whitney property* if the following holds: for every sequence (x_i) of points in X converging to y and every sequence (y_i) of points in Y converging to y with $x_i \neq y_i$ for all i such that the sequence $(T_{x_i}X)$ converges to some $\tau \in G_k(\mathbb{R}^m)$ and the sequence of secant lines $(\mathbb{R}.(x_i - y_i))$ converges to a line $\ell \in G_1(\mathbb{R}^m)$, we have $\ell \subseteq \tau$. We now extend this definition to arbitrary ambient manifolds M. Let X, Y be C^1 submanifolds of M with $\dim X = k$ and let $y \in Y$. We say that the triple (X, Y, y) has the Whitney property if for some (equivalently, for every) C^1 diffeomorphism φ of an open neighborhood U of y onto an open subset $\varphi(U) \subseteq \mathbb{R}^m$, the triple $(\varphi(U \cap X), \varphi(U \cap Y), \varphi(y))$ has the Whitney property. Put

$$W(X, Y) := \{\, y \in Y : (X, Y, y) \text{ has the Whitney property} \,\}.$$

We say that the pair (X, Y) has the Whitney property if $W(X, Y) = Y$.

Remark. What we call the Whitney property is often referred to as "Whitney's condition (b)".

A C^p *stratification* of a closed subset S of M is a locally finite partition $\mathcal{P}$ of S into C^p submanifolds of M, called *strata*, such that if $X, Y \in \mathcal{P}$ with $X \neq Y$ and $\mathrm{cl}(X) \cap Y \neq \emptyset$, then $Y \subseteq \operatorname{fr}(X)$ and $\dim Y < \dim X$. A single member of a stratification is called a *stratum*. For $p = \omega$ we also say "analytic stratification". (We are mainly interested in the case that $S \in \mathcal{C}(M)$ and $\mathcal{P} \subseteq \mathcal{C}(M)$, and in that case the condition "$\dim Y < \dim X$" in the definition is automatic by 1.16(3).) A C^p *Whitney stratification* of a closed set $S \subseteq M$ is a C^p stratification $\mathcal{W}$ of S such that for all $X, Y \in \mathcal{W}$, if $Y \subseteq \operatorname{fr}(X)$, then (X, Y) has the Whitney property.

Given a C^1 map $f : X \to Y$ between C^1 manifolds X and Y, we define the function $\operatorname{rk} f : X \to \mathbb{N}$ by $\operatorname{rk} f(x) := \operatorname{rank}(T_x f)$, where $T_x f : T_x X \to T_{f(x)} Y$ is the induced linear map between the tangent spaces.

Given a closed set $S \subseteq M$ and $f : S \to N$, a C^p *Whitney stratification of* f is a pair $(\mathcal{S}, \mathcal{T})$, where $\mathcal{S}$ and $\mathcal{T}$ are C^p Whitney stratifications of S and N respectively, such that for each $P \in \mathcal{S}$, the map $f|P : P \to N$ is C^p with $f(P) \in \mathcal{T}$ and $(\operatorname{rk} f|P)(x) = \dim f(P)$ for all $x \in P$.

Given collections $\mathcal{A}$, $\mathcal{B}$ of subsets of a set C, we say that $\mathcal{A}$ is *compatible* with $\mathcal{B}$ if for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$, either $A \cap B = \emptyset$ or $A \subseteq B$.

1.19. Whitney stratification. *Let $S \in \mathcal{C}(M)$ be closed and p be a positive integer.*

(1) *For every locally finite $\mathcal{A} \subseteq \mathcal{C}(M)$ there is a C^p Whitney stratification $\mathcal{P} \subseteq \mathcal{C}(M)$ of S, compatible with $\mathcal{A}$, with each stratum connected and relatively compact.*

(2) *Let $f : S \to N$ be a proper $\mathcal{C}$-map and $\mathcal{F} \subseteq \mathcal{C}(M)$, $\mathcal{G} \subseteq \mathcal{C}(N)$ be locally finite. Then there is a C^p Whitney stratification $(\mathcal{S}, \mathcal{T})$ of f with connected strata such that $\mathcal{S} \subseteq \mathcal{C}(M)$ is compatible with $\mathcal{F}$ and $\mathcal{T} \subseteq \mathcal{C}(N)$ is compatible with $\mathcal{G}$.*

Remarks.

In (2) above, we may require $f|A$ to be injective for each A in $\mathcal{S}$ with $\operatorname{rk} f|A = \dim A$.

Whitney stratification holds with $p = \omega$ for the analytic-geometric categories $\mathcal{C}_{\text{an}}$, $\mathcal{C}_{\text{an}}^{\mathbb{R}}$ and $\mathcal{C}_{\text{an,exp}}$. We don't know if there are analytic-geometric categories for which 1.19 fails with $p = \omega$.

1.20. *If $A \in \mathcal{C}(M)$ is closed and p is a positive integer, then there is a $\mathcal{C}$-map $f : M \to \mathbb{R}$ of class C^p with $A = Z(f) := \{ x \in M : f(x) = 0 \}$.*

Bierstone, Milman and Pawłucki gave a proof of this for the case $\mathcal{C} = \mathcal{C}_{\text{an}}$, which generalizes to arbitrary $\mathcal{C}$; see Appendices C and D for details.

Remark. We do not know if the uniformization and rectilinearization properties of subanalytic sets (see 0.1, 0.2 of [1]) have suitable analogs for $\mathcal{C}$-sets.

2. Structures on $(\mathbb{R}, +, \cdot)$

Proofs of properties of $\mathcal{C}$-sets often involve charts to reduce to the case where the ambient manifold is $\mathbb{R}^m$. For the cartesian spaces $\mathbb{R}^m$ there is available a notion of "globally nice" set, which is more convenient to deal with than the strictly local notion of $\mathcal{C}$-set, and is also better behaved: we don't need properness or (relative) compactness assumptions, and "locally finite" can often be replaced by "finite". "Semialgebraic set in $\mathbb{R}^m$" is an example of such a notion of "globally nice" set, but "subanalytic set in $\mathbb{R}^m$" is not.

In developing the theory of analytic-geometric categories from scratch it is most efficient to first deal systematically with this equivalent but better-behaved *global* notion of nice set (restricting oneself to ambient spaces $\mathbb{R}^m$). Once the properties of these "nice" sets are available, one can then use these sets as "affine models" to define the $\mathcal{C}$-sets of analytic-geometric categories and obtain all relevant properties. Proceeding this way is also more convenient for actually *constructing* analytic-geometric categories. In this sense, things in this paper are partly done in reverse; the next three sections introduce tools that enable us to prove the properties listed in the previous section. (In the appendices, we follow the strictly "logical" order.)

Definition. A *structure on the real field* $(\mathbb{R}, +, \cdot)$ is a sequence $\mathfrak{S} = (\mathfrak{S}_n)_{n\in\mathbb{N}}$ such that for each $n \in \mathbb{N}$:

S1. $\mathfrak{S}_n$ is a boolean algebra of subsets of $\mathbb{R}^n$, with $\mathbb{R}^n \in \mathfrak{S}_n$.
S2. $\mathfrak{S}_n$ contains the diagonals $\{ (x_1, \ldots, x_n) \in \mathbb{R}^n : x_i = x_j \}$ for $1 \le i < j \le n$.
S3. If $A \in \mathfrak{S}_n$, then $A \times \mathbb{R}$ and $\mathbb{R} \times A$ belong to $\mathfrak{S}_{n+1}$.
S4. If $A \in \mathfrak{S}_{n+1}$, then $\pi(A) \in \mathfrak{S}_n$, where $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is the projection on the first n coordinates.
S5. $\mathfrak{S}_3$ contains the graphs of addition and multiplication.

Below, $\mathfrak{S}$ denotes a structure on $(\mathbb{R}, +, \cdot)$. We say that a set $A \subseteq \mathbb{R}^n$ *belongs to* $\mathfrak{S}$ if $A \in \mathfrak{S}_n$, and that a (not necessarily continuous) map $f : A \to \mathbb{R}^n$ with $A \subseteq \mathbb{R}^m$ *belongs to* $\mathfrak{S}$ if its graph $\Gamma(f) \subseteq \mathbb{R}^{m+n}$ belongs to $\mathfrak{S}$. Instead of "A belongs to $\mathfrak{S}$" we also say "$\mathfrak{S}$ contains A", and similarly for maps.

2.1. Although the *definition* of "structure on $(\mathbb{R}, +, \cdot)$" is not symmetric with respect to the coordinates $x_1, x_2, x_3, \ldots$, we obtain this symmetry and other basic facts by very elementary 'logical' arguments in Appendix B. Here we just state some of these facts.

If $B \in \mathfrak{S}_n$ and $i(1), \ldots, i(n) \in \{1, \ldots, m\}$ (repetitions allowed), then the set

$$\{ (x_1, \ldots, x_m) \in \mathbb{R}^m : (x_{i(1)}, \ldots, x_{i(n)}) \in B \}$$

belongs to $\mathfrak{S}$. (Thus, we can permute and identify variables.) If $A \subseteq \mathbb{R}^m$ and $f = (f_1, \ldots, f_n) : A \to \mathbb{R}^n$ is a map, then f belongs to $\mathfrak{S}$ if and only if $f_1, \ldots, f_n$ belong to $\mathfrak{S}$; in that case also $A \in \mathfrak{S}_m$, $f(A) \in \mathfrak{S}_n$, $f^{-1}(B) \in \mathfrak{S}_m$ for all $B \in \mathfrak{S}_n$, and each restriction $f|A' : A' \to \mathbb{R}^n$ with $A' \subseteq A$ and $A' \in \mathfrak{S}_m$ belongs to $\mathfrak{S}$. If $A = A_1 \cup \cdots \cup A_k$ with each $A_i \in \mathfrak{S}_m$, then a map $f : A \to \mathbb{R}^n$ belongs to $\mathfrak{S}$ if and only if all $f|A_i$ belong to $\mathfrak{S}$.

If $\mathfrak{S}$ contains the set $S \subseteq \mathbb{R}^{m+n}$ and the singleton $\{a\}$ with $a \in \mathbb{R}^m$, then the fiber $S_a := \{ y \in \mathbb{R}^n : (a, y) \in S \}$ belongs to $\mathfrak{S}$.

Let $A \subseteq \mathbb{R}^m$ and $B \subseteq \mathbb{R}^n$. If $f : A \to \mathbb{R}^n$ and $g : B \to \mathbb{R}^q$ belong to $\mathfrak{S}$, then the composition $g \circ f : f^{-1}(B) \to \mathbb{R}^q$ belongs to $\mathfrak{S}$. If $f : A \to \mathbb{R}^n$ belongs to $\mathfrak{S}$ and is injective, then its compositional inverse $f^{-1} : f(A) \to \mathbb{R}^m$ belongs to $\mathfrak{S}$. If $A \in \mathfrak{S}_m$, then for each rational number r the constant function $x \mapsto r : A \to \mathbb{R}$ belongs to $\mathfrak{S}$ and the set $\{ f : A \to \mathbb{R} : f \text{ belongs to } \mathfrak{S} \}$ is a ring under pointwise addition and multiplication of functions with multiplicative identity $x \mapsto 1 : A \to \mathbb{R}$. For each polynomial $f(X_1, \ldots, X_n) \in \mathbb{Q}[X_1, \ldots, X_n]$ the corresponding function $x \mapsto f(x) : \mathbb{R}^n \to \mathbb{R}$ belongs to $\mathfrak{S}$. The order relation $<$ belongs to $\mathfrak{S}$, that is, $\{ (x, y) \in \mathbb{R}^2 : x < y \} \in \mathfrak{S}_2$.

If $A \in \mathfrak{S}_m$, then $\mathrm{cl}(A), \mathrm{int}(A) \in \mathfrak{S}_m$. Given a function $f : U \to \mathbb{R}$ belonging to $\mathfrak{S}$ with U open in $\mathbb{R}^n$, the set of points in U where f is differentiable belongs to $\mathfrak{S}$, and if f is differentiable on U, then each partial derivative also belongs to $\mathfrak{S}$.

For the next two results we fix a set $A \subseteq \mathbb{R}^n$. We also let p be a positive integer and $k \in \{0, \dots, n\}$. We view $\mathbb{R}^n$ as the ambient manifold, and identify both its tangent bundle $T\mathbb{R}^n$ and its cotangent bundle $T^*\mathbb{R}^n$ with $\mathbb{R}^{2n}$ in the obvious way. In particular, the point $(a, b) \in \mathbb{R}^{2n} = T^*\mathbb{R}^n$ corresponds to the linear form $x \mapsto b.x$ on $\mathbb{R}^n = T_a\mathbb{R}^n$. If A is a C^1 submanifold of $\mathbb{R}^n$ then these identifications make its tangent bundle TA and conormal bundle $T^*_A\mathbb{R}^n$ subsets of $\mathbb{R}^{2n}$.

2.2. *If A belongs to $\mathfrak{S}$ then $\mathrm{Reg}^p_k(A)$ belongs to $\mathfrak{S}$.*

2.3. *If A belongs to $\mathfrak{S}$ and is a C^1 submanifold of $\mathbb{R}^n$ then the tangent bundle TA and the conormal bundle $T^*_A\mathbb{R}^n$ belong to $\mathfrak{S}$.*

2.4. *Let X, Y be C^1 submanifolds of $\mathbb{R}^n$ belonging to $\mathfrak{S}$ with $Y \subseteq \mathrm{fr}(X)$. Then the set $W(X, Y)$ (as in §1) belongs to $\mathfrak{S}$.*

Given structures $\mathfrak{S} = (\mathfrak{S}_n)$ and $\mathfrak{S}' = (\mathfrak{S}'_n)$ on $(\mathbb{R}, +, \cdot)$ we put $\mathfrak{S} \subseteq \mathfrak{S}'$ if $\mathfrak{S}_n \subseteq \mathfrak{S}'_n$ for all $n \in \mathbb{N}$; this defines a partial order on the set of all structures on $(\mathbb{R}, +, \cdot)$. Given functions $f_j : \mathbb{R}^{n(j)} \to \mathbb{R}$ (j in some index set J) we let $(\mathbb{R}, +, \cdot, (f_j)_{j\in J})$ denote the real field equipped with the functions f_j as extra "basic operations", and we let $\mathfrak{S}(\mathbb{R}, +, \cdot, (f_j)_{j\in J})$ denote the smallest structure on $(\mathbb{R}, +, \cdot)$ containing the graphs of all functions f_j; we call $\mathfrak{S}(\mathbb{R}, +, \cdot, (f_j)_{j\in J})$ the *structure on* $(\mathbb{R}, +, \cdot)$ *generated by the* f_j*'s*. (A function $f : \mathbb{R}^0 = \{0\} \to \mathbb{R}$ is identified with the corresponding real constant $f(0)$.)

Note. In model theory, $(\mathbb{R}, +, \cdot, (f_j)_{j\in J})$ itself is called a structure, and $A \subseteq \mathbb{R}^m$ is said to be *definable in* $(\mathbb{R}, +, \cdot, (f_j)_{j\in J})$ if A belongs to $\mathfrak{S}(\mathbb{R}, +, \cdot, (f_j)_{j\in J})$.

2.5. Examples.

(1) There is evidently a largest structure on $(\mathbb{R}, +, \cdot)$, namely the structure obtained by letting $\mathfrak{S}_n$ be the collection of all subsets of $\mathbb{R}^n$, for each $n \in \mathbb{N}$. This structure is of no further relevance in this paper.

(2) The smallest structure on $(\mathbb{R}, +, \cdot)$ is by definition $\mathfrak{S}(\mathbb{R}, +, \cdot)$. By results stated above, $\mathfrak{S}(\mathbb{R}, +, \cdot)_n$ must contain all finite unions of sets of the form

$$\{\, x \in \mathbb{R}^n : f(x) = 0, g_1(x) > 0, \dots, g_k(x) > 0 \,\}$$

with $f, g_1, \dots, g_k \in \mathbb{Q}[X_1, \dots, X_n]$. The collection of these finite unions (for $n \in \mathbb{N}$) clearly satisfies axioms S1, S2, S3 and S5, and by Tarski's theorem, also axiom S4. Hence $\mathfrak{S}(\mathbb{R}, +, \cdot)_n$ consists exactly of these finite unions. (One might call these sets "semialgebraic sets defined over $\mathbb{Q}$".) Note that a singleton $\{r\}$ with $r \in \mathbb{R}$ belongs to this structure if and only if r is algebraic.

(3) The smallest structure on $(\mathbb{R}, +, \cdot)$ that contains all singletons $\{r\}$ with $r \in \mathbb{R}$ is (by definition) $\mathfrak{S}(\mathbb{R}, +, \cdot, (r)_{r\in\mathbb{R}})$. (Note that $(\mathbb{R}, +, \cdot, (r)_{r\in\mathbb{R}})$ is just $(\mathbb{R}, +, \cdot)$ equipped with the constant functions $\mathbb{R}^0 = \{0\} \to \mathbb{R}$ as additional basic operations.) By the Tarski-Seidenberg theorem, $\mathfrak{S}(\mathbb{R}, +, \cdot, (r)_{r\in\mathbb{R}})_n$ is precisely the collection of all semialgebraic sets in $\mathbb{R}^n$ for $n \in \mathbb{N}$.

(4) Let $\mathbb{R}_{\text{an}} := (\mathbb{R}, +, \cdot, (f))$ where f ranges over all *restricted analytic functions*, that is, over all functions $\mathbb{R}^n \to \mathbb{R}$ (n ranging over $\mathbb{N}$) that vanish identically off $[-1,1]^n$ and whose restrictions to $[-1,1]^n$ are analytic. We call $\mathbb{R}_{\text{an}}$ the *field of real numbers with restricted analytic functions*; $\mathfrak{S}(\mathbb{R}_{\text{an}})$ consists of the so-called *finitely subanalytic* sets introduced in [3], and again in [13] under the name of "globally subanalytic" sets. The reason for this terminology is that a set $A \subseteq \mathbb{R}^n$ belongs to $\mathfrak{S}(\mathbb{R}_{\text{an}})$ if and only if A is subanalytic in the projective space $\mathbb{P}^n(\mathbb{R})$, where we identify $\mathbb{R}^n$ with an open subset of $\mathbb{P}^n(\mathbb{R})$ via

$$(y_1, \ldots, y_n) \mapsto (1 : y_1 : \cdots : y_n) : \mathbb{R}^n \to \mathbb{P}^n(\mathbb{R}).$$

In the next section, we will see that this fact is a special case of a 1-1 correspondence between "o-minimal" structures on $\mathbb{R}_{\text{an}}$ and analytic-geometric categories. (A *structure on* $\mathbb{R}_{\text{an}}$ is a structure $\mathfrak{S}$ on $(\mathbb{R}, +, \cdot)$ with $\mathfrak{S}(\mathbb{R}_{\text{an}}) \subseteq \mathfrak{S}$.)

(5) Let $\mathbb{R}_{\text{an}}^{\mathbb{R}} := (\mathbb{R}, +, \cdot, (f), (x^r)_{r \in \mathbb{R}})$, where f ranges over all restricted analytic functions as in (4), and the function $x^r : \mathbb{R} \to \mathbb{R}$ is given by

$$a \mapsto \begin{cases} a^r, & a > 0 \\ 0, & a \leq 0. \end{cases}$$

(6) Let $\mathbb{R}_{\text{an,exp}} := (\mathbb{R}, +, \cdot, (f), \exp)$, where f ranges over all restricted analytic functions as before, and $\exp : \mathbb{R} \to \mathbb{R}$ is given by $\exp(x) = e^x$. Note that $\mathfrak{S}(\mathbb{R}_{\text{an,exp}})$ contains the logarithm function $\log : (0, \infty) \to \mathbb{R}$, as well as each function x^r from (5), since $a^r = \exp(r \log a)$ for $a > 0$.

All of $\mathfrak{S}(\mathbb{R}_{\text{an}})$, $\mathfrak{S}(\mathbb{R}_{\text{an}}^{\mathbb{R}})$ and $\mathfrak{S}(\mathbb{R}_{\text{an,exp}})$ are structures on $\mathbb{R}_{\text{an}}$, and we have

$$\mathfrak{S}(\mathbb{R}, +, \cdot) \subseteq \mathfrak{S}(\mathbb{R}, +, \cdot, (r)_{r \in \mathbb{R}}) \subseteq \mathfrak{S}(\mathbb{R}_{\text{an}}) \subseteq \mathfrak{S}(\mathbb{R}_{\text{an}}^{\mathbb{R}}) \subseteq \mathfrak{S}(\mathbb{R}_{\text{an,exp}}).$$

These inclusions are strict: $\{e\}$ does not belong to $\mathfrak{S}(\mathbb{R}, +, \cdot)$; $\exp|[-1,1]$ is not semialgebraic; the function $x^{\sqrt{2}}$ is not finitely subanalytic, since its graph is not subanalytic at the origin; exp does not belong to $\mathfrak{S}(\mathbb{R}_{\text{an}}^{\mathbb{R}})$, since by [19] every function $f : \mathbb{R} \to \mathbb{R}$ belonging to $\mathfrak{S}(\mathbb{R}_{\text{an}}^{\mathbb{R}})$ either ultimately vanishes identically or is asymptotic at $+\infty$ to some function cx^r, $c \neq 0$. We conjecture that there are no structures on $(\mathbb{R}, +, \cdot)$ lying strictly between $\mathfrak{S}(\mathbb{R}_{\text{an}}^{\mathbb{R}})$ and $\mathfrak{S}(\mathbb{R}_{\text{an,exp}})$.

If a structure $\mathfrak{S}$ on $(\mathbb{R}, +, \cdot)$ contains each real singleton $\{r\}$, then it contains all intervals of all kinds (that is, all nonempty connected subsets of $\mathbb{R}$), and hence all finite unions of intervals of all kinds.

Definition. A structure $\mathfrak{S}$ on $(\mathbb{R}, +, \cdot)$ is *o-minimal* (short for "order-minimal") if $\mathfrak{S}_1$ consists exactly of the finite unions of intervals of all kinds (including singletons).

Sets and functions belonging to o-minimal structures on $(\mathbb{R}, +, \cdot)$ have many nice topological and geometric and metric properties, many of which we list in §4. The structure $\mathfrak{S}(\mathbb{R}_{\text{an,exp}})$ is o-minimal, hence (by the inclusions listed above) so are the structures $\mathfrak{S}(\mathbb{R}, +, \cdot, (r)_{r \in \mathbb{R}})$, $\mathfrak{S}(\mathbb{R}_{\text{an}})$ and $\mathfrak{S}(\mathbb{R}_{\text{an}}^{\mathbb{R}})$.

We also say that $(\mathbb{R}, +, \cdot, (f_j)_{j \in J})$ is o-minimal if $\mathfrak{S}(\mathbb{R}, +, \cdot, (f_j)_{j \in J}, (r)_{r \in \mathbb{R}})$ is o-minimal. Thus, $(\mathbb{R}, +, \cdot)$, $(\mathbb{R}, +, \cdot, (r)_{r \in \mathbb{R}})$, $\mathbb{R}_{\text{an}}$, $\mathbb{R}_{\text{an}}^{\mathbb{R}}$ and $\mathbb{R}_{\text{an,exp}}$ are all o-minimal, even though $\mathfrak{S}(\mathbb{R}, +, \cdot)$ is not (since it doesn't contain $\{e\}$).

3. Analytic-geometric categories correspond to o-minimal structures on $\mathbb{R}_{\text{an}}$

From an analytic-geometric category $\mathcal{C}$ we obtain an o-minimal structure $\mathfrak{S} = \mathfrak{S}(\mathcal{C})$ on $\mathbb{R}_{\text{an}}$ by defining

$$\mathfrak{S}_n = \mathfrak{S}(\mathcal{C})_n := \{\, X \subseteq \mathbb{R}^n : X \in \mathcal{C}(\mathbb{P}^n(\mathbb{R}))\,\},$$

where we identify the analytic manifold $\mathbb{R}^n$ with an open subset of the projective space $\mathbb{P}^n(\mathbb{R})$ via

$$(y_1, \dots, y_n) \mapsto (1 : y_1 : \cdots : y_n) : \mathbb{R}^n \to \mathbb{P}^n(\mathbb{R}).$$

In particular, $\mathfrak{S}_n \subseteq \mathcal{C}(\mathbb{R}^n)$ and all bounded $\mathcal{C}$-sets in $\mathbb{R}^n$ (as well as their complements) belong to $\mathfrak{S}$. An equivalent way of defining $\mathfrak{S}(\mathcal{C})$ is by means of the semialgebraic maps $\tau_n : \mathbb{R}^n \to \mathbb{R}^n$ given by

$$\tau_n(x_1, \dots, x_n) := \left(\frac{x_1}{\sqrt{1+x_1^2}}, \dots, \frac{x_n}{\sqrt{1+x_n^2}} \right).$$

Note that τ_n is an analytic isomorphism of $\mathbb{R}^n$ onto $(-1,1)^n$. Then for $A \subseteq \mathbb{R}^n$ we have:

$$A \in \mathfrak{S}(\mathcal{C})_n \Leftrightarrow \tau_n(A) \in \mathcal{C}(\mathbb{R}^n).$$

Conversely, from an o-minimal structure $\mathfrak{S} = (\mathfrak{S}_n)$ on $\mathbb{R}_{\text{an}}$ we obtain an analytic-geometric category $\mathcal{C} = \mathcal{C}(\mathfrak{S})$ by defining the $\mathcal{C}$-sets in an m-dimensional manifold M to be those sets $A \subseteq M$ such that for each point $x \in M$ there is an open neighborhood U of x, an open $V \subseteq \mathbb{R}^m$ and an analytic isomorphism $h : U \to V$ such that $h(U \cap A) \in \mathfrak{S}_m$. (In that case, we can always take $V = \mathbb{R}^m$.)

These operations $\mathcal{C} \mapsto \mathfrak{S}(\mathcal{C})$ and $\mathfrak{S} \mapsto \mathcal{C}(\mathfrak{S})$ are inverse to each other: for each analytic-geometric category $\mathcal{C}$ and each o-minimal structure $\mathfrak{S}$ on $\mathbb{R}_{\text{an}}$ we have $\mathcal{C}(\mathfrak{S}(\mathcal{C})) = \mathcal{C}$ and $\mathfrak{S}(\mathcal{C}(\mathfrak{S})) = \mathfrak{S}$.

(For proofs of the above statements see Appendix D.)

This correspondence allows us to establish facts (such as 1.7 through 1.20) about the analytic-geometric category $\mathcal{C}$ by passing to the o-minimal structure $\mathfrak{S}(\mathcal{C})$. Under this correspondence the analytic-geometric category $\mathcal{C}_{\text{an}}$ of subanalytic sets corresponds to $\mathfrak{S}(\mathbb{R}_{\text{an}})$, the class of finitely (or globally) subanalytic sets. We let $\mathcal{C}_{\text{an}}^{\mathbb{R}}$ and $\mathcal{C}_{\text{an,exp}}$ denote respectively the analytic-geometric categories $\mathcal{C}(\mathfrak{S}(\mathbb{R}_{\text{an}}^{\mathbb{R}}))$ and $\mathcal{C}(\mathfrak{S}(\mathbb{R}_{\text{an,exp}}))$.

Let $\mathfrak{S}$ be an o-minimal structure on $(\mathbb{R}, +, \cdot)$, not necessarily extending $\mathfrak{S}(\mathbb{R}_{\text{an}})$. Then we can still define a "geometric" category of "$\mathfrak{S}$-manifolds", remaining moreover in a strictly finite setting (in contrast to the "locally finite" setting of analytic-geometric categories). To be precise, define an $\mathfrak{S}$-*atlas* on a manifold M to be an atlas $(g_i)_{i \in I}$ with finite index set I such that each chart $g_i : U_i \to V_i$ is an analytic isomorphism from open $U_i \subseteq M$ onto open $V_i \subseteq \mathbb{R}^m$ with $V_i \in \mathfrak{S}_m$ and such that all transition maps

$$g_{ij} = g_j \circ g_i^{-1} : g_i(U_i \cap U_j) \to g_j(U_j \cap U_i) \qquad (i, j \in I)$$

belong to $\mathfrak{S}$ as well. (In particular, the domains $g_i(U_i \cap U_j)$ of these transition maps belong to $\mathfrak{S}$.) Two $\mathfrak{S}$-atlases (g_i) and (h_j) are said to be $\mathfrak{S}$-*equivalent* if all "mixed" transition maps $h_j \circ g_i^{-1}$ belong to $\mathfrak{S}$; then "$\mathfrak{S}$-equivalence" is an equivalence relation on the collection of $\mathfrak{S}$-atlases on M. An $\mathfrak{S}$-*manifold* is a manifold M equipped with an $\mathfrak{S}$-equivalence class of $\mathfrak{S}$-atlases. Each space $\mathbb{R}^m$ is considered as an $\mathfrak{S}$-manifold by taking the $\mathfrak{S}$-atlas consisting of just one chart, the identity map on $\mathbb{R}^m$. Each projective space $\mathbb{P}^n(\mathbb{R})$ is considered as an $\mathfrak{S}$-manifold by taking as $\mathfrak{S}$-atlas the usual atlas of $n+1$ affine coordinate charts

$$g_i : U_i = \{\, x = (x_0 : \cdots : x_n) \in \mathbb{P}^n(\mathbb{R}) : x_i \neq 0 \,\} \to \mathbb{R}^n \qquad (i = 0, \ldots, n)$$

given by

$$g_i(x) = \left(\frac{x_0}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_n}{x_i} \right).$$

Let M, N be $\mathfrak{S}$-manifolds given by an $\mathfrak{S}$-atlas (g_i) on M and an $\mathfrak{S}$-atlas (h_j) on N. Then $M \times N$ is an $\mathfrak{S}$-manifold given by the $\mathfrak{S}$-atlas $(g_i \times h_j)$. The tangent bundle TM is made into an $\mathfrak{S}$-manifold by taking the $\mathfrak{S}$-atlas consisting of the charts $Tg_i : TU_i \to TV_i$, where $g_i : U_i \to V_i \subseteq \mathbb{R}^m$ and TV_i is identified with a subset of $\mathbb{R}^{2m}$ (as in §2). In the same way, we make the cotangent bundle T^*M into an $\mathfrak{S}$-manifold. Define a set $A \subseteq M$ to be an $\mathfrak{S}$-*set in* M if $g_i(U_i \cap A) \in \mathfrak{S}_m$ for all $i \in I$; given $\mathfrak{S}$-sets A in M and B in N, define a map $f : A \to B$ to be an $\mathfrak{S}$-*map* if it is continuous and its graph is an $\mathfrak{S}$-set in $M \times N$. (These notions are independent of the choice of $\mathfrak{S}$-atlases (g_i) and (h_j) that make M and N into $\mathfrak{S}$-manifolds.)

The $\mathfrak{S}$-sets in $\mathfrak{S}$-manifolds are the objects of a category with the $\mathfrak{S}$-maps between them as morphisms, and composition of morphisms given by the usual composition of maps. The $\mathfrak{S}$-sets in $\mathbb{R}^m$ are exactly the sets belonging to $\mathfrak{S}_m$.

Theorem. *Each $\mathfrak{S}$-set in an $\mathfrak{S}$-manifold M is isomorphic in this category to an $\mathfrak{S}$-set in $\mathbb{R}^n$ for some n.*

(See Ch. 10 of [4].)

If $\mathfrak{S}(\mathbb{R}_{\text{an}}) \subseteq \mathfrak{S}$ and M is compact—for example, $M = \mathbb{P}^n(\mathbb{R})$—then the $\mathfrak{S}$-sets in M are exactly the $\mathcal{C}(\mathfrak{S})$-sets in M.

Example. Let $\mathfrak{S}$ consist of the semialgebraic sets in $\mathbb{R}^m$ for $m \in \mathbb{N}$. Then the $\mathfrak{S}$-manifolds are exactly the so-called (analytic) Nash manifolds (see [23]), and the $\mathfrak{S}$-sets in a Nash manifold are exactly its semialgebraic subsets.

4. Some properties of o-minimal structures on $(\mathbb{R}, +, \cdot)$

We now list a few of the important properties of o-minimal structures on $(\mathbb{R}, +, \cdot)$; unless otherwise stated, proofs can be found in [4]. Several results in this section are new and proved in Appendix C.

Throughout this section $\mathfrak{S}$ denotes some fixed, but arbitrary, o-minimal structure on $(\mathbb{R}, +, \cdot)$, and, unless indicated otherwise, p denotes a positive integer.

4.1. Monotonicity theorem. *Let $f : (a,b) \to \mathbb{R}$ belong to $\mathfrak{S}$, $-\infty \le a < b \le \infty$. Then there are $a_0, a_1, \dots, a_{k+1}$ with $a = a_0 < a_1 < \cdots < a_k < a_{k+1} = b$ such that $f|(a_i, a_{i+1})$ is C^p, and either constant or strictly monotone, for $i = 0, \dots, k$.*

Remark. For every presently-known o-minimal structure on $(\mathbb{R}, +, \cdot)$, the above holds with "analytic" in place of "C^p".

Cells and cell decomposition. We define the C^p *cells* in $\mathbb{R}^n$ as certain C^p submanifolds of $\mathbb{R}^n$ belonging to $\mathfrak{S}_n$; the definition is by induction on n:

(1) The C^p cells in $\mathbb{R}$ ($= \mathbb{R}^1$) are just the points $\{r\}$ and the open intervals (a,b), $-\infty \le a < b \le +\infty$;

(2) Let $D \in \mathfrak{S}_n$ be a C^p cell. Then $D \times \mathbb{R}$ is a C^p cell in $\mathbb{R}^{n+1}$. Let $f : D \to \mathbb{R}$ of class C^p belong to $\mathfrak{S}$; then the sets $\Gamma(f)$, $\{\,(x,r) \in D \times \mathbb{R} : r < f(x)\,\}$ and $\{\,(x,r) \in D \times \mathbb{R} : f(x) < r\,\}$ are C^p cells in $\mathbb{R}^{n+1}$. Let $g : D \to \mathbb{R}$ of class C^p belong to $\mathfrak{S}$ such that $f(x) < g(x)$ for all $x \in D$; then

$$\{\,(x,r) \in D \times \mathbb{R} : f(x) < r < g(x)\,\}$$

is a C^p cell in $\mathbb{R}^{n+1}$.

(We also consider $\mathbb{R}^0 = \{0\}$ as a cell in $\mathbb{R}^0$; so (2) even holds for $n = 0$.)

For $p = \omega$ we usually say "analytic cells" instead of "C^ω cells".

A C^p *decomposition* of $\mathbb{R}^n$ is a special kind of partition of $\mathbb{R}^n$ into finitely many C^p cells. Definition is by induction on n:

(1) A C^p decomposition of $\mathbb{R}$ is a collection of intervals and points of the form

$$\{(-\infty, a_1), (a_1, a_2), \dots, (a_k, +\infty), \{a_1\}, \dots, \{a_k\}\},$$

with $a_1 < \cdots < a_k$ real numbers. (For $k = 0$ this is just $\{(-\infty, \infty)\}$.)

(2) A C^p decomposition of $\mathbb{R}^{n+1}$ is a finite partition $\mathcal{D}$ of $\mathbb{R}^{n+1}$ into C^p cells such that the set of projections $\{\,\pi(D) : D \in \mathcal{D}\,\}$ is a decomposition of $\mathbb{R}^n$, where $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is the projection on the first n coordinates.

Note that each C^p cell D in $\mathbb{R}^n$ is connected, and there exist integers $i_1, \dots, i_m$ with $1 \le i_1 < \cdots < i_m \le n$ such that the map $x \mapsto (x_{i_1}, \dots, x_{i_m}) : D \to \mathbb{R}^m$ is a C^p diffeomorphism onto an open cell in $\mathbb{R}^m$ (where $m = \dim D$).

Note. Cells and decompositions are always defined relative to some particular structure on $(\mathbb{R}, +, \cdot)$ (the structure $\mathfrak{S}$ throughout this section).

4.2. Cell decomposition.

(1) *Given $A_1, \dots, A_k \in \mathfrak{S}_n$, there is a C^p decomposition of $\mathbb{R}^n$ compatible with $\{A_1, \dots, A_k\}$.*

(2) *For every function $f : A \to \mathbb{R}$ belonging to $\mathfrak{S}$, $A \subseteq \mathbb{R}^n$, there is a C^p decomposition $\mathcal{D}$ of $\mathbb{R}^n$ compatible with $\{A\}$ such that $f|D : D \to \mathbb{R}$ is of class C^p for each $D \in \mathcal{D}$ with $D \subseteq A$.*

We also say that $\mathfrak{S}$ has C^∞ decomposition if the above holds with $p = \infty$, and that $\mathfrak{S}$ has analytic decomposition if the above holds with $p = \omega$. The structures $\mathfrak{S}(\mathbb{R}_{\text{an}}^{\mathbb{R}})$ and $\mathfrak{S}(\mathbb{R}_{\text{an,exp}})$ have analytic decomposition; see [7] and [19].

4.3. Component theorem. *Every A belonging to $\mathfrak{S}$ has finitely many connected components, each belonging to $\mathfrak{S}$.*

4.4. Uniform bounds on fibers. *Let $A \subseteq \mathbb{R}^{m+n}$ belong to $\mathfrak{S}$. Then there exists $N \in \mathbb{N}$ such that for all $x \in \mathbb{R}^m$ the set $A_x := \{ y \in \mathbb{R}^n : (x,y) \in A \}$ has at most N connected components.*

4.5. Definable choice. *Let $A \subseteq \mathbb{R}^{m+n}$ belong to $\mathfrak{S}$, and let $\pi A \subseteq \mathbb{R}^m$ be the projection of A onto the first m coordinates. Then there is a map $f : \pi A \to \mathbb{R}^n$ belonging to $\mathfrak{S}$ such that $\Gamma(f) \subseteq A$. In particular, if $B \subseteq \mathbb{R}^m$ and $g : B \to \mathbb{R}^n$ belongs to $\mathfrak{S}_{m+n}$ then there exists $f : g(B) \to \mathbb{R}^m$ belonging to $\mathfrak{S}$ such that $f(g(x)) = x$ for all $x \in B$.*

4.6. Curve selection (with parameters). *Let $A \in \mathfrak{S}_n$. Then $\operatorname{fr}(A)$ is a disjoint union of connected C^p submanifolds $B_1, \dots, B_l$ of $\mathbb{R}^n$, each belonging to $\mathfrak{S}$, and there is a map $f : \operatorname{fr}(A) \times (0,1) \to A$ belonging to $\mathfrak{S}$ such that $f|(B_i \times (0,1))$ is C^p for $1 = 1, \dots, l$, and for each $x \in \operatorname{fr}(A)$ the function $t \mapsto f(x,t) : (0,1) \to \mathbb{R}^n$ is injective, with $\lim_{t \to 0^+} f(x,t) = x$.*

4.7. Dimension is well behaved. *Let $A \in \mathfrak{S}_n$ be nonempty. Then:*

(1) $\dim \operatorname{fr}(A) < \dim A$.

(2) *If $f : A \to \mathbb{R}^m$ belongs to $\mathfrak{S}$, then* $\dim f(A) \leq \dim A$.

4.8. Whitney stratification.

(1) *Given $A_1, \dots, A_k \in \mathfrak{S}_m$, there is a finite C^p Whitney stratification of $\mathbb{R}^m$ compatible with $\{A_1, \dots, A_k\}$, with each stratum a C^p cell in $\mathbb{R}^m$.*

(2) *Let $f : A \to \mathbb{R}^n$ belong to $\mathfrak{S}$ with $A \subseteq \mathbb{R}^m$ closed; let $\mathcal{F}$ and $\mathcal{G}$ be finite subcollections of $\mathfrak{S}_m$ and $\mathfrak{S}_n$ respectively. Then there is a finite C^p Whitney stratification $(\mathcal{S}, \mathcal{T})$ of f with connected strata such that $\mathcal{S} \subseteq \mathfrak{S}_m$ is compatible with $\mathcal{F}$ and $\mathcal{T} \subseteq \mathfrak{S}_n$ is compatible with $\mathcal{G}$. The strata of $\mathcal{T}$ can be taken to be C^p cells in $\mathbb{R}^n$.*

(See Appendix D.18.)

Remarks.

(1) If $\mathfrak{S}$ has C^∞ decomposition then 4.8 holds with $p = \infty$; similarly with $p = \omega$ if $\mathfrak{S}$ has analytic decomposition.

(2) In 4.8(2), we may require in addition that $f|A$ be injective for each $A \in \mathcal{S}$ with $\operatorname{rk} f|A = \dim A$.

4.9. Good directions. *Let $A \in \mathfrak{S}_n$ with $\dim A \leq k \leq n$. Then there exist a k-dimensional linear subspace U of $\mathbb{R}^n$ and $N \in \mathbb{N}$ such that $\operatorname{card}(\pi^{-1}(x) \cap A) \leq N$ for all $x \in U$, where π is the orthogonal projection from $\mathbb{R}^n$ onto U.*

Remark. In fact, the set of all such U as above is dense in the Grassmannian $G_k(\mathbb{R}^n)$.

4.10. Triangulation. *Let $A, A_1, \dots, A_l \in \mathfrak{S}_n$ with $A_1, \dots, A_l \subseteq A$. Then there exist a finite simplicial complex K in $\mathbb{R}^n$ and a map $\varphi : A \to \mathbb{R}^n$ belonging to $\mathfrak{S}$ such that φ maps A and each A_i homeomorphically onto a union of open simplices of K.*

Definition. Let $A \subseteq \mathbb{R}^m$. A map $f : A \to \mathbb{R}^n$ in $\mathfrak{S}_{m+n}$ is called $\mathfrak{S}$-*trivial* if there exist $k \in \mathbb{N}$ and $g : A \to \mathbb{R}^k$ in $\mathfrak{S}_{m+k}$ such that $a \mapsto (f(a), g(a))$ is a homeomorphism of A onto $f(A) \times g(A)$.

4.11. Generic triviality. *Let $A \subseteq \mathbb{R}^m$ and $f : A \to \mathbb{R}^n$ in $\mathfrak{S}_{m+n}$ be continuous. Then there is a partition $\{C_1, \dots, C_l\} \subseteq \mathfrak{S}_n$ of $f(A)$ such that $f|f^{-1}(C_i)$ is $\mathfrak{S}$-trivial for $i = 1, \dots, l$.*

Remark. In the semialgebraic case, generic triviality is due to Hardt [10].

A striking feature of o-minimal structures on $(\mathbb{R}, +, \cdot)$ has to do with the possibilities for asymptotic behaviour of functions $f : \mathbb{R} \to \mathbb{R}$ belonging to the structure.

Definition. A structure on $(\mathbb{R}, +, \cdot)$ is *polynomially bounded* if for every function $f : \mathbb{R} \to \mathbb{R}$ belonging to the structure, there exists some $N \in \mathbb{N}$ (depending on f) such that $f(t) = O(t^N)$ as $t \to +\infty$. A structure on $(\mathbb{R}, +, \cdot)$ is *exponential* if it contains exp. We also say that $(\mathbb{R}, +, \cdot, (f_j)_{j\in J})$ is polynomially bounded if $\mathfrak{S}(\mathbb{R}, +, \cdot, (f_j)_{j\in J})$ is polynomially bounded; similarly with "exponential".

4.12. Growth dichotomy. *Either $\mathfrak{S}$ is polynomially bounded, or it is exponential. If $\mathfrak{S}$ is polynomially bounded, then for every $f : \mathbb{R} \to \mathbb{R}$ belonging to $\mathfrak{S}$, either f is ultimately identically equal to 0, or there exist $c, r \in \mathbb{R}$, $c \neq 0$, such that $x \mapsto x^r : (0, \infty) \to \mathbb{R}$ belongs to $\mathfrak{S}$ and $f(t) = ct^r + o(t^r)$ as $t \to +\infty$.*

(See [18].)

This dichotomy appears in many guises in the study of o-minimal structures on $(\mathbb{R}, +, \cdot)$.

In 4.13 through 4.17 we assume moreover that $\mathfrak{S}$ is polynomially bounded.

4.13. Piecewise uniform asymptotics. *Let $f : A \times \mathbb{R} \to \mathbb{R}$ belong to $\mathfrak{S}$, $A \subseteq \mathbb{R}^m$. Then there exist $r_1, \dots, r_l \in \mathbb{R}$ such that for all $x \in A$, either $f(x,t) = 0$ for all sufficiently small (depending on x) positive t, or $f(x,t) = ct^{r_i} + o(t^{r_i})$ as $t \to 0^+$ for some $i \in \{1, \dots, l\}$ and $c = c(x) \in \mathbb{R} \setminus \{0\}$.*

(See [19].)

It follows immediately that there exists $N \in \mathbb{N}$ depending only on f such that for every $x \in A$ we have $|f(x,t)| \leq t^N$ for all sufficiently large (depending on x) positive t. Some consequences of this are the following closely related results, all generalizations of certain well-known metric properties of subanalytic subsets of euclidean spaces.

Notation. We put $|x| := \sup\{\, |x_i| : i = 1, \dots, n \,\}$ and $||x|| := \sqrt{x_1^2 + \cdots + x_n^2}$ for $x \in \mathbb{R}^n$.

Given $f : A \to \mathbb{R}$, $A \subseteq \mathbb{R}^m$, put $Z(f) := \{\, a \in A : f(a) = 0 \,\}$. Note that if f belongs to $\mathfrak{S}$, then so does $Z(f)$.

4.14. Hölder continuity and Łojasiewicz inequality. *Let $A \in \mathfrak{S}_n$ be compact, and let $f : A \to \mathbb{R}$ be a continuous function belonging to $\mathfrak{S}$.*

1. *There exist $r, C > 0$ such that $|f(x) - f(y)| \leq C|x - y|^r$ for all $x, y \in A$.*
2. *Let $g : A \to \mathbb{R}$ belonging to $\mathfrak{S}$ be continuous with $Z(f) \subseteq Z(g)$. Then there exist $N > 0$ and $C > 0$ such that $|g(x)|^N \leq C|f(x)|$ for all $x \in A$.*

(This follows from 4.20 below; see also Appendix C.15.)

Paths.

A path in $\mathbb{R}^n$ is a continuous map $g : [a,b] \to \mathbb{R}^n$ ($-\infty < a < b < +\infty$); we then say that g is a path from x to y where $g(a) = x$ and $g(b) = y$. The length of such a path, denoted by $\text{length}(g)$, is by definition the supremum of all sums of the form $\sum_{i=1}^{k} ||g(a_i) - g(a_{i-1})||$, taken over all finite partitions $a = a_0 < a_1 < \cdots < a_k = b$ of $[a,b]$. The path g is said to be rectifiable if $\text{length}(g) < +\infty$. If g belongs to $\mathfrak{S}$ then g is rectifiable. (See Appendix C.)

4.15. Whitney regularity. *Let $A \in \mathfrak{S}_n$ be nonempty, compact and connected. Then there exist $r, C > 0$ and a map $\gamma : A^2 \times [0,1] \to A$ belonging to $\mathfrak{S}$ such that for every $x, y \in A$,*

$$\gamma_{x,y} := t \mapsto \gamma(x,y,t) : [0,1] \to A$$

is a path from x to y, with $\text{length}(\gamma_{x,y}) \le C||x-y||^r$.

(See 4.21 below.)

Definition. A subring $S \subseteq C^\infty(U, \mathbb{R})$ with U an open connected subset of $\mathbb{R}^n$ is *quasianalytic* if for each nonzero $f \in S$ and $x \in U$ the Taylor series at x of f is not zero (for example, the set of all real analytic functions $f : U \to \mathbb{R}$ is quasianalytic). Note that then S is necessarily an integral domain.

4.16. Quasianalyticity. *Let $U \in \mathfrak{S}_m$ be open and connected. Then the ring of all C^∞ functions $f : U \to \mathbb{R}$ belonging to $\mathfrak{S}$ is quasianalytic.*

(See [20].)

4.17. Descending chain condition for zero sets. *Suppose that $\mathfrak{S}$ has C^∞ decomposition. Let $(f_i : U \to \mathbb{R})_{i \in \mathbb{N}}$ be a family of C^∞ functions, each belonging to $\mathfrak{S}$, U open in $\mathbb{R}^n$. Then there exists $N \in \mathbb{N}$ such that*

$$\bigcap_{i \in \mathbb{N}} Z(f_i) = \bigcap_{i \le N} Z(f_i).$$

(See [8].)

Note. By results of Tougeron [26], 4.17 holds for families of *analytic* functions belonging to $\mathfrak{S}$ *without* the assumptions that $\mathfrak{S}$ be polynomially bounded or have C^∞ decomposition.

The remaining results of this section are new. They are all closely related, and generalize certain important metric properties of subanalytic subsets of euclidean spaces. (See Appendix C for proofs and further information.) Note that in these results we make no assumption of polynomial boundedness on $\mathfrak{S}$.

4.18. Uniform bounds on growth. *Let $A \subseteq \mathbb{R}^m$ and $g : A \times \mathbb{R} \to \mathbb{R}$ belong to $\mathfrak{S}$. Then there exist functions $\psi : \mathbb{R} \to \mathbb{R}$ and $\rho : A \to \mathbb{R}$ belonging to $\mathfrak{S}$ such that $|g(x,t)| < \psi(t)$ for all $x \in A$ and $t > \rho(x)$.*

Notation. Let $\Phi^p_{\mathfrak{S}}$ denote the set of all odd, strictly increasing bijections $\phi : \mathbb{R} \to \mathbb{R}$ belonging to $\mathfrak{S}$ that are C^p on $\mathbb{R}$ and p-flat at 0 (that is, $\phi^{(k)}(0) = 0$ for $k = 0, \dots, p$).

4.19. C^p multipliers. *Let $f, g : \mathbb{R}^n \to \mathbb{R}$ belonging to $\mathfrak{S}$ be continuous, of class C^p on $\mathbb{R}^n \setminus Z(g)$, with $Z(f) \subseteq Z(g)$. Then there exists $\phi \in \Phi^p_{\mathfrak{S}}$ and a C^p function $h : \mathbb{R}^n \to \mathbb{R}$ belonging to $\mathfrak{S}$ such that $\phi \circ g = hf$.*

4.20. Generalized Łojasiewicz inequality. *Let $f, g : A \to \mathbb{R}$ belonging to $\mathfrak{S}$ be continuous, with $Z(f) \subseteq Z(g)$ and $A \subseteq \mathbb{R}^n$ compact. Then there exists $\phi \in \Phi^p_{\mathfrak{S}}$ such that $|\phi(g(x))| \leq |f(x)|$ for all $x \in A$.*

4.21. Uniform path-connectedness. *Let $A \in \mathfrak{S}_n$ be nonempty and connected.*

(1) *There exists a map $\gamma : A^2 \times [0,1] \to A$ belonging to $\mathfrak{S}$ such that for every $x, y \in A$, $\gamma_{x,y}$ (notation as in 4.15) is a path from x to y.*
(2) *If A is compact, then there exists γ as in (1) and $\phi \in \Phi^p_{\mathfrak{S}}$ such that for every $x, y \in A$, $\operatorname{length}(\gamma_{x,y}) \leq \phi^{-1}(||x - y||)$.*

Remark. 4.14 and 4.15 are easy corollaries of 4.20 and 4.21.

4.22. Closed sets are zero sets. *Let $A \in \mathfrak{S}_n$ be closed. Then there exists a C^p function $f : \mathbb{R}^n \to \mathbb{R}$ belonging to $\mathfrak{S}$ with $A = Z(f)$.*

Remark. Clearly, 4.22 fails with $p = \omega$ for $A = [0,1] \subseteq \mathbb{R}$. It also fails with $p = \infty$ and $A = [0,1]$ if $\mathfrak{S}$ is polynomially bounded, by quasianalyticity (4.16). We do not know if 4.22 holds with $p = \infty$ if $\mathfrak{S}$ is exponential.

5. Some special properties of $\mathfrak{S}(\mathbb{R}^{\mathbb{R}}_{\text{an}})$ and $\mathfrak{S}(\mathbb{R}_{\text{an,exp}})$

Let functions $f_j : \mathbb{R}^{n(j)} \to \mathbb{R}$ (j in some index set J) be given, and let $\mathfrak{R}$ denote $(\mathbb{R}, +, \cdot, (f_j)_{j \in J})$, the field of real numbers equipped with the functions f_j for $j \in J$.

We define the *$\mathfrak{R}$-functions on $\mathbb{R}^n$* inductively as follows:

(1) The projection functions $x \mapsto x_i : \mathbb{R}^n \to \mathbb{R}$ ($i = 1, \dots, n$) are $\mathfrak{R}$-functions on $\mathbb{R}^n$.
(2) If $g, h : \mathbb{R}^n \to \mathbb{R}$ are $\mathfrak{R}$-functions, then $-g$, $g + h$ and $g \cdot h$ are $\mathfrak{R}$-functions on $\mathbb{R}^n$.
(3) If $j \in J$ and $g_1, \dots, g_{n(j)}$ are $\mathfrak{R}$-functions on $\mathbb{R}^n$, then $f_j(g_1, \dots, g_{n(j)})$ is an $\mathfrak{R}$-function on $\mathbb{R}^n$.

Note that all $\mathfrak{R}$-functions belong to $\mathfrak{S}(\mathbb{R}, +, \cdot, (f_j)_{j \in J})$.

5.1. *Let $\mathfrak{R}$ be either $\mathbb{R}^{\mathbb{R}}_{\text{an}}$ or $(\mathbb{R}_{\text{an,exp}}, \log)$ (where we put $\log(x) := 0$ for $x \leq 0$); put $\mathfrak{S} = \mathfrak{S}(\mathfrak{R})$ and let $n \in \mathbb{N}$.*

(1) *Every set in $\mathfrak{S}_n$ is a finite union of sets of the form*

$$\{\, x \in \mathbb{R}^n : f(x) = 0, g_1(x) < 0, \dots, g_l(x) < 0 \,\}$$

where $f, g_1, \dots, g_l$ are $\mathfrak{R}$-functions on $\mathbb{R}^n$.

(2) *Given $f : A \to \mathbb{R}$ in $\mathfrak{S}_{n+1}$, there are $\mathfrak{R}$-functions $f_1, \dots, f_l$ on $\mathbb{R}^n$ such that for every $x \in \mathbb{R}^n$ there is an $i \in \{1, \dots, l\}$ with $f(x) = f_i(x)$; that is, f is given piecewise by $\mathfrak{R}$-functions.*
(3) *$\mathfrak{S}$ has analytic decomposition.*

(See [7], [5] and [19].)

For $\mathbb{R}_{\mathrm{an}}^{\mathbb{R}}$, we have an analog of Puiseux expansion of 1-variable subanalytic functions:

5.2. *Let $f : (0, \epsilon) \to \mathbb{R}$ belong to $\mathfrak{S}(\mathbb{R}_{\mathrm{an}}^{\mathbb{R}})$ with $f(t) \neq 0$ for all $t \in (0, \epsilon)$. Then there exist $d \in \mathbb{N}$, a convergent real power series $F(Y_1, \ldots, Y_d)$ with $F(0) \neq 0$, and $r_0, r_1, \ldots, r_d \in \mathbb{R}$ with $r_1, \ldots, r_d > 0$ such that $f(t) = t^{r_0} F(t^{r_1}, \ldots, t^{r_d})$ for all sufficiently small positive t.*

(See [19].)

There is also a stronger version of this fact in which f depends on parameters (see [8]); it is used to establish the following extension of Tamm's theorem [25]:

5.3. *Let $f : A \to \mathbb{R}$ belong to $\mathfrak{S}(\mathbb{R}_{\mathrm{an}}^{\mathbb{R}})$, $A \subseteq \mathbb{R}^{m+n}$. Then there exists $N \in \mathbb{N}$ such that for all $x \in \mathbb{R}^m$ and all open sets $U \subseteq \mathbb{R}^n$ with*

$$U \subseteq A_x := \{\, y \in \mathbb{R}^n : (x, y) \in A \,\},$$

if f_x is C^N on U, then f_x is analytic on U.

(Here, f_x denotes the function $y \mapsto f(x, y) : A_x \to \mathbb{R}$.)

This result extends Tamm's theorem simultaneously in two ways: (i) the domain A_x and the function f_x varies with x, with an N independent of x; (ii) the function f need not be subanalytic. It follows easily—see Appendix B.8(5)—that the set $\{\, (x, y) \in A : f_x \text{ is analytic at } y \,\}$ belongs to $\mathfrak{S}(\mathbb{R}_{\mathrm{an}}^{\mathbb{R}})$. Using this fact, it is not hard to show that every structure $\mathfrak{S}$ on $(\mathbb{R}, +, \cdot)$ with $\mathfrak{S} \subseteq \mathfrak{S}(\mathbb{R}_{\mathrm{an}}^{\mathbb{R}})$ has analytic decomposition.

5.4. Note. The preceding type of result *never* holds in o-minimal structures $\mathfrak{S}$ on $(\mathbb{R}, +, \cdot)$ which are *not* polynomially bounded, even with "C^∞" in place of "analytic": By growth dichotomy (4.12), the exponential function belongs to every such $\mathfrak{S}$, and thus the function $F : (-1, 1)^2 \to \mathbb{R}$ given by

$$F(x, y) := \begin{cases} |y|^{1/|x|} \cdot \exp\left(-1/(x^2 + y^2)\right), & xy \neq 0 \\ 0, & \text{otherwise} \end{cases}$$

belongs to $\mathfrak{S}$. For $x \in (-1, 1)$ we have that F_x is C^∞ at $y = 0$ if and only if $x = 0$ or $x = 1/(2k)$ with k a nonzero integer, so the set

$$\{\, (x, y) \in (-1, 1)^2 : F_x \text{ is } C^\infty \text{ at } y \,\}$$

does not belong to $\mathfrak{S}$. Also note that for each positive integer k, F is C^k on some open neighborhood U_k of $(0, 0)$, but is not C^∞ on any neighborhood of $(0, 0)$; consequently,

$$\mathrm{Reg}^1(\Gamma(F)) \supset \mathrm{Reg}^2(\Gamma(F)) \supset \cdots \supset \mathrm{Reg}^k(\Gamma(F)) \supset \cdots$$

is a strictly decreasing chain, whose intersection strictly contains $\mathrm{Reg}^\infty(\Gamma(F))$. This provides a counterexample to a conjecture of Shiota; see §2 of [24].

5.5. Exponential bounds. *Let* $f : \mathbb{R} \to \mathbb{R}$ *belong to* $\mathfrak{S}(\mathbb{R}_{\mathrm{an,exp}})$. *Then there exists some compositional iterate* $\exp_N$ *of* $\exp$ *such that* $f(t) = O(\exp_N(t))$ *as* $t \to +\infty$.

(See [6].)

Applying this together with 4.18, 4.20 and 4.21, one obtains for $\mathfrak{S}(\mathbb{R}_{\mathrm{an,exp}})$ "exponential" versions of the Łojasiewicz inequality and Hölder continuity (4.14) as well as Whitney regularity (4.15). (See also Loi [14] for various refinements and applications of exponential Łojasiewicz inequalities.)

Remark. Let $\mathfrak{S}$ be a structure on $(\mathbb{R}, +, \cdot)$ and suppose that the statement of 5.5 holds with "$\mathfrak{S}$" in place of "$\mathfrak{S}(\mathbb{R}_{\mathrm{an,exp}})$"; we then say that $\mathfrak{S}$ is *exponentially bounded*. We do not know if there are o-minimal structures on $(\mathbb{R}, +, \cdot)$ that are not exponentially bounded.

Appendices

We here provide proofs of assertions made earlier for which no references were given. We do not prove these assertions in quite the same form or order in which they were made.

Recall that $|x|$ denotes $\sup\{\, |x_i| : i = 1, \dots, n\,\}$ for $x \in \mathbb{R}^n$. For $x, y \in \mathbb{R}^n$ we let $x.y$ denote the dot product $x_1 y_1 + \cdots + x_n y_n$.

Appendix A. Logical formalism

There is one easy technique that we use over and over again to show that certain constructions on sets belonging to a structure on $(\mathbb{R}, +, \cdot)$ give again sets belonging to the same structure, and hence (via the correspondence of §3) that certain constructions on $\mathcal{C}$-sets give again $\mathcal{C}$-sets. (For example, we will apply this technique in a routine way to show that the conormal bundle of a smooth $\mathcal{C}$-set is again a $\mathcal{C}$-set.) To explain this, let x and y be variables ranging over sets X and Y respectively and let $\varphi(x, y)$ and $\psi(x, y)$ be formulas (conditions on (x, y)) defining subsets Φ and Ψ of $X \times Y$. Then we have the following correspondence between *formulas* on the left and *sets* on the right:

$$\begin{array}{rll}
\neg\varphi(x,y) & \text{defines} & \text{the complement of } \Phi \text{ in } X \times Y, \\
\varphi(x,y) \vee \psi(x,y) & \text{defines} & \text{the union } \Phi \cup \Psi, \\
\varphi(x,y) \wedge \psi(x,y) & \text{defines} & \text{the intersection } \Phi \cap \Psi, \\
\exists x \varphi(x,y) & \text{defines} & \text{the projection } \pi\Phi \subseteq Y, \text{ where } \pi \text{ is the} \\
 & & \text{projection map } X \times Y \to Y, \\
\forall y \psi(x,y) & \text{defines} & \{\, x \in X : \{x\} \times Y \subseteq \Psi\}.
\end{array}$$

What is the point of using logical symbols, when the more standard set notation serves the same purpose? An advantage of the logical notation is that it appeals to our natural linguistic and logical abilities. For instance, given a function $f : X \to Y$ the set $f(X)$ is defined by the equivalence

$$y \in f(X) \Leftrightarrow \exists x[\, f(x) = y\,].$$

Since the formula "$f(x) = y$" defines $\Gamma(f)$, this equivalence exhibits $f(X)$ as the image of $\Gamma(f)$ under the projection map $X \times Y \to Y$ according to the correspondence above between formulas and sets. This reduction of *arbitrary* maps to *projection* maps is used all the time; the bland set notation "$f(X)$" fails to suggest this reduction. Note also that the familiar equivalence

$$\forall y \psi(x,y) \Leftrightarrow \neg\exists y \neg\psi(x,y)$$

shows the subset of X defined by $\forall y \psi(x, y)$ to be obtained from Ψ by first taking the complement of Ψ, then projecting this to X, and again taking the complement. The logical formalism does part of our thinking for us, if we pay attention! This technique is particularly useful when dealing with logically complicated notions like continuity and differentiability, which we simply express in the usual way with ϵ's and δ's, and quantifiers over them. In such cases we often deal with formulas with more than two variables, and repeated quantifiers, like in $\forall y \exists z \xi(x,y,z)$, where z ranges over some set Z: this formula defines the set

$$\{\, x \in X : \text{ for all } y \in Y \text{ there is } z \in Z \text{ such that } \xi(x,y,z) \text{ holds} \,\}.$$

The correspondence above between formulas and sets shows this set to be obtained by a series of projections and complementations from the set $\Xi \subseteq X \times Y \times Z$ defined by $\xi(x,y,z)$.

The following notational conventions are also employed: A condition $\theta(x)$ on elements $x \in X$ defining a set $\Theta \subseteq X$ may in some cases be more conveniently viewed as a condition on pairs $(x,y) \in X \times Y$, so that it then defines the set $\Theta \times Y$. Instead of $\varphi(x,y) \wedge \psi(x,y)$ we also write $\varphi(x,y) \;\;\&\;\; \psi(x,y)$. We use the implication sign, as in $\varphi(x,y) \to \theta(x)$, to abbreviate $(\neg\varphi(x,y)) \vee \theta(x)$. We write $\exists v_1 \cdots v_k$ instead of $\exists v_1 \cdots \exists v_k$ and similarly for the universal quantifier $\forall$.

Appendix B. Results about structures on $(\mathbb{R}, +, \cdot)$

Throughout this appendix, let $\mathfrak{S}$ be a given structure on $(\mathbb{R}, +, \cdot)$.

B.1. *If $A \in \mathfrak{S}_m$ and $B \in \mathfrak{S}_n$, then $A \times B \in \mathfrak{S}_{m+n}$.*

Proof. Note that $A \times B = (A \times \mathbb{R}^n) \cap (\mathbb{R}^m \times B)$ and use S1 and S2 of §2. □

B.2. *Let $B \in \mathfrak{S}_n$, and let $i(1), \ldots, i(n)$ be a sequence in $\{1, \ldots, m\}$ (possibly with repetitions). Then the set $A \subseteq \mathbb{R}^m$ defined by the condition*

$$(x_1, \ldots, x_m) \in A \Leftrightarrow \big(x_{i(1)}, \ldots, x_{i(n)}\big) \in B$$

belongs to $\mathfrak{S}$. ("Permuting and identifying variables is allowed.")

Proof. Note that

$$\begin{gathered} (x_1, \ldots, x_m) \in A \\ \Leftrightarrow \\ \exists y_1 \cdots y_n \big[\, x_{i(1)} = y_1 \;\;\&\;\; \cdots \;\;\&\;\; x_{i(n)} = y_n \;\;\&\;\; (y_1, \ldots, y_n) \in B \,\big]. \end{gathered}$$ □

B.3. *Let $f = (f_1, \ldots, f_n) : A \to \mathbb{R}^n$ with $A \subseteq \mathbb{R}^m$ belong to $\mathfrak{S}$. Then:*

(1) *$A \in \mathfrak{S}_n$.*
(2) *$f(A) \in \mathfrak{S}_m$.*
(3) *If $B \in \mathfrak{S}_n$, then $f^{-1}(B) \in \mathfrak{S}_m$.*
(4) *If $A' \in \mathfrak{S}_m$ and $A' \subseteq A$, then $f|A'$ belongs to $\mathfrak{S}$.*

Proof.

(1) For all $x \in \mathbb{R}^m$,

$$x \in A \Leftrightarrow \exists y_1 \cdots y_n \, [\, (x, y_1, \ldots, y_n) \in \Gamma(f) \,] \, .$$

(2) For all $y \in \mathbb{R}^n$,

$$y \in f(A) \Leftrightarrow \exists x_1 \cdots x_m \, [\, (x_1, \ldots, x_m, y) \in \Gamma(f) \,] \, .$$

(3) For all $x \in \mathbb{R}^m$,

$$x \in f^{-1}(B) \Leftrightarrow \exists y \, [\, (x, y) \in \Gamma(f) \;\; \& \;\; y \in B \,] \, .$$

(4) For all $(x, y) \in \mathbb{R}^{m+n}$,

$$(x, y) \in \Gamma(f|A') \Leftrightarrow x \in A' \;\; \& \;\; (x, y) \in \Gamma(f). \quad \square$$

B.4. *A map $f = (f_1, \ldots, f_n) : A \to \mathbb{R}^n$ with $A \subseteq \mathbb{R}^m$ belongs to $\mathfrak{S}$ if and only if each $f_i : A \to \mathbb{R}$ belongs to $\mathfrak{S}$.*

Proof. Suppose that $f = (f_1, \ldots, f_n) : A \to \mathbb{R}^n$ belongs to $\mathfrak{S}$, $A \subseteq \mathbb{R}^m$. Let $i \in \{1, \ldots, n\}$; then for all $(x, y) \in \mathbb{R}^{m+1}$ we have

$$\begin{gathered} y = f_i(x) \\ \Leftrightarrow \\ x \in A \;\; \& \;\; \exists z_1 \cdots z_n \, [\, y = z_i \;\; \& \;\; (x, z_1, \ldots, z_n) \in \Gamma(f) \,] \, . \end{gathered}$$

If $f_1, \ldots, f_n : A \to \mathbb{R}$ belong to $\mathfrak{S}$, then for all $(x, y_1, \ldots, y_n) \in \mathbb{R}^{m+n}$ we have

$$\begin{gathered} (x, y_1, \ldots, y_n) \in \Gamma(f) \\ \Leftrightarrow \\ x \in A \;\; \& \;\; (x, y_1) \in \Gamma(f_1) \;\; \& \;\; \cdots \;\; \& \;\; (x, y_n) \in \Gamma(f_n). \quad \square \end{gathered}$$

The other facts listed in the first four paragraphs of 2.1 are proved similarly and are left as exercises.

B.5. *If $A \in \mathfrak{S}_m$, then* $\mathrm{cl}(A), \mathrm{int}(A) \in \mathfrak{S}_m$.

Proof. Note that

$$\begin{gathered} (x_1, \ldots, x_m) \in \mathrm{cl}(A) \\ \Leftrightarrow \\ \forall y_1 \cdots y_m \forall z_1 \cdots z_m [\, (y_1 < x_1 < z_1 \;\; \& \;\; \cdots \;\; \& \;\; y_m < x_m < z_m) \to \\ \exists a_1 \cdots a_m (y_1 < a_1 < z_1 \;\; \& \;\; \cdots \;\; \& \;\; y_m < a_m < z_m \;\; \& \;\; (a_1, \ldots, a_m) \in A) \,] . \end{gathered}$$

Now use the interpretation of the logical symbols in terms of operations on sets. $\square$

B.6. *Let A, B belong to $\mathfrak{S}$, with $A \subseteq B \subseteq \mathbb{R}^m$ and A (relatively) open in B. Then there is an open $U \subseteq \mathbb{R}^m$ that belongs to $\mathfrak{S}$ with $U \cap B = A$.*

Proof. Let U be the union of all open boxes in $\mathbb{R}^m$ whose intersection with B is contained in A. The equivalence

$$
\begin{gathered}
(x_1, \ldots, x_m) \in U \\
\Leftrightarrow \\
\exists y_1 \cdots y_m \exists z_1 \cdots z_m [\, \forall a_1 \cdots a_m (y_1 < a_1 < z_1 \ \& \ \cdots \ \& \ y_m < a_m < z_m) \to \\
(a_1, \ldots, a_m) \in A) \ \& \ (y_1 < x_1 < z_1 \ \& \ \cdots \ \& \ y_m < x_m < z_m) \,]
\end{gathered}
$$

shows that U belongs to $\mathfrak{S}$. $\square$

In the following we identify an $\mathbb{R}$-linear map from $\mathbb{R}^n$ into $\mathbb{R}^m$ with its $m \times n$ matrix with respect to the standard bases, and identify the $\mathbb{R}$-linear space of real $m \times n$ matrices with $\mathbb{R}^{mn}$ via some linear ordering of the set of pairs (i, j) with $1 \le i \le m$ and $1 \le j \le n$. In particular, if a function $f : U \to \mathbb{R}^m$ with $U \subseteq \mathbb{R}^n$ is differentiable at a point $x \in \operatorname{int}(U)$, its derivative $Df(x)$ is considered as an element of $\mathbb{R}^{mn}$.

B.7. *Let $U \subseteq \mathbb{R}^n$ and let $f : U \to \mathbb{R}^m$ belong to $\mathfrak{S}$ (in particular $U \in \mathfrak{S}_n$).*

(1) *The set $\{\, x \in U : \ f$ is continuous at $x \,\}$ belongs to $\mathfrak{S}$.*
(2) *The set $U' := \{\, x \in \operatorname{int}(U) : \ f$ is differentiable at $x \,\}$ belongs to $\mathfrak{S}$.*
(3) *The derivative $Df : U' \to \mathbb{R}^{mn}$ belongs to $\mathfrak{S}$.*
(4) *The set consisting of all $x \in U$ such that f is C^1 on an open neighborhood of x contained in U belongs to $\mathfrak{S}$.*
(5) *For each positive integer p the set consisting of all $x \in U$ such that f is C^p on an open neighborhood of x contained in U belongs to $\mathfrak{S}$.*
(6) *If $f : U \to \mathbb{R}^m$ is C^1, then the sets $\{\, x \in U : \operatorname{rk} f(x) = i \,\}$ belong to $\mathfrak{S}$ for $i = 1, \ldots, m$.*

Proof. Let x, y range over U. Then (1) follows from the equivalence

$$
\begin{gathered}
f \text{ is continuous at } x \\
\Leftrightarrow \\
\forall \epsilon \,[\, \epsilon > 0 \to \exists \delta \{ \delta > 0 \ \& \ \forall y (|y - x| < \delta \to |f(y) - f(x)| < \epsilon)\} \,].
\end{gathered}
$$

(From now on we write "$\forall \epsilon > 0 \ldots$", et cetera, to abbreviate "$\forall \epsilon [\, \epsilon > 0 \to \ldots]$".) To prove (2) and (3) we may reduce to the case $m = 1$ (which is notationally simpler), and in this case we note that (2) and (3) follow from the equivalence

$$
\begin{gathered}
Df(x) = r \\
\Leftrightarrow \\
x \in \operatorname{int}(U) \ \& \\
\forall \epsilon > 0 \ \exists \delta > 0 \ \forall y [\, |y - x| < \delta \to |f(y) - f(x) - r.(y - x)| < \epsilon |y - x| \,].
\end{gathered}
$$

Note that the set defined in (4) is simply the interior of the set of points $x \in U'$ at which Df is continuous; hence (4) follows from applying (1) to Df. For (5), use (4) p times. For (6), note that by (3) the map $Df : U \to \mathbb{R}^{mn}$ belongs to $\mathfrak{S}$. Now use that for all $x \in U$ and $i \in \mathbb{N}$ we have

$$\begin{aligned} \operatorname{rk} f(x) = i \Leftrightarrow & \text{ some } i \times i \text{ minor of the } m \times n \text{ matrix } Df(x) \text{ is nonzero} \\ & \text{and all } (i+1) \times (i+1) \text{ minors of } Df(x) \text{ are zero.} \quad \square \end{aligned}$$

We actually need these results "with parameters".

Notation. Given $U \subseteq \mathbb{R}^{N+n}$ and $a \in \mathbb{R}^N$, put $U_a := \{\, x \in \mathbb{R}^n : (a,x) \in U \,\}$ and given a map $f : U \to \mathbb{R}^m$ define $f_a : U_a \to \mathbb{R}^m$ by $f_a(x) := f(a,x)$.

B.8. *Let $f : U \to \mathbb{R}^m$ belong to $\mathfrak{S}$, $U \subseteq \mathbb{R}^{N+n}$.*

(1) *The set $\{\, (a,x) \in U : f_a$ is continuous at $x \,\}$ belongs to $\mathfrak{S}$.*
(2) *The set $U' := \{\, (a,x) \in U : x \in \operatorname{int}(U_a)$ and f_a is differentiable at $x \,\}$ belongs to $\mathfrak{S}$.*
(3) *The map $(a,x) \mapsto D(f_a)(x) : U' \to \mathbb{R}^{mn}$ belongs to $\mathfrak{S}$.*
(4) *The set of all $(a,x) \in U$ such that f_a is C^1 on an open neighborhood of x contained in U_a belongs to $\mathfrak{S}$.*
(5) *For each positive integer p the set of all $(a,x) \in U$ such that f_a is C^p on an open neighborhood of x contained in U_a belongs to $\mathfrak{S}$.*

The proofs are just like those for B.7.

For the next three results we fix a set $A \subseteq \mathbb{R}^n$. We also let p be a positive integer and $k \in \{0, \dots, n\}$.

B.9. *If A belongs to $\mathfrak{S}$ then $\operatorname{Reg}^p_k(A)$ belongs to $\mathfrak{S}$.*

Proof. Let λ range over the strictly increasing functions $\{1, \dots, k\} \to \{1, \dots, n\}$, and let $\pi_\lambda : \mathbb{R}^n \to \mathbb{R}^k$ be given by

$$\pi_\lambda(x_1, \dots, x_n) := \big(x_{\lambda(1)}, \dots, x_{\lambda(k)}\big).$$

Then

$$\operatorname{Reg}^p_k(A) = \bigcup_\lambda \operatorname{Reg}^p_{k,\lambda}(A)$$

where $\operatorname{Reg}^p_{k,\lambda}(A)$ is the set of all $a \in A$ for which there is an $\epsilon > 0$ such that the map

$$\pi_\lambda | B(a,\epsilon) \cap A : B(a,\epsilon) \cap A \to \mathbb{R}^k$$

is injective with open image $\pi_\lambda(B(a,\epsilon) \cap A)$ in $\mathbb{R}^k$ and that the inverse from $\pi_\lambda(B(a,\epsilon) \cap A)$ into $\mathbb{R}^n$ is C^p. Routine "logical" arguments show that $\mathfrak{S}$ contains the set U_λ of all $(\epsilon, a, x) \in \mathbb{R}^{1+n+k}$ such that $\epsilon > 0$, $a \in A$,

$$\pi_\lambda | B(a,\epsilon) \cap A : B(a,\epsilon) \cap A \to \mathbb{R}^k$$

is injective with open image $\pi_\lambda(B(a,\epsilon)\cap A)$ in $\mathbb{R}^k$, and $x \in \pi_\lambda(B(a,\epsilon)\cap A)$. Define the function $f_\lambda : U_\lambda \to \mathbb{R}^n$ by letting $f_\lambda(\epsilon, a, x)$ be the unique $y \in B(a,\epsilon)\cap A$ with $\pi_\lambda(y) = x$. Then f_λ belongs to $\mathfrak{S}$. The above description of $\mathrm{Reg}^p_{k,\lambda}(A)$ amounts to the following equivalence, where the variable a ranges over $\mathbb{R}^n$:

$$a \in \mathrm{Reg}^p_{k,\lambda}(A) \Leftrightarrow a \in A \ \ \& \ \ \exists \epsilon > 0 \left[\, (f_\lambda)_{(\epsilon,a)} \text{ is } C^p \text{ on } \mathrm{int}((U_\lambda)_{(\epsilon,a)}) \,\right].$$

By B.8(5) we obtain $\mathrm{Reg}^p_{k,\lambda}(A) \in \mathfrak{S}_n$, and thus $\mathrm{Reg}^p_k(A)$ belongs to $\mathfrak{S}$. $\square$

By a similar argument, we have:

B.10. *If $A \subseteq \mathbb{R}^{N+n}$ belongs to $\mathfrak{S}$ then $\{\,(a,x) \in \mathbb{R}^{N+n} : x \in \mathrm{Reg}^p_k(A_a)\,\}$ belongs to $\mathfrak{S}$.*

Note. Tamm's theorem [25] implies that B.9 holds with $p = \omega$ for $\mathfrak{S}(\mathbb{R}_{\mathrm{an}})$, and in [8] this fact is extended to show that B.10 holds with $p = \omega$ for $\mathfrak{S}(\mathbb{R}^{\mathbb{R}}_{\mathrm{an}})$.

B.11. *Suppose that A belongs to $\mathfrak{S}$ and is a C^1 submanifold of $\mathbb{R}^n$. Then the tangent bundle TA and the conormal bundle $T^*_A\mathbb{R}^n$ belong to $\mathfrak{S}$.*

Proof. We use the notation of the preceding proof. Let $\dim A = k$. Note that $A = \bigcup_\lambda A(\lambda)$, where $A(\lambda) := \mathrm{Reg}^1_{k,\lambda}(A)$ is clearly open in A. Hence $TA = \bigcup_\lambda TA(\lambda)$. With variables a, b ranging over $\mathbb{R}^n$ we have:

$$(a,b) \in TA(\lambda) \Leftrightarrow a \in A(\lambda) \ \ \& \ \ b = Df_\lambda(\pi_\lambda(a)),$$

which shows that $TA(\lambda)$ belongs to $\mathfrak{S}$. Hence TA belongs to $\mathfrak{S}$. Similarly,

$$T^*_A\mathbb{R}^n = \bigcup_\lambda T^*_{A(\lambda)}\mathbb{R}^n,$$

and with a, b and x ranging over $\mathbb{R}^n$ we have the equivalence

$$(a,b) \in T^*_{A(\lambda)}\mathbb{R}^n \Leftrightarrow a \in A(\lambda) \ \ \& \ \ \forall x \left[\, (a,x) \in TA(\lambda) \to b.x = 0 \,\right],$$

which shows that $T^*_{A(\lambda)}\mathbb{R}^n$ belongs to $\mathfrak{S}$. Hence $T^*_A\mathbb{R}^n$ belongs to $\mathfrak{S}$. $\square$

B.12. *Let X, Y be C^1 submanifolds of $\mathbb{R}^n$ belonging to $\mathfrak{S}$ with $Y \subseteq \mathrm{fr}(X)$. Then the set $W(X,Y)$ belongs to $\mathfrak{S}$.*

The proof in [15] for $\mathfrak{S} = \mathfrak{S}(\mathbb{R}, +, \cdot, \exp, (r)_{r\in\mathbb{R}})$ goes through for arbitrary structures on $(\mathbb{R}, +, \cdot)$.

Appendix C. Results about o-minimal structures on $(\mathbb{R}, +, \cdot)$

Throughout this appendix we let p be a fixed positive integer and $\mathfrak{S}$ be an o-minimal structure on $(\mathbb{R}, +, \cdot)$. The reader is advised to recall the monotonicity and cell decomposition theorems (4.1 and 4.2); we will use these results often.

The next two results are essential for proving Whitney stratifiability in the o-minimal and analytic-geometric contexts.

C.1. Proposition. *Let X, Y be nonempty C^1 submanifolds of $\mathbb{R}^n$ belonging to $\mathfrak{S}$ with $Y \subseteq \operatorname{fr}(X)$. Then $\dim(Y \setminus W(X,Y)) < \dim Y$.*

The proof of this in [15] for the special case that $\mathfrak{S} = \mathfrak{S}(\mathbb{R}, +, \cdot, \exp, (r)_{r\in\mathbb{R}})$ goes through for arbitrary o-minimal structures on $(\mathbb{R}, +, \cdot)$.

C.2. Lemma. *Let $f : S \to \mathbb{R}^n$ belong to $\mathfrak{S}$, $\emptyset \neq S \subseteq \mathbb{R}^m$. Then there is a finite collection $\mathcal{A} \subseteq \mathfrak{S}_m$ of disjoint C^p cells in $\mathbb{R}^m$, each contained in S, such that $\dim(S \setminus \bigcup \mathcal{A}) < \dim S$, and for each $A \in \mathcal{A}$ the map $f|A : A \to \mathbb{R}^n$ is C^p and $\operatorname{rk} f|A$ is constant.*

Proof. First, by cell decomposition we take a finite partition $\mathcal{P} \subseteq \mathfrak{S}_m$ of S into C^p cells in $\mathbb{R}^m$ such that for each $P \in \mathcal{P}$ the map $f|P : P \to \mathbb{R}^n$ is C^p. Note that for each $P \in \mathcal{P}$ we have $P = P_0 \cup \cdots \cup P_n$, where

$$P_i := \{\, x \in P : (\operatorname{rk} f|P)(x) = i \,\};$$

by B.7(6) each such P_i belongs to $\mathfrak{S}$. Next (again by cell decomposition) we refine $\mathcal{P}$ to a finite partition $\mathcal{P}' \subseteq \mathfrak{S}_m$ of S into disjoint C^p cells in $\mathbb{R}^m$ such that $\mathcal{P}'$ is compatible with $\{\, P_i : P \in \mathcal{P} \ \& \ i \in \{0, \ldots, n\} \,\}$. Then

$$\mathcal{A} := \{\, A \in \mathcal{P}' : \dim A = \dim S \,\}$$

has the desired properties. To see this, let $A \in \mathcal{A}$ and take $P \in \mathcal{P}$ and $i \in \{0, \ldots, n\}$ such that $A \subseteq P_i$. Then A and P are submanifolds of $\mathbb{R}^m$ of equal dimension $\dim S$, so A is open in P and thus $(\operatorname{rk} f|A)(x) = i$ for all $x \in A$. $\square$

Remarks.

(1) In C.2 we may require in addition that $f|A$ be injective for each $A \in \mathcal{A}$ with $\operatorname{rank} f|A = \dim A$. To see this, let $A \in \mathcal{A}$ and $\operatorname{rk} f|A = \dim A$. Then the fibers $A \cap f^{-1}(y)$ ($y \in \mathbb{R}^n$) of $f|A$ are discrete subsets of A, hence by 4.4 there exists $N \in \mathbb{N}$ such that $\operatorname{card}(A \cap f^{-1}(y)) \leq N$ for all $y \in \mathbb{R}^n$. An N-fold application of "definable choice" (4.5), followed by a cell decomposition, gives a finite partition $\mathcal{P}_A$ of A into C^p cells on each of which f is injective. Now replace each A as above by the cells in $\mathcal{P}_A$ that have the same dimension as A.

(2) The rest of Whitney stratification in the o-minimal context is more efficiently treated in Appendix D, since the arguments are just variants of those needed in the setting of analytic-geometric categories; see D.18.

C.3. Note. Let $A \subseteq \mathbb{R}^m$ be locally closed but not closed. Then

$$x \mapsto (x, 1/\operatorname{d}(x, \operatorname{fr}(A))) : A \to \mathbb{R}^{m+1}$$

maps A homeomorphically onto the closed set

$$B := \{\, (x,y) \in \mathbb{R}^{m+1} : x \in \operatorname{cl}(A) \ \& \ y > 0 \ \& \ \operatorname{d}(x, \operatorname{fr}(A)) = 1/y \,\}.$$

(Here, $\operatorname{d}(x, Y) := \inf\{\, |x - y| : y \in Y \,\}$ for $x \in \mathbb{R}^n$ and $\emptyset \neq Y \subseteq \mathbb{R}^n$.) Note that A is the projection of B on the first m coordinates and that if A belongs to $\mathfrak{S}$ then so does B.

The next result grew from correspondence between Miller and T.L. Loi.

C.4. Proposition. *Let $g : A \times \mathbb{R} \to \mathbb{R}$ belong to $\mathfrak{S}$ with $A \subseteq \mathbb{R}^m$. Then there exist functions $\psi : \mathbb{R} \to \mathbb{R}$ and $\rho : A \to \mathbb{R}$ belonging to $\mathfrak{S}$ such that $|g(x,t)| < \psi(t)$ for all $x \in A$ and $t > \rho(x)$.*

Proof. Replacing g by $|g|$, we assume that g is everywhere nonnegative. The set

$$\{\, x \in A : \lim_{t \to +\infty} g(x,t) = +\infty \,\}$$

belongs to $\mathfrak{S}$, so we may reduce to the case that $\lim_{t\to+\infty} g(x,t) = +\infty$ for every $x \in A$. By monotonicity, for each $x \in A$ there exists $s > 0$ (depending on x) such that $t \mapsto g(x,t)$ is strictly increasing on (s,∞); let $\xi(x)$ be the infimum of all such s. Note that $\xi : A \to \mathbb{R}$ belongs to $\mathfrak{S}$. By cell decomposition, we may reduce to the case that ξ is continuous, g is continuous on $\{\,(x,t) \in A \times \mathbb{R} : t > \xi(x)\,\}$ and A is locally closed; indeed by C.3 we may even assume that A is closed. Define $\psi : (0,\infty) \to \mathbb{R}$ by

$$\psi(t) := \sup\{\, g(x, \xi(x) + t) : x \in A \;\;\&\;\; |x| \le t \,\}.$$

Then ψ belongs to $\mathfrak{S}$, and for all $x \in A$ and $t > \rho(x) := \max\{\xi(x), |x|\}$ we have

$$g(x,t) < g(x, \xi(x) + t) \le \psi(t). \quad \square$$

Recall that $\Phi^p_{\mathfrak{S}}$ denotes the set of all odd, increasing bijections $f : \mathbb{R} \to \mathbb{R}$ belonging to $\mathfrak{S}$ that are C^p on $\mathbb{R}$ and p-flat at 0.

C.5. Lemma. *Let $a > 0$ and $f : [0,a] \to \mathbb{R}$ belonging to $\mathfrak{S}$ be continuous and strictly increasing with $f(0) = 0$. Then there exists $\phi \in \Phi^p_{\mathfrak{S}}$ such that $\phi(t) < f(t)$ for all sufficiently small positive t.*

Proof. If there exists $n \in \mathbb{N}$ such that $f(t) \ge t^n$ as $t \to 0^+$, then take $\phi(t) = t^m$ where m is any odd integer strictly greater than n and p. So suppose that $\lim_{t\to 0^+} t^{-n} f(t) = 0$ for every $n \in \mathbb{N}$. By cell decomposition, we may assume that f is C^p on $(0,a)$ and that $a < 1$. Put $\phi(t) := t f(at^2/(1+t^2))$; then ϕ is clearly C^p on $\mathbb{R} \setminus \{0\}$, $\phi(t) < f(t)$ for all sufficiently small $t > 0$, and $\lim_{t\to 0} \phi(t)/t^p = 0$. By l'Hospital's rule, we have $\lim_{t\to 0} \phi^{(k)}(t) = 0$ for $k = 0, \dots, p$, and hence $\phi \in \Phi^p_{\mathfrak{S}}$. $\square$

C.6. Corollary. *Let $f : \mathbb{R} \to \mathbb{R}$ belong to $\mathfrak{S}$. Then there exists $\phi \in \Phi^p_{\mathfrak{S}}$ such that $|f(t)| < 1/\phi(1/t)$ for all sufficiently large positive t.*

Proof. If f is bounded as $t \to +\infty$ then the result is clear, so suppose (by monotonicity) that $\lim_{t\to+\infty} |f(t)| = +\infty$. Apply the preceding lemma to $t \mapsto |1/f(1/t)|$ (which is defined, continuous and strictly increasing on some interval $(0,a)$ and belongs to $\mathfrak{S}$). $\square$

C.7. Lemma. *Let $f : A \times (0,\infty) \to \mathbb{R}$ belong to $\mathfrak{S}$, $A \subseteq \mathbb{R}^m$. Then there exists $\phi \in \Phi^p_{\mathfrak{S}}$ such that $\lim_{t\to 0^+} \phi(t) f(x,t) = 0$ for each $x \in A$.*

Proof. Applying C.4 and C.6 to the function

$$(x,t) \mapsto f(x, 1/t) : A \times (0,\infty) \to \mathbb{R},$$

we obtain $\theta \in \Phi^p_{\mathfrak{S}}$ such that $\lim_{t\to 0^+} \theta(t) f(x,t) \in [-1,1]$ for every $x \in A$. Now put $\phi := \theta^3$ (where θ^3 denotes the cube of θ). $\square$

C.8. Lemma. *Let $g : A \to \mathbb{R}$, $f_1, \dots, f_l : A \setminus Z(g) \to \mathbb{R}$ belong to $\mathfrak{S}$ and be continuous, with A locally closed in $\mathbb{R}^m$. Then there exists $\phi \in \Phi^p_{\mathfrak{S}}$ such that $\phi(g(x))f_i(x) \to 0$ as $x \to y$, $x \in A \setminus Z(g)$, for each $y \in Z(g)$ and $i = 1, \dots, l$.*

Proof. By C.3 we may suppose that A is closed. For $y \in Z(g)$ and $t > 0$ put

$$A(y,t) := \{\, x \in A : |x - y| \leq 1 \;\;\&\;\; |g(x)| = t \,\}.$$

Note that $A(y,t)$ is compact and belongs to $\mathfrak{S}$, and if $x \in A(y,t)$ then $x \in A \setminus Z(g)$. Define $G : Z(g) \times (0, \infty) \to \mathbb{R}$ by

$$G(y,t) := \begin{cases} \max\{\, |(f_1(x), \dots, f_l(x))| : x \in A(y,t) \,\}, & \text{if } A(y,t) \neq \emptyset \\ 0, & \text{otherwise.} \end{cases}$$

Take ϕ as in C.7 and note that $|\phi \circ g| = \phi \circ |g|$. $\square$

Notation. Let $C^p_{\mathfrak{S}}(\mathbb{R}^n)$ denote the set of all C^p functions $\mathbb{R}^n \to \mathbb{R}$ belonging to $\mathfrak{S}$.

C.9. Proposition. *Let $f, g : \mathbb{R}^n \to \mathbb{R}$ belonging to $\mathfrak{S}$ be continuous, of class C^p on $\mathbb{R}^n \setminus Z(g)$, with $Z(f) \subseteq Z(g)$. Then there exist $\phi \in \Phi^p_{\mathfrak{S}}$ and $h \in C^p_{\mathfrak{S}}(\mathbb{R}^n)$ such that $\phi \circ g = hf$.*

Proof. We just do the case $p = 1$; the proof for p an arbitrary positive integer is similar but notationally cumbersome.

Put $U := \mathbb{R}^n \setminus Z(g)$. By C.8, there exists $\theta \in \Phi^1_{\mathfrak{S}}$ such that for each function $\sigma : U \to \mathbb{R}$ from the collection

$$\{1/f|U, D_1 f|U, \dots, D_n f|U, D_1 g|U, \dots, D_n g|U\}$$

and every $y \in \operatorname{bd}(Z(g))$ we have $\lim_{x \to y} \theta(g(x))\sigma(x) = 0$. (Note that $Z(f) \cap U = \emptyset$.) Put $\phi := \theta^3$ and define $h : \mathbb{R}^n \to \mathbb{R}$ by

$$h(x) := \begin{cases} (\phi(g(x))/f(x), & x \in U \\ 0, & \text{otherwise.} \end{cases}$$

Then for $i = 1, \dots, n$ and $x \in U$ we have

$$D_i h(x) = \frac{f(x) 3(\theta(g(x)))^2 \theta'(g(x)) D_i g(x) - D_i f(x) (\theta(g(x)))^3}{f(x)^2}.$$

Let $y \in \operatorname{bd}(Z(g))$. Since $\theta'(0) = 0$ and $g(x) \to 0$ as $x \to y$, it follows that $D_i h(x) \to 0$ as $x \to y$, $x \in U$. Hence, by monotonicity and l'Hospital's rule, each partial of h exists at y with value 0. Thus, $h \in C^1_{\mathfrak{S}}(\mathbb{R}^n)$. $\square$

(In general, for a fixed positive integer p one specifies a finite subset S of the ring of functions on U generated over $\mathbb{Z}$ by $1/f|U$ and all partials of $f|U$ and $g|U$ of order less than or equal to p. One then obtains some $\theta \in \Phi^p_{\mathfrak{S}}$ as in C.8 applied to S and then puts $\phi := \theta^M$ for some suitable odd integer $M > p$.)

C.10. Corollary. *Let $g : \mathbb{R}^n \to \mathbb{R}$ belonging to $\mathfrak{S}$ be continuous and C^p on $\mathbb{R}^n \setminus Z(g)$. Then there exists $\phi \in \Phi^p_{\mathfrak{S}}$ such that $\phi \circ g \in C^p_{\mathfrak{S}}(\mathbb{R}^n)$.*

Proof. Apply the preceding result with $f = 1$. $\square$

The next result was established by Bierstone, Milman and Pawłucki for the subanalytic category [unpublished]. The proof below is patterned after theirs; we thank Deirdre Haskell for pointing out an error in an earlier version.

C.11. Theorem. *Let $A \in \mathfrak{S}_n$ be closed. Then there exists $f \in C^p_{\mathfrak{S}}(\mathbb{R}^n)$ with $A = Z(f)$.*

Proof. We proceed by induction on $n \geq 0$ and $\dim A$. The case $n = 0$ is trivial. Assume now that $n > 0$ and $A \in \mathfrak{S}_n$ is closed with $\dim A = d \geq 0$.

First, suppose that $d = n$. Then $\dim \operatorname{bd}(A) < n$, so inductively we may assume there exists $g \in C^p_{\mathfrak{S}}(\mathbb{R}^n)$ with $Z(g) = \operatorname{bd}(A)$. By C.10, there exists $\phi \in \Phi^p_{\mathfrak{S}}$ such that the function

$$f(x) := \begin{cases} \phi(g(x)), & x \notin \operatorname{int}(A) \\ 0, & x \in \operatorname{int}(A) \end{cases}$$

belongs to $C^p_{\mathfrak{S}}(\mathbb{R}^n)$ and $Z(f) = A$.

Now suppose that $d < n$. Replacing A with $\operatorname{cl}(\tau_n(A))$ (τ_n as in §3), we reduce to the case that A is compact. By cell decomposition there are finitely many C^p maps

$$\psi_1 : U_1 \to \mathbb{R}^{n-d(1)}, \ldots, \psi_k : U_k \to \mathbb{R}^{n-d(k)}$$

belonging to $\mathfrak{S}$, each U_i open in $\mathbb{R}^{d(i)}$ with $0 \leq d(i) \leq d$, such that

$$A = \operatorname{cl}(\Gamma(\psi_1)) \cup \cdots \cup \operatorname{cl}(\Gamma(\psi_k)).$$

It suffices to consider the case that $k = 1$, for if we have $f_1, \ldots, f_k \in C^p_{\mathfrak{S}}(\mathbb{R}^n)$ with $Z(f_i) = \operatorname{cl}(\Gamma(\psi_i))$ for $i = 1, \ldots, k$ then the product $f_1 \cdots f_k$ belongs to $C^p_{\mathfrak{S}}(\mathbb{R}^n)$ and has zero set equal to A.

We have now reduced to the case that A is the closure of the graph of a C^p map $\psi : U \to \mathbb{R}^e$ belonging to $\mathfrak{S}$ with $e = n - d$ and U open in $\mathbb{R}^d$, $U \neq \emptyset$. Inductively, there exists $g \in C^p_{\mathfrak{S}}(\mathbb{R}^d)$ with $Z(g) = \operatorname{bd}(U)$. Squaring, we may assume that g is everywhere nonnegative. For $(x, y) \in \mathbb{R}^d \times \mathbb{R}^e$ put

$$G(x, y) := \begin{cases} ||y - \psi(x)||g(x), & x \in U \\ g(x), & \text{otherwise.} \end{cases}$$

(Recall that $||v||$ denotes the euclidean norm of $v \in \mathbb{R}^n$.) Note that G belongs to $\mathfrak{S}$, and is continuous, nonnegative and C^p off its zero set $Z(G) = (\operatorname{bd}(U) \times \mathbb{R}^e) \cup A$. Applying C.10, there exists $F_1 \in C^p_{\mathfrak{S}}(\mathbb{R}^n)$ with

$$Z(F_1) = (\operatorname{bd}(U) \times \mathbb{R}^e) \cup A.$$

To finish the proof, it now suffices to find $F_2 \in C^p_{\mathfrak{S}}(\mathbb{R}^n)$ such that $\Gamma(\psi) \subseteq Z(F_2)$ and $F_2(x, y) > 0$ for all $(x, y) \in (\operatorname{bd}(U) \times \mathbb{R}^e) \setminus A$; by continuity we then have $A \subseteq Z(F_2)$ and thus $A = Z(F_1^2 + F_2^2)$. Since $A \setminus \Gamma(\psi) = \mathrm{fr}(\Gamma(\psi))$ is closed, by the inductive assumptions and 4.7 there exists $h \in C^p_{\mathfrak{S}}(\mathbb{R}^n)$ with $h \geq 0$ and

$$Z(h) = A \setminus \Gamma(\psi) = (\mathrm{bd}(U) \times \mathbb{R}^e) \setminus A.$$

Define $H : \mathbb{R}^d \to \mathbb{R}$ by

$$H(x) := \begin{cases} h(x, \psi(x)), & x \in U \\ 0, & x \in \mathbb{R}^d \setminus U. \end{cases}$$

Applying C.10, we obtain $\phi \in \Phi^p_{\mathfrak{S}}$ such that $\phi \circ H \in C^p_{\mathfrak{S}}(\mathbb{R}^d)$. Then $F_2 : \mathbb{R}^n \to \mathbb{R}$ given by $(x, y) \mapsto \phi(h(x, y)) - \phi(H(x))$ has the required properties. $\square$

C.12. Corollary. *Let $A, B \in \mathfrak{S}_n$ be disjoint and closed. Then there exists $g \in C^p_{\mathfrak{S}}(\mathbb{R}^n)$ with $Z(g) = A$, $Z(g - 1) = B$ and $0 \leq g \leq 1$.*

Proof. There exist $f_1, f_2 \in C^p_{\mathfrak{S}}(\mathbb{R}^n)$ with $Z(f_1) = A$ and $Z(f_2) = B$. Now put $g_i := f_i^2/(1 + f_i^2)$ for $i = 1, 2$, and $g := (g_1 + g_1 g_2)/(g_1 + g_2)$. $\square$

An argument similar to (but easier than) that of C.9 yields the following:

C.13. Proposition. *Let $f, g : A \to \mathbb{R}$ belonging to $\mathfrak{S}$ be continuous, $\emptyset \neq A \subseteq \mathbb{R}^n$ locally closed, with $Z(f) \subseteq Z(g)$. Then there exists $\phi \in \Phi^p_{\mathfrak{S}}$ and $h : A \to \mathbb{R}$ continuous belonging to $\mathfrak{S}$ with $\phi \circ g = hf$.*

C.14. Generalized Łojasiewicz inequality. *Let $f, g : A \to \mathbb{R}$ belonging to $\mathfrak{S}$ be continuous, with $Z(f) \subseteq Z(g)$ and $\emptyset \neq A \subseteq \mathbb{R}^n$ compact. Then there exists $\phi \in \Phi^p_{\mathfrak{S}}$ such that $|\phi(g(x))| \leq |f(x)|$ for all $x \in A$.*

Proof. There exist $\theta \in \Phi^p_{\mathfrak{S}}$ and $h : A \to \mathbb{R}$ continuous such that $\theta \circ g = hf$. Put $C := 1 + \max\{\, |h(x)| : x \in A \,\}$ and $\phi := \theta/C$. $\square$

C.15. Generalized Hölder continuity. *Let $f : A \to \mathbb{R}$ belonging to $\mathfrak{S}$ be continuous, $\emptyset \neq A \subseteq \mathbb{R}^n$ compact. Then there exists $\phi \in \Phi^p_{\mathfrak{S}}$ such that*

$$|f(x) - f(y)| \leq \phi^{-1}(|x - y|)$$

for all $x, y \in A$.

Proof. Applying C.14 to the functions

$$(x, y) \mapsto |x - y| : A^2 \to \mathbb{R}, \quad (x, y) \mapsto |f(x) - f(y)| : A^2 \to \mathbb{R}$$

we obtain $\theta \in \Phi^p_{\mathfrak{S}}$ and $C > 0$ such that $\theta(|f(x) - f(y)|) \leq C|x - y|$ for all $x, y \in A$. Then $|f(x) - f(y)| \leq \theta^{-1}(C|x - y|)$ for all $x, y \in A$. Put $\phi := \theta/C$. $\square$

We now set out to prove the results on paths of 4.21.

C.16. Lemma. *Let $g = (g_1 \dots, g_n) : [a,b] \to \mathbb{R}^n$ be a path.*

(1) *If each g_i is monotone, then g is rectifiable with* $\text{length}(g) \le ||g(b) - g(a)||_1$, *where* $||x||_1 := \sum_{i=1}^n |x_i|$ *for* $x = (x_1, ..., x_n) \in \mathbb{R}^n$.

(2) *If g belongs to $\mathfrak{S}$, then g is rectifiable.*

Proof. For (1), we may reduce to the case that each g_i is increasing, so that with $a = a_0 < a_1 < \cdots < a_k = b$ we have:

$$\sum_{i=1}^{k} ||g(a_i) - g(a_{i-1})|| \le \sum_{i=1}^{k} ||g(a_i) - g(a_{i-1})||_1 = ||g(b) - g(a)||_1.$$

Now (2) is immediate from (1) and the monotonicity theorem. $\square$

We need a slightly more precise result. Let $g = (g_1, \dots, g_n) : [a,b] \to \mathbb{R}^n$ be a path belonging to $\mathfrak{S}$. Define

$$\begin{aligned} \text{Mono}(g) := \{\, t \in (a,b) : & \text{ there exist } a', b' \text{ with } a < a' < t < b' < b \text{ such that} \\ & \text{some } g_i \text{ is increasing on } (a',t) \text{ and strictly decreasing on } (t,b'), \\ & \text{or some } g_i \text{ is decreasing on } (a',t) \text{ and strictly increasing on } (t,b') \,\}. \end{aligned}$$

Note that $\text{Mono}(g)$ is finite, and that if $a = t_0 < t_1 < \cdots < t_k = b$ and all points of $\text{Mono}(g)$ are among $t_1, \dots, t_{k-1}$, then $\text{length}(g) \le \sum_{i=1}^k ||g(t_i) - g(t_{i-1})||_1$.

C.17. Lemma. *Let $B \in \mathfrak{S}_n$ be compact and $h : B \to \mathbb{R}^m$ be a continuous map belonging to $\mathfrak{S}$. Then there exists $\psi \in \Phi^p_{\mathfrak{S}}$ such that* $\text{length}(h \circ g) \le N\psi^{-1}(\text{length}(g))$ *for all paths $g : [a,b] \to B$ belonging to $\mathfrak{S}$, where* $N = 1 + \text{card}(\text{Mono}(h \circ g))$.

Proof. By C.15, there exists $\psi \in \Phi^p_{\mathfrak{S}}$ such that $||h(x) - h(y)||_1 \le \psi^{-1}(||x - y||)$ for all $x, y \in B$. Let $g : [a,b] \to B$ be a path belonging to $\mathfrak{S}$. Let $t_1 < \cdots < t_{N-1}$ be the elements of $\text{Mono}(h \circ g)$ and put $t_0 := a$ and $t_N := b$. Then

$$\begin{aligned} \text{length}(h \circ g) &\le \sum_{i=1}^{N} ||h(g(t_i)) - h(g(t_{i-1}))||_1 \\ &\le \sum_{i=1}^{N} \psi^{-1}(||g(t_i) - g(t_{i-1})||) \le N\psi^{-1}(\text{length}(g)). \quad \square \end{aligned}$$

C.18. Generalized Whitney regularity. *Let $A \in \mathfrak{S}_n$ be nonempty, compact and connected. Then there exist $\phi \in \Phi^p_{\mathfrak{S}}$ and a map $\gamma : A^2 \times [0,1] \to A$ belonging to $\mathfrak{S}$ such that for every $x, y \in A$,*

$$\gamma_{x,y} := t \mapsto \gamma(x,y,t) : [0,1] \to A$$

is a path from x to y, with $\text{length}(\gamma_{x,y}) \le \phi^{-1}(||x - y||)$.

Proof. By the triangulation theorem there is a finite simplicial complex K in $\mathbb{R}^n$ spanning the polyhedron $|K|$ and a homeomorphism $h : |K| \to A$ belonging to $\mathfrak{S}$. By an elementary argument there is a semialgebraic map $\sigma : |K| \times |K| \times [0,1] \to |K|$ and a positive constant C such that for all $p, q \in |K|$ the map $\sigma_{p,q} : [0,1] \to |K|$ given by $t \mapsto \sigma(p,q,t)$ is a path from p to q with $\text{length}(\sigma_{p,q}) \le C||p-q||$. In particular, for all $x, y \in A$ we have

$$\text{length}\left(\sigma_{h^{-1}(x),h^{-1}(y)}\right) \le C||h^{-1}(x) - h^{-1}(y)||,$$

so by C.15 there exists $\phi_1 \in \Phi^p_{\mathfrak{S}}$ such that

$$\text{length}\left(\sigma_{h^{-1}(x),h^{-1}(y)}\right) \le \phi_1^{-1}(||x-y||)$$

for all $x, y \in A$. Define $\gamma : A \times A \times [0,1] \to A$ by

$$\gamma(x,y,t) := h(\sigma(h^{-1}(x), h^{-1}(y), t)).$$

Now γ belongs to $\mathfrak{S}$, so the set

$$\{ (x,y,t) \in A \times A \times [0,1] : t \in \text{Mono}\,(\gamma_{x,y}) \}$$

also belongs to $\mathfrak{S}$. Since for every $x, y \in A$ we have that $\text{card}(\text{Mono}(\gamma_{x,y}))$ is finite, by uniform bounds (4.4) there exists $N \in \mathbb{N}$ such that $1 + \text{card}(\text{Mono}(\gamma_{x,y})) \le N$ for all $x, y \in A$. Applying C.17 (with $B = |K|$) we get $\phi_2 \in \Phi^p_{\mathfrak{S}}$ such that for $x, y \in A$:

$$\begin{aligned} \text{length}(\gamma_{x,y}) &= \text{length}\left(h \circ \sigma_{h^{-1}(x),h^{-1}(y)}\right) \\ &\le N\phi_2^{-1}\left(\text{length}\left(\sigma_{h^{-1}(x),h^{-1}(y)}\right)\right) \\ &\le N\phi_2^{-1}\left(\phi_1^{-1}(||x-y||\right). \end{aligned}$$

Now put $\phi(t) := \phi_1(\phi_2(t/N))$ for $t \in \mathbb{R}$. $\square$

Remarks.

(1) Proposition 4.21(1) is obtained by a similar use of triangulation (4.10).
(2) Various refinements of 4.21 are easily obtained by choosing the map σ in the proof to have extra properties; see e.g. [11].
(3) The argument above is somewhat more elementary than the proofs for the subanalytic case in [1] and [11].

C.19. *Let $A \in \mathfrak{S}_n$, $\dim A = 1$. Then there exist an open neighborhood $U \in \mathfrak{S}_n$ of 0 and injective C^p maps $\gamma_1, \dots, \gamma_k : [0,1) \to \mathbb{R}^n$ belonging to $\mathfrak{S}$ such that $U \cap A \setminus \{0\}$ is the disjoint union of $\gamma_1(0,1), \dots, \gamma_k(0,1)$.*

Proof. By cell decomposition there is an open neighborhood $U \in \mathfrak{S}_n$ of 0 such that $U \cap A \setminus \{0\}$ is a finite disjoint union of 1-dimensional C^p cells $D_1, \dots, D_k \subseteq \mathbb{R}^n$ with $0 \in \text{fr}(D_i)$ for $i = 1, \dots, k$. Fix $D := D_i$ for some $i \in \{1, \dots, k\}$. It suffices to show that there is an injective C^p map $\gamma : [0,1) \to \mathbb{R}^n$ belonging to $\mathfrak{S}$ with $\gamma(0) = 0$ and $\gamma(0,1) = D$. By the definition of 1-dimensional C^p cells there is a C^p diffeomorphism $\sigma : (0,1) \to D$ in $\mathfrak{S}$ ith $\lim_{t\to 0^+} \sigma(t) = 0$. Now use C.5 to obtain a C^p diffeomorphism $\varphi : (0,1) \to (0,1)$ belonging to $\mathfrak{S}$ such that $\gamma := \sigma \circ \varphi : (0,1) \to D$ satisfies $|\gamma(t)| = o(t^p)$ as $t \to 0^+$. Extending γ to 0 by setting $\gamma(0) = 0$ we obtain a C^p map $\gamma : [0,1) \to \mathbb{R}^n$ as required (cf. the proof of C.5). $\square$

Remark. It is clear that this proof provides $\gamma_1, \dots, \gamma_k$ such that each restriction $\gamma_i|(0,1)$ is a C^p diffeomorphism onto a C^p cell.

Appendix D. Results about analytic-geometric categories

Let $\mathcal{C}$ be an analytic-geometric category; M, N denote arbitrary manifolds of dimensions m and n respectively; the variable x will range over M until further notice. Recall that we identify the analytic manifold $\mathbb{R}^n$ with an open subset of the projective space $\mathbb{P}^n(\mathbb{R})$ via

$$(y_1, \dots, y_n) \mapsto (1 : y_1 : \cdots : y_n) : \mathbb{R}^n \to \mathbb{P}^n(\mathbb{R}).$$

D.1. *All subanalytic sets are $\mathcal{C}$-sets; in particular, $\mathbb{R}^n \in \mathcal{C}(\mathbb{P}^n(\mathbb{R}))$.*

Proof. By the local definition of subanalytic sets it suffices to show that semianalytic sets are $\mathcal{C}$-sets; since this question is again local it suffices to consider an analytic function $f : M \to \mathbb{R}$ and show that the sets $\{\, x : f(x) = 0 \,\}$ and $\{\, x : f(x) > 0 \,\}$ belong to $\mathcal{C}(M)$. For the first set, use the equivalence

$$f(x) = 0 \Leftrightarrow \exists y \in \mathbb{P}^1(\mathbb{R})[\,(x, y) \in \Gamma(f) \;\;\&\;\; y = 0\,];$$

the fact that $\Gamma(f), M \times \{0\} \in \mathcal{C}(M \times \mathbb{P}^1(\mathbb{R}))$ (because

$$x \mapsto (x, f(x)) : M \to M \times \mathbb{P}^1(\mathbb{R})$$

and

$$x \mapsto (x, 0) : M \to M \times \mathbb{P}^1(\mathbb{R})$$

are proper analytic maps); and the properness of the projection map

$$(x, y) \mapsto x : M \times \mathbb{P}^1(\mathbb{R}) \to M.$$

To show that $\{\, x : f(x) > 0 \,\} \in \mathcal{C}(M)$, note the equivalence

$$f(x) > 0 \Leftrightarrow f(x) \neq 0 \;\;\&\;\; \exists y = (y_0 : y_1) \in \mathbb{P}^1(\mathbb{R})[\, f(x).y_1^2 - y_0^2 = 0\,],$$

and use that the set $\{\,(x, y) \in M \times \mathbb{P}^1(\mathbb{R}) : f(x).y_1^2 - y_0^2 = 0\,\}$ is locally given around every point of $M \times \mathbb{P}^1(\mathbb{R})$ by the vanishing of an analytic function, hence belongs to $\mathcal{C}(M \times \mathbb{P}^1(\mathbb{R}))$ by AG4 of §1 and what we showed earlier. Note that $\mathbb{R}^n = \mathbb{P}^n(\mathbb{R}) \setminus \{\, y \in \mathbb{P}^n(\mathbb{R}) : y_0 = 0\,\}$, and that $\{\, y \in \mathbb{P}^n(\mathbb{R}) : y_0 = 0\,\}$ is the hyperplane at infinity. This hyperplane is analytic in $\mathbb{P}^n(\mathbb{R})$, that is, locally given around each point of $\mathbb{P}^n(\mathbb{R})$ by the vanishing of an analytic function. It follows that $\mathbb{R}^n$ is subanalytic in $\mathbb{P}^n(\mathbb{R})$, and hence a $\mathcal{C}$-set in $\mathbb{P}^n(\mathbb{R})$. □

D.2. *Each analytic map $f : M \to N$ is a $\mathcal{C}$-map.*

Proof. This follows from D.1 since $\Gamma(f)$ is analytic, hence subanalytic in $M \times N$. □

D.3. *Given an open covering (U_i) of M and $A \in \mathcal{C}(M)$, a map $f : A \to N$ is a $\mathcal{C}$-map if and only if each restriction $f|U_i \cap A : U_i \cap A \to N$ is a $\mathcal{C}$-map.*

Proof. This follows from the definition of $\mathcal{C}$-map by applying axiom AG4 to the open covering $(U_i \times N)$ of $M \times N$. □

D.4. *Let $A \in \mathcal{C}(M)$ and $B \in \mathcal{C}(N)$. Then $A \times B \in \mathcal{C}(M \times N)$ and the projection maps $A \times B \to A$ and $A \times B \to B$ are $\mathcal{C}$-maps.*

Proof. Note first that $A \times B = (A \times N) \cap (M \times B)$. To see that $A \times N \in \mathcal{C}(M \times N)$, one can by axiom AG4 reduce to the case that $N = \mathbb{R}^n$, and this case follows by n-fold application of axiom AG2. Using the symmetry $M \times N \cong N \times M$ it also follows that $M \times B$ is a $\mathcal{C}$-set.

Let $\pi_A : A \times B \to A$ and $\pi_M : M \times N \to M$ be the projection maps. Then inside the manifold $(M \times N) \times M$ we have $\Gamma(\pi_A) = \Gamma(\pi_M) \cap ((A \times B) \times M)$. By D.2, we have $\Gamma(\pi_M) \in \mathcal{C}((M \times N) \times M)$. Hence π_A is a $\mathcal{C}$-map. $\square$

D.5. *Let $A \in \mathcal{C}(M)$, $B \in \mathcal{C}(N)$, $f : A \to B$ be a $\mathcal{C}$-map, and let $A' \subseteq A$ and $B' \subseteq B$ be $\mathcal{C}$-sets in M and N respectively, such that $f(A') \subseteq B'$. Then $f|A' : A' \to B'$ is a $\mathcal{C}$-map.*

Proof. Use D.4 and that $\Gamma(f|A') = \Gamma(f) \cap (A' \times B')$. $\square$

D.6. *Let $A, A' \in \mathcal{C}(M)$ with $A' \subseteq A$, and let $f : A \to N$ be a proper $\mathcal{C}$-map. Then $f(A') \in \mathcal{C}(N)$.*

Proof. Using the properness of f, axiom AG4 and D.5, one easily reduces to the case that $M = \mathbb{R}^m$ and A is compact, and then, with x ranging over $\mathbb{P}^m(\mathbb{R})$ and y over N, we have

$$y \in f(A') \Leftrightarrow \exists x[\, x \in A' \ \ \& \ \ (x, y) \in \Gamma(f)\,].$$

Since A is bounded in $\mathbb{R}^m$, the sets $A' \times N$ and $\Gamma(f)$ are not only $\mathcal{C}$-sets of $\mathbb{R}^m \times N$ but also of $\mathbb{P}^m(\mathbb{R}) \times N$. Now use the properness of the (analytic) projection map $\mathbb{P}^m(\mathbb{R}) \times N \to N$ and axiom AG3. $\square$

Remark. Assertion D.6 becomes false if the assumption that $f : A \to N$ is a proper $\mathcal{C}$-map is replaced by the assumption that $f : A \to B$ is a proper $\mathcal{C}$-map, for some $B \in \mathcal{C}(N)$. To see this, take $M = N = \mathbb{R}^2$,

$$A = B = \{\, (r, s) \in \mathbb{R}^2 : 0 < r < 1 \ \ \& \ \ s < 0 \,\},$$
$$A' = \{\, (r, \log r) : 0 < r < 1 \,\},$$

and define $f : A \to B$ by $f(r, s) = (r, 1/s)$; then A, A' and B are all subanalytic in $\mathbb{R}^2$, and f is a proper subanalytic map, but $f(A')$ is not subanalytic (at the origin) in $\mathbb{R}^2$.

D.7. *Let $A \in \mathcal{C}(M)$ be closed in M, $f : A \to N$ be a $\mathcal{C}$-map and $B \in \mathcal{C}(N)$. Then $f^{-1}(B) \in \mathcal{C}(M)$.*

Proof. With y ranging over N, we have

$$x \in f^{-1}(B) \Leftrightarrow \exists y[(\, x, y) \in \Gamma(f) \ \ \& \ \ y \in B\,],$$

so $f^{-1}(B)$ is the image of the $\mathcal{C}$-set $\Gamma(f) \cap (M \times B)$ under the map

$$(x, y) \mapsto x : \Gamma(f) \to M.$$

Now use D.4 and the fact that this map is a proper $\mathcal{C}$-map. $\square$

Remark. Assertion D.7 becomes false if we omit the assumption that A is closed. This is shown by the same example as in the remark following D.6, except that now the target space of f should be $N = \mathbb{R}^2$ and $B = \{\, (r, \log r) : 0 < r < 1 \,\}$.

D.8. *Let $f : A \to B$ and $g : B \to C$ be $\mathcal{C}$-maps. Then the composition $g \circ f : A \to C$ is a $\mathcal{C}$-map. The identity map on A is a $\mathcal{C}$-map. The map f is an isomorphism in the category $\mathcal{C}$ if and only if f is a homeomorphism onto B.*

Proof. Let C be a $\mathcal{C}$-set in the manifold P. Using D.3 and the continuity of f we may reduce to the case that $N = \mathbb{R}^n$ and B is bounded in $\mathbb{R}^n$. Then $\mathit{\Gamma}(f) \in \mathcal{C}(M \times \mathbb{P}^n(\mathbb{R}))$ and $\mathit{\Gamma}(g) \in \mathcal{C}(\mathbb{P}^n(\mathbb{R}) \times P)$. Let the variables y and z range over $\mathbb{P}^n(\mathbb{R})$ and P respectively. Then we have the equivalence

$$(x, z) \in \mathit{\Gamma}(g \circ f) \Leftrightarrow \exists y[\,(x, y) \in \mathit{\Gamma}(f) \quad \& \quad (y, z) \in \mathit{\Gamma}(g)\,],$$

which exhibits $\mathit{\Gamma}(g \circ f)$ as the image of the set $(\mathit{\Gamma}(f) \times P) \cap (M \times \mathit{\Gamma}(g))$ under the projection map $M \times \mathbb{P}^n(\mathbb{R}) \times P \to M \times P$, which is a proper analytic map. Hence $\mathit{\Gamma}(g \circ f) \in \mathcal{C}(M \times P)$ by D.4 and AG3. That the identity on A is a $\mathcal{C}$-map follows from D.5 and the fact that the identity on M is a $\mathcal{C}$-map by D.2. $\square$

D.9. *Let $A, B_1, \ldots, B_k$ be $\mathcal{C}$-sets. Then a map*

$$f = (f_1, \ldots, f_k) : A \to B_1 \times \cdots \times B_k$$

is a $\mathcal{C}$-map if and only if each component f_i is a $\mathcal{C}$-map.

Proof. The "if" direction is an easy consequence of D.4, while the "only if" direction follows from D.4 and D.8. $\square$

D.10. *Let $\mathcal{C}$ be an analytic-geometric category and $\mathfrak{S}$ an o-minimal structure on $\mathbb{R}_{\mathrm{an}}$. Then*:

1. *For each $A \subseteq \mathbb{R}^n$, $A \in \mathfrak{S}(\mathcal{C})_n$ if and only if $\tau_n(A) \in \mathcal{C}(\mathbb{R}^n)$.*
2. *$\mathfrak{S}(\mathcal{C})$ is an o-minimal structure on $\mathbb{R}_{\mathrm{an}}$.*
3. *$\mathcal{C}(\mathfrak{S})$ is an analytic-geometric category.*
4. *$\mathcal{C}(\mathfrak{S}(\mathcal{C})) = \mathcal{C}$ and $\mathfrak{S}(\mathcal{C}(\mathfrak{S})) = \mathfrak{S}$.*

Proof of (1).

Assume first that $A \in \mathfrak{S}(\mathcal{C})$. With x ranging over $\mathbb{P}^n(\mathbb{R})$ and y over $\mathbb{R}^n$ we have:

$$y \in \tau_n(A) \Leftrightarrow \exists x[\,x \in A \quad \& \quad (x, y) \in \mathit{\Gamma}(\tau_n)\,],$$

which exhibits $\tau_n(A)$ as the image of $(A \times \mathbb{R}^n) \cap \mathit{\Gamma}(\tau_n)$ under the projection map $\mathbb{P}^n(\mathbb{R}) \times \mathbb{R}^n \to \mathbb{R}^n$, which is proper. But $\mathit{\Gamma}(\tau_n)$ is semialgebraic in $\mathbb{R}^n \times \mathbb{R}^n$, hence semianalytic in $\mathbb{P}^n(\mathbb{R}) \times \mathbb{R}^n$, in particular a $\mathcal{C}$-set in $\mathbb{P}^n(\mathbb{R}) \times \mathbb{R}^n$. It follows that $\tau_n(A) \in \mathcal{C}(\mathbb{R}^n)$.

Conversely, suppose that $\tau_n(A) \in \mathcal{C}(\mathbb{R}^n)$. Then, with x and y both ranging over $\mathbb{P}^n(\mathbb{R})$, we have:

$$x \in A \Leftrightarrow \exists y[\,y \in \tau_n(A) \;\; \& \;\; (x, y) \in \mathit{\Gamma}(\tau_n)\,],$$

which exhibits A as the image of $(\tau_n(A) \times \mathbb{P}^n(\mathbb{R})) \cap \mathit{\Gamma}(\tau_n)$ under the projection map $\mathbb{P}^n(\mathbb{R}) \times \mathbb{P}^n(\mathbb{R}) \to \mathbb{P}^n(\mathbb{R})$ on the first factor, which is a proper map. Since $\tau_n(A)$ is bounded in $\mathbb{R}^n$, it is also a $\mathcal{C}$-set in $\mathbb{P}^n(\mathbb{R})$. It follows that $A \in \mathcal{C}(\mathbb{P}^n(\mathbb{R}))$, that is, $A \in \mathfrak{S}(\mathcal{C})_n$.

Proof of (2). Note first that $\mathbb{R}^n \in \mathfrak{S}(\mathcal{C})_n$ and that $\mathfrak{S}(\mathcal{C})_n$ is a boolean algebra.

The various diagonals and the graphs of addition and multiplication belong to $\mathfrak{S}(\mathcal{C})_n$, since these sets are Zariski-closed in their ambient real affine space and therefore semianalytic in the corresponding real projective space.

For $A \subseteq \mathbb{R}^n$ we have

$$\tau_{n+1}(A \times \mathbb{R}) = \tau_n(A) \times (-1,1) \quad \text{and} \quad \tau_{n+1}(\mathbb{R} \times A) = (-1,1) \times \tau_n(A),$$

so that if $A \in \mathfrak{S}(\mathcal{C})_n$, then $A \times \mathbb{R}$ and $\mathbb{R} \times A$ belong to $\mathfrak{S}(\mathcal{C})_{n+1}$ by (1). Let $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ be the projection map. It follows easily then from (1) and the fact that $\tau_n(\pi(A)) = \pi(\tau_{n+1}(A))$ that if $A \in \mathfrak{S}(\mathcal{C})_{n+1}$, then $\pi(A) \in \mathfrak{S}(\mathcal{C})_n$.

If $f : [-1,1] \to \mathbb{R}$ is analytic, then $\Gamma(f)$ is clearly a bounded semianalytic subset of $\mathbb{R}^{n+1}$, hence $\Gamma(f) \in \mathfrak{S}(\mathcal{C})_{n+1}$.

If $A \in \mathfrak{S}(\mathcal{C})_1$, then $\tau_1(A) \in \mathcal{C}(\mathbb{R})$, hence has finite boundary. Thus, A itself has finite boundary.

Proof of (3).

The axioms AG1, AG2 and AG4 are clearly satisfied for the $\mathcal{C}(\mathfrak{S})$-sets of a manifold. To verify AG3, consider a proper analytic map $f : M \to N$ and a $\mathcal{C}(\mathfrak{S})$-set A in M. We must show that $f(A)$ is a $\mathcal{C}(\mathfrak{S})$-set in N. Let $y \in N$. Take an open neighborhood V of y and an analytic isomorphism $h : V \to h(V)$ onto an open set in $\mathbb{R}^n$ containing $[-1,1]^n$. Let $V' := h^{-1}([-1,1]^n)$, so that V' and $f^{-1}(V')$ are compact. Let any $x \in f^{-1}(V')$ be given and take an open neighborhood U_x of x and an analytic isomorphism $g_x : U_x \to g_x(U_x)$ onto an open set in $\mathbb{R}^m$ containing $[-1,1]^m$, such that $g_x(A \cap U_x)$ belongs to $\mathfrak{S}$. Shrinking U_x if necessary, we may assume that $f(U_x) \subseteq V$. Let $U'_x := g_x^{-1}([-1,1]^m)$; then $\mathfrak{S}$ contains $g_x(A \cap U'_x) = g_x(A \cap U_x) \cap [-1,1]^m$. The map

$$a \mapsto h(f(g_x^{-1}(a))) : [-1,1]^m \to h(V) \subseteq \mathbb{R}^n$$

is analytic. The image of $g_x(A \cap U'_x)$ under this map is $h(f(A \cap U'_x))$, and so $h(f(A \cap U'_x))$ belongs to $\mathfrak{S}$. Since $f^{-1}(V')$ is compact there are finitely many points $x(1), \ldots, x(k) \in f^{-1}(V')$ such that

$$f^{-1}(V') \subseteq U' := U'_{x(1)} \cup \cdots \cup U'_{x(k)}.$$

Hence $h(f(A \cap U'))$ belongs to $\mathfrak{S}$, and so does $h(f(A \cap U')) \cap (-1,1)^n$. Let $V'' := h^{-1}((-1,1)^n)$, so that

$$f(A) \cap V'' = h^{-1}[h(f(A \cap U')) \cap (-1,1)^n].$$

So V'' is an open neighborhood of y in N and $h|V'' : V'' \to (-1,1)^n$ is an analytic isomorphism mapping $f(A) \cap V''$ onto a set belonging to $\mathfrak{S}$. Since y was arbitrary this shows that $f(A)$ is a $\mathcal{C}(\mathfrak{S})$-set in N.

To verify AG5 we consider a bounded $\mathcal{C}(\mathfrak{S})$-set A in $\mathbb{R}$ and show that $\mathrm{bd}(A)$ is finite. Since $\mathrm{bd}(A)$ is compact, it suffices to show that each $x \in \mathrm{bd}(A)$ is isolated in $\mathrm{bd}(A)$. Let $x \in \mathrm{bd}(A)$. Take an open neighborhood U of x in $\mathbb{R}$ and an analytic isomorphism h from U onto the open interval $(-1,1)$ such that $h(U \cap A)$ belongs to $\mathfrak{S}$; then $h(x) \in \mathrm{bd}(h(U \cap A))$.

Hence, there is an open neighborhood $V \subseteq (-1,1)$ of $h(x)$ such that $V \cap \mathrm{bd}(h(U \cap A)) = \{h(x)\}$. Thus, $h^{-1}(V)$ is an open neighborhood of x and $h^{-1}(V) \cap \mathrm{bd}(A) = \{x\}$.

Proof of (4). First, we show that $\mathcal{C}(\mathfrak{S}(\mathcal{C})) = \mathcal{C}$. Let A be a $\mathcal{C}$-set in the m-dimensional manifold M. Let $x \in M$ and take a chart around x (i.e., an open neighborhood U of x and an analytic isomorphism $h : U \to V$ onto an open set V in $\mathbb{R}^m$). Then $U \cap A \in \mathcal{C}(U)$, hence $h(U \cap A) \in \mathcal{C}(V)$. Choose $\epsilon > 0$ such that the closure in $\mathbb{R}^m$ of the euclidean ball B of radius ϵ and center $h(x)$ is contained in V. Then $h(U \cap A) \cap B$ is a bounded $\mathcal{C}$-set in $\mathbb{R}^m$, hence $h(U \cap A) \cap B \in \mathfrak{S}(\mathcal{C})_m$. As x was arbitrary this shows that A is a $\mathcal{C}(\mathfrak{S}(\mathcal{C}))$-set in M. Conversely, assume that A is a $\mathcal{C}(\mathfrak{S}(\mathcal{C}))$-set in the m-dimensional manifold M. Let $x \in M$. Then there is an open neighborhood U_x of x and an analytic isomorphism $h_x : U_x \to \mathbb{R}^m$ onto $\mathbb{R}^m$ such that $h_x(U_x \cap A) \in \mathcal{C}(\mathbb{R}^m)$. Hence, $U_x \cap A \in \mathcal{C}(U)$. But $\bigcup_{x \in M} U_x$ covers M, so $A \in \mathcal{C}(M)$ by AG3.

We next show that $\mathfrak{S}(\mathcal{C}(\mathfrak{S})) = \mathfrak{S}$. Let $A \subseteq \mathbb{R}^m$ belong to $\mathfrak{S}$. To show that A belongs to $\mathfrak{S}(\mathcal{C}(\mathfrak{S}))$ it suffices by (1) to show that $\tau_m(A) \in \mathcal{C}(\mathbb{R}^m)$. But $\tau_m(A)$ belongs to $\mathfrak{S}$, hence $\tau_m(A)$ is indeed a $\mathcal{C}(\mathfrak{S})$-set in $\mathbb{R}^m$. Conversely, suppose $A \subseteq \mathbb{R}^m$ belongs to $\mathfrak{S}(\mathcal{C}(\mathfrak{S}))$. Then $\tau_m(A)$ is a $\mathcal{C}(\mathfrak{S})$-set in $\mathbb{R}^m$ by (1). For each $x = (x_1, \ldots, x_m) \in \mathbb{R}^m$ there exist U_x open in $\mathbb{R}^m$ and an analytic isomorphism $h_x : U_x \to V_x$ onto an open subset V_x of $\mathbb{R}^m$ such that $h_x(U_x \cap \tau_m(A)))$ belongs to $\mathfrak{S}$. Take $\epsilon > 0$ such that the closed box

$$B_x := [x_1 - \epsilon, x_1 + \epsilon] \times \cdots \times [x_m - \epsilon, x_m + \epsilon]$$

is contained in U_x. Note that the map $h_x | B_x : B_x \to \mathbb{R}^m$ belongs to $\mathfrak{S}$, and that

$$B_x \cap \tau_m(A) = (h_x|B_x)^{-1}(h_x(U_x \cap \tau_m(A))).$$

Hence $B_x \cap \tau_m(A)$ belongs to $\mathfrak{S}$. Since $\tau_m(A)$ is bounded, finitely many boxes B_x cover $\tau_m(A)$. It follows that $\tau_m(A)$ belongs to $\mathfrak{S}$. Hence A belongs to $\mathfrak{S}$. $\square$

Having now established the correspondence between o-minimal structures on $\mathbb{R}_{\mathrm{an}}$ and analytic-geometric categories, note that 1.7, 1.8 and 1.9 follow easily from 2.1 through 2.4; and that 1.10 through 1.18 are obtained in a routine way from various facts in §4 and Appendix C.

We next proceed to establish Whitney stratification. This is accomplished in D.16 below, after some lemmas.

We recall here some easy facts on locally finite collections. Let $\mathcal{F}$ be a locally finite collection of subsets of M. If $\mathcal{G}$ is a locally finite collection of subsets of M, then so is $\mathcal{F} \cup \mathcal{G}$. If for each $F \in \mathcal{F}$ we have a locally finite collection $\mathcal{A}_F$ of subsets of M such that $F = \bigcup \mathcal{A}_F$, then $\{\, A : A \in \mathcal{A}_F$ for some $F \in \mathcal{F} \,\}$ is locally finite. The collection $\{\, \mathrm{cl}(F) : F \in \mathcal{F} \,\}$ is locally finite and $\bigcup \{\, \mathrm{cl}(F) : F \in \mathcal{F} \,\} = \mathrm{cl}(\bigcup \mathcal{F})$.

We will repeatedly use the following facts about the Whitney property:

(1) Let X, Y, X', Y' be C^1-submanifolds of M with X' open in X and $Y' \subseteq Y$. If $y \in Y'$ and $y \in W(X,Y)$, then $y \in W(X', Y')$.

(2) Let $X, Y \in \mathcal{C}(M)$ be nonempty C^1 submanifolds of M with $Y \subseteq \mathrm{fr}(X)$. Then $W(X,Y) \in \mathcal{C}(M)$ and $\dim(Y \setminus W(X,Y)) < \dim Y$.

(Item (1) is immediate from the definitions; (2) follows easily from B.12 and C.1 using the correspondence of §3.)

Throughout the rest of this paper put $\mathfrak{S} := \mathfrak{S}(\mathcal{C})$ and let p denote a positive integer (unless stated otherwise).

D.11. *Let $\mathcal{A} \subseteq \mathcal{C}(M)$ be locally finite. Then there is a locally finite partition $\mathcal{P} \subseteq \mathcal{C}(M)$ of M, compatible with $\mathcal{A}$, consisting of connected, relatively compact C^p submanifolds of M.*

Proof. Since M has a countable basis for its topology, there exist an open covering $(U_i)_{i\in\mathbb{N}}$ of M, analytic isomorphisms $(\varphi_i : U_i \to \mathbb{R}^m)_{i\in\mathbb{N}}$ and a sequence $(V_i)_{i\in\mathbb{N}}$ of compact $\mathcal{C}$-sets in M such that $(\mathrm{int}(V_i))_{i\in\mathbb{N}}$ covers M and $V_i \subseteq U_i$ for all $i \in \mathbb{N}$. For each $i \in \mathbb{N}$, take (by cell decomposition) a finite partition $\mathcal{D}_i \subseteq \mathfrak{S}_m$ of $\varphi_i(V_i)$ into C^p cells in $\mathbb{R}^m$ compatible with

$$\left\{\varphi_i\left(A \cap \left(V_i \setminus \bigcup_{k<i} V_k\right)\right) : A \in \mathcal{A}\right\},$$

a finite collection of bounded sets belonging to $\mathfrak{S}_m$. Now, for $i \in \mathbb{N}$ put

$$\mathcal{P}_i := \left\{\varphi_i^{-1}(D) : D \in \mathcal{D}_i \ \ \& \ \ D \subseteq \varphi_i\left(V_i \setminus \bigcup_{k<i} V_k\right)\right\},$$

a finite partition of $V_i \setminus \bigcup_{k<i} V_k$. Then $\mathcal{P} := \bigcup_{i\in\mathbb{N}} \mathcal{P}_i$ has the desired properties. (To check that $\mathcal{P}$ is locally finite, use that $(\mathrm{int}(V_i))_{i\in\mathbb{N}}$ covers M.) $\square$

D.12. *Let $\mathcal{A} \subseteq \mathcal{C}(M)$ be locally finite. Then there is a C^p stratification $\mathcal{S} \subseteq \mathcal{C}(M)$ of M that is compatible with $\mathcal{A}$.*

Proof. By D.11 there is a locally finite partition $\mathcal{P}_m \subseteq \mathcal{C}(M)$ of M, compatible with $\mathcal{A}$, consisting of C^p submanifolds of M. Assume that for a certain $k \in \{1, \dots, m\}$ we have constructed a locally finite partition $\mathcal{P}_k \subseteq \mathcal{C}(M)$ of M consisting of C^p submanifolds of M and compatible with $\mathcal{A} \cup \{\,\mathrm{fr}(X) : X \in \mathcal{P}_k, \dim X > k\,\}$. (Note that this is trivially the case for $k = m$ if $m > 0$.) By D.11 there is a locally finite partition $\mathcal{H} \subseteq \mathcal{C}(M)$ of $\bigcup\{\,X \in \mathcal{P}_k : \dim X < k\,\}$ compatible with $\mathcal{P}_k$ and with $\{\,\mathrm{fr}(X) : X \in \mathcal{P}_k, \dim X = k\,\}$ and consisting of C^p submanifolds of M. Then

$$\mathcal{P}_{k-1} := \{\,X \in \mathcal{P}_k : \dim X \geq k\,\} \cup \mathcal{H}$$

is a locally finite partition of M consisting of C^p submanifolds of M belonging to $\mathcal{C}(M)$ and compatible with $\mathcal{A} \cup \{\,\mathrm{fr}(X) : X \in \mathcal{P}_{k-1}, \dim X > k-1\,\}$. This inductive construction leads in m steps to a C^p stratification $\mathcal{S} = \mathcal{P}_0$ as required. $\square$

D.13. *Let $S \in \mathcal{C}(M)$ and $f : S \to N$ be a $\mathcal{C}$-map. Then there is a locally finite collection $\mathcal{A} \subseteq \mathcal{C}(M)$ of disjoint C^p submanifolds of M contained in S such that $\dim(S \setminus \bigcup \mathcal{A}) < \dim S$, and such that for each $A \in \mathcal{A}$ the map $f|A : A \to N$ is C^p and $\mathrm{rk}\, f|A$ is constant.*

This is proved much along the lines of D.11 using C.2.

By induction on dimension the preceding result leads to:

D.14. *Let $S \in \mathcal{C}(M)$ and $f : S \to N$ be a $\mathcal{C}$-map. Then there is a locally finite partition $\mathcal{P} \subseteq \mathcal{C}(M)$ of S into C^p submanifolds of M such that for each $P \in \mathcal{P}$ the map $f|P : P \to N$ is C^p and* $\operatorname{rk} f|P$ *is constant.*

D.15. *Let $\mathcal{B} \subseteq \mathcal{C}(M)$ be a C^p Whitney stratification of the closed set $S \subseteq M$, with $p \geq 2$. Then the collection $\mathcal{A}$ of connected components of strata in $\mathcal{B}$ is a C^p Whitney stratification of S.*

Proof. It is clear that $\mathcal{A}$ is a partition of S into C^p submanifolds of M. Given $x \in M$, take a compact neighborhood $K \in \mathcal{C}(M)$ of x. Then K intersects only finitely many $B \in \mathcal{B}$, and for every such B the set $K \cap B$ has only finitely many connected components, and each of these components is contained in only one set $A \in \mathcal{A}$. This shows that $\mathcal{A}$ is locally finite.

Suppose now that $A_1, A_2 \in \mathcal{A}$ with $\operatorname{cl}(A_1) \cap A_2 \neq \emptyset$ and $A_1 \neq A_2$. We must show that: (i) (A_1, A_2, x) has the Whitney property for all $x \in A_2$, and (ii) $A_2 \subseteq \operatorname{fr}(A_1)$. Take $B_1, B_2 \in \mathcal{B}$ with A_1 a component of B_1 and A_2 a component of B_2. If $B_1 = B_2 = B$, then A_1 and A_2 are connected components of B, hence closed in B, so $\operatorname{cl}(A_1) \cap A_2 = \emptyset$; contradiction. Hence, we have $B_2 \subseteq \operatorname{fr}(B_1)$. Then (i) follows from the fact that (B_1, B_2, x) has the Whitney property for all $x \in B_2$. Applying Proposition 8.7 from [17] now yields (ii). $\square$

D.16. *Let $S \in \mathcal{C}(M)$ be closed.*

1. *For every locally finite $\mathcal{F} \subseteq \mathcal{C}(M)$ there is a C^p Whitney stratification $\mathcal{P} \subseteq \mathcal{C}(M)$ of S compatible with $\mathcal{F}$ consisting of connected strata.*
2. *Let $f : S \to N$ be a proper $\mathcal{C}$-map and $\mathcal{F} \subseteq \mathcal{C}(M)$, $\mathcal{G} \subseteq \mathcal{C}(N)$ be locally finite. Then there is a C^p Whitney stratification $(\mathcal{S}, \mathcal{T})$ of f with connected strata such that $\mathcal{S} \subseteq \mathcal{C}(M)$ is compatible with $\mathcal{F}$ and $\mathcal{T} \subseteq \mathcal{C}(N)$ is compatible with $\mathcal{G}$.*

Remark. The proof uses methods of Łojasiewicz [16] and Hardt [9], but there are enough differences to justify writing out the details.

Proof. We proceed directly to establish (2), during the course of which we also obtain (1). We may and shall assume $p \geq 2$. Let $m = \dim M$ as usual.

For inductive purposes we define a *k-nice stratification* for $k = 0, \ldots, m$ to be a C^p stratification $\mathcal{P} \subseteq \mathcal{C}(M)$ of S such that: (i) $\mathcal{P}$ is compatible with $\mathcal{F}$; (ii) $f|Q$ is C^p of constant rank for all $Q \in \mathcal{P}$ with $\dim Q \geq k$; (iii) for all $P, Q \in \mathcal{P}$ with $Q \subseteq \operatorname{fr}(P)$ and $\dim Q \geq k$, the pair (P, Q) has the Whitney property.

We first show that there exist m-nice stratifications. By D.14 there is a locally finite partition $\mathcal{A} \subseteq \mathcal{C}(M)$ of S compatible with $\mathcal{F}$ and consisting of C^p submanifolds of M such that $f|A$ is C^p of constant rank for each $A \in \mathcal{A}$. By D.12 there is a C^p stratification $\mathcal{P} \subseteq \mathcal{C}(M)$ of S compatible with $\mathcal{A}$. Then $\mathcal{P}$ is an m-nice stratification.

Assume that for a certain $k \in \{1, \ldots, m\}$ we have constructed a k-nice stratification $\mathcal{P}_k$. By D.14 we can choose for each $Y \in \mathcal{P}_k$ with $\dim Y = k - 1$ a locally finite partition $\mathcal{P}_Y \subseteq \mathcal{C}(M)$ of Y into C^p submanifolds of M such that for each $H \in \mathcal{P}_Y$ the map $f|H : H \to N$ is C^p of constant rank. Let

$$\mathcal{W}_k := \{\, W(X, Y) : X, Y \in \mathcal{P}_k,\ \dim Y = k - 1,\ Y \subseteq \operatorname{fr}(X) \,\};$$

note that $\mathcal{W}_k \subseteq \mathcal{C}(M)$ is locally finite. By D.12 there is a C^p stratification $\mathcal{H} \subseteq \mathcal{C}(M)$ of the closed $\mathcal{C}$-set $\bigcup\{\, Y : Y \in \mathcal{P}_k,\ \dim Y < k \,\}$ compatible with $\mathcal{P}_k$, with $\mathcal{W}_k$, and with $\mathcal{P}_Y$ for each $Y \in \mathcal{P}_k$ such that $\dim Y = k-1$. One easily checks that this gives the $(k-1)$-nice stratification

$$\mathcal{P}_{k-1} := \{\, P \in \mathcal{P}_k : \dim P \geq k \,\} \cup \mathcal{H}.$$

This inductive construction leads in m steps to a 0-nice stratification $\mathcal{P}_0 \subseteq \mathcal{C}(M)$, which is clearly a C^p Whitney stratification of S compatible with $\mathcal{F}$. By D.15 the collection of connected components of the strata of $\mathcal{P}_0$ is then a stratification as required in (1). This proves (1) since we could have taken $N = M$ and f to be the inclusion map $S \to M$. Continuing now with the proof of (2), the collection $\{\, f(P) : P \in \mathcal{P}_0 \,\} \subseteq \mathcal{C}(N)$ is locally finite (since f is proper). Hence we can choose by D.16(1) a C^p Whitney stratification $\mathcal{T} \subseteq \mathcal{C}(N)$ of N with connected strata compatible with $\mathcal{G} \cup \{\, f(P) : P \in \mathcal{P}_0,\ P \subseteq S \,\}$. Let

$$\mathcal{S}' := \{\, P \cap f^{-1}(T) : P \in \mathcal{P}_0,\ T \in \mathcal{T},\ T \subseteq f(P) \,\}.$$

It is clear that $\mathcal{S}' \subseteq \mathcal{C}(M)$ is a locally finite partition of S. Simple arguments using the rank theorem show that in addition $\mathcal{S}'$ is a C^p Whitney stratification of S. Let $\mathcal{S}$ be the collection of connected components of the strata in $\mathcal{S}'$. Then it follows again from the rank theorem and D.15 that the pair $(\mathcal{S}, \mathcal{T})$ has the desired properties. *But this is not true; there are counterexamples. A corrigendum will eventually be available.* $\square$

D.17. Improvements of D.16.

(1). Define a $(C^p, \mathcal{C})$ *cell in* M to be a relatively compact $\mathcal{C}$-set A in M for which there is an open neighborhood U of $\mathrm{cl}(A)$ and an analytic isomorphism $\varphi : U \to \mathbb{R}^m$ such that $\varphi(A)$ is a C^p cell in $\mathbb{R}^m$ with regard to $\mathfrak{S} = \mathfrak{S}(\mathcal{C})$ (see §§3,4). In particular, a $(C^p, \mathcal{C})$ cell A in M is a C^p submanifold of M that is C^p diffeomorphic to $\mathbb{R}^d$, $d = \dim A$. We may now require in D.16(1) in addition that the strata of $\mathcal{P}$ be $(C^p, \mathcal{C})$ cells in M. In D.16(2) we may require in addition that the strata of $\mathcal{T}$ be $(C^p, \mathcal{C})$ cells in N and that the strata of $\mathcal{S}$ be contained in $(C^p, \mathcal{C})$ cells in M. To see this, note that the construction in the proof of D.11 yields $\mathcal{P} \subseteq \mathcal{C}(M)$ whose members are $(C^p, \mathcal{C})$ cells in M. Consequently, in D.12, D.13 and D.14 the members of $\mathcal{S}$, $\mathcal{A}$ and $\mathcal{P}$ respectively can all be taken to be $(C^p, \mathcal{C})$ cells in M as well. In the proof of D.16 we merely have to require at appropriate places that certain sets are $(C^p, \mathcal{C})$ cells in M or N.

(2). In D.16(2) we may require, in addition to the property mentioned in D.17(1), that $f|A$ is injective for each A in $\mathcal{S}$ with $\mathrm{rk}\, f|A = \dim A$. To see this, note that by Remark (1) following C.2 we may add this requirement in D.13, and hence in D.14. In the proof of D.16 we then add the requirement for the restrictions $f|Q$ in the definition of "k-nice stratification".

D.18. Whitney stratification in the o-minimal setting.

The proof of 4.8 (that is, Whitney stratification in the o-minimal setting) is quite similar to the proof of D.16: Using C.1, C.2 and the obvious o-minimal version of D.14, one repeats the proof of D.16, working throughout with finite collections of C^p cells in $\mathbb{R}^m$ and $\mathbb{R}^n$, instead of locally finite collections contained in $\mathcal{C}(M)$ and $\mathcal{C}(N)$.

We remark that 4.8 is not a special case of D.16: the o-minimal structures do not have to be of the form $\mathfrak{S}(\mathcal{C})$; the maps f of 4.8(2) are not assumed to be proper; and we obtain finite—rather than locally finite—stratifications.

Note. If $\mathfrak{S}$ admits analytic decomposition, then D.11 through D.17 hold with $p = \omega$ for $\mathcal{C}$; in particular, if $\mathcal{C} = \mathcal{C}_{\text{an,exp}}$ or $\mathcal{C} = \mathcal{C}_{\text{an}}^{\mathbb{R}}$ (by 5.1(3)). Similarly, if a given o-minimal structure on $(\mathbb{R}, +, \cdot)$ has analytic (or C^∞) decomposition, then D.18 holds with $p = \omega$ (or $p = \infty$).

D.19. *Let $A \in \mathcal{C}(M)$ be closed. Then there is a $\mathcal{C}$-map $f : M \to \mathbb{R}$ of class C^p with $A = Z(f) := \{ x \in M : f(x) = 0 \}$.*

Proof. By the correspondence of §3 (and because M is σ-compact and Lindelöf), there exist a locally finite, countable open covering $(U_i)_{i\in\mathbb{N}}$ of M and analytic isomorphisms $h_i : U_i \to \mathbb{R}^m$ with $h_i(U_i \cap A) \in \mathfrak{S}_m$. By C.12, for each $i \in \mathbb{N}$ there exists $g_i \in C^p_{\mathfrak{S}}(\mathbb{R}^m)$ such that $Z(g_i) = \text{cl}(B(1)) \cap h_i(U_i \cap A)$ and $Z(g_i - 1) = \mathbb{R}^m \setminus B(2)$—since $\text{cl}(B(1)) \cap h_i(U_i \cap A)$ is compact—where $B(1)$ and $B(2)$ denote respectively the euclidean open balls about the origin of radii 1 and 2. For each $i \in \mathbb{N}$, define $f_i : M \to \mathbb{R}$ by

$$f_i(x) := \begin{cases} g_i(h_i(x)), & x \in U_i \\ 1, & \text{otherwise.} \end{cases}$$

Clearly, each f_i is a $\mathcal{C}$-map of class C^p. Note that each $x \in M$ has a neighborhood U such that $f_i|U = 1$ for all but finitely many $i \in \mathbb{N}$. Hence the map $f : M \to \mathbb{R}$ given by $f(x) := \prod_{i\in\mathbb{N}} f_i(x)$ is a $\mathcal{C}$-map of class C^p with $A = Z(f)$. $\square$

References

[1] E. Bierstone and P. Milman, *Semianalytic and subanalytic sets*, Inst. Hautes Études Sci. Publ. Math. **67** (1988), 5–42.

[2] J. Bochnak, M. Coste and M.-F. Roy, *Géométrie algébrique réelle*, Ergeb. Math. Grenzgeb. (3) **12**, Springer-Verlag, Berlin, 1987.

[3] L. van den Dries, *A generalization of the Tarski-Seidenberg theorem and some nondefinability results*, Bull. Amer. Math. Soc. (N.S.) **15** (1986), 189–193.

[4] ———, *Tame Topology and O-minimal Structures*, London Math. Soc. Lecture Note Series **248**, Cambridge University Press, 1998.

[5] L. van den Dries, A. Macintyre and D. Marker, *The elementary theory of restricted analytic fields with exponentiation*, Ann. of Math. **140** (1994), 183–205.

[6] ———, *Logarithmic-exponential power series*, J. London Math. Soc. (2) **56** (1997), 417–434.

[7] L. van den Dries and C. Miller, *On the real exponential field with restricted analytic functions*, Israel J. Math. **85** (1994), 19–56.

[8] ———, *Extending Tamm's theorem*, Ann. Inst. Fourier (Grenoble) **44** (1994), 1367–1395.

[9] R. Hardt, *Stratification of real analytic maps and images*, Inv. Math **28** (1975), 193–208.

[10] ———, *Semi-algebraic local triviality in semi-algebraic mappings*, Amer. J. Math. **102** (1980), 291–302.

[11] ———, *Some analytic bounds for subanalytic sets*, Differential Geometric Control Theory, Progr. Math. **27**, Birkhäuser, Boston, 1983, pp. 259–267.

[12] M. Kashiwara and P. Schapira, *Sheaves on Manifolds*, Grundlehren Math. Wiss. **292**, Springer, 1990.

[13] T. Kayal and G. Raby, *Ensembles sous-analytiques: quelques propriétés globales*, C. R. Acad. Sci. Paris Ser. I Math. **308** (1989), 521–523.

[14] T. Loi, *On the global Łojasiewicz inequalities for the class of analytic logarithmic-exponential functions*, C. R. Acad. Sci. Paris Ser. I Math. **318** (1994), 543-548.

[15] ———, *Whitney stratification of sets definable in the structure* $\mathbb{R}_{\exp}$, Banach Center Publications **33**, 1996, pp. 401–409.

[16] S. Łojasiewicz, *Ensembles semi-analytiques*, Institut des Haute Études Scientifiques, Bures-sur-Yvette, 1965.

[17] J. Mather, *Stratifications and mappings*, Dynamical Systems (M. Peixoto, ed.), Academic Press, 1973.

[18] C. Miller, *Exponentiation is hard to avoid*, Proc. Amer. Math. Soc. **122** (1994), 257–259.

[19] ———, *Expansions of the real field with power functions*, Ann. Pure Appl. Logic **68** (1994), 79–94.

[20] ———, *Infinite differentiability in polynomially bounded o-minimal structures*, Proc. Amer. Math. Soc. **123** (1995), 2551–2555.

[21] J.-P. Ressayre, *Integer parts of real closed exponential fields*, Arithmetic, Proof Theory and Computational Complexity, Oxford Logic Guides **23**, Oxford University Press, New York, 1993, pp. 278–288.

[22] W. Schmid and K. Vilonen, *Characteristic cycles of constructible sheaves*, Invent. Math. **124** (1996), 451–502.

[23] M. Shiota, *Nash Manifolds*, Lecture Notes in Math. 1269, Springer-Verlag, New York.

[24] ———, *Geometry of subanalytic and semialgebraic sets: abstract*, Real Analytic and Algebraic Geometry, W. de Gruyter, Berlin, 1995, pp. 251–275.

[25] M. Tamm, *Subanalytic sets in the calculus of variations*, Acta Math. **146** (1981), 167–199.

[26] J.-Cl. Tougeron, *Algèbres analytiques topologiquement noethériennes. Théorie de Khovanskii*, Ann. Inst. Fourier (Grenoble) **41** (1991), 823–840.

[27] A. Wilkie, *Model completeness results for expansions of the ordered field of real numbers by restricted Pfaffian functions and the exponential function*, J. Amer. Math. Soc. **9** (1996), 1051–1094.

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