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Geometric complexity theory: an introduction for geometers

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Abstract This article is survey of recent developments in, and a tutorial on, the approach to **P** v. **NP** and related questions called geometric complexity theory (GCT). It is written to be accessible to graduate students. Numerous open questions in algebraic geometry and representation theory relevant for GCT are presented.

Keywords Geometric complexity theory · Determinant · Permanent · Secant variety · Dual variety · Foulkes-Howe conjecture · Depth 3 circuit

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1 Introduction

This is a survey of problems dealing with the separation of complexity classes that translate to questions in algebraic geometry and representation theory. I will refer to these translations as *geometric complexity theory* (GCT), although this term has been used both more broadly and more narrowly. I do not cover topics such as the complexity of matrix multiplication (see [45] for an overview and [48,53] for the state of the art), matrix rigidity (see [28,41]), or the GCT approach to the complexity of tensors (see [11]), although these topics in complexity theory have interesting algebraic geometry and representation theory associated to them.

The basic problem (notation is explained in Sect. 1.2 below): Let V be a vector space, let $G \subset GL(V)$ be a reductive group, and let $v, w \in V$. Consider the orbit closures $\overline{G \cdot [v]}, \overline{G \cdot [w]} \subset \mathbb{P}V$. Determine if $\overline{G \cdot [v]} \subset \overline{G \cdot [w]}$.

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In more detail: First, for computer science one is interested in asymptotic geometry, so one has a sequence of vector spaces V_n, and sequences of vectors and groups, and one wants to know if the inclusion fails for infinitely many (or even all) n greater than some n_0. Second, usually $G_n = GL(W_n)$, where $W_n = \mathbb{C}^{f(n)}$ for some function $f(n)$ (usually $f(n) = n^2$) and $V_n = S^n W_n$ is a space of polynomials on W_n^*. Third, the points v, w will be of a very special nature - they will (usually) be *characterized by their stabilizers* (see Definition 2.17).

The most important example will be $V = S^n\mathbb{C}^{n^2}$, $G = GL_{n^2}$, $w = \det_n$, the determinant, and $v = \ell^{n-m}\operatorname{perm}_m$, the *padded permanent* (see Sect. 2.1 for the definition).

The article is part a survey of recent developments and part tutorial directed at graduate students. The level of difficulty of the sections varies considerably and is not monotone (for example Sect.11 is elementary). I have placed the most emphasis on areas where there are open questions that appear to be both tractable and interesting. The numerous open questions scattered throughout the article are labeled by "**Problem**". Most of the sections can be read independently of the others.

1.1 Overview

Section 2 serves as a detailed introduction to the rest of the paper. In it I describe the flagship conjecture on determinant versus permanent and related conjectures, introduce relevant algebraic varieties, and establish basic information about GCT. In Sect. 3, I cover background from representation theory. GCT has deep connections to classical algebraic geometry - a beautiful illustration of this is how solving an old question regarding dual varieties led to lower bounds for the flagship conjecture, which is discussed in Sect. 4, along with a use of differential geometry to get lower bounds for a conjecture of Valiant. The boundary of the variety $\mathcal{D}et_n := \overline{GL_{n^2} \cdot \det_n}$ is discussed in Sect. 5. The classical problem of determining the symmetries of a polynomial and how it relates to the GCT program is discussed in Sect. 6, including geometric computations of the stabilizers of the determinant and permanent polynomials. I believe the *Chow variety* of polynomials that decompose into a product of linear factors will play a central role in advancing GCT, so I discuss it in detail in Sect. 7, including: unpublished results of Ikenmeyer and Mkrtchyan on the kernel of the Hermite-Hadamard-Howe map, a history of what is called the *Foulkes-Howe Conjecture* (essentially due to Hadamard), recent work with S. Kumar related to the *Alon-Tarsi Conjecture*, a longstanding conjecture in combinatorics, and an exposition of Brion's proof of an asymptotic version of the Foulkes-Howe Conjecture. In Sect. 8 I translate recent results in computer science [31] to geometric language—they allow for two new, completely different formulations of Valiant's conjecture **VP** $\neq$ **VNP**, one involving secant varieties of the Chow variety, and another involving secant varieties of Veronese re-embeddings of secant varieties of Veronese varieties. An exposition of S. Kumar's results on the non-normality of $\mathcal{D}et_n$ and $\overline{GL_{n^2} \cdot \ell^{n-m}\operatorname{perm}_n}$ is given in Sect. 9. In Sect. 10, I present unpublished results of Li and Zhang, using work of Maulik and Pandharipande [55], that the degree of the hypersurface of determinantal quartic surfaces is 640, 224. My feeling is that any near-term lower bounds for the flagship conjecture 2.1 will come from classical geometry and linear algebra. I discuss this perspective in Sect. 11 which consists of unpublished joint work with L. Manivel and N. Ressayre. Finally Sect. 12 is an appendix of very basic complexity theory: the origin of **P** v. **NP**, definitions regarding circuits, and Valiant's conjectures.

1.2 Notation

Throughout V, W are complex vector spaces of dimensions $\mathbf{v}$, $\mathbf{w}$. The group of invertible linear maps $W \to W$ is denoted $GL(W)$, and $SL(W)$ denotes the maps with determinant one. Since we are dealing with $GL(W)$-varieties, their ideals and coordinate rings will be $GL(W)$-modules. The $GL(W)$-modules appearing in the tensor algebra of W are indexed by partitions, $\pi = (p_1, \dots, p_q)$, where if π is a partition of d, i.e., $p_1 + \dots + p_q = d$ and $p_1 \geq p_2 \geq \dots \geq p_q \geq 0$, the module $S_\pi W$ appears in $W^{\otimes d}$ and in no other degree. In particular the d-th symmetric power is $S^d V = S_{(d)} V$ and the d-th exterior power is $\Lambda^d V = S_{(1,\dots,1)} V =: S_{(1)^d} V$. Write $|\pi| = d$ and $\ell(\pi) = q$. The symmetric algebra is denoted $Sym(V) := \oplus_d S^d V$. For a group G, a G-module V, and $v \in V$, $G_v := \{g \in G \mid gv = v\} \subset G$ denotes its stabilizer, and for a subgroup $H \subset G$, $V^H := \{v \in V \mid hv = v\} \subset V$ denotes the H-invariants in V. The irreducible representations of the permutation group on n elements $\mathfrak{S}_n$ are also indexed by partitions, and $[\pi]$ denotes the $\mathfrak{S}_n$-module associated to π. Repeated numbers in partitions are sometimes expressed as exponents when there is no danger of confusion, e.g. $(3, 3, 1, 1, 1, 1) = (3^2, 1^4)$.

Projective space is $\mathbb{P}V = (V \backslash 0)/\mathbb{C}^*$. For $v \in V$, $[v] \in \mathbb{P}V$ denotes the corresponding point in projective pace and for any subset $Z \subset \mathbb{P}V$, $\hat{Z} \subset V$ is the corresponding cone in V. For a variety $X \subset \mathbb{P}V$, $I(X) \subset Sym(V^*)$ denotes its ideal, $\mathbb{C}[\hat{X}] = Sym(V^*)/I(X)$ is the ring of regular functions on $\hat{X}$, which is also $\mathbb{C}[X]$, the homogeneous coordinate ring of X. The singular locus of X is denoted X_{sing} and X_{smooth} denotes its smooth points. More generally, for an affine variety Z, $\mathbb{C}[Z]$ denotes its ring of regular functions. For $x \in X_{smooth}$, $\hat{T}_x X \subset V$ denotes its affine tangent space. For a subset $Z \subset V$ or $Z \subset \mathbb{P}V$, its Zariski closure is denoted $\overline{Z}$.

For $P \in S^d V$, and $1 \leq k \leq \lfloor \frac{\mathbf{v}}{2} \rfloor$, the linear map $P_{k,d-k} : S^k V^* \to S^{d-k} V$ is called the *polarization* of P, where $P_{k,d-k} \in S^k V \otimes S^{d-k} V$ is P considered as a bilinear form, see Sect. 2.3 for more details. I write $\overline{P}$ for the complete polarization of P, i.e. considering P as a multilinear form and $Z(P) \subset \mathbb{P}V^*$ for the zero set.

Repeated indices appearing up and down are to be summed over.

Let $T_V \subset SL(V)$ denote a torus (diagonal matrices), and I write $T = T_V$ if V is understood. When $\dim V = n$, let $\Gamma_n := T \rtimes \mathfrak{S}_n = \{g \in SL(V) \mid ghg^{-1} \in T \forall h \in T\}$ denote its normalizer in $SL(V)$, where $\mathfrak{S}_n$ acts as permutation matrices.

For a reductive group G, Λ_G^+ denotes the set of finite dimensional irreducible G-modules. Since I work exclusively over $\mathbb{C}$, a group is reductive if and only if every G-module admits a decomposition into a direct sum of irreducible G-modules.

The set $\{1, \dots, m\}$ will be denoted $[m]$. log denotes $\log_2$.

Let $f, g : \mathbb{R} \to \mathbb{R}$ be functions. Write $f = \Omega(g)$ (resp. $f = O(g)$) if and only if there exists $C > 0$ and x_0 such that $|f(x)| \geq C|g(x)|$ (resp. $|f(x)| \leq C|g(x)|$) for all $x \geq x_0$. Write $f = \omega(g)$ (resp. $f = o(g)$) if and only if for all $C > 0$ there exists x_0 such that $|f(x)| \geq C|g(x)|$ (resp. $|f(x)| \leq C|g(x)|$) for all $x \geq x_0$. These definitions are used for any ordered range and domain, in particular $\mathbb{Z}$. In particular, for a function $f(n)$, $f = \omega(1)$ means f goes to infinity as $n \to \infty$.

Exercise 1.1 Show that asymptotically, for any constants $C, D, E > 1$,

$$n^C < n^{\sqrt{n}} = 2^{\sqrt{n}\log n} < D^n < n^n < E^{n^2}.$$

2 Geometric complexity theory

2.1 The flagship conjecture

Let $W = \mathbb{C}^{n^2}$, and let $\det_n \in S^n W$ denote the determinant polynomial. Let $n > m$ and let $\mathrm{perm}_m \in S^m\mathbb{C}^{m^2}$ denote the permanent. In coordinates,

$$\det_n = \sum_{\sigma \in \mathfrak{S}_n} \mathrm{sgn}(\sigma) x^1_{\sigma(1)} \cdots x^n_{\sigma(n)}$$
$$\mathrm{perm}_m = \sum_{\sigma \in \mathfrak{S}_m} y^1_{\sigma(1)} \cdots y^m_{\sigma(m)}.$$

Let ℓ be a linear coordinate on $\mathbb{C}^1$ and consider any linear inclusion $\mathbb{C}^1 \oplus \mathbb{C}^{m^2} \to W$, so in particular $\ell^{n-m}\,\mathrm{perm}_m \in S^n W$. Let

$$\mathcal{D}et_n := \overline{GL(W) \cdot [\det_n]}$$

and let

$$\mathcal{P}erm^m_n := \overline{GL(W) \cdot [\ell^{n-m}\,\mathrm{perm}_m]}.$$

Conjecture 2.1 (Mulmuley-Sohoni [62]) *Let $n = m^c$ for any constant c. Then for all sufficiently large n,*

$$\mathcal{P}erm^m_n \not\subset \mathcal{D}et_n.$$

While this flagship conjecture appears to be out of reach, I hope to convince the reader that there are many interesting intermediate problems that are tractable and that these questions have deep connections to geometry, representation theory, combinatorics, and other areas of mathematics.

It is convenient to introduce the following notation: For a homogeneous polynomial P of degree m, write $\overline{dc}(P)$ for the smallest n such that $[\ell^{n-m}P] \in \mathcal{D}et_n$, called the *border determinental complexity* of P. Define $dc(P)$ to be the smallest n such that $\ell^{n-m}P \in \mathrm{End}(W) \cdot \det_n$, so $\overline{dc}(P) \leq dc(P)$. Conjecture 2.1 can be restated that $\overline{dc}(\mathrm{perm}_m)$ grows faster than any polynomial in m. For example, $\overline{dc}(\mathrm{perm}_2) = 2$, and it is known (respectively [30] and [49]) that $5 \leq \overline{dc}(\mathrm{perm}_3) \leq dc(\mathrm{perm}_3) \leq 7$. The known general lower bound is

Theorem 2.2 [49] $\overline{dc}(\mathrm{perm}_m) \geq \frac{m^2}{2}$.

See Sect. 4.2 for a discussion.

Conjecture 2.1 is a stronger version of a conjecture of L. Valiant [75] that $dc(\mathrm{perm}_m)$ grows faster than any polynomial in m. The best lower bound for $dc(\mathrm{perm}_m)$ is

Theorem 2.3 [57] $dc(\mathrm{perm}_m) \geq \frac{m^2}{2}$.

See Sect. 4.1 for a discussion.

Problem 2.4 Determine $\overline{dc}(\mathrm{perm}_3)$.

If you were to prove either Valiant's conjecture or Conjecture 2.1, it would be by far the most significant result since the dawn of complexity theory. Proving Conjecture 2.1 for $c = 3$ would already be a huge accomplishment. If you disprove Valiant's conjecture plus (1) the projections from $\det_n$ to perm_m use only rational constants of polynomial bit-length, and (2) the projection (for some $n = m^k$) is computable by a polynomial time algorithm, then you can claim the Clay prize for showing $\mathbf{P} = \mathbf{NP}$.

A geometer's first reaction to Conjecture 2.1 might be: "well, the determinant is wonderful, it has a nice geometric description, but what about this permanent? It is not so wonderful at first sight".

In fact that was my first reaction. If you had this reaction, you probably think of the determinant, not in terms of its formula, but, letting $A, B = \mathbb{C}^n$, as the unique point in $\mathbb{P}S^n(A\otimes B)$ invariant under $SL(A) \times SL(B)$, i.e., a point in the trivial $SL(A) \times SL(B)$-module $\Lambda^n A\otimes\Lambda^n B \subset S^n(A\otimes B)$. If you think this way, then consider, instead of the permanent, the four factor *Pascal Determinant* (also called the *combinatorial determinant*): and let $A_j = \mathbb{C}^m$ for $j = 1, \dots, 4$. The 4-factor Pascal determinant $\mathrm{Pasdet}_{4,m}$ spans the unique trivial $SL(A_1) \times \cdots \times SL(A_4)$-module in $S^m(A_1\otimes \cdots \otimes A_4)$, namely $\Lambda^m A_1\otimes \cdots \otimes \Lambda^m A_4$. Assume $n > m^4$, choose a linear embedding $\mathbb{C} \oplus A_1\otimes \cdots \otimes A_4 \subset W$, and define $\mathcal{P}asdet^m_n := \overline{GL(W) \cdot [\ell^{n-m}\mathrm{Pasdet}_{4,m}]}$. Then, a consequence of an observation of Gurvits [32] is that Conjecture 2.1 is equivalent to:

Conjecture 2.5 *Let $n = m^c$ for some constant c. Then for all sufficiently large n,*

$$\mathcal{P}asdet^m_n \not\subset \mathcal{D}et_n.$$

That being said, I have since changed my perspective and have come around to admiring the beauty of the permanent as well. In Remark 6.15 we will see it is the "next best" polynomial in $S^n(\mathbb{C}^n\otimes\mathbb{C}^n)$ after the determinant.

There are many similarities between the permanent and the 4-factor Pascal determinant. Two examples: for both, the dimension of the hypersurfaces they define is roughly the dimension of the symmetry group G_P squared (in contrast to the determinant where the dimension of the symmetry group is roughly the same as the dimension of the hypersurface), and in both cases the tangent space $T_P(GL(W)\cdot P)$ is a reducible G_p-module (for the determinant it is irreducible).

Remark 2.6 For all even k one can define the k-factor Pascal determinant as a point in the unique trivial $SL(A_1)\times\cdots\times SL(A_k)$-module in $S^m(A_1\otimes\cdots\otimes A_k)$, namely $\Lambda^m A_1\otimes\cdots\otimes\Lambda^m A_k$. When $k=2$ this is just the usual determinant.

Exercise 2.7 For $P = \det_n, \mathrm{perm}_m$, and $\mathrm{Pasdet}_{4,m}$, determine the structure of $T_P(GL(W)\cdot P)$ as a G_P-module. Hint: for any orbit, $G\cdot v = G/H$, one has $T_v G/H \simeq \mathfrak{g}/\mathfrak{h}$ as an $\mathfrak{h}$-module.

2.2 Relevant algebraic varieties

Two important varieties for our study will be the *Veronese variety* $v_n(\mathbb{P}W)\subset\mathbb{P}S^nW$ and a certain *Chow variety* $Ch_n(W)\subset\mathbb{P}S^nW$. These are defined as

$$v_n(\mathbb{P}W)=\{[z]\in\mathbb{P}S^nW \mid z=w^n \text{ for some } w\in W\} \tag{1}$$
$$Ch_n(W)=\{[z]\in\mathbb{P}S^nW \mid z=w_1\cdots w_n \text{ for some } w_j\in W\}. \tag{2}$$

Note that the first variety is a subvariety of the second, and if we consider the Segre variety

$$Seg(\mathbb{P}W\times\cdots\times\mathbb{P}W):=\{[T]\in\mathbb{P}(W^{\otimes n}) \mid T=w_1\otimes\cdots\otimes w_n \text{ for some } w_j\in W\}\subset\mathbb{P}(W^{\otimes n}),$$

then $v_n(\mathbb{P}W)=Seg(\mathbb{P}W\times\cdots\times\mathbb{P}W)\cap\mathbb{P}(S^nW)$ and $Ch_n(W)=proj_{\mathbb{P}S^dW^c}(Seg(\mathbb{P}W\times\cdots\times\mathbb{P}W))$, where $S^dW^c\subset W^{\otimes n}$ is the $GL(W)$-complement to S^dW, and $proj_L$ denotes linear projection from the linear space L. (Here I am respectively considering S^nW as a subspace and as a quotient of $W^{\otimes n}$.) The Veronese is homogeneous, so in particular its ideal and coordinate ring are well understood. The Chow variety is an orbit closure when $n\leq\mathbf{w}$. Determining information about its ideal is a topic of current research, and has surprising connections to different areas of mathematics, including a longstanding conjecture in combinatorics, see Sect. 7.9. There is a natural map $h_{d,n}: S^d(S^nW^*)\to S^n(S^dW^*)$, dating back to Hermite and Hadamard, such that $I_d(Ch_n(W))=\ker(h_{d,n})$, see Sect. 7.

The Chow variety is a good testing ground for GCT, so it is discussed in detail in Sect. 7. In particular, the coordinate rings of the Chow variety, its normalization, and the orbit $GL(W)\cdot(x_1,\ldots,x_n)$ are compared. Since $Ch_n(W)\subset\mathcal{D}et_n$, we can get some information about the coordinate ring of $\mathcal{D}et_n$ from the coordinate ring of $Ch_n(W)$.

We will often construct auxiliary varieties from our original varieties. Let $X\subset\mathbb{P}V$ be a variety, which we assume to be irreducible and reduced.

Define the *dual variety* of X:

$$X^\vee:=\overline{\{H\in\mathbb{P}V^*\mid \exists x\in X_{smooth},\ \mathbb{P}\hat{T}_xX\subseteq H\}}\subset\mathbb{P}V^*.$$

In the special case $V=S^nW^*$ and $X=v_n(\mathbb{P}W^*)$ is the Veronese variety, then the hypersurface $v_n(\mathbb{P}W^*)^\vee\subset\mathbb{P}S^nW$ may be identified with the variety of hypersurfaces of degree n in $\mathbb{P}W^*$ that are singular. To see this, for a hypersurface $Z(P) \subset \mathbb{P}W^*$ (the zero set of the polynomial P), $[x] \in Z(P)_{sing}$ if and only if $\overline{P}(x^{n-1}y) = 0$ for all $y \in W^*$. But $\hat{T}_{[x^n]}v_n(\mathbb{P}W^*) = \{x^{n-1}y \mid y \in W^*\}$. See [45, §8.2.1] for more details.

The set

$$\sigma_r^0(X) := \bigcup_{x_1,\ldots,x_r \in X} \langle x_1, \ldots, x_r\rangle \subset \mathbb{P}V,$$

where $\langle x_1, \ldots, x_r\rangle$ denotes the (projective) linear span of the points $x_1, \ldots, x_r$, is called the set of points of *X-rank* at most r. The variety $\sigma_r(X) := \overline{\sigma_r^0(X)}$ is called the r-th *secant variety* of X (or *the variety of secant* $\mathbb{P}^{r-1}$*'s to* X). Assume X is not contained in a hyperplane. Given $z \in \mathbb{P}V$, define the *X-border rank* of z to be the smallest r such that $z \in \sigma_r(X)$, and one writes $\underline{\mathbf{R}}_X(z) = r$. Similarly, if z has X-rank r, one writes $\mathbf{R}_X(z) = r$.

When $X = v_n(\mathbb{P}W)$, the $v_n(\mathbb{P}W)$-rank is called the *Waring rank* (or *symmetric tensor rank*) and the Waring rank and border rank of a polynomial are first measures of its complexity. One writes $\mathbf{R}_S = \mathbf{R}_{v_n(\mathbb{P}W)}$ and $\underline{\mathbf{R}}_S = \underline{\mathbf{R}}_{v_n(\mathbb{P}W)}$. We call the $Ch_n(W)$-rank the *Chow rank*. The Chow rank is an important measure of complexity, it is related to the size of the smallest *homogeneous depth 3 circuit* (sometimes called a homogeneous $\Sigma\Pi\Sigma$ circuit) that can compute a polynomial, and even more importantly, as the smallest depth 3 circuit that can compute a padded polynomial, see Sect. 8.

2.3 First equations

Equations for the secant varieties of Chow varieties are mostly unknown, and even for the Veronese very little is known. One class of equations is obtained from the so-called *flattenings* or *catalecticants*, which date back to Sylvester: for $P \in S^dV$, and $1 \leq k \leq \lfloor \frac{\mathbf{v}}{2} \rfloor$, consider the linear map $P_{k,d-k} : S^kV^* \to S^{d-k}V$, obtained from the *polarization* of P, where, from a tensorial point of view, $P_{k,d-k} \in S^kV \otimes S^{d-k}V$ is P considered as a bilinear form on $S^kV^* \times S^{d-k}V^*$. The image of $P_{k,d-k}$, considered as a map $S^kV^* \to S^{d-k}V$, is the space of all k-th order partial derivatives of P, and is studied frequently in the computer science literature under the name the *method of partial derivatives* (see, e.g., [14] and the references therein). To see this description of the image, note that S^kV^* may be identified with the space of k-th order constant coefficient homogeneous degree k differential operators on S^nV. In bases, if $x^1, \ldots, x^{\mathbf{v}}$ is a basis of V, then $\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^{\mathbf{v}}}$ is a basis of V^*. The kernel and image of $P_{k,n-k}$ is often easy to compute, or at least estimate.

If $[P] \in v_d(\mathbb{P}V)$, the rank of $P_{k,d-k}$ is one, so the size $(r+1)$-minors of $P_{k,d-k}$ furnish some equations in $I_{r+1}(\sigma_r(v_d(\mathbb{P}V)))$. The only other equations I am aware of come from *Young flattenings*, see [18,46] for a discussion of the Young flattenings and the state of the art. If $P \in Ch_d(V)$, then the rank of $P_{k,d-k}$ is $\binom{d}{k}$, so the size $r\binom{d}{k}+1$ minors furnish some equations for $\sigma_r(Ch_d(V))$.

For $P \in S^dV$, the Young flattening, $P_{k,d-k[\ell]} : S^kV^* \otimes S^\ell V \to S^{d-k+\ell}V$ obtained by tensoring $P_{k,d-k}$ with the identity map $Id_{S^\ell V} : S^\ell V \to S^\ell V$, and projecting (symmetrizing) the image in $S^{d-k}V \otimes S^\ell V$ to $S^{d-k+\ell}V$, goes under the name "method of shifted partial derivatives" in the computer science literature. It is the main tool for proving the results discussed in Sect. 8.4. It's skew cousin led to the current best lower bound for the border rank of matrix multiplication in [48].

2.4 Problems regarding secant varieties related to Valiant's conjectures

Problem 2.8 Find equations in the ideal of $\sigma_r(Ch_n(W))$. This would enable one to prove lower complexity bounds for depth 3 circuits.

The motivation comes from:

Conjecture 2.9 *For all but a finite number of m, for all r, n with $rn = 2^{\sqrt{m}\log(m)\omega(1)}$,*

$$[\ell^{n-m}\,\mathrm{perm}_m] \notin \sigma_r(Ch_n(\mathbb{C}^{m^2+1})). \tag{3}$$

As explained in Sect. 8.4, Conjecture 2.9 would imply Valiant's conjecture that **VP** $\neq$ **VNP**. (Valiant's conjecture is explained in Sect. 12.)

Another variety of interest is $\sigma_\rho(v_\delta(\sigma_r(v_n(\mathbb{P}W)))) \subset \mathbb{P}S^{\delta n}W$. If $\dim W = r\rho$, and W has basis x_{is}, $1 \le i \le r$, $1 \le s \le \rho$, this variety is the $GL(W)$-orbit closure of the polynomial $\sum_{s=1}^{\rho}(x_{1s}^n + \cdots + x_{rs}^n)^\delta$.

Problem 2.10 Find equations in the ideal of $\sigma_\rho(v_\delta(\sigma_r(v_n(\mathbb{P}W))))$.

Such equations would enable one to prove lower complexity bounds for the $\Sigma\Lambda\Sigma\Lambda\Sigma$ circuits defined in Sect. 8. The motivation comes from:

Conjecture 2.11 *For all but a finite number of m, for all $\delta \simeq \sqrt{m}$ and all r, ρ with $r\rho = 2^{\sqrt{m}log(m)\omega(1)}$,*

$$[\mathrm{perm}_m] \notin \sigma_\rho(v_\delta(\sigma_r(v_{\frac{m}{\delta}}(\mathbb{P}^{m^2-1})))). \tag{4}$$

As explained in Sect. 8.4 Conjecture 2.11 would also imply Valiant's conjecture that **VP** $\neq$ **VNP**.

Note that although the variety appearing in (4) is more complicated than the one appearing in (3), we do not have to deal with *cones* and *padding*, which I discuss next.

2.5 Cones and padding

The inclusion $\mathbb{C}^{m^2+1} \subset \mathbb{C}^{n^2}$, indicates we should consider the variety of cones, or *subspace variety*

$$Sub_k(S^nW) := \{[P] \in \mathbb{P}S^nW \mid \exists U^k \subset W,\, P \in S^nU\},$$

and the ℓ^{n-m} factor in both (3) and Conjecture 2.1 indicates we should consider the *variety of padded polynomials*

$$Pad_t(S^nW) := \{[P] \in \mathbb{P}S^nW \mid P = \ell^tQ \text{ for some } \ell \in W,\, Q \in S^{n-t}W\}.$$

The ideal of $Sub_k(S^n W)$ in degree d consists of the isotypic components of all $S_\pi W^*$ with $\ell(\pi) > k$, see, e.g. [45, §7.1]. The ideal is generated in degree $k+1$ by the minors of flattenings [47]. The ideal of $Pad_t(S^n W)$ is not known completely. We do know:

Theorem 2.12 [39] *For all d, $I_d(Pad_t(S^n W^*))$ contains the isotypic component of $S_\pi W$ in $S^d(S^n W)$ for all $\pi = (p_1, \ldots, p_d)$ (so $|\pi| = nd$) with $p_1 < dt$. It does not contain a copy of any $S_\pi W$ where $p_1 \geq \min\{d(n-1), dn-(n-t)\}$.*

Although we know for dimension reasons that $Pad_{n-m}(Sub_{m^2+1}(S^m\mathbb{C}^{n^2})) \not\subset \mathcal{D}et_n$ asymptotically when $n = m^c$ by counting dimensions, it would be useful to have a proof using equations.

2.6 GCT useful modules

One could break down the problem of separating the determinant from the padded permanent into three steps: separating the determinant from a generic cone, separating the determinant from a cone over a padded polynomial, and finally separating the determinant from the cone over the padded permanent. That is, to separate $\mathcal{D}et_n$ from $\mathcal{P}erm_n^m$, we should not just look for modules in the ideal of $\mathcal{D}et_n$, but modules in the ideal that are not in the ideal of $Sub_{m^2+1}(S^n W)$ or $Pad_{n-m}(S^n\mathbb{C}^{m^2+1})$.

Definition 2.13 A $GL(W)$-module module M such that $M \subset I(\mathcal{D}et_n)$ and $M \not\subset I(Sub_{m^2+1}(S^n W))$ and not known to be in the ideal of $Pad_{n-m}(S^n\mathbb{C}^{m^2+1})$, is called *$(n,m)$-GCT useful.*

More precisely, one should speak of modules that are, e.g. "April 2013 GCT useful", since what is known will change over time, but I ignore this in the notation. To summarize:

Theorem 2.14 [39] *Necessary conditions for a module $S_\pi W$ with $|\pi| = dn$ to be (n,m)-GCT useful are*

(1) $\ell(\pi) \leq m+1$ *and*
(2) $p_1 \geq d(n-m)$.

Problem 2.15 Find a (5, 3)-GCT useful module.

2.7 The program of [62]

The algebraic Peter-Weyl theorem (see Sect. 3.1) implies that for a reductive algebraic group G and a subgroup H, that the ring of regular functions on G/H, denoted $\mathbb{C}[G/H]$, as G-module is simply

$$\mathbb{C}[G/H] = \bigoplus_{\lambda \in \Lambda_G^+} V_\lambda \otimes (V_\lambda^*)^H.$$

Here Λ_G^+ indexes the irreducible G-modules, V_λ is the irreducible module associated to λ, and for a G-module W, $W^H := \{w \in W \mid h \cdot w = w \forall h \in H\}$ denotes the subspace of H-invariants. Here G acts on the V_λ and $(V_\lambda^*)^H$ is just a vector space whose dimension records the multiplicity of V_λ in $\mathbb{C}[G/H]$.

Let $v \in V$ and consider the homogeneous space $G \cdot v = G/G_v \subset V$. Then there is an injection $\mathbb{C}[\overline{G \cdot v}] \to \mathbb{C}[G/G_v]$ by restriction of functions. Thus if we can find a module V_λ that occurs in $Sym(V^*)$ that does not occur in $\mathbb{C}[G/G_v]$, the isotypic component of V_λ in $Sym(V^*)$ must be in the ideal of $\overline{G \cdot v}$. More generally, if the multiplicity of V_λ in $Sym(V^*)$ is higher than its multiplicity in $\mathbb{C}[G/G_v]$, at least some copy of it must occur in $I(\overline{G \cdot v})$.

Definition 2.16 Let $v \in V$ as above. An irreducible G-module V_λ is an *orbit occurrence obstruction* for $\overline{G \cdot v}$ if $V_\lambda \subset Sym(V^*)$ and $(V_\lambda)^{*G_v} = 0$. The module V_λ is an *orbit representation-theoretic obstruction* if $\mathrm{mult}(V_\lambda, Sym(V^*)) > \dim(V_\lambda)^{*G_v}$. More generally, an irreducible G-module V_λ is an *occurrence obstruction* if $V_\lambda \subset Sym(V^*)$ and $V_\lambda \not\in \mathbb{C}[\overline{G \cdot v}]$. The module V_λ is a *representation-theoretic obstruction* if $\mathrm{mult}(V_\lambda, Sym(V^*)) > \mathrm{mult}(V_\lambda \mathbb{C}[\overline{G \cdot v}])$.

Note the implications: M is an orbit occurrence obstruction implies M is an orbit representation-theoretic obstruction implies M is a representation-theoretic obstruction and M is an occurrence obstruction implies M is a representation-theoretic obstruction.

To summarize: The isotypic component of an occurrence obstruction in $Sym(V^*)$ is in the ideal of $\overline{G \cdot v}$, and at least some copy of a representation-theoretic obstruction must be in the ideal of $\overline{G \cdot v}$.

The program initiated in [62] and continued in [61,63] and other articles, was to find such obstructions via representation theory, perhaps using canonical bases for nonstandard quantum groups, see especially [59,60].

Definition 2.17 $P \in S^d V$ is *characterized by* G_P if any $Q \in S^d V$ with $G_Q \supseteq G_P$ is of the form $Q = cP$ for some constant c.

In our situations (where G_P is reductive), the orbit closure of a polynomial characterized by its symmetry group is essentially determined by multiplicity data, which makes one more optimistic for representation-theoretic, or even occurrence obstructions.

In the negative direction, C. Ikenmeyer [38, Conj. 8.1.2] made numerous computations that lead him to conjecture that all modules that occur in $Sym(S^n W^*)$ when n and the partitions are both even, also occur in $\mathbb{C}[GL(W) \cdot \det_n]$.

2.8 The boundary of $\mathcal{D}et_n$

When Conjecture 2.1 was first proposed, it was not known if the inclusion $End(W) \cdot det_n \subset \mathcal{D}et_n$ was proper. In Sect. 5, I describe an explicit component of $\partial \mathcal{D}et_n$ (found in [49]) that is not contained in $End(W) \cdot det_n$. Determining the components of the boundary should be very useful for GCT. It also relates to a classical question in linear algebra: determine the unextendable linear spaces on $\{\det_n = 0\}$.

2.9 Bad news

Hartog's theorem states that a holomorphic function defined off of a codimension two subset of a complex manifold extends to be defined on the complex manifold. Its analog in algebraic geometry, for say affine varieties, is true in the sense that a function defined off of a codimension two subvariety of an affine variety Z extends to be defined on all of Z as long as the affine variety Z is *normal* (see Sect. 7.6 for the definition of normal). When studying a normal orbit closure, the only difference between $\mathbb{C}[G \cdot v]$ and $\mathbb{C}[\overline{G \cdot v}]$ comes from functions having poles along a component of the boundary. With non-normal varieties the situation is far subtler. The following theorem and its proof are discussed in Sect. 9.

Theorem 2.18 (Kumar [43]) *$\mathcal{D}et_n$ is not normal for $n \geq 3$. $\mathcal{P}erm_n^m$ is not normal for $n > 2m$.*

Remark 2.19 In [65] an algorithm is described that in principle can distinguish when one orbit closure is contained in another.

3 Representation theory

3.1 The algebraic Peter-Weyl theorem

Let G be a reductive algebraic group and V a G-module. Given $\alpha \in V^*$ and $v \in V$ we get an algebraic function

$$\begin{aligned} f_{\alpha\otimes v} : G &\to \mathbb{C} \\ g &\mapsto \alpha(gv). \end{aligned}$$

Note this is linear in V and V^* (e.g. $f_{(\alpha_1+\alpha_2)\otimes v} = f_{\alpha_1\otimes v} + f_{\alpha_2\otimes v}$ etc..), and commutes with the action of G, so we obtain an injective G-module map $V^* \otimes V \to \mathbb{C}[G]$.

Exercise 3.1 Show the map $V^* \otimes V \to \mathbb{C}[G]$ is indeed injective.

The linearity shows that it is sufficient to consider irreducible modules to avoid redundancies. We have shown: $\mathbb{C}[G] \supseteq \oplus_{\lambda\in\Lambda_G^+} V_\lambda^* \otimes V_\lambda$.

Theorem 3.2 (Algebraic Peter-Weyl) *(see, e.g* [66, Ch. 7, §3.1.1]) *As a left-right $G \times G$ module, $\mathbb{C}[G] = \oplus_{\lambda\in\Lambda_G^+} V_\lambda^* \otimes V_\lambda$.*

The $G \times G$ module structure is given by $(g_1, g_2) f(g) = f(g_1 g g_2)$. For the proof of the equality (which is not difficult), see [66, pp. 160, 180].

We will need the following Corollary:

Corollary 3.3 *Let $H \subset G$ be a closed subgroup. Then, as a G-module,*

$$\mathbb{C}[G/H] = \mathbb{C}[G]^H = \oplus_{\lambda\in\Lambda_G^+} V_\lambda \otimes (V_\lambda^*)^H = \oplus_{\lambda\in\Lambda_G^+} V_\lambda^{\oplus \dim(V_\lambda^*)^H}.$$

3.2 Representations of $GL(V)$

The irreducible representations of $GL(V)$ are indexed by sequences $\pi = (p_1, \dots, p_l)$ of non-increasing integers with $l \leq \dim V$. Those that occur in $V^{\otimes d}$ are partitions of d, and we write $|\pi| = d$ and $S_\pi V$ for the module. $V^{\otimes d}$ is also an $\mathfrak{S}_d$ module, and the groups $GL(V)$ and $\mathfrak{S}_d$ are the commutants of each other in $V^{\otimes d}$ which implies the famous Schur-Weyl duality that $V^{\otimes d} = \oplus_{|\pi|=d, \ell(\pi)\leq \mathbf{v}} [\pi] \otimes S_\pi V$ as a $(\mathfrak{S}_d \times GL(V))$-module, where $[\pi]$ is the irreducible $\mathfrak{S}_d$-module associated to π. Repeated numbers in partitions are sometimes expressed as exponents when there is no danger of confusion, e.g. $(3, 3, 1, 1, 1, 1) = (3^2, 1^4)$. For example, $S_{(d)}V = S^d V$ and $S_{(1^d)}V = \Lambda^d V$. The modules $S_{s^{\mathbf{v}}} V = (\Lambda^{\mathbf{v}} V)^{\otimes s}$ are exactly the $SL(V)$-trivial modules. The module $S_{(22)}V$ is the home of the Riemann curvature tensor in Riemannian geometry. See any of [45, Chap. 6], [24, Chap 6] or [66, Chap. 9] for more details on the representations of $GL(V)$, Schur-Weyl duality, and what follows.

Assuming $\mathbf{v}$, $\mathbf{w}$ are sufficiently large, we may write:

$$S_\pi(V \oplus W) = \bigoplus_{|\mu|+|\nu|=|\pi|} (S_\mu V \otimes S_\nu W)^{\oplus c^\pi_{\mu\nu}} \tag{5}$$

$$S_\pi(V \otimes W) = \bigoplus_{|\mu|=|\nu|=|\pi|} (S_\mu V \otimes S_\nu W)^{\oplus k_{\pi\mu\nu}} \tag{6}$$

for some non-negative integers $c^\pi_{\mu\nu}$, $k_{\pi\mu\nu}$. On the left hand side one respectively has $GL(V \oplus W)$ and $GL(V \otimes W)$ modules and on the right hand side $GL(V) \times GL(W)$-modules. The constants $c^\pi_{\mu\nu}$ are called *Littlewood-Richardson coefficients* and the $k_{\pi\mu\nu}$ are called *Kronecker coefficients*. They are independent of the dimensions of the vector spaces as long as $\mathbf{v}$, $\mathbf{w}$ are larger than the lengths of the partitions. They also (via Schur-Weyl duality) admit descriptions in terms of the symmetric group:

$$c^\pi_{\mu\nu} = \dim(\mathrm{Hom}_{\mathfrak{S}_{|\mu|} \times \mathfrak{S}_{|\nu|}}([\pi], [\mu] \otimes [\nu]) \tag{7}$$

$$k_{\pi\mu\nu} = \dim([\pi] \otimes [\mu] \otimes [\nu])^{\mathfrak{S}_d} \tag{8}$$

where in the first line $|\pi| = |\mu| + |\nu|$ and in the second line $|\pi| = |\mu| = |\nu| = d$, so in particular Kronecker coefficients are symmetric in their three indices.

Often one writes partitions in terms of *Young diagrams*, where $\pi = (p_1, \dots, p_t)$ is represented by a collection of boxes, left justified, with p_j boxes in the j-th row (Fig. 1). There is a nice pictorial recipe for computing Littlewood-Richardson coefficients in terms of Young diagrams (see, e.g., [23, Chap. 5]).

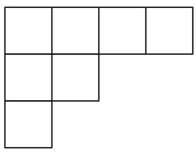

Fig. 1 Young diagram for $\pi = (4, 2, 1)$

A useful special case of the Littlewood Richardson coefficients is the *Pieri formula*

$$c^{\nu}_{\lambda,(d)} = \begin{cases} 1 & \text{if } \nu \text{ is obtained from } \lambda \text{ by adding } d \text{ boxes to} \\ & \text{the rows of } \lambda \text{ with no two in the same column;} \\ 0 & \text{otherwise.} \end{cases} \tag{9}$$

Exercise 3.4 Show that $(S^n V)^{\otimes d}$ does not contain any $SL(V)$-invariants for $d < \mathbf{v}$.

Exercise 3.5 Show that $k_{\pi\mu\nu} = \dim(\mathrm{Hom}_{\mathfrak{S}_d}([\pi], [\mu]\otimes[\nu]))$.

3.3 A duality theorem for weight zero spaces and plethysms

For any $S_\pi V$, the Weyl group $\mathfrak{S}_{\mathbf{v}}$ acts on the $\mathfrak{sl}$-*weight zero space*, which I will denote $(S_\pi V)_0$. This is by definition the subspace of $S_\pi V$ on which the torus T_V acts trivially. Recall that $\mathfrak{S}_d$ acts on $V^{\otimes d}$ and is the commutator of $GL(V)$.

Exercise 3.6 Show that the weight zero space of $V^{\otimes d}$ is zero unless $\mathbf{v}$ divides d, in which case we write $d = \mathbf{v}s$.

Note that $S_\mu(S^s V) \subset V^{\otimes s|\mu|}$. We have the following duality theorem:

Theorem 3.7 [26] *For* $|\pi| = d = \mathbf{v}s$ *and* $|\mu| = \mathbf{v}$,

$$\mathrm{mult}_{\mathfrak{S}_{\mathbf{v}}}([\mu], (S_\pi V)_0) = \mathrm{mult}_{GL(V)}(S_\pi V, S_\mu(S^s V)).$$

In particular,

Corollary 3.8 *Let* $|\pi| = d$.

(1) When $d = \mathbf{v}$, $(S_\pi V)_0 = [\pi]$.
(2) For any $d = \mathbf{v}s$, $\dim[(S_\pi V)_0]^{\mathfrak{S}_d} = \mathrm{mult}(S_\pi V, S^{\mathbf{v}}(S^s V))$.

To get an idea of the proof, note that $S_\pi V = \mathrm{Hom}_{\mathfrak{S}_d}([\pi], V^{\otimes d})$ and thus $(S_\pi V)_0 = \mathrm{Hom}_{\mathfrak{S}_d}([\pi], (V^{\otimes d})_0)$, so the left hand side is $\mathrm{mult}_{\mathfrak{S}_{\mathbf{v}}}([\mu], \mathrm{Hom}_{\mathfrak{S}_d}([\pi], (V^{\otimes d})_0))$. On the other hand, $S^s V = (V^{\otimes s})^{\mathfrak{S}_s}$, and $(S^s V)^{\otimes \mathbf{v}} = (V^{\otimes \mathbf{v}s})^{\mathfrak{S}_s\times\cdots\times\mathfrak{S}_s}$, where we have $\mathbf{v}$ copies of $\mathfrak{S}_s$. So the right hand side is $\mathrm{mult}_{\mathfrak{S}_d}([\pi], \mathrm{Hom}_{\mathfrak{S}_{\mathbf{v}}}([\mu], (V^{\otimes \mathbf{v}s})^{\mathfrak{S}_s\times\cdots\times\mathfrak{S}_s})$. Now $(V^{\otimes d})_0$ is an $\mathfrak{S}_d$ and an $\mathfrak{S}_{\mathbf{v}}$-module and has a basis $e_{i_1}\otimes\cdots\otimes e_{i_d}$ with $\{i_1, \ldots, i_d\} = [\mathbf{v}]^s$ where $d = \mathbf{v}s$. Moreover the $\mathfrak{S}_d$ and $\mathfrak{S}_{\mathbf{v}}$ actions commute, and the $\mathfrak{S}_d$ action is transitive on the set of basis elements. Letting $H = \mathfrak{S}_s^{\times\mathbf{v}} \subset \mathfrak{S}_d$, D. Gay shows the normalizer of H divided by H is $Nor(H)/H = \mathfrak{S}_{\mathbf{v}}$ and the centralizer of $\mathfrak{S}_d$ in $\mathfrak{S}_{\dim(V^{\otimes d})_0}$ is $\mathfrak{S}_{\mathbf{v}}$. The result follows by applying a combination of Frobenius reciprocity and Schur-Weyl duality to go from $\mathfrak{S}_d$-modules to $GL(V)$-modules. A key point is noting that H is also the stabilizer of the vector $x := e_1\otimes\cdots\otimes e_1\otimes e_2\otimes\cdots\otimes e_2\otimes\cdots\otimes e_{\mathbf{v}}\otimes\cdots\otimes e_{\mathbf{v}} = e_1^{\otimes s}\otimes\cdots\otimes e_{\mathbf{v}}^{\otimes s}$.

Exercise 3.9 We may realize $S_{(s^{\mathbf{v}})}V$ as $\mathbb{C}\{(e_1\wedge\cdots\wedge e_{\mathbf{v}})^{\otimes s}\}$. Show that $\mathfrak{S}_{\mathbf{v}}$ acts on $S_{(s^{\mathbf{v}})}V$ by the sign representation when s is odd and acts trivially when s is even. Conclude $S^{\mathbf{v}}(S^s V)$ has a unique $SL(V)$-invariant when s is even and none when s is odd, and that $[S^{2\mathbf{v}}(S^s V)]^{SL(V)} \neq 0$ for all $s > 1$. This had been observed in [36, Prop. 4.3a].

Exercise 3.10 Show that the $SL(V)$-invariant $P \in S^{\mathbf{v}}(S^s V)$ from the previous problem has the following expression. Let $z = (x_1^1 \cdots x_s^1) \cdots (x_1^{\mathbf{v}} \cdots x_s^{\mathbf{v}})$. Then

$$\langle \overline{P}, z \rangle = \sum_{\sigma_1, \ldots, \sigma_{\mathbf{v}} \in \mathfrak{S}_s} \overline{\det_{\mathbf{v}}}(x^1_{\sigma_1(1)}, \ldots, x^{\mathbf{v}}_{\sigma_{\mathbf{v}}(1)}) \cdots \overline{\det_{\mathbf{v}}}(x^s_{\sigma_1(s)}, \ldots, x^{\mathbf{v}}_{\sigma_{\mathbf{v}}(s)}). \qquad (10)$$

To compute $P(u)$ for any $u \in S^s V^*$, consider $u^{\mathbf{v}}$ and expand it out as a sum of terms of the form z. Then $P(u) = \langle \overline{P}, u^{\mathbf{v}} \rangle$.

4 Lower bounds via geometry

4.1 The second fundamental form and the $\frac{m^2}{2}$ bound for Valiant's conjecture

For hypersurfaces in affine space, one can attach a differential invariant, the *second fundamental form*, to each point. This form is essentially the quadratic term in an adapted Taylor series for the hypersurface graphed over its tangent space at that point. The rank of this quadratic form gives an invariant that can not increase on the image of general points under affine linear projections. It is straight-forward to compute that for smooth points of $\{\det_n = 0\}$ the rank of the quadratic form is $2n - 2$ whereas, if one chooses a judicious point of $\{\mathrm{perm}_m = 0\}$ one finds the rank is the maximal $m^2 - 2$. Combining these two gives:

Theorem 4.1 [57] *$dc(\mathrm{perm}_m) \geq \frac{m^2}{2}$, i.e., if $\mathrm{perm}_m \in \mathrm{End}(\mathbb{C}^{n^2} \cdot \det_n)$, then $n \geq \frac{m^2}{2}$.*

Valiant's conjecture [75] that motivated the work of Mulmuley and Sohoni is that n must grow faster than any polynomial in m to have $\mathrm{perm}_m \in \mathrm{End}(\mathbb{C}^{n^2} \cdot \det_n)$.

4.2 Dual varieties and the $\frac{m^2}{2}$ lower bound for Conjecture 2.1

Define $\mathcal{D}ual_{k,d,N} \subset \mathbb{P}(S^d W^*)$ as the Zariski closure of the set of irreducible hypersurfaces of degree d in $\mathbb{P}W \simeq \mathbb{P}^{N-1}$ whose dual variety has dimension at most k. (I identify a hypersurface (as a scheme) with its equation.)

It had been a classically studied problem to determine set-theoretic equations for $\mathcal{D}ual_{k,d,N}$. Motivated by GCT, Manivel, Ressayre and I were led to solve it. I follow [49] in this subsection.

Let $P \in S^d W^*$ be irreducible. The B. Segre dimension formula [68] states that for $[w] \in Z(P)_{general}$,

$$\dim Z(P)^\vee = \mathrm{rank}(P_{d-2,1,1}(w^{d-2})) - 2.$$

The bilinear form $P_{d-2,1,1}(w^{d-2})$ is called the *Hessian* of P at w. Write H_P for $P_{d-2,1,1}$; in bases it is an $n \times n$ symmetric matrix whose entries are polynomials of degree $d - 2$.

Thus $\dim(Z(P)^\vee) \leq k$ if and only if, for all $w \in \hat{Z}(P)$ and $F \in G(k+3, W)$,

$$\det{}_{k+3}(H_P(w)|_F) = 0.$$

Equivalently, P must divide $\det_{k+3}(H_P|_F)$. Note that $P \mapsto \det_{k+3}(H_P|_F)$ is a polynomial of degree $(k+3)(d-2)$ on $S^d W^*$.

By restricting first to a projective line $L \subset \mathbb{P}W$, and then to an affine line $\mathbb{A}^1 \subset L$ within the projective line, one can test divisibility by Euclidean division. The remainder will depend on our choice of coordinates on $\mathbb{A}^1$, but the leading coefficient of the remainder only depends on the choice of point in L that distinguishes the affine line.

Set theoretically, the equations obtained from the invariant part of the remainder as one varies $\mathbb{A}^1$, L, F suffice to define $\mathcal{D}ual_{k,d,N}$ on the open subset parameterizing irreducible hypersurfaces, as once the plane $\hat{L}$ is fixed, by varying the line $\mathbb{A}^1$ one obtains a family of equations expressing the condition that $P|_L$ divides $\det(H_P|_F)|_L$. A polynomial P divides Q if and only if when restricted to each plane P divides Q, so the conditions imply that the dual variety of the irreducible hypersurface $Z(P)$ has dimension at most k.

By keeping track of weights along the flag $\mathbb{A}^1 \subset \hat{L}^2 \subset F^{k+3}$ one concludes:

Theorem 4.2 [49] *The variety $\mathcal{D}ual_{k,d,N} \subset \mathbb{P}(S^d(\mathbb{C}^N)^*)$ has equations given by a copy of the GL_N-module $S_{\pi(k,d)}\mathbb{C}^N$, where*

$$\pi(k,d) = ((k+2)(d^2-2d)+1,\ d(k+2)-2k-3,\ 2^{k+1}).$$

Since $|\pi| = d(k+2)(d-1)$, these equations have degree $(k+2)(d-1)$.

If P is not reduced, then these equations can vanish even if the dual of the reduced polynomial with the same zero set as P is non-degenerate. For example, if $P = R^2$ where R is a quadratic polynomial of rank $2s$, then $\det(H_P)$ is a multiple of R^{2s}. The polynomial $\ell^{n-m}\operatorname{perm}_m$ is neither reduced nor irreducible, but fortunately we have the following lemma:

Lemma 4.3 [49] *Let $U = \mathbb{C}^M$ and $L = \mathbb{C}$, let $R \in S^m U^*$ be irreducible, let $\ell \in L^*$ be nonzero, let $U^* \oplus L^* \subset W^*$ be a linear inclusion, and let $P = \ell^{d-m} R \in S^d W^*$.*

If $[R] \in \mathcal{D}ual_{\kappa,m,M}$ and $[R] \notin \mathcal{D}ual_{\kappa-1,m,M}$, then $[P] \in \mathcal{D}ual_{\kappa,d,N}$ and $[P] \notin \mathcal{D}ual_{\kappa-1,d,N}$.

Checking that $\{\operatorname{perm}_m = 0\}^\vee$ is indeed a hypersurface by computing the second fundamental form of $\{\operatorname{perm}_m = 0\}$ is of full rank at the matrix all of whose entries are 1 except, e.g., the $(1,1)$ slot which is $1-n$ (the kernel of the second fundamental form has the same dimension as the kernel of the Hessian), it follows $\mathcal{P}erm^m_{\frac{m^2}{2}} \not\subset \mathcal{D}et_{\frac{m^2}{2}}$ proving Theorem 2.2.

The main theorem of [49] is:

Theorem 4.4 [49] *The scheme $\mathcal{D}ual_{2n-2,n,n^2}$ is smooth at $[\det_n]$, and $\mathcal{D}et_n$ is an irreducible component of $\mathcal{D}ual_{2n-2,n,n^2}$.*

For polynomials in $N' < N$ variables, the maximum rank of the Hessian is N' so the determinant of the Hessian will vanish on any F of dimension $N'+1$. Thus $Sub_{k+2}(S^nW) \subset \mathcal{D}ual_{k,n,N}$. The subspace variety $Sub_{k+2}(S^d\mathbb{C}^N)$, which has dimension $\binom{k+d+1}{d}+(k+2)(N-k-2)-1$, also forms an irreducible component of $\mathcal{D}ual_{k,n,N}$ (see [49]), so $\mathcal{D}ual_{2n-2,n,n^2}$ is not irreducible.

Theorem 4.4 is proved by computing the Zariski tangent space to both varieties at $[\det_n]$. To carry out the computation, one uses that the Zariski tangent spaces are $G_{\det_n}$-modules, so one just needs to single out a vector in each $S_\pi E \otimes S_\pi F$. On then uses immanants (see Sect. 6.15) to get a preferred vector in each module to test.

In particular, Theorem 4.4 implies that the $GL(W)$-module of highest weight $\pi(2n-2,n)$ given by Theorem 4.2 gives local equations at $[\det_n]$ of $\overline{GL_{n^2}\cdot[\det_n]}$, of degree $2n(n-1)$.

5 The boundary of $\mathcal{D}et_n$

It is expected that understanding the components of the boundary of $\mathcal{D}et_n$ will be useful for GCT. There is the obvious component obtained by eliminating a variable, which is contained in $\mathrm{End}(\mathbb{C}^{n^2}\cdot\det_n)$, and is related to Valiant's conjecture. To understand the difference between Valiant's conjecture and the Conjecture 2.1, one needs to examine the other components of the boundary.

Determining additional components of the boundary relates to yet another classical question: determine unextendable linear spaces on the hypersurface $\{\det_n = 0\}$. Roughly speaking, given one such, call it $L \subset \mathbb{C}^{n^2}$, write $\mathbb{C}^{n^2} = L \oplus L^c$ where L^c is some choice of complement to L. Then compose the determinant with a (suitably normalized) curve $f_t = Id_L + tId_{L^c} \in GL_{n^2}$. In the limit as $t \to 0$ one may arrive at a new component of the boundary.

For an explicit example, write $\mathbb{C}^{n^2} = W = W_S \oplus W_\Lambda$, where we split up the $n\times n$ matrices into symmetric and skew-symmetric matrices. When n is odd, the curve

$$g(t) = \frac{1}{t}(Id_{W_\Lambda} + tId_{W_S})$$

determines a polynomial $P_\Lambda := \lim_{t\to 0} g(t)\cdot\det_n$. To see P_Λ explicitly, decompose a matrix M into its symmetric and skew-symmetric parts M_S and M_Λ. Then

$$P_\Lambda(M) = \overline{\det_n}(M_\Lambda,\ldots,M_\Lambda,M_S).$$

More explicitly, P_Λ can be expressed as follows. Let $Pf_i(M_\Lambda)$ denote the Pfaffian of the skew-symmetric matrix, of even size, obtained from M_Λ by suppressing its i-th row and column. Then

$$P_\Lambda(M) = \sum_{i,j}(M_S)_{ij}Pf_i(M_\Lambda)Pf_j(M_\Lambda).$$

Proposition 5.1 [49] *The orbit closure $\overline{GL(W)\cdot P_\Lambda}$ is an irreducible codimension one component of $\partial\mathcal{D}et_n$ that is not contained in* $\mathrm{End}(W)\cdot[\det_n]$. *In particular* $\overline{dc}(P_{\Lambda,m}) = m < dc(P_{\Lambda,m})$.

Proposition 5.1 indicates that Conjecture 2.1 could be strictly stronger than Valiant's conjecture. To prove the second assertion, one computes the stabilizer G_{P_Λ} explicitly and sees it has dimension one less than the dimension of $G_{\det_n}$.

The hypersurface $Z(P_\Lambda)\subset\mathbb{P}W$ has interesting properties, for example:

Proposition 5.2 [49]

$$Z(P_\Lambda)^\vee = \overline{\mathbb{P}\{v^2\oplus v\wedge w\in S^2\mathbb{C}^n\oplus\Lambda^2\mathbb{C}^n,\ v,w\in\mathbb{C}^n\}}\subset\mathbb{P}W^*.$$

Note that $Z(P_\Lambda)^\vee$ resembles $Seg(\mathbb{P}^{n-1}\times\mathbb{P}^{n-1})$. It can be defined as the image of the projective bundle $\pi:\mathbb{P}(E)\to\mathbb{P}^{n-1}$, where $E=\mathcal{O}(-1)\oplus Q$ is the sum of the tautological and quotient bundles on $\mathbb{P}^{n-1}$, by a sub-linear system of $\mathcal{O}_E(1)\otimes\pi^*\mathcal{O}(1)$. This sub-linear system contracts the divisor $\mathbb{P}(Q)\subset\mathbb{P}(E)$ to the Grassmannian $G(2,n)\subset\mathbb{P}\Lambda^2\mathbb{C}^n$.

The only other components of $\partial\mathcal{D}et_n$ that I am aware of were found by J. Brown, N. Bushek, L. Oeding, D. Torrance and Y. Qi, as part of an AMS Mathematics Research Community in June 2012. They found two additional components of $\partial\mathcal{D}et_4$.

Problem 5.3 Find additional components of $\partial\mathcal{D}et_n$.

Problem 5.4 Determine all components of $\partial\mathcal{D}et_3$.

6 Symmetries of polynomials and coordinate rings of orbits

Throughout this section $G=GL(V)$ and $\dim V=n$. Given $P\in S^dV$, let

$$G_P := \{g\in GL(V)\mid g\cdot P=P\}=\{g\in GL(V)\mid P(g\cdot x)=P(x)\forall x\in V^*\}$$

denote the symmetry group of P. We let $G_{[P]}:=\{g\in GL(V)\mid g\cdot[P]=[P]\}$. Determining the connected component of the identity G_P^0 is simply a matter of linear algebra, as the computation of $\mathfrak{g}_P$ is a linear problem. However one can compute G_P directly in only a few simple cases.

Throughout this section, let $V=\mathbb{C}^n$ and use index ranges $1\le i,j,k\le n$. Examples 6.1, 6.2, 6.3, and 6.4 follow [14].

6.1 Two easy examples

Example 6.1 Let $P=x_1^d\in S^dV$. Let $g=(g^i_j)\in GL(V)$. Then $g\cdot(x_1^d)=(g^j_1x_j)^d$ so if $g\cdot(x_1^d)=x_1^d$, then $g^j_1=0$ for $j>1$ and g^1_1 must be a d-th root of unity. There are no other restrictions, thus

$$G_P = \left\{ g \in GL(V) \mid g = \begin{pmatrix} \omega & * & \cdots & * \\ 0 & * & \cdots & * \\ & \vdots & & \\ 0 & * & \cdots & * \end{pmatrix}, \omega^d = 1 \right\},$$

$$G_{[P]} = \left\{ g \in GL(V) \mid g = \begin{pmatrix} * & * & \cdots & * \\ 0 & * & \cdots & * \\ & \vdots & & \\ 0 & * & \cdots & * \end{pmatrix} \right\}.$$

The $GL(V)$ orbit of $[x_1^d]$ is closed and equal to the Veronese variety $v_d(\mathbb{P}V)$.

Exercise 6.2 Use Corollary 3.3 to determine $\mathbb{C}[v_d(\mathbb{P}V)]$ (even if you already know it by a different method).

Example 6.3 Let $P = \mathrm{chow}_n = x_1 \cdots x_n \in S^n V$, which I will call the "Chow polynomial". It is clear $\Gamma_n := T_n^{SL} \rtimes \mathfrak{S}_n \subset G_{\mathrm{chow}_n}$, we need to determine if the stabilizer is larger. Again, we can work by brute force: $g \cdot \mathrm{chow}_n = (g_1^j x_j) \cdots (g_n^j x_j)$. In order that this be equal to $x_1 \cdots x_n$, by unique factorization of polynomials, there must be a permutation $\sigma \in \mathfrak{S}_n$ such that for each k, we have $g_k^j x_j = \lambda_k x_{\sigma(k)}$ for some $\lambda_k \in \mathbb{C}^*$. Composing with the inverse of this permutation we have $g_k^j = \delta_k^j \lambda_j$, and finally we see that we must further have $\lambda_1 \cdots \lambda_n = 1$, which means it is an element of T_n^{SL}, so the original g is an element of Γ_n. Thus $G_{\mathrm{chow}_n} = \Gamma_n$.

The orbit closure of chow_n is the Chow variety $Ch_n(V) \subset \mathbb{P}S^n V$. The coordinate ring of $GL(V) \cdot \mathrm{chow}_n$ is discussed in Sect. 7.

6.2 Techniques

We can usually guess a large part of G_P. We then form auxiliary objects from P which have a symmetry group H that one can compute, and by construction H contains G_P. If $H = G_P$, we are done, and if not, we simply have to examine the difference between the groups.

Remark 6.4 The very recent preprint [25] describes further techniques for determining stabilizers of points.

Consider the hypersurface $Z(P) := \{[v] \in \mathbb{P}V^* \mid P(v) = 0\} \subset \mathbb{P}V^*$. If all the irreducible components of P are reduced, then $G_{Z(P)} = G_{[P]}$, as a reduced polynomial may be recovered up to scale from its zero set, and in general $G_{Z(P)} \supseteq G_{[P]}$. Moreover, we can consider its singular set $Z(P)_{sing}$, which may be described as the zero set of the image of $P_{1,d-1}$ (which is essentially the exterior derivative dP). If $P = a_I x^I$, where $a_{i_1,\dots,i_d}$ is symmetric in its lower indices, then $Z(P)_{sing} = \{[v] \in \mathbb{P}V^* \mid a_{i_1,i_2,\dots,i_d} x^{i_2}(v) \cdots x^{i_d}(v) = 0 \ \forall i_1\}$. While we could consider the singular locus of the singular locus etc.., it turns out to be easier to work with what I will call the *very singular* loci. For an arbitrary variety $X \subset \mathbb{P}V$, define $X_{verysing} = X_{verysing,1} := \{x \in \mathbb{P}V \mid dP_x = 0 \forall P \in I(X)\}$. If X is a hypersurface, then $X_{sing} = X_{verysing}$

but in general they can be different. Define $X_{verysing,k} := (X_{verysing,k-1})_{verysing}$. Algebraically, if $X = Z(P)$ for some $P \in S^d V$, then the ideal of $Z(P)_{verysing,k}$ is generated by the image of $P_{k,n-k} : S^k V^* \to S^{n-k} V$. The symmetry groups of these varieties all contain G_P.

6.3 The Fermat

Let $\mathrm{fermat}_n^d := x_1^d + \cdots + x_n^d$. The $GL(V)$-orbit closure of $[\mathrm{fermat}_n^d]$ is the n-th secant variety of the Veronese variety $\sigma_n(v_d(\mathbb{P}V)) \subset \mathbb{P}S^n V$, see Sect. 2.2. It is clear $\mathfrak{S}_n \subset G_{\mathrm{fermat}}$, as well as the diagonal matrices whose entries are d-th roots of unity. We need to see if there is anything else. The first idea, to look at the singular locus, does not work, as the zero set is smooth, so we consider $\mathrm{fermat}_{2,d-2} = x_1^2 \otimes x^{d-2} + \cdots + x_n^2 \otimes x^{d-2}$. Write the further polarization $P_{1,1,d-2}$ as a symmetric matrix whose entries are homogeneous polynomials of degree $d-2$ (the Hessian matrix). We get

$$\begin{pmatrix} x_1^{d-2} & & \\ & \ddots & \\ & & x_n^{d-2} \end{pmatrix}.$$

Were the determinant of this matrix $GL(V)$-invariant, we could proceed as we did with chow_n, using unique factorization. Although it is not, it is close enough as follows: Recall that for a linear map $f : W \to V$, where $\dim W = \dim V = n$, we have $f^{\wedge n} \in \Lambda^n W^* \otimes \Lambda^n V$ and an element $(h, g) \in GL(W) \times GL(V)$ acts on $f^{\wedge n}$ by $(h, g) \cdot f^{\wedge n} = (\det(h))^{-1}(\det(g)) f^{\wedge n}$. In our case $W = V^*$ so $P^{\wedge n}_{2,d-2}(x) = \det(g)^2 P^{\wedge n}_{2,d-2}(g \cdot x)$, and the polynomial obtained by the determinant of the Hessian matrix is invariant up to scale.

Arguing as above, $(g_1^j x_j)^{d-2} \cdots (g_n^j x_j)^{d-2} = x_1^{d-2} \cdots x_n^{d-2}$ and we conclude again by unique factorization that g is in Γ_n. Composing with a permutation matrix to make $g \in T$, we see that, by acting on the Fermat itself, that the entries on the diagonal are d-th roots of unity.

Exercise 6.5 Show that the Fermat is characterized by its symmetries.

6.4 The sum-product polynomial

The following polynomial, called the *sum-product polynomial*, will be important when studying depth-3 circuits. Its $GL(mn)$-orbit closure is the m-th secant variety of the Chow variety $\sigma_m(Ch_n(\mathbb{C}^{nm}))$:

$$S_m^n := \sum_{i=1}^m \Pi_{j=1}^n x_{ij} \in S^n(\mathbb{C}^{nm}).$$

Exercise 6.6 Determine $G_{S_m^n}$ and show that S_m^n is characterized by its symmetries.

6.5 The determinant

I follow [15] in this section. Write $\mathbb{C}^{n^2} = E \otimes F$ with $E, F = \mathbb{C}^n$.

Theorem 6.7 (Frobenius [22]) *Let $\phi \in \rho(Gl_{n^2}) \subset GL(S^n\mathbb{C}^{n^2})$ be such that $\phi(\det_n) = \det_n$. Then, identifying $\mathbb{C}^{n^2} \simeq Mat_{n\times n}$,*

$$\phi(X) = \begin{cases} X \mapsto gXh \\ X \mapsto gX^T h \end{cases}$$

where $g, h \in GL_n$, and $\det_n(g)\det_n(h) = 1$. Here X^T denotes the transpose of X.

Corollary 6.8 *Let μ_n denote the n-th roots of unity embedded diagonally in $SL(E) \times SL(F)$. Then $G_{\det_n} = (SL(E) \times SL(F))/\mu_n \rtimes \mathbb{Z}_2$*

To prove the Corollary, just note that the $\mathbb{C}^*$ and μ_n are in the kernel of the map $\mathbb{C}^* \times SL(E) \times SL(F) \to GL(E\otimes F)$.

Exercise 6.9 Prove the $n = 2$ case of the theorem. Hint: in this case the determinant is a smooth quadric.

Write $\mathbb{C}^{n^2} = W = A\otimes B = \mathrm{Hom}(A^*, B)$. The following lemma is standard, its proof is left as an exercise:

Lemma 6.10 *Let $U \subset W$ be a linear subspace such that $U \subset \{\det_n = 0\}$. Then $\dim U \leq n^2 - n$ and the subvariety of the Grassmannian $G(n^2 - n, W)$ consisting of maximal linear spaces on $\{\det_n = 0\}$ has two components, call them Σ_α and Σ_β, where*

$$\Sigma_\alpha = \{X \mid \ker(X) = \hat{L} \text{ for some } L \in \mathbb{P}A\}, \textit{and} \tag{11}$$

$$\Sigma_\beta = \{X \mid \mathrm{Image}(X) = \hat{H} \text{ for some } H \in \mathbb{P}B^*\}. \tag{12}$$

Moreover, for any two distinct $X_j \in \Sigma_\alpha$, $j = 1, 2$, and $Y_j \in \Sigma_\beta$ we have

$$\dim(X_1 \cap X_2) = \dim(Y_1 \cap Y_2) = n^2 - 2n, \textit{and} \tag{13}$$

$$\dim(X_i \cap Y_j) = n^2 - 2n + 1. \tag{14}$$

Proof of Theorem 6.7 Let $\Sigma = \Sigma_\alpha \cup \Sigma_\beta$. Then the map on $G(n^2 - n, W)$ induced by ϕ must preserve Σ. By the conditions (13), (14) of Lemma 6.10, in order to preserve dimensions of intersections, every $X \in \Sigma_\alpha$ must map to a point of Σ_α or every $X \in \Sigma_\alpha$ must map to a point of Σ_β, and similarly for Σ_β. If we are in the second case, replace ϕ by $\phi \circ T$, where $T(X) = X^T$, so we may now assume ϕ preserves both Σ_α and Σ_β.

Now $\Sigma_\alpha \simeq \mathbb{P}A$, so ϕ induces an algebraic map $\phi_A : \mathbb{P}A \to \mathbb{P}A$. If $L_1, L_2, L_3 \in \mathbb{P}A$ lie on a $\mathbb{P}^1$, in order for ϕ to preserve the dimensions of triple intersections, the images of the L_j under ϕ_A must also lie on a $\mathbb{P}^1$. By Exercise 6.9 we may assume $n \geq 3$ so the above condition is non-vacuous. But then, by classical projective geometry

$\phi_A \in PGL(A)$, and similarly, $\phi_B \in PGL(B)$, where $\phi_B : \mathbb{P}B^* \to \mathbb{P}B^*$ is the corresponding map. Write $\hat{\phi}_A \in GL(A)$ for any choice of lift and similarly for B.

Consider the map $\tilde{\phi} \in \rho(GL(W))$ given by $\tilde{\phi}(X) = \hat{\phi}_A{}^{-1}\phi(X)\hat{\phi}_B{}^{-1}$. The map $\tilde{\phi}$ sends each $X_j \in \Sigma_\alpha$ to itself as well as each $Y_j \in \Sigma_\beta$, in particular it does the same for all intersections. Hence it preserves $Seg(\mathbb{P}A \times \mathbb{P}B) \subset \mathbb{P}(A\otimes B)$ point-wise, so it is up to scale the identity map. □

Remark 6.11 For those familiar with Picard groups, M. Brion points out that there is a shorter proof of Theorem 6.7 as follows: In general, if a polynomial P is reduced and irreducible, then $G_{Z(P)^\vee} = G_{Z(P)} = G_{[P]}$. (This follows as $(Z(P)^\vee)^\vee = Z(P)$.) The dual of $Z(\det_n)$ is the Segre $Seg(\mathbb{P}^{n-1} \times \mathbb{P}^{n-1})$. Now the automorphism group of $\mathbb{P}^{n-1} \times \mathbb{P}^{n-1} = \mathbb{P}E \times \mathbb{P}F$ acts on the Picard group which is $\mathbb{Z} \times \mathbb{Z}$ and preserves the two generators $\mathcal{O}_{\mathbb{P}E\times\mathbb{P}F}(1, 0)$ and $\mathcal{O}_{\mathbb{P}E\times\mathbb{P}F}(0, 1)$ coming from the generators on $\mathbb{P}E, \mathbb{P}F$. Thus, possibly composing with $\mathbb{Z}_2$ swapping the generators (corresponding to transpose in the ambient space), we may assume each generator is preserved. But then we must have an element of $Aut(\mathbb{P}E) \times Aut(\mathbb{P}F) = PGL(E) \times PGL(F)$. Passing back to the ambient space, we obtain the result.

6.6 The coordinate ring of $GL(W) \cdot \det_n$

For $\pi = (p_1, \ldots, p_{n^2})$ with $p_1 \geq \cdots \geq p_{n^2}$, recall that the multiplicity of $S_\pi W$ in $\mathbb{C}[GL(W) \cdot \det_n]$ is $\dim(S_\pi W)^{G_{\det_n}}$, where $G_{\det_n} = (SL(E) \times SL(F))/\mu_n \rtimes \mathbb{Z}_2$. Following Sect. 3.2, write the $SL(E) \times SL(F)$-decomposition $S_\pi(E\otimes F) = \oplus(S_\mu E\otimes S_\nu F)^{\oplus k_{\pi\mu\nu}}$. To have $SL(E)\times SL(F)$-trivial modules, we need $\mu = \nu = (\delta^n)$ for some $\delta \in \mathbb{Z}$. Recalling the interpretation $k_{\pi\mu\nu} = \dim(\mathrm{Hom}_{\mathfrak{S}_d}([\pi], [\mu]\otimes[\nu]))$, when $\mu = \nu$, $[\mu]\otimes[\mu] = S^2[\mu]\otimes\Lambda^2[\mu]$. Define the *symmetric Kronecker coefficient* $sk^\pi_{\mu\mu} := \dim \mathrm{Hom}_{\mathfrak{S}_d}([\pi], S^2[\mu])$. It is not hard to check (see [12]) that

$$((S_{\delta^n} E\otimes S_{\delta^n} F)^{\oplus k_{\pi\delta^n\delta^n}})^{\mathbb{Z}_2} = (S_{\delta^n} E\otimes S_{\delta^n} F)^{\oplus sk^\pi_{\delta^n\delta^n}}.$$

We conclude:

Proposition 6.12 [12] *Let* $W = \mathbb{C}^{n^2}$. *The coordinate ring of the* $GL(W)$*-orbit of* $\det_n$ *is*

$$\mathbb{C}[GL(W) \cdot \det_n] = \bigoplus_{\delta\in\mathbb{Z}} \bigoplus_{\pi \mid |\pi|=n\delta} (S_\pi W^*)^{\oplus sk^\pi_{\delta^n\delta^n}}.$$

Thus partitions π of dn such that $sk^\pi_{\delta^n\delta^n} < \mathrm{mult}(S_\pi W, S^d(S^n W))$ are representation-theoretic obstructions, and if moreover $\mathrm{mult}(S_\pi W, S^d(S^n W)) = 0$, $S_\pi W$ is an occurrence obstruction. C. Ikenmeyer [38] has examined the situation for $\mathcal{D}et_3$. He found on the order of 3,000 representation-theoretic obstructions, of which on the order of 100 are occurrence obstructions in degrees up to $d = 15$. There are two such partitions with seven parts, $(13^2, 2^5)$ and $(15, 5^6)$. The rest consist of partitions with at least 8 parts (and many with 9). Also of interest is that for approximately 2/3 of the partitions $sk^\pi_{\delta^3\delta^3} < k_{\pi\delta^3\delta^3}$. The lowest degree of an occurrence obstruction is $d = 10$, where $\pi = (9^2, 2^6)$ has $sk^{\pi}_{10^3 10^3} = k_{\pi 10^3 10^3} = 0$ but $\mathrm{mult}(S_\pi W, S^{10}(S^3 W)) = 1$. In degree 11, $\pi = (11^2, 2^5, 1)$ is an occurrence obstruction where $\mathrm{mult}(S_\pi W, S^{11}(S^3 W)) = k_{\pi 11^3 11^3} = 1 > 0 = sk^{\pi}_{11^3 11^3}$.

6.7 The permanent

Write $\mathbb{C}^{n^2} = E \otimes F$. Then it is easy to see $(\Gamma^E_n \times \Gamma^F_n) \rtimes \mathbb{Z}_2 \subseteq G_{\mathrm{perm}_n}$, where the nontrivial element of $\mathbb{Z}_2$ acts by sending a matrix to its transpose and recall $\Gamma^E_n = T_E \rtimes \mathfrak{S}_n$. We would like to show this is the entire symmetry group. However, it is not when $n = 2$.

Exercise 6.13 What is G_{perm_2}? Hint: $\{\mathrm{perm}_2 = 0\}$ is a smooth quadric.

Theorem 6.14 [52] *For $n \geq 3$, $G_{\mathrm{perm}_n} = (\Gamma^E_n \times \Gamma^F_n)/\mu_n \rtimes \mathbb{Z}_2$.*

Remark 6.15 From Theorem 6.14, one can begin to appreciate the beauty of the permanent. Since $\det_n$ is the only polynomial invariant under $SL(E) \times SL(F)$, to find other interesting polynomials on spaces of matrices, we will have to be content with subgroups of this group. But what could be a more natural subgroup than the product of the normalizer of the tori? In fact, say we begin by asking simply for a polynomial invariant under the action of $T_E \times T_F$. We need to look at $S^n(E \otimes F)_0$, where the 0 denotes the $\mathfrak{sl}$-weight zero subspace. This decomposes as $\oplus_\pi (S_\pi E)_0 \otimes (S_\pi F)_0$. By Corollary 3.8(i), these spaces are the $\mathfrak{S}^E_n \times \mathfrak{S}^F_n$-modules $[\pi] \otimes [\pi]$. Only one of these is trivial, and that corresponds to the permanent! More generally, if we consider the diagonal $\mathfrak{S}_n \subset \mathfrak{S}^E_n \times \mathfrak{S}^F_n$, then both $[\pi]$'s are modules for the same group, and since $[\pi] \simeq [\pi]^*$, there is then a preferred vector corresponding to the identity map. These vectors are Littlewood's *immanants*, of which the determinant and permanent are special cases.

Consider $Z(\mathrm{perm}_n)_{sing} \subset \mathbb{P}(E \otimes F)^*$. It consists of the matrices all of whose size $n-1$ submatrices have zero permanent. (To see this, note the permanent has Laplace type expansions.) This seems even more complicated than the hypersurface $Z(\mathrm{perm}_n)$ itself. Continuing, $Z(\mathrm{perm}_n)_{verysing,k}$ consists of the matrices all of whose sub-matrices of size $n-k$ have zero permanent. In particular $Z(\mathrm{perm}_n)_{verysing,n-2}$ is defined by quadratic equations. Its zero set has many components, but each component is easy to describe:

Lemma 6.16 *Let A be an $n \times n$ matrix all of whose size 2 submatrices have zero permanent. Then one of the following hold:*

(1) all the entries of A are zero except those in a single size 2 submatrix, and that submatrix has zero permanent.
(2) all the entries of A are zero except those in the j-th row for some j. Call the associated component C^j.
(3) all the entries of A are zero except those in the j-th column for some j. Call the associated component C_j.

The proof is straight-forward. Take a matrix with entries that don't fit that pattern, e.g., one that begins

$$\begin{matrix} a & b & e \\ * & d & * \end{matrix}$$

and note that it is not possible to fill in the two unknown entries and have all size two sub-permanents, even in this corner, zero. There are just a few such cases since we are free to act by $\mathfrak{S}_n \times \mathfrak{S}_n$.

Proof of Theorem 6.14 (I follow [78].) Any linear transformation preserving the permanent must send a component of $Z(\mathrm{perm}_n)_{verysing,n-2}$ of type (1) to another of type (1). It must send a component C^j either to some C^k or some C_i. But if $i \neq j, C^j \cap C^i = 0$ and for all $i, j, \dim(C^i \cap C_j) = 1$. Since intersections must be mapped to intersections, either all components C^i are sent to components C_k or all are permuted among themselves. By composing with an element of $\mathbb{Z}_2$, we may assume all the C^i's are sent to C^i's and the C_j's are sent to C_j's. Similarly, by composing with an element of $\mathfrak{S}_n \times \mathfrak{S}_n$ we may assume each C_i and C^j is sent to itself. But then their intersections are sent to themselves. So we have, for all i, j,

$$(x^i_j) \mapsto (\lambda^i_j x^i_j) \tag{15}$$

for some λ^i_j and there is no summation in the expression. Consider the image of a size 2 submatrix, e.g.,

$$\begin{matrix} x^1_1 & x^1_2 \\ x^2_1 & x^2_2 \end{matrix} \mapsto \begin{matrix} \lambda^1_1 x^1_1 & \lambda^1_2 x^1_2 \\ \lambda^2_1 x^2_1 & \lambda^2_2 x^2_2 \end{matrix}. \tag{16}$$

In order that the map (15) be in G_{perm_n}, when $(x^i_j) \in Z(\mathrm{perm}_n)_{verysing,n-2}$, the permanent of the matrix on the right hand side of (16) must be zero. The permanent of the right hand side of (16) when $(x^i_j) \in Z(\mathrm{perm}_n)_{verysing,n-2}$ is $\lambda^1_1\lambda^2_2 x^1_1 x^2_2 + \lambda^2_1\lambda^1_2 x^1_2 x^2_1 = x^1_1 x^2_2(\lambda^1_1\lambda^2_2 - \lambda^2_1\lambda^1_2)$ which implies $\lambda^1_1\lambda^2_2 - \lambda^1_2\lambda^2_1 = 0$, thus all the 2×2 minors of the matrix (λ^i_j) are zero, so it has rank one and is the product of a column vector and a row vector, but then it is an element of $T_E \times T_F$. □

6.8 Iterated matrix multiplication

Let $IMM^k_n \in S^n(\mathbb{C}^{k^2 n})$ denote the iterated matrix multiplication operator for $k \times k$ matrices, $(X_1, \ldots, X_n) \mapsto \mathrm{trace}(X_1 \cdots X_n)$. Letting $V_j = \mathbb{C}^k$, invariantly

$$\begin{aligned} IMM^k_n = Id_{V_1} \otimes \cdots \otimes Id_{V_n} &\in (V_1 \otimes V_2^*) \otimes (V_2 \otimes V_3^*) \otimes \cdots \otimes (V_{n-1} \otimes V_n^*) \otimes (V_n \otimes V_1^*) \\ &\subset S^n((V_1 \otimes V_2^*) \oplus (V_2 \otimes V_3^*) \oplus \cdots \oplus (V_{n-1} \otimes V_n^*) \oplus (V_n \otimes V_1^*)), \end{aligned}$$

and the connected component of the identity of $G_{IMM^k_n} \subset GL(\mathbb{C}^{k^2 n})$ is clear.

Problem 6.17 Determine $G_{IMM_n^3}$.

The case of IMM_n^3 is important as this sequence is complete for the complexity class $\mathbf{VP}_e$, see Sect. 12. Moreover IMM_n^n is complete for the class $\mathbf{VP}_{ws}$.

Problem 6.18 Find equations in the ideal of $\overline{GL_{9n} \cdot IMM_n^3}$. Determine lower bounds for the inclusions $\mathcal{P}erm_n^m \subset \overline{GL_{9n} \cdot IMM_n^3}$ and $\mathcal{D}et_n^m \subset \overline{GL_{9n} \cdot IMM_n^3}$.

7 The Chow variety

If one specializes the determinant or permanent to diagonal matrices and takes the orbit closure, one obtains the Chow variety defined in Sect. 2.2. Thus $I(\mathcal{D}et_n) \subset I(Ch_n(W))$. The ideal of the Chow variety has been studied for some time, dating back at least to Gordan and Hadamard. The history is rife with rediscoveries and errors that only make the subject more intriguing.

The secant varieties of the Chow variety are also important for the study of depth 3 circuits, as described in Sect. 8. It is easy to see that $\sigma_2(Ch_n(W)) \subset \mathcal{D}et_n$, and a consequence of the equations described in Sect. 4.2 is that $\sigma_3(Ch_n(W))$ is *not* contained in $\mathcal{D}et_n$. I do not know if it is contained in $\mathcal{P}erm_n^n$.

Problem 7.1 Determine if $\sigma_3(Ch_n(W)) \subset \mathcal{P}erm_n^n$.

Problem 7.2 Determine equations in the ideal of $\sigma_2(Ch_n(W))$. Which modules are in the ideal of $\mathcal{D}et_n$?

7.1 History

A map, which, following a suggestion of A. Abdessalem, I now call the *Hermite-Hadamard-Howe* map, $h_{d,n} : S^d(S^n W) \to S^n(S^d W)$ was defined by Hermite [35] when $\dim W = 2$, and Hermite proved the map is an isomorphism in this case. His celebrated reciprocity theorem (Theorem 7.22) is this isomorphism. Hadamard [33] defined the map in general and observed that its kernel is $I_d(Ch_n(W^*))$, the degree d component of the ideal of the Chow variety (see Sect. 7.2). Originally he mistakenly thought the map was always of maximal rank, but in [34] he proved the map is an isomorphism when $d = n = 3$ and posed determining if injectivity holds in general when $d \leq n$ as a open problem. (Injectivity for $d \leq n$ is equivalent to surjectivity when $d \geq n$, see Exercise 7.5.) Brill wrote down set-theoretic equations for the Chow variety of degree $n+1$, via a map that I denote $Brill : S_{n,n} W \otimes S^{n^2-n} W \to S^{n+1}(S^n W)$, see [27] or [45]. There was a gap in Brill's argument, that was repeated in [27] and finally fixed by E. Briand in [7]. The map $h_{d,n}$ was rediscovered by Howe in [36] where he also wrote "it is reasonable to expect" that $h_{d,n}$ is always of maximal rank. This reasonable expectation dating back to Hadamard has become known as the "Foulkes-Howe conjecture". Howe had been investigating a conjecture of Foulkes [21] that for $d > n$, the irreducible modules counted with multiplicity occurring in $S^n(S^d W)$ also occur in $S^d(S^n W)$. Howe's conjecture is now known to be false, and Foulkes' original conjecture is still open. An asymptotic version of Foulke's conjecture was proved by Manivel [51], and asymptotic versions of Howe's conjecture by Brion [8,9] as discussed below. The proof that Howe's conjecture is false follows from a computer calculation of Müller and Neunhöffer [58] related to the symmetric group. A. Abdessalem realized their computation showed the map $h_{5,5}$ is not injective. (In [58] they mistakenly say the result comes from [6] rather than their own paper.) This computation was mysterious, in particular, the modules in the kernel were not determined. As part of an AMS Mathematics Research Community in June 2012 and follow-up to it, C. Ikenmeyer and S. Mkrtchyan determined the modules in the kernel explicitly. In particular the kernel does not consist of isotypic components. In his PhD thesis [6], Briand announced a proof that if $h_{d,n}$ is surjective, then $h_{d+1,n}$ is also surjective. Then A. Abdesselam found a gap in Briand's argument. Fortunately this result follows from results of T. McKay [56], see Sect. 7.4. Brion [8,9], and independently Weyman and Zelevinsky (unpublished) proved that the Foulkes-Howe conjecture is true asymptotically (see Corollary 7.18), with Brion giving an explicit, but very large bound for d in terms of n and $\dim W$, see Equation (19).

Problem 7.3 What is the kernel of $Brill : S_{n,n}W \otimes S^{n^2-n}W \to S^{n+1}(S^nW)$?

7.2 The ideal of the Chow variety

Consider the map $h_{d,n} : S^d(S^nW) \to S^n(S^dW)$ defined as follows: First include $S^d(S^nW) \subset W^{\otimes nd}$. Next, regroup the copies of W and symmetrize the blocks to $(S^dW)^{\otimes n}$. Finally, thinking of S^dW as a single vector space, symmetrize again.

For example, putting subscripts on W to indicate position:

$$\begin{aligned} S^2(S^3W) \subset W^{\otimes 6} &= W_1 \otimes W_2 \otimes W_3 \otimes W_4 \otimes W_5 \otimes W_6 \\ &= (W_1 \otimes W_4) \otimes (W_2 \otimes W_5) \otimes (W_3 \otimes W_6) \\ &\to S^2W \otimes S^2W \otimes S^2W \\ &\to S^3(S^2W) \end{aligned}$$

Note that $h_{d,n}$ is a linear map, in fact a $GL(W)$-module map.

Exercise 7.4 Show that $h_{d,n}(x_1^n \cdots x_d^n) = (x_1 \cdots x_d)^n$.

Note that the definition of $h_{d,n}$ depends on one's conventions for symmetrization (whether or not to divide by a constant). Take the definition of $h_{d,n}$ so that this exercise is true.

Exercise 7.5 Show that $h_{d,n} : S^d(S^nV) \to S^n(S^dV)$ is "self-dual" in the sense that $h_{d,n}^T = h_{n,d} : S^n(S^dV^*) \to S^d(S^nV^*)$. Conclude that $h_{d,n}$ surjective if and only if $h_{n,d}$ is injective.

Proposition 7.6 [33] $\ker h_{d,n} = I_d(Ch_n(W^*))$.

Proof Say $P = \sum_j x_{1j}^n \cdots x_{dj}^n$. Let $\ell^1, \ldots, \ell^n \in W^*$.

$$\begin{aligned}P(\ell^1\cdots\ell^n) &= \langle \overline{P}, (\ell^1\cdots\ell^n)^d\rangle\\ &= \sum_j \langle x^n_{1j}\cdots x^n_{dj}, (\ell^1\cdots\ell^n)^d\rangle\\ &= \sum_j \langle x^n_{1j}, (\ell^1\cdots\ell^n)\rangle\cdots\langle x^n_{dj}, (\ell^1\cdots\ell^n)\rangle\\ &= \sum_j \Pi^n_{s=1}\Pi^d_{i=1} x_{ij}(\ell_s)\\ &= \sum_j \langle x_{1j}\cdots x_{dj}, (\ell^1)^d\rangle\cdots\langle x_{1j}\cdots x_{dj}, (\ell^n)^d\rangle\\ &= \langle h_{d,n}(P), (\ell^1)^d\cdots(\ell^n)^d\rangle\end{aligned}$$

If $h_{d,n}(P)$ is nonzero, there will be some monomial it will pair with to be nonzero. On the other hand, if $h_{d,n}(P) = 0$, then P annihilates all points of $Ch_n(W^*)$. $\square$

Exercise 7.7 Show that if $h_{d,n} : S^d(S^n\mathbb{C}^m) \to S^n(S^d\mathbb{C}^m)$ is not surjective, then $h_{d,n} : S^d(S^n\mathbb{C}^k) \to S^n(S^d\mathbb{C}^k)$ is not surjective for all $k > m$, and that the partitions describing the kernel are the same in both cases if $d \le m$.

Exercise 7.8 Show that if $h_{d,n} : S^d(S^n\mathbb{C}^m) \to S^n(S^d\mathbb{C}^m)$ is surjective, then $h_{d,n}$: $S^d(S^n\mathbb{C}^k) \to S^n(S^d\mathbb{C}^k)$ is surjective for all $k < m$.

Proposition 7.9 (Ikenmeyer, Mkrtchyan)

(1) The kernel of $h_{5,5} : S^5(S^5\mathbb{C}^5) \to S^5(S^5\mathbb{C}^5)$ consists of irreducible modules corresponding to the following partitions:

$$\{(14, 7, 2, 2), (13, 7, 2, 2, 1), (12, 7, 3, 2, 1), (12, 6, 3, 2, 2),\\ (12, 5, 4, 3, 1), (11, 5, 4, 4, 1), (10, 8, 4, 2, 1), (9, 7, 6, 3)\}.$$

All these occur with multiplicity one in the kernel, but not all occur with multiplicity one in $S^5(S^5\mathbb{C}^5)$, so in particular, the kernel is not an isotypic component.

(2) The kernel of $h_{6,6} : S^6(S^6\mathbb{C}^6) \to S^6(S^6\mathbb{C}^6)$ contains, with high probability, a module corresponding to the partition $(20, 7, 6, 1, 1, 1)$.

The phrase "with high probability" means the result was obtained numerically, not symbolically.

7.3 Multi-symmetric function formulation

Given any $GL(V)$-module map $f : U \to W$, where U, W are modules with support in the root lattice of $GL(V)$, e.g., $U, W \subset V^{\otimes a\mathbf{v}}$ for some $a \in \mathbb{Z}_{>0}$, the injectivity (or surjectivity) of f is equivalent to the injectivity (or surjectivity) of f restricted to the $\mathfrak{sl}$-weight zero subspaces $f|_0 : U_0 \to W_0$, that is the subspaces of $GL(V)$-weight $(a^{\mathbf{v}})$. On these subspaces the Weyl group $\mathfrak{S}_{\mathbf{v}}$ acts, and so the assertion about a $GL(V)$-module map can be converted to an assertion about an $\mathfrak{S}_{\mathbf{v}}$-module map, and vice-versa. This was Briand's approach in [6].

7.4 $\mathfrak{S}_{dn}$-formulation

Foulke's conjecture has been well-studied in the combinatorics literature in the following form: One compares the multiplicities of the $\mathfrak{S}_{dn}$-module induced from the trivial representation of $\mathfrak{S}_d^{\times n}$ with the $\mathfrak{S}_{dn}$-module induced from the trivial representation of $\mathfrak{S}_n^{\times d}$. Moreover there is an explicit map between these modules whose kernel in terms of $\mathfrak{S}_{dn}$-modules corresponds to the kernel of $h_{d,n}$ as long as the dimension of V is sufficiently large, as this map between $\mathfrak{S}_{dn}$-modules is just $h_{d,n}$ restricted to the $\mathfrak{sl}$-weight zero subspace. Some results, such as Hermite reciprocity 7.22 and Exercises 7.7, 7.8 are less transparent from this perspective and are the subject of research articles in combinatorics. However I do not know of a "GL"- proof of the following Theorem of T. McKay [56]:

Theorem 7.10 [56, Thm. 8.1] *If $h_{d,n}$ is surjective, then $h_{d',n}$ is surjective for all $d' > d$. In other words, if $h_{n,d}$ is injective, then $h_{n,d'}$ is injective for all $d' > d$.*

The two statements are equivalent by Exercise 7.5.

7.5 Coordinate ring of the orbit

Recall from Sect. 2.2, that if $\dim W \geq n$, then $\hat{C}h_n(W) = \overline{GL(W) \cdot x_1 \cdots x_n}$. Assume $\dim W = n$, then $G_{x_1 \cdots x_n} = T_n^{SL} \rtimes \mathfrak{S}_n =: \Gamma_n$. By the algebraic Peter-Weyl theorem 3.1,

$$\mathbb{C}[GL(W) \cdot (x_1 \cdots x_n)] = \bigoplus_{\ell(\pi) \leq n} (S_\pi W^*)^{\oplus \dim (S_\pi W)^{\Gamma_n}},$$

where $\pi = (p_1, \ldots, p_n)$ is such that $p_1 \geq p_2 \geq \cdots \geq p_n$ and $p_j \in \mathbb{Z}$. We are only interested in those π that are partitions, i.e., where $p_n \geq 0$, as only those could occur in the coordinate ring of the orbit closure. Define the GL-degree of a module $S_\pi W$ to be $p_1 + \cdots + p_n$ and for a $GL(W)$-module M, define M_{poly} to be the sum of the isotypic components of the $S_\pi W$ in M with π a partition. The space of T^{SL} invariants is the $\mathfrak{sl}(W)$-weight zero space, so we need to compute $(S_\pi W)_0^{\mathfrak{S}_n}$. By Corollary 3.8(ii) this is $\mathrm{mult}(S_\pi W, S^n(S^s W))$. If we consider all the π's together, we conclude

$$\mathbb{C}[GL(W) \cdot (x_1 \cdots x_n)]_{poly} = \oplus_s S^n(S^s W^*).$$

In particular, $\oplus_s S^n(S^s W^*)$ inherits a ring structure.

7.6 Coordinate ring of the normalization

In this section I follow [8]. There is another variety whose coordinate ring is as computable as the coordinate ring of the orbit, the normalization of the Chow variety. We work in affine space.

An affine variety Z is *normal* if $\mathbb{C}[Z]$ is integrally closed, that is if every element of $\mathbb{C}(Z)$, the field of fractions of $\mathbb{C}[Z]$, that is integral over $\mathbb{C}[Z]$ (i.e., that satisfies a monic polynomial with coefficients in $\mathbb{C}[Z]$) is in $\mathbb{C}[Z]$. To every affine variety Z one may associate a unique normal affine variety $Nor(Z)$, called the *normalization* of Z, such that there is a finite map $Nor(Z) \to Z$ (i.e. $\mathbb{C}[Nor(Z)]$ is integral over $\mathbb{C}[Z]$) that is generically one to one, in particular it is one to one over the smooth points of Z. For details see [69, Chap II.5].

In particular, there is an inclusion $\mathbb{C}[Z] \to \mathbb{C}[Nor(Z)]$. If the non-normal points of Z form a finite set, then the cokernel is finite dimensional. If Z is a G-variety, then $Nor(Z)$ will be too.

Recall $Ch_n(W)$ is the projection of the Segre variety, but since we want to deal with affine varieties, we will deal with the cone over it. So instead consider the product map

$$\begin{aligned} \phi_n : W^{\times n} &\to S^n W \\ (u_1, \dots, u_n) &\mapsto u_1 \cdots u_n \end{aligned}$$

Note that i) the image of ϕ_n is $\hat{C}h_n(W)$, ii) ϕ_n is $\Gamma_n = T_W \ltimes \mathfrak{S}_n$ equivariant.

For any affine algebraic group Γ and any Γ-variety Z, one can define the *GIT quotient* $Z//\Gamma$ which by definition is the affine algebraic variety whose coordinate ring is $\mathbb{C}[Z]^\Gamma$. (When Γ is finite, this is just the usual set-theoretic quotient. In the general case, Γ-orbits will be identified in the quotient when there are no Γ-invariant regular functions that can distinguish them.) If Z is normal, then so is $Z//\Gamma$ (see, e.g. [16, Prop 3.1]). In our case $W^{\times n}$ is an affine Γ_n-variety and ϕ_n factors through the GIT quotient because it is Γ_n-equivariant, so we obtain a map

$$\psi_n : W^{\times n}//\Gamma_n \to S^n W$$

whose image is still $\hat{C}h_n(W)$. Also note that by unique factorization, ψ_n is generically one to one. (Elements of $W^{\times n}$ of the form $(0, u_2, \dots, u_n)$ cannot be distinguished from $(0, \dots, 0)$ by Γ_n invariant functions, so they are identified with $(0, \dots, 0)$ in the quotient, which is consistent with the fact that $\phi_n(0, u_2, \dots, u_n) = 0$.) Observe that ϕ_n and ψ_n are $GL(W) = SL(W) \times \mathbb{C}^*$ equivariant.

Consider the induced map on coordinate rings:

$$\psi_n^* : \mathbb{C}[S^n W] \to \mathbb{C}[W^{\times n}//\Gamma_n] = \mathbb{C}[W^{\times n}]^{\Gamma_n}.$$

Recall that for affine varieties, $\mathbb{C}[Y \times Z] = \mathbb{C}[Y] \otimes \mathbb{C}[Z]$, so

$$\begin{aligned} \mathbb{C}[W^{\times n}] &= \mathbb{C}[W]^{\otimes n} \\ &= Sym(W^*) \otimes \cdots \otimes Sym(W^*) \\ &= \bigoplus_{i_1, \dots, i_n \in \mathbb{Z}_{\geq 0}} S^{i_1} W^* \otimes \cdots \otimes S^{i_n} W^*. \end{aligned}$$

Taking torus invariants gives

$$\mathbb{C}[W^{\times n}]^{T_n^{SL}} = \bigoplus_i S^i W^* \otimes \cdots \otimes S^i W^*,$$

and finally

$$(\mathbb{C}[W^{\times n}]^{T_n^{SL}})^{\mathfrak{S}_n} = S^n(S^i W^*).$$

In summary,

$$\psi_n^* : Sym(S^n W^*) \to \oplus_i (S^n(S^i W^*)),$$

and this map respects GL-degree, so it gives rise to maps $\tilde{h}_{d,n} : S^d(S^n W^*) \to S^n(S^d W^*)$.

Proposition 7.11 $\tilde{h}_{d,n} = h_{d,n}$.

Proof Since elements of the form $x_1^n \cdots x_d^n$ span $S^d(S^n W)$ it will be sufficient to prove the maps agree on such elements. By Exercise 7.4, $h_{d,n}(x_1^n \cdots x_d^n) = (x_1 \cdots x_d)^n$. On the other hand, in the algebra $\mathbb{C}[W]^{\otimes n}$, the multiplication is $(f_1 \otimes \cdots \otimes f_n) \circledcirc (g_1 \otimes \cdots \otimes g_n) = f_1 g_1 \otimes \cdots \otimes f_n g_n$ and this descends to the algebra $(\mathbb{C}[W]^{\otimes n})^{\Gamma_n}$ which is the target of the algebra map ψ_n^*, i.e.,

$$\begin{aligned}\tilde{h}_{d,n}(x_1^n \cdots x_d^n) &= \psi_n^*(x_1^n \cdots x_d^n)\\ &= \psi_n^*(x_1^n) \circledcirc \cdots \circledcirc \psi_n^*(x_d^n)\\ &= x_1^n \circledcirc \cdots \circledcirc x_d^n\\ &= (x_1 \cdots x_d)^n.\end{aligned}$$

□

Proposition 7.12 $\psi_n : W^{\times n} // \Gamma_n \to \hat{C}h_n(W)$ *is the normalization of* $\hat{C}h_n(W)$.

Recall (see, e.g. [69, p. 61]) that a regular (see, e.g. [69, p.27] for the definition of regular) map between affine varieties $f : X \to Y$ such that $f(X)$ is dense in Y is *finite* if $\mathbb{C}[X]$ is integral over $\mathbb{C}[Y]$. To prove the proposition, we will need a lemma:

Lemma 7.13 *Let* X, Y *be affine varieties equipped with polynomial* $\mathbb{C}^*$*-actions with unique fixed points* $0_X \in X, 0_Y \in Y$*, and let* $f : X \to Y$ *be a* $\mathbb{C}^*$*-equivariant morphism such that as sets,* $f^{-1}(0_Y) = \{0_X\}$*. Then* f *is finite.*

Proof of Proposition 7.12 Since $W^{\times n} // \Gamma_n$ is normal and ψ_n is regular and generically one to one, it just remains to show ψ_n is finite.

Write $[0] = [0, \ldots, 0]$. To show finiteness, by Lemma 7.13, it is sufficient to show ${\psi_n}^{-1}(0) = [0]$ as a set, as $[0]$ is the unique $\mathbb{C}^*$ fixed point in $W^{\times n} // \Gamma_n$, and every $\mathbb{C}^*$ orbit closure contains $[0]$. Now $u_1 \cdots u_n = 0$ if and only if some $u_j = 0$, say $u_1 = 0$. The T-orbit closure of $(0, u_2, \ldots, u_n)$ contains the origin so $[0, u_2, \ldots, u_n] = [0]$.

□

Proof of Lemma 7.13 $\mathbb{C}[X]$, $\mathbb{C}[Y]$ are $\mathbb{Z}_{\geq 0}$-graded, and the hypothesis $f^{-1}(0_Y) = \{0_X\}$ states that $\mathbb{C}[X]/f^*(\mathbb{C}[Y]_{>0})\mathbb{C}[X]$ is a finite dimensional vector space. We want to show that $\mathbb{C}[X]$ is integral over $\mathbb{C}[Y]$. This is a graded version of Nakayama's Lemma (the algebraic implicit function theorem). $\square$

In more detail (see, e.g. [43, Lemmas 3.1,3.2], or [19, p136, Ex. 4.6a]):

Lemma 7.14 *Let R, S be $\mathbb{Z}_{\geq 0}$-graded, finitely generated domains over $\mathbb{C}$ such that $R_0 = S_0 = \mathbb{C}$, and let $f^* : R \to S$ be an injective graded algebra homomorphism. If $f^{-1}(R_{>0}) = \{S_{>0}\}$ as sets, where $f : Spec(S) \to Spec(R)$ is the induced map on the associated schemes, then S is a finitely generated R-module. In particular, it is integral over R.*

Proof The hypotheses on the sets says that $S_{>0}$ is the only maximal ideal of S containing the ideal $\mathfrak{m}$ generated by $f^*(R_{>0})$, so the radical of $\mathfrak{m}$ must equal $S_{>0}$, and in particular $S^d_{>0}$ must be contained in it for all $d > d_0$, for some d_0. So $S/\mathfrak{m}$ is a finite dimensional vector space, and by the next lemma, S is a finitely generated R-module. $\square$

Lemma 7.15 *Let S be as above, and let M be a $\mathbb{Z}_{\geq 0}$-graded S-module. Assume $M/(S_{>0} \cdot M)$ is a finite dimensional vector space over $S/S_{>0} \simeq \mathbb{C}$. Then M is a finitely generated S-module.*

Proof Choose a set of homogeneous generators $\{\overline{x}_1, \ldots, \overline{x}_n\} \subset M/(S_{>0} \cdot M)$ and let $x_j \in M$ be a homogeneous lift of $\overline{x}_j$. Let $N \subset M$ be the graded S-submodule $Sx_1 + \cdots + Sx_n$. Then $M = S_{>0}M + N$, as let $a \in M$, consider $\overline{a} \in M/(S_{>0}M)$ and lift it to some $b \in N$, so $a - b \in S_{>0}M$, and $a = (a - b) + b$. Now quotient by N to obtain

$$S_{>0} \cdot (M/N) = M/N. \tag{17}$$

If $M/N \neq 0$, let d_0 be the smallest degree such that $(M/N)^{d_0} \neq 0$. But $S_{>0} \cdot (M/N)^{\geq d_0} \subset (M/N)^{\geq d_0+1}$ so there is no way to obtain $(M/N)^{d_0}$ on the right hand side. Contradiction. $\square$

Remark 7.16 The referee points out that one can avoid the use of the graded Nakayama lemma in the proof of Lemma 7.13 by first observing that the map is finite to one as a set map, which is clear, and then using Zariski's main theorem in the form that a quasi-finite morphism between affine varieties is open in a finite morphism. Then the assumption that there is only one fixed point implies that the open immersion is an isomorphism.

Theorem 7.17 [8] *For all $n \geq 1$, ψ_n restricts to a map*

$$\psi_n^o : (W^{\times n}//\,\Gamma_n)\backslash[0] \to S^n W\backslash 0 \tag{18}$$

such that $\psi_n^{o*} : \mathbb{C}[S^n W\backslash 0] \to \mathbb{C}[(W^{\times n}//\,\Gamma_n)\backslash[0]]$ is surjective.

Corollary 7.18 [8] *The Hermite-Hadamard-Howe map*

$$h_{d,n} : S^d(S^n W^*) \to S^n(S^d W^*)$$

is surjective for d sufficiently large.

Proof of Corollary Theorem 7.17 implies $(\psi_n^*)_d$ is surjective for d sufficiently large, because the cokernel of ψ_n^* is supported at a point and thus must vanish in large degree. □

The proof of Theorem 7.17 will give a second proof that the kernel of ψ_n^* is indeed the ideal of $Ch_n(W)$.

Proof of Theorem Since ψ_n is $\mathbb{C}^*$-equivariant, we can consider the quotient to projective space

$$\underline{\psi}_n : ((W^{\times n}//\Gamma_n)\backslash[0])/\mathbb{C}^* \to (S^n W\backslash 0)/\mathbb{C}^* = \mathbb{P}S^n W$$

and show that $\underline{\psi}_n^*$ is surjective. Note that $((W^{\times n}//\Gamma_n)\backslash[0])/\mathbb{C}^*$ is $GL(V)$-isomorphic to $(\mathbb{P}W)^{\times n}/\mathfrak{S}_n$, as

$$(W^{\times n}//\Gamma_n)\backslash[0] = (W\backslash 0)^{\times n}/\Gamma_n$$

and $\Gamma_n \times \mathbb{C}^* = (\mathbb{C}^*)^{\times n} \rtimes \mathfrak{S}_n$. So we have

$$\underline{\psi}_n : (\mathbb{P}W)^{\times n}/\mathfrak{S}_n \to \mathbb{P}S^n W.$$

It will be sufficient to show $\underline{\psi}_n^*$ is surjective on affine open subsets that cover the spaces. Let $w_1, \ldots, w_{\mathbf{w}}$ be a basis of W and consider the affine open subset of $\mathbb{P}W$ given by elements where the coordinate on w_1 is nonzero, and the corresponding induced affine open subsets of $(\mathbb{P}W)^{\times n}$ and $\mathbb{P}S^n W$, call these $(\mathbb{P}W)_1^{\times n}$ and $(\mathbb{P}S^n W)_1$. We will show that the algebra of $\mathfrak{S}_n$-invariant functions on $(\mathbb{P}W)_1^{\times n}$ is in the image of $(\mathbb{P}S^n W)_1$. The restriction of the quotient by $\mathfrak{S}_n$ of $(\mathbb{P}W)^{\times n}$ composed with $\underline{\psi}_n$ to these open subsets in coordinates is just

$$\left(w_0 + \sum_{s=2}^{\mathbf{w}} x_s^i w_s\right)_{1\le i\le n} \mapsto \Pi_{i=1}^n \left(w_0 + \sum_{s=2}^{\mathbf{w}} x_s^i w_s\right).$$

Finally, by e.g., [77, §II.3], the coordinates on the right hand side generate the algebra of $\mathfrak{S}_n$-invariant functions in the n sets of variables $(x_s^i)_{i=1,\ldots,n}$. □

With more work, in [9, Thm 3.3], Brion obtains an explicit (but enormous) function $d_0(n, \mathbf{w})$ which is

$$d_0(n,\mathbf{w}) = (n-1)(\mathbf{w}-1)((n-1)\left\lfloor \frac{\binom{n+\mathbf{w}-1}{\mathbf{w}-1}}{\mathbf{w}} \right\rfloor - n) \tag{19}$$

for which the $h_{d,n}$ is surjective for all $d > d_0$ where $\dim W = \mathbf{w}$.

Problem 7.19 Improve Brion's bound to say, a polynomial bound in n when $n = \mathbf{w}$.

Problem 7.20 Note that $\mathbb{C}[Nor(Ch_n(W))] = \mathbb{C}[GL(W) \cdot (x_1 \cdots x_n)]_{\geq 0}$ and that the the boundary of the orbit closure is irreducible. Is it true that whenever a $GL(W)$-orbit closure with reductive stabilizer has an irreducible boundary, that the coordinate ring of the normalization of the orbit closure equals the positive part of the coordinate ring of the orbit?

Remark 7.21 An early use of geometry in the study of plethysm was in [76] where J. Wahl used his *Gaussian maps* (local differential geometry) to study the decomposition of tensor products of representations of reductive groups. Then in [51], Manivel used these maps to determine "stable" multiplicities in $S^d(S^n W)$, where one fixes either d or n and allows the other to grow. Brion then developed more algebraic versions of these techniques to obtain the results above.

7.7 The case $\dim W = 2$

When $\dim W = 2$, every polynomial decomposes as a product of linear factors, so the ideal of $Ch_n(\mathbb{C}^2)$ is zero. We recover the following theorem of Hermite:

Theorem 7.22 (Hermite reciprocity) *The map $h_{d,n} : S^d(S^n\mathbb{C}^2) \to S^n(S^d\mathbb{C}^2)$ is an isomorphism for all d, n. In particular $S^d(S^n\mathbb{C}^2)$ and $S^n(S^d\mathbb{C}^2)$ are isomorphic GL_2-modules.*

Often in modern textbooks only the "In particular" is stated.

7.8 The case $d = n = 3$

Theorem 7.23 (Hadamard [34]) *The map $h_{3,3} : S^3(S^3\mathbb{C}^n) \to S^3(S^3\mathbb{C}^n)$ is an isomorphism.*

Proof Without loss of generality, assume $n = 3$ and x_1, x_2, x_3 are independent. Say we had $P \in I_3(Ch_3(\mathbb{C}^3))$. Consider $P(\mu(x_1^3 + x_2^3 + x_3^3) - \lambda x_1x_2x_3)$ as a cubic polynomial on $\mathbb{P}^1$ with coordinates $[\mu, \lambda]$. Note that it vanishes at the four points $[0, 1], [1, 3], [1, 3\omega], [1, 3\omega^2]$ where ω is a primitive third root of unity. Thus it must vanish identically on the $\mathbb{P}^1$, in particular, at $[1, 0]$, i.e., on $x_1^3 + x_2^3 + x_3^3$. Hence it must vanish identically on $\sigma_3(v_3(\mathbb{P}^2))$. But $\sigma_3(v_3(\mathbb{P}^2)) \subset \mathbb{P}S^3\mathbb{C}^3$ is a hypersurface of degree four. A cubic polynomial vanishing on a hypersurface of degree four is identically zero. □

Remark 7.24 The above proof is due to A. Abdesselam (personal communication). It is a variant of Hadamard's original proof, where instead of $x_1^3 + x_2^3 + x_3^3$ one uses an arbitrary cubic f, and generalizing $x_1x_2x_3$ one uses the Hessian $H(f)$. Then the curves $f = 0$ and $H(f) = 0$ intersect in 9 points (the nine flexes of $f = 0$) and there are four groups of three lines going through these points, i.e. four places where the polynomial becomes a product of linear forms.

7.9 The Chow variety and a conjecture in combinatorics

From Exercise 3.9, the trivial SL_n-module $S_{n^n}\mathbb{C}^n$ occurs in $S^n(S^n\mathbb{C}^n)$ with multiplicity one when n is even and zero when n is odd.

Conjecture 7.25 (Kumar [42]) *Let n be even, then for all $i \leq n$, $S_{n^i}\mathbb{C}^n \subset \mathbb{C}[Ch_n(\mathbb{C}^n)]$.*

It is not hard to see that the $i = n$ case implies the others. Adopt the notation that if $\pi = (p_1, \dots, p_k)$, then $m\pi = (mp_1, \dots, mp_k)$. By taking Cartan products in the coordinate ring, the conjecture would imply:

Conjecture 7.26 (Kumar [42]) *For all partitions π with $\ell(\pi) \leq n$, the module $S_{n\pi}\mathbb{C}^n$ occurs in $\mathbb{C}[Ch_n(\mathbb{C}^n)]$. In particular, $S_{n\pi}\mathbb{C}^{n^2}$ occurs in $\mathbb{C}[\mathcal{D}et_n]$ and $\mathbb{C}[\mathcal{P}erm_n^n]$.*

Conjecture 7.25 turns out to be related to a famous conjecture in combinatorics: an $n \times n$ matrix such that each row and column consists of the integers $\{1, \dots, n\}$ is called a *Latin square*. To each row and column one can associate an element $\sigma \in \mathfrak{S}_n$ based on the order the integers appear. Call the products of all the signs of these permutations the sign of the Latin square.

Conjecture 7.27 (Alon-Tarsi [2]) *Let n be even. The number of sign -1 Latin squares of size n is not equal to the number of sign $+1$ Latin squares of size n.*

In joint work, Kumar and I have shown:

Proposition 7.28 *Fix n even. The following are equivalent:*

(1) The Alon-Tarsi conjecture for n.
(2) Conjecture 7.25 for n with $i = n$.
(3) $\int_{g\in SU(n)} (\mathrm{perm}_n(g))^n d\mu \neq 0$, where $d\mu$ is Haar measure.
(4) Let $\mathbb{C}^{n^2}$ have coordinates x^i_j and the dual space coordinates y^i_j, then

$$\langle (\mathrm{perm}_n(y))^n, (\det_n(x))^n \rangle \neq 0$$

which may be thought of as a pairing between homogeneous polynomials of degree n^2 and homogeneous differential operators of order n^2.

The following two statements are equivalent and would imply the above are true:

(i) $\int_{g\in SU(n)} \Pi_{1\leq i,j\leq n} g^i_j d\mu \neq 0$, where $d\mu$ is Haar measure.
(ii) $\langle \Pi_{ij} y^i_j, \det_n(x)^n \rangle \neq 0$.

Currently the Alon-Tarsi conjecture is known to be true for $n = p \pm 1$, where p is a prime number [17,29].

To see the equivalence of (1) and (2), in [37] they showed that the Latin square conjecture is true for even n if and only if the "column sign" Latin square conjecture holds, where one instead computes the the products of the signs of the permutations of the columns. Then expression (10) gives the equivalence. The equivalence of (3) and (4) comes from the Peter-Weyl theorem and the equivalence of (2) and (3) from the fact that one can restrict to a maximal compact, and integration over the group picks out the trivial modules.

Problem 7.29 Find explicit modules that either are or are not in the kernel of the Hermite-Hadamard-Howe map. For example any module with at most two parts is clearly not in the kernel.

8 Secant varieties of the Chow variety and depth three circuits

Recently there has been substantial progress regarding shallow circuits. I first define a circuit, which is the model of computation generally used in algebraic complexity theory, and then I describe the varieties associated to shallow circuits as well as recent results and conjectures regarding shallow circuits in geometric language.

Definition 8.1 An *arithmetic circuit* $\mathcal{C}$ is a finite, acyclic, directed graph with vertices of in-degree 0 or 2 and exactly one vertex of out-degree 0. The vertices of in-degree 0 are labeled by elements of $\mathbb{C} \cup \{x_1, \dots, x_n\}$, and called *inputs*. Those of in-degree 2 are labeled with $+$ or $*$ and are called *gates*. If the out-degree of v is 0, then v is called an *output gate*. The *size* of $\mathcal{C}$ is the number of edges. From a circuit $\mathcal{C}$, one can construct a polynomial $p_{\mathcal{C}}$ in the variables $x_1, \dots, x_n$ (Fig. 2).

Exercise 8.2 Show that if one instead uses the number of gates to define the size, the asymptotic size estimates are the same. (Size is sometimes defined as the number of gates.)

To each vertex v of a circuit $\mathcal{C}$ we associate the polynomial that is computed at v, which will be denoted $\mathcal{C}_v$. In particular the polynomial associated with the output gate is the polynomial computed by $\mathcal{C}$. The *depth* of $\mathcal{C}$ is the length of (i.e., the number of

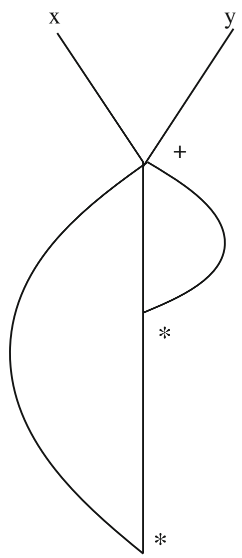

Fig. 2 Circuit for $(x + y)^3$

edges in) the longest path in $\mathcal{C}$ from an input to an output. If a circuit has small depth, the polynomial it computes can be computed quickly in parallel.

The *formula size* of f is the smallest tree circuit computing f. Tree circuits are called *formulas*.

Circuits of bounded depth (called *shallow circuits*) are used to study the complexity of calculations done in parallel. When one studies circuits of bounded depth, one must allow gates to have an arbitrary number of edges coming in to them ("unbounded fanin"). For such circuits, multiplication by constants is considered "free."

There is a substantial literature dedicated to showing that given any circuit computing a polynomial, there is a "slightly larger" shallow circuit that computes the same polynomial. Recently there have been significant advances for circuits of depths 3 [31] and 4 [1,40,73] and a special class of circuits of depth 5 [31]. The circuits of bounded depth that are trees have a nice variety associated to them which I now describe. In the literature they deal with inhomogeneous circuits, but, as I describe below (following a suggestion of K. Efremenko), this can be avoided, so we will deal exclusively with homogeneous circuits, that is, those computing homogeneous polynomials at each step along the way.

Following [44], for varieties $X \subset \mathbb{P}S^aW$ and $Y \subset \mathbb{P}S^bW$, defined the *multiplicative join* of X and Y, $MJ(X,Y) := \{[xy] \mid [x] \in X,\ [y] \in Y\} \subset \mathbb{P}S^{a+b}W$, and define $MJ(X_1,\dots,X_k)$ similarly. Let $\mu_k(X) = MJ(X_1,\dots,X_k)$ when all the $X_j = X$, which is a multiplicative analog of the secant variety. Note that $\mu_k(\mathbb{P}W) = Ch_k(W)$. The varieties associated to the polynomials computable by bounded depth formulas are of the form $\sigma_{r_k}(\mu_{d_{k-1}}(\sigma_{r_{k-2}}(\cdots\mu_{d_1}(\mathbb{P}W)\cdots)))$, and $\mu_{d_{k+1}}(\sigma_{r_k}(\mu_{d_{k-1}}(\sigma_{r_{k-2}}(\cdots\mu_{d_1}(\mathbb{P}W)\cdots))))$.

Remark 8.3 For those interested in circuits, note that if the first level consists of addition gates, this is "free" from the perspective of algebraic geometry, as since we are not choosing coordinates, linear combinations of basis vectors are not counted. More on this below.

Useful depth three circuits are always trees where the first level consists of additions, the second multiplications, and the third an addition that adds all the outputs of the second level together. Such are called $\Sigma\Pi\Sigma$ circuits.

A circuit is *homogeneous* if the polynomial produced by each gate is homogeneous, and otherwise it is *inhomogeneous*. The relation between secant varieties of Chow varieties and depth three circuits is as follows:

Proposition 8.4 *A polynomial $P \in S^nW$ in $\sigma_r^0(Ch_n(W))$ is computable by a homogeneous circuit of size $r+nr(1+\mathbf{w})$. If $P \notin \sigma_r^0(Ch_n(W))$, then P cannot be computed by a homogeneous circuit of size $n(r+1)+(r+1)$.*

Proof In the first case, $P = \sum_{j=1}^r (x_j^1\cdots x_j^n)$ for some $x_j^i \in W$. Expressed in terms of a fixed basis of W, each x_j^i is a linear combination of at worst $\mathbf{w}$ basis vectors, thus to create each one requires at worst $nr\mathbf{w}$ additions. Then to multiply them in groups of n is nr multiplications, and finally to add these together is r further additions. In the second case, at best P is in $\sigma_{r+1}^0(Ch_n(W))$, in which case, even if each of the x_j^i's is a basis vector (so no initial additions are needed), we still must perform $n(r+1)$ multiplications and $r+1$ additions. $\square$

I first explain why the computer science literature generally allows inhomogeneous depth three circuits, and then why one does not need to do so.

8.1 Why homogeneous depth three circuits do not appear useful at first glance

Using the flattening (see Sect. 2.2), $(\det_n)_{\lceil\frac{n}{2}\rceil,\lfloor\frac{n}{2}\rfloor} : S^{\lceil\frac{n}{2}\rceil}W \to S^{\lfloor\frac{n}{2}\rfloor}W$ and writing $W = E\otimes F = \mathbb{C}^n\otimes\mathbb{C}^n$, the image of this map is easily seen to be $\Lambda^{\lfloor\frac{n}{2}\rfloor}E\otimes\Lambda^{\lfloor\frac{n}{2}\rfloor}F$, the minors of size $\lfloor\frac{n}{2}\rfloor$. For the permanent one similarly gets sub-permanents. Thus, in the notation of Sect. 2.2,

$$\underline{\mathbf{R}}_S(\det_n) \geq \binom{n}{\lfloor\frac{n}{2}\rfloor}^2, \quad \underline{\mathbf{R}}_S(\mathrm{perm}_n) \geq \binom{n}{\lfloor\frac{n}{2}\rfloor}^2.$$

Recalling that $\binom{2m}{m} \sim \frac{4^m}{\sqrt{\pi m}}$, we have $[\det_n], [\mathrm{perm}_n] \notin \sigma_{O(\frac{4^n}{n})}v_n(\mathbb{P}W)$.

In [67] they showed

$$\mathbf{R}_S(x_1\cdots x_n) = 2^{n-1}. \tag{20}$$

The upper bound on $\mathbf{R}_S(x_1\cdots x_n)$ follows from the expression

$$x_1\cdots x_n = \frac{1}{2^{n-1}n!}\sum_{\epsilon\in\{-1,1\}^{n-1}}(x_1+\epsilon_1x_2+\cdots+\epsilon_{n-1}x_n)^n\epsilon_1\cdots\epsilon_{n-1}, \tag{21}$$

a sum with 2^{n-1} terms. (This expression dates at least back to [20].) In particular

$$\sigma_r(Ch_n(W)) \subset \sigma_{r2^n}(v_n(\mathbb{P}W)).$$

We conclude, for any constant C and n sufficiently large, that

$$\det_n \notin \sigma_{C\frac{2^n}{n}}(Ch_n(W)),$$

and similarly for the permanent. By Proposition 8.4, we conclude:

Proposition 8.5 [64] *The polynomial sequences* $\det_n$ *and* perm_n *do not admit depth three circuits of size* 2^n.

(In [64] they consider all partial derivatives of all orders simultaneously, but the bulk of the dimension is concentrated in the middle order flattening, so one does not gain very much this way.) Thus homogeneous depth three circuits at first sight do not seem that powerful because a polynomial sized homogeneous depth 3 circuit cannot compute the determinant.

To make matters worse, consider the polynomial corresponding to iterated matrix multiplication of three by three matrices $IMM_k^3 \in S^k(\mathbb{C}^{9k})$. It is complete for $\mathbf{VP}_e$, polynomials with small formula sizes (see Sect. 12), and also has an exponential lower bound for its Chow border rank.

Exercise 8.6 Use flattenings to show $\underline{\mathbf{R}}_S(IMM_k^3) \geq (const.)3^k$, and conclude $IMM_k^3 \notin \sigma_{poly(k)}(Ch_k(W))$.

By Exercise 8.6, homogeneous depth three circuits (naïvely applied) cannot even capture sequences of polynomials admitting small formulas.

Another benchmark in complexity theory are the elementary symmetric functions

$$e_n^k := \sum_{I\subset[n],|I|=k} x_{i_1}\cdots x_{i_k}.$$

To fix ideas, set $n = 4k$. Let $k = 2p$. Consider the flattening:

$$(e_{4k}^k)_{p,p} : S^p\mathbb{C}^{2k*} \to S^p\mathbb{C}^{2k}$$

It has image all monomials $x_{i_1}\cdots x_{i_p}$ with the i_j distinct, so its rank is $\binom{4k}{\frac{k}{2}}$ and since $\binom{4k}{\frac{k}{2}}/\binom{k}{\frac{k}{2}}$ grows faster than any polynomial in k, we conclude even the elementary symmetric function e_{4k}^k cannot be computed by a homogeneous depth three circuit of polynomial size. This last assertion is [64, Thm. 0], where they show more generally (by the same method) that $e_n^{2d} \notin \sigma^0_{\Omega((\frac{n}{4d})^d)}(Ch_{2d}(\mathbb{C}^n))$.

Remark 8.7 Strassen [72] proved a lower bound of $\Omega(n\log n)$ for the size of *any* arithmetic circuit computing all the e_n^j simultaneously.

8.2 Upper bounds for homogeneous depth three circuits

The most famous homogeneous depth three circuit is probably Ryser's formula for the permanent:

$$\mathrm{perm}_n = 2^{-n+1} \sum_{\substack{\epsilon\in\{-1,1\}^n \\ \epsilon_1=1}} \prod_{1\leq i\leq n} \sum_{1\leq j\leq n} \epsilon_i\epsilon_j x_{i,j}, \tag{22}$$

the outer sum is taken over n-tuples $\epsilon = (\epsilon_1 = 1, \epsilon_2, \ldots, \epsilon_n)$. Note that each term in the outer sum is a product of n independent linear forms and there are 2^{n-1} terms. In particular $[\mathrm{perm}_n] \in \sigma^0_{2^{n-1}}(Ch_n(\mathbb{C}^{n^2}))$, and since $Ch_n(\mathbb{C}^{n^2}) \subset \sigma^0_{2^{n-1}}(v_n(\mathbb{P}^{n^2-1}))$, we obtain $\mathbf{R}_S(\mathrm{perm}_n) \leq 4^{n-1}$.

8.3 Homogeneous depth three circuits for padded polynomials

At first glance it seems polynomial sized depth 3 circuits are useless, as they cannot compute even simple sequences of polynomials as we just saw. However, if one allows padded polynomials, the situation changes dramatically. (As mentioned above, in [31] and elsewhere they consider inhomogeneous polynomials and circuits instead of padding.) The following geometric version of a result of Ben-Or (presented below as a Corollary) was suggested by K. Efremenko:

Proposition 8.8 *Let* $\mathbb{C}^{m+1}$ *have coordinates* $\ell, x_1, \ldots, x_m$ *and let* $e^k_m = e^k_m(x_1, \ldots, x_m)$. *For all* $k \leq m$, $\ell^{m-k} e^k_m \in \sigma^0_m(Ch_m(\mathbb{C}^{m+1}))$.

Proof Fix an integer $u \in \mathbb{Z}$ and define

$$\begin{aligned} g_u(x, \ell) &= \prod_{i=1}^{m} (x_i + u\ell) \\ &= \sum_k u^{m-k} e^k_m(x) \ell^{m-k} \end{aligned}$$

Note $g_u(x, \ell) \in Ch_m(\mathbb{C}^{m+1})$. Letting $u = 1, \ldots, m$, we may use the inverse of the Vandermonde matrix to write each $\ell^{m-k} e^k_m$ as a sum of m points in $Ch_m(\mathbb{C}^{m+1})$ because

$$\begin{pmatrix} 1^0 & 1^1 & \cdots & 1^m \\ 2^0 & 2^1 & \cdots & 2^m \\ & \vdots & & \\ m^0 & m^1 & \cdots & m^m \end{pmatrix} \begin{pmatrix} \ell^{m-1} e^1_m \\ \ell^{m-2} e^2_m \\ \vdots \\ \ell^0 e^m_m \end{pmatrix} = \begin{pmatrix} g_1(x, \ell) \\ g_2(x, \ell) \\ \vdots \\ g_m(x, \ell) \end{pmatrix}.$$

$\square$

Corollary 8.9 (Ben-Or) $\ell^{m-k} e^k_m$ *can be computed by a homogeneous depth three circuit of size* $3m^2 + m$.

Proof As remarked above, for any point of $\sigma_r Ch_n(\mathbb{C}^{m+1})$ one gets a circuit of size at most $r + nr + rn(m+1)$, but here at the first level all the addition gates have fanin two (i.e., there are two inputs to each addition gate) instead of the possible $m+1$. $\square$

Problem 8.10 ([14] *Open problem* 11.1) Find an explicit sequence of polynomials $P_m \in S^m \mathbb{C}^{\mathbf{w}-1}$ such that $\ell^{n-m} P_m \notin \sigma_r(Ch_n(W))$, whenever r, $\mathbf{w}$, n are polynomials in m and m is sufficiently large.

Remark 8.11 The best lower bound for computing the e^k_n via a $\Sigma\Pi\Sigma$ circuit is $\Omega(n^2)$ [70], so Corollary 8.9 is very close to (and may well be) sharp.

8.4 Depth reduction

The following theorem combines results of [1,5,31,40,73] as explained in the discussion below. (The circuit bounds stated in the theorem come from [73].) A $\Sigma\Lambda\Sigma\Lambda\Sigma$ circuit is a depth 5 circuit where the first level consists of additions, the second of "powering gates", where a powering gate takes f to f^δ for some δ (the size of the circuit takes the size of δ into account), the third additions, the fourth powering gates and the fifth an addition. See [31] for more details. The $\Sigma\Lambda\Sigma\Lambda\Sigma$ circuits are related to the variety $\sigma_{r_1}(v_{\frac{d}{\delta}}(\sigma_{r_2}(v_\delta(\mathbb{P}V))) \subset \mathbb{P}S^d V$ in the same way that the $\Sigma\Pi\Sigma$ circuits are related to $\sigma_r(Ch_n(V))$.

Theorem 8.12 *Let $d = n^{O(1)}$ and let $P \in S^d\mathbb{C}^n$ be a polynomial that can be computed by a circuit of size s.*
Then:

(1) P is computable by a homogeneous $\Sigma\Pi\Sigma\Pi$ circuit of size $2^{O(\sqrt{d\log(ds)\log(n)})}$.
(2) P is computable by a $\Sigma\Pi\Sigma$ circuit of size $2^{O(\sqrt{d\log(n)\log(ds)})}$. In particular, $[\ell^{N-d}P] \in \sigma_r(Ch_N(\mathbb{C}^{n+1}))$ with $rN = 2^{O(\sqrt{d\log(n)\log(ds)})}$.
(3) P is computable, for some $\delta \simeq \sqrt{d}$, by a homogeneous $\Sigma\Lambda\Sigma\Lambda\Sigma$ circuit of size $2^{O(\sqrt{d\log(ds)\log(n)})}$. In particular, $[P] \in \sigma_{r_1}(v_{\frac{d}{\delta}}(\sigma_{r_2}(v_\delta(\mathbb{P}^{n-1}))))$ with $r_1r_2(\delta + 1) = 2^{O(\sqrt{d\log(ds)\log(n)})}$.

The "in particular" of (2) follows by setting the circuit size equal to $r + Nr$ (the smallest, i.e., worst case size of a circuit for a point of $\sigma_r(Ch_N(\mathbb{C}^{n+1}))$ that is not in a smaller variety). The "in particular" of (3) follows similarly, as the smallest circuit for a point of $\sigma_{r_1}(v_{d-\delta}(\sigma_{r_2}(v_\delta(\mathbb{P}^{n-1}))))$ not in a smaller variety is $r_1r_2(\delta + 1) + \frac{d}{\delta}r_1$.

Corollary 8.13 [31] $[\ell^{n-m}\det_m] \in \sigma_r(Ch_n(\mathbb{C}^{m^2+1}))$ *where* $rn = 2^{O(\sqrt{m}\log m)}$.

Proof The determinant admits a circuit of size m^4, so it admits a $\Sigma\Pi\Sigma$ circuit of size

$$2^{O(\sqrt{m\log(m)\log(m*m^4)})} \sim 2^{O(\sqrt{m}\log m)},$$

so its padded version lies in $\sigma_r(Ch_n(\mathbb{C}^{m^2+1}))$ where $rn = 2^{O(\sqrt{m}\log m)}$. □

Corollary 8.14 [31] *If for all but finitely many m and all r, n with $rn = 2^{\sqrt{m}\log(m)\omega(1)}$, one has $[\ell^{n-m}\,\mathrm{perm}_m] \notin \sigma_r(Ch_n(\mathbb{C}^{m^2+1}))$, then there is no circuit of polynomial size computing the permanent, i.e.,* **VP** $\neq$ **VNP**.

Proof In this case the s in (2) cannot be a polynomial. □

Corollary 8.15 [31] *If for all but finitely many m, $\delta \simeq \sqrt{m}$, and all r_1, r_2 such that $r_1r_2 = 2^{\sqrt{m}\log(m)\omega(1)}$, one has $[\mathrm{perm}_m] \notin \sigma_{r_1}(v_{m/\delta}(\sigma_{r_2}(v_\delta(\mathbb{P}^{m^2-1}))))$, then there is no circuit of polynomial size computing the permanent, i.e.,* **VP** $\neq$ **VNP**.

Proof In this case the s in (3) cannot be a polynomial. □

These Corollaries give rise to Conjectures 2.9 and 2.11 stated in Sect. 2.4.

The results above follow from an extensive amount of research. Here is an overview:

In [74] it was show that if a polynomial of degree d can be computed by a circuit of size s, then it can be computed by a circuit of depth $O(\log d * \log s)$ and size $s^{O(1)}$. In [31] they prove their upper bounds for the size of an inhomogeneous depth three circuit computing a polynomial, in terms of the size of an arbitrary circuit computing the polynomial, by first applying the work of [40,1], which allows one to reduce an arbitrary circuit of size s computing a polynomial of degree d in n variables to a formula of size $2^{O(\log s \log d)}$ and depth d. Next they reduce to a depth four circuit of size $s' = 2^{O(\sqrt{d\log d\log s\log n})}$. This second passage is via iterated matrix multiplication.

From the depth four circuit, they use (21) to convert all multiplication gates to sums of elements of the Veronese (what they call $\Sigma\Lambda\Sigma$ circuits), to have a depth five circuit of size $O(s')$ and of the form $\Sigma\Lambda\Sigma\Lambda\Sigma$. Finally, they use Newton's identities to convert power sums to elementary symmetric functions which keeps the size at $O(s')$ and drops the depth to three.

Remark 8.16 In [31], they also show that, for a similar price, one can convert a depth three circuit to a $\Sigma\Lambda\Sigma\Lambda\Sigma$ circuit by using the inverse identities without substantially increasing the size.

Remark 8.17 Ultimately, if one wants to separate $\mathbf{VP}_{ws}$ from $\mathbf{VNP}$, one will have to find polynomials that separate $\det_n$ from $\ell^{n-m}\operatorname{perm}_m$. These auxiliary varieties arising from shallow circuits should be viewed as a guide to how to look for such equations, not as a way to avoid finding them.

Remark 8.18 Note the expected dimension of $\sigma_r(Ch_d(W))$ is $rd\mathbf{w}+r-1$. If we take $d' = d2^m$ and work instead with padded polynomials $\ell^{2^m}P$, the expected dimension of $\sigma_r(Ch_{d'}(W))$ is $2^m rd\mathbf{w} + r - 1$. In contrast, the expected dimension of $\sigma_r(v_{d-a}(\sigma_\rho(v_a(\mathbb{P}W))))$ does not change when one increases the degree, which gives some insight as to why padding is so useful for homogeneous depth three circuits but not for $\Sigma\Lambda\Sigma\Lambda\Sigma$ circuits.

9 Non-normality

I follow [43] in this section. Throughout this section I make the following assumptions and adopt the following notation:

$$\textbf{Assumptions}: \tag{23}$$

(1) V is a $GL(W)$-module,
(2) $P \in V$ is such that the $SL(W)$-orbit of P is closed.
(3) Let $\mathcal{P}^0 := GL(W)\cdot P$ and $\mathcal{P} := \overline{GL(W)\cdot P} \subset V$ denote its orbit and orbit closure, and let $\partial\mathcal{P} = \mathcal{P}\backslash\mathcal{P}^0$ denote its boundary, which we assume to be more than zero (otherwise $[\mathcal{P}]$ is homogeneous).
(4) Assume the stabilizer $G_P \subset GL(W)$ is reductive, which is equivalent (by a theorem of Matsushima [54]) to requiring that $\mathcal{P}^0$ is an affine variety.

This situation holds when $V = S^n W$, $\dim W = n^2$ and $P = \det_n$ or perm_n as well as when $\dim W = rn$ and $P = S_n^r := \sum_{j=1}^r x_1^j \cdots x_n^j$, the "sum-product polynomial", in which case $\mathcal{P} = \hat{\sigma}_r(Ch_n(W))$.

Lemma 9.1 [43] *Assumptions as in (23). Let $M \subset \mathbb{C}[\mathcal{P}]$ be a nonzero $GL(W)$-module, and let $Z(M) = \{y \in \mathcal{P} \mid f(y) = 0\ \forall f \in M\}$ denote its zero set. Then $0 \subseteq Z(M) \subseteq \partial\mathcal{P}$.*

If moreover $M \subset I(\partial\mathcal{P})$, then as sets, $Z(M) = \partial\mathcal{P}$.

Proof Since $Z(M)$ is a $GL(W)$-stable subset, if it contains a point of $\mathcal{P}^0$ it must contain all of $\mathcal{P}^0$ and thus M vanishes identically on $\mathcal{P}$, which cannot happen as M

is nonzero. Thus $Z(M) \subseteq \partial\mathcal{P}$. For the second assertion, since $M \subset I(\partial\mathcal{P})$, we also have $Z(M) \supseteq \partial\mathcal{P}$. □

Proposition 9.2 [43] *Assumptions as in* (23). *The space of $SL(W)$-invariants of positive degree in the coordinate ring of $\mathcal{P}$, $\mathbb{C}[\mathcal{P}]^{SL(W)}_{>0}$, is non-empty and contained in $I(\partial\mathcal{P})$. Moreover,*

(1) any element of $\mathbb{C}[\mathcal{P}]^{SL(W)}_{>0}$ cuts out $\partial\mathcal{P}$ set-theoretically, and
(2) the components of $\partial\mathcal{P}$ all have codimension one in $\mathcal{P}$.

Proof To study $\mathbb{C}[\mathcal{P}]^{SL(W)}$, consider the GIT quotient $\mathcal{P}//SL(W)$ whose coordinate ring, by definition, is $\mathbb{C}[\mathcal{P}]^{SL(W)}$. It parametrizes the closed $SL(W)$-orbits in $\mathcal{P}$, so it is non-empty. Thus $\mathbb{C}[\mathcal{P}]^{SL(W)}$ is nontrivial.

We will show that every $SL(W)$-orbit in ∂P contains $\{0\}$ in its closure, i.e., that $\partial\mathcal{P}$ maps to zero in the GIT quotient. This will imply any $SL(W)$-invariant of positive degree is in $I(\partial\mathcal{P})$ because any non-constant function on the GIT quotient vanishes on the inverse image of $[0]$. Then (1) follows from Lemma 9.1. The zero set of a single polynomial, if it is not empty, has codimension one, which implies the components of $\partial\mathcal{P}$ are all of codimension one, proving (2).

It remains to show $\partial\mathcal{P}$ maps to zero in $\mathcal{P}//SL(W)$, where $\rho : GL(W) \to GL(V)$ is the representation. This GIT quotient inherits a $\mathbb{C}^*$ action via $\rho(\lambda Id)$, for $\lambda \in \mathbb{C}^*$. Its normalization is just the affine line $\mathbb{A}^1 = \mathbb{C}$. To see this, consider the $\mathbb{C}^*$-equivariant map $\sigma : \mathbb{C} \to \mathcal{P}$ given by $z \mapsto \rho(zId) \cdot P$, which descends to a map $\overline{\sigma} : \mathbb{C} \to \mathcal{P}//SL(W)$. Since the $SL(W)$-orbit of P is closed, for any $\lambda \in \mathbb{C}^*$, $\rho(\lambda Id)P$ does not map to zero in the GIT quotient, so we have $\overline{\sigma}^{-1}([0]) = \{0\}$ as a set. Lemma 7.13 applies so $\overline{\sigma}$ is finite and gives the normalization. Finally, were there a closed nonzero orbit in $\partial\mathcal{P}$, it would have to equal $SL(W) \cdot \sigma(\lambda)$ for some $\lambda \in \mathbb{C}^*$ since $\overline{\sigma}$ is surjective. But $SL(W) \cdot \sigma(\lambda) \subset \mathcal{P}^0$. □

Remark 9.3 That each irreducible component of $\partial\mathcal{P}$ is of codimension one in $\mathcal{P}$ is due to Matsushima [54]. It is a consequence of his result mentioned above.

The key to proving non-normality of $\hat{\mathcal{D}et}_n$ and $\hat{\mathcal{P}erm}^n_n$ is to find an $SL(W)$-invariant in the coordinate ring of the normalization (which has a $GL(W)$-grading), which does not occur in the corresponding graded component of the coordinate ring of S^nW, so it cannot occur in the coordinate ring of any $GL(W)$-subvariety.

Lemma 9.4 *Assumptions as in* (23). *Let $P \in S^nW$ be such that $SL(W) \cdot P$ is closed and G_P is reductive. Let d be the smallest positive $GL(W)$-degree such that $\mathbb{C}[\mathcal{P}^0]^{SL(W)}_d \neq 0$. If n is even and $d < n\mathbf{w}$ (resp. n is odd and $d < 2n\mathbf{w}$) then $\mathcal{P}$ is not normal.*

Proof Since $\mathcal{P}^0 \subset \mathcal{P}$ is a Zariski open subset, we have the equality of $GL(W)$-modules $\mathbb{C}(\mathcal{P}) = \mathbb{C}(\mathcal{P}^0)$. By restriction of functions $\mathbb{C}[\mathcal{P}] \subset \mathbb{C}[\mathcal{P}^0]$ and thus $\mathbb{C}[\mathcal{P}]^{SL(W)} \subset \mathbb{C}[\mathcal{P}^0]^{SL(W)}$. Now $\mathcal{P}^0//SL(W) = \mathcal{P}^0/SL(W) \simeq \mathbb{C}^*$, so $\mathbb{C}[\mathcal{P}^0]^{SL(W)} \simeq \oplus_{k\in\mathbb{Z}}\mathbb{C}\{z^k\}$. Under this identification, z has $GL(W)$-degree d. By Proposition 9.2, $\mathbb{C}[\mathcal{P}]^{SL(W)} \neq 0$. Let $h \in \mathbb{C}[\mathcal{P}]^{SL(W)}$ be the smallest element in positive degree. Then $h = z^k$ for some k. Were $\mathcal{P}$ normal, we would have $k = 1$.

But now we also have a surjection $\mathbb{C}[S^nW] \to \mathbb{C}[\mathcal{P}]$, and by Exercise 3.9 the smallest possible $GL(W)$-degree of an $SL(W)$-invariant in $\mathbb{C}[S^nW]$ when n is even (resp. odd) is $\mathbf{w}n$ (resp. $2\mathbf{w}n$) which would occur in $S^{\mathbf{w}}(S^nW)$ (resp. $S^{2\mathbf{w}}(S^nW)$). We obtain a contradiction. □

Theorem 9.5 (Kumar [43]) *For all $n \geq 3$, $\mathcal{D}et_n$ and $\mathcal{P}erm_n^n$ are not normal. For all $n \geq 2m$ (the range of interest), $\mathcal{P}erm_n^m$ is not normal.*

I give the proof for $\mathcal{D}et_n$, the case of $\mathcal{P}erm_n^n$ is an easy exercise. Despite the variety being much more singular, the proof for $\mathcal{P}erm_n^m$ with $m > n$ is more difficult, see [43].

Proof We will show that when n is congruent to 0 or 1 mod 4, $\mathbb{C}[\mathcal{D}et_n^0]_{n-GL}^{SL(W)} \neq 0$ and when n is congruent to 2 or 3 mod 4, $\mathbb{C}[\mathcal{D}et_n^0]_{2n-GL}^{SL(W)} \neq 0$. Since $n, 2n < (n^2)n$ Lemma 9.4 applies.

The $SL(W)$-trivial modules are $(\Lambda^{n^2}W)^{\otimes s} = S_{s^{n^2}}W$. Write $W = E\otimes F$. We want to determine the lowest degree trivial $SL(W)$-module that has a $G_{det_n} = (SL(E) \times SL(F)/\mu_n) \rtimes \mathbb{Z}_2$ invariant. We have the decomposition $(\Lambda^{n^2}W)^{\otimes s} = (\bigoplus_{|\pi|=n^2} S_\pi E\otimes S_{\pi'}F)^{\otimes s}$, where π' is the conjugate partition to π. Thus $(\Lambda^{n^2}W)^{\otimes s}$ contains the trivial $SL(E) \times SL(F)$ module $(\Lambda^n E)^{\otimes ns}\otimes(\Lambda^n F)^{\otimes ns}$ with multiplicity one. (In the language of Sect. 3.2, $k_{s^{n^2},(sn)^n,(sn)^n} = 1$.) Now we consider the effect of the $\mathbb{Z}_2 \subset G_{\det_n}$ with generator $\tau \in GL(W)$. It sends $e_i\otimes f_j$ to $e_j\otimes f_i$, so acting on W it has $+1$ eigenspace $e_i\otimes f_j + e_j\otimes f_i$ for $i \leq j$ and -1 eigenspace $e_i\otimes f_j - e_j\otimes f_i$ for $1 \leq i < j \leq n$. Thus it acts on the one-dimensional vector space $(\Lambda^{n^2}W)^{\otimes s}$ by $((-1)^{\binom{n}{2}})^s$, i.e., by -1 if $n \equiv 2, 3 \bmod 4$ and s is odd and by 1 otherwise. We conclude that there is an invariant as asserted above. (In the language of Sect. 6.6, $sk^{s^{n^2}}_{(sn)^n,(sn)^n} = 1$ for all s when $\binom{n}{2}$ is even, and $sk^{s^{n^2}}_{(sn)^n,(sn)^n} = 1$ for even s when $\binom{n}{2}$ is odd and is zero for odd s.) □

Exercise 9.6 Write out the proof of the non-normality of $\mathcal{P}erm_n^n$.

Exercise 9.7 Show the same method gives another proof that $Ch_n(W)$ is not normal.

Exercise 9.8 Show that the proof of Theorem 9.5 holds for any reductive group with a nontrivial center (one gets a $\mathbb{Z}^k$-grading of modules if the center is k-dimensional), in particular it holds for $G = GL(A) \times GL(B) \times GL(C)$. Use this to show that $\sigma_r(Seg(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C))$ is not normal when $\dim A = \dim B = \dim C = r > 2$.

10 Determinantal hypersurfaces

Classically, there was interest in determining which smooth hypersurfaces of degree d were expressible as a $d \times d$ determinant. The result in the first nontrivial case shows how daunting GCT might be.

Theorem 10.1 (Letao Zhang and Zhiyuan Li) *The variety* $\mathbb{P}\{P \in S^4\mathbb{C}^4 \mid [P] \in \mathcal{D}et_4\} \subset \mathbb{P}S^4\mathbb{C}^4$ *is a hypersurface of degree* 640, 224.

The following "folklore" theorem was made explicit in [3, Cor. 1.12]:

Theorem 10.2 *Let $U = \mathbb{C}^{n+1}$, let $P \in S^d U$, and let $Z = Z(P) \subset \mathbb{CP}^n$ be the corresponding hypersurface of degree d. Assume Z is smooth and choose any inclusion $U \subset \mathbb{C}^{d^2}$.*

If $P \in \mathrm{End}(\mathbb{C}^{d^2}) \cdot [\det_d]$, we may form a map between vector bundles $M : \mathcal{O}_{\mathbb{P}^n}(-1)^d \to \mathcal{O}^d_{\mathbb{P}^n}$ whose cokernel is a line bundle $L \to Z$ with the properties:

i) $H^i(Z, L(j)) = 0$ for $1 \leq i \leq n-2$ and all $j \in \mathbb{Z}$
ii) $H^0(X, L(-1)) = H^{n-1}(X, L(j)) = 0$

Conversely, if there exists $L \to Z$ satisfying properties (i) and (ii), then Z is determinantal via a map M as above whose cokernel is L.

If we are concerned with the hypersurface being in $\mathcal{D}et_n$, the first case where this is not automatic is for quartic surfaces, where it is a codimension one condition:

Proposition 10.3 [3, Cor. 6.6] *A smooth quartic surface is determinantal if and only if it contains a nonhyperelliptic curve of genus* 3 *embedded in $\mathbb{P}^3$ by a linear system of degree* 6.

Proof of 10.1 From Proposition 10.3, the hypersurface is the locus of quartic surfaces containing a (Brill-Noether general) genus 3 curve C of degree six. This translates into the existence of a lattice polarization

$$\begin{array}{ccc} & h & C \\ h & 4 & 6 \\ C & 6 & 4 \end{array}$$

of discriminant $-(4^2-6^2) = 20$. By the Torelli theorems, the $K3$ surfaces with such a lattice polarization have codimension one in the moduli space of quartic $K3$ surfaces.

Let $D_{3,6}$ denote the locus of quartic surfaces containing a genus 3 curve C of degree six in $\mathbb{P}^{34} = \mathbb{P}(S^4\mathbb{C}^4)$. It corresponds to the Noether-Lefschetz divisor NL_{20} in the moduli space of the degree four $K3$ surfaces. Here NL_d denotes the Noether-Lefschetz divisor, parameterizing the degree 4 $K3$ surfaces whose Picard lattice has a rank 2 sub-lattice containing h with discriminant $-d$. (h is the polarization of the degree four $K3$ surface, $h^2 = 4$.)

The Noether-Lefschetz number n_{20}, which is defined by the intersection number of NL_{20} and a line in the moduli space of degree four $K3$ surfaces, equals the degree of $D_{3,6}$ in $\mathbb{P}^{34} = \mathbb{P}(S^4\mathbb{C}^4)$.

The key fact is that n_d can be computed via the modularity of the generating series for any integer d. More precisely, the generating series $F(q) := \sum_d n_d q^{d/8}$ is a modular form of level 8, and can be expressed by a polynomial of $A(q) = \sum_n q^{n^2/8}$ and $B(q) = \sum_n (-1)^n q^{n^2/8}$.

The explicit expression of $F(q)$ is in [55, Thm 2]. As an application, the Noether-Lefschetz number n_{20} is the coefficient of the term $q^{20/8} = q^{5/2}$, which is 640, 224. $\square$

11 Classical linear algebra and GCT

One potential source of new equations for $\mathcal{D}et_n$ is to exploit classical identities the determinant satisfies. What follows are ideas in this direction. This section is joint unpublished work with L. Manivel and N. Ressayre.

11.1 Cayley's identity

Let $\mathbb{C}^{n^2}$ have coordinates x^i_j and the dual space coordinates y^i_j. The classical Cayley identity (apparently first due to Vivanti, see [13]) is

$$\langle (\det_n(y)), (\det_n(x))^{s+1}\rangle = \frac{(s+n)!}{s!}(\det_n(x))^s$$

which may be thought of as a pairing between homogeneous polynomials of degree $n(s+1)$ and homogeneous differential operators of order n (compare with Proposition 7.28). This and more general Bernstein-Sato type identities (again, see [13]) appear as if they could be used to obtain equations for $\mathcal{D}et_n$. So far we have only found rational equations in this manner.

In more detail, "$\det_n(y)$" depends on the choice of identification of $\mathbb{C}^{n^2}$ with $\mathbb{C}^{n^2*}$ given by the coordinates, but one could, e.g. ask for polynomials $P \in S^nW$ such that there exists some $Q \in S^nW^*$, with G_P and G_Q isomorphic and $\langle Q, P^{s+1}\rangle = \frac{(s+n)!}{s!}P^s$.

11.2 A generalization of the Sylvester-Franke Theorem

Let $f : V \to V$ be a diagonalizable linear map with distinct eigenvalues $\lambda_1, \dots, \lambda_{\mathbf{v}}$. The induced linear map $f^{\wedge k} : \Lambda^k V \to \Lambda^k V$ has eigenvalues $\lambda_{i_1}\cdots\lambda_{i_k}$, $1 \le i_1 < \cdots < i_k \le \mathbf{v}$. In particular $f^{\wedge\mathbf{v}} : \Lambda^{\mathbf{v}}V \to \Lambda^{\mathbf{v}}V$ is multiplication by the scalar $\det(f) = \lambda_1\cdots\lambda_{\mathbf{v}}$. Now consider $\Lambda^k V$ as a vector space (ignoring its extra structure), and

$$[f^{\wedge k}]^{\wedge s} : \Lambda^s(\Lambda^k V) \to \Lambda^s(\Lambda^k V).$$

Let

$$\begin{aligned} cp_s : V{\otimes}V^* &\to \mathbb{C} \\ f &\mapsto \mathrm{trace}(f^{\wedge s}), \end{aligned} \tag{24}$$

denote the s-th coefficient of the characteristic polynomial. We may consider $cp_s = Id_{\Lambda^s V} \in \Lambda^s V{\otimes}\Lambda^s V^* \subset S^s(V{\otimes}V^*)$. Recall that $cp_{\mathbf{v}} = \det$.

Proposition 11.1 *The degree $\mathbf{v}p$ polynomial on $V{\otimes}V^*$ given by $f \mapsto (\det)^p(f)$ divides the degree $(\binom{\mathbf{v}-1}{k} + p)k$ polynomial $f \mapsto cp_{\binom{\mathbf{v}-1}{k}+p}(f^{\wedge k})$.*

In other words, for a $\mathbf{v} \times \mathbf{v}$ *matrix* A *with indeterminate entries, the degree* $\mathbf{v}p$ *polynomial* $\det(A)^p$ *divides the trace of the* $[\binom{\mathbf{v}-1}{k} + p]$*-th companion matrix of the* k*-th companion matrix of* A*.*

The Sylvester-Franke theorem is the special case $p = \binom{\mathbf{v}-1}{k-1}$.

Proof Assume f has $\mathbf{v}$ distinct eigenvalues. The eigenvalues of $[f^{\wedge k}]^{\wedge s}$ are sums of terms of the form $\sigma_{J_1} \cdots \sigma_{J_s}$ where $\sigma_{J_m} = \lambda_{j_{m,1}} \cdots \lambda_{j_{m,k}}$ and the $\lambda_{j_{m,1}}, \ldots, \lambda_{j_{m,k}}$ are distinct eigenvalues of f. Once every λ_j appears in a monomial to a power p, $\det^p$ divides the monomial. The result now follows for linear maps with distinct eigenvalues by the pigeonhole principle. Since the subset of linear maps with distinct eigenvalues forms a Zariski opens subset of $V \otimes V^*$, the equality of polynomials holds everywhere. □

11.3 A variant of Proposition 11.1 for the Hessian

Say $g : \Lambda^2 V^* \to \Lambda^2 V^*$ is a linear map such that there exists a basis $v_1, \ldots, v_{\mathbf{v}}$ of V with dual basis $\alpha^1, \ldots, \alpha^{\mathbf{v}}$ such that

$$g = \sum_{i<j} \lambda_{ij} \alpha^i \wedge \alpha^j \otimes v_i \wedge v_j,$$

so λ_{ij} are the eigenvalues of g. We will be concerned with the case $g = f^{\wedge(\mathbf{v}-2)}$, where $f : V \to V$ is a linear map with distinct eigenvalues $\lambda_1, \ldots, \lambda_{\mathbf{v}}$, $v_1, \ldots, v_{\mathbf{v}}$ is an eigenbasis of V with dual basis $\alpha^1, \ldots, \alpha^{\mathbf{v}}$, so $f = \lambda_1 \alpha^1 \otimes v_1 + \cdots + \lambda_{\mathbf{v}} \alpha^{\mathbf{v}} \otimes v_{\mathbf{v}}$. Then $\lambda_{ij} = \lambda_1 \cdots \lambda_{i-1} \lambda_{i+1} \cdots \lambda_{j-1} \lambda_{j+1} \cdots \lambda_{\mathbf{v}}$.

Consider the inclusion $in : \Lambda^2 V^* \otimes \Lambda^2 V \subset S^2(V \otimes V^*)$. On decomposable elements it is given by

$$\begin{aligned} \alpha \wedge \beta \otimes v \wedge w \mapsto\ & (\alpha \otimes v) \otimes (\beta \otimes w) - (\alpha \otimes w) \otimes (\beta \otimes v) \\ & - (\beta \otimes v) \otimes (\alpha \otimes w) + (\beta \otimes w) \otimes (\alpha \otimes v) \end{aligned}$$

The space $V \otimes V^*$ is self-dual as a $GL(V)$-module, with the natural quadratic form $Q(\alpha \otimes v) = \alpha(v)$, so we may identify $S^2(V \otimes V^*)$ as a subspace of $\mathrm{End}(V \otimes V^*)$ via the linear map $Q^\flat : V^* \otimes V \to V \otimes V^*$ given by $\alpha^i \otimes v_j \mapsto v_i \otimes \alpha^j$.

Say we have a map g as above. Consider $g^\flat := Q^\flat \circ in(g) : V \otimes V^* \to V \otimes V^*$, then

$$\begin{aligned} g^\flat = \sum_{i<j} \lambda_{ij} [& (v_i \otimes \alpha^i) \otimes (\alpha^j \otimes v_j) - (v_i \otimes \alpha^j) \otimes (\alpha^j \otimes v_i) - (v_j \otimes \alpha^i) \otimes (\alpha^i \otimes v_j) \\ & + (v_j \otimes \alpha^j) \otimes (\alpha^i \otimes v_i)], \end{aligned}$$

so,

$$\begin{aligned} g^\flat(v_i \otimes \alpha^j) &= -\lambda_{ij} v_j \otimes \alpha^i \ i \neq j, \\ g^\flat(v_i \otimes \alpha^i) &= \sum_{j \neq i} \lambda_{ij} v_j \otimes \alpha^j. \end{aligned}$$

Thus $g^\flat$ may be thought of as a sum of two linear maps, one preserving the subspace $D := \langle v_1 \otimes \alpha^1, \dots, v^{\mathbf{v}} \otimes \alpha^{\mathbf{v}} \rangle$ and another preserving the subspace $D^c := \langle v_i \otimes \alpha^j \mid i \neq j \rangle$.

The $2\binom{\mathbf{v}}{2}$ eigenvalues of $g^\flat|_{D^c}$ are $\pm\lambda_{ij}$. Write ψ_s for the coefficients of the characteristic polynomial of $g^\flat|_{D^c}$. Since the eigenvalues come paired with their negatives, $\psi_s = 0$ when s is odd.

With respect to the given basis, the matrix for $g^\flat|_D$ is a symmetric matrix with zeros on the diagonal, whose off diagonal entries are the λ_{ij}. Write the coefficients of the characteristic polynomial of $g^\flat|_D$ as $\zeta_1, \dots, \zeta_{\mathbf{v}}$, and note that $\zeta_1 = 0, \zeta_2 = \sum_{i<j} \lambda_{ij}^2$, $\zeta_3 = 2\sum_{i<j<k} \lambda_{ij}\lambda_{ik}\lambda_{jk}$.

Now let $g = f^{\wedge(\mathbf{v}-2)}$ as above and we compare the determinant of f with the coefficients of the characteristic polynomial of the Hessian $H(\det(f))$. (Invariantly, $\det(f) = f^{\wedge n}$ and $H : S^n(V \otimes V^*) \to S^2(V \otimes V^*) \otimes S^{n-2}(V \otimes V^*)$ is the $(2, n)$-polarization, so $H(\det(f)) = in(f^{\wedge 2}) \otimes in(f^{\wedge n-2})$.)

Observe that $\det(f)^{2(s+1-\mathbf{v})}$ divides ψ_{2s} and $\det(f)^k$ divides ζ_{k+2}. Also note that $\zeta_{\mathbf{v}-1} = 2Q\det_{\mathbf{v}}^{\mathbf{v}-2}$, $\zeta_{\mathbf{v}} = (\mathbf{v}-1)(\det_{\mathbf{v}})^{\mathbf{v}-2}$, and $\psi_{\mathbf{v}^2-\mathbf{v}} = (-1)^{\binom{\mathbf{v}}{2}}(\det_{\mathbf{v}})^{(\mathbf{v}-1)(\mathbf{v}-2)}$.

Recall $cp_j(A_1 + A_2) = \sum_{\alpha=0}^{j} cp_\alpha(A_1)cp_{j-\alpha}(A_2)$. Thus

$$\begin{aligned} cp_{2k}(H(det(f))) &= \zeta_{2k} + \zeta_{2k-2}\psi_2 + \zeta_{2k-4}\psi_4 + \cdots + \zeta_2\psi_{2k-2} + \psi_{2k}, \\ cp_{2k+1}(H(det(f))) &= \zeta_{2k+1} + \zeta_{2k-1}\psi_2 + \zeta_{2k-3}\psi_4 + \cdots + \zeta_3\psi_{2k-2}. \end{aligned}$$

We conclude:

Theorem 11.2 *Let $Q \in S^2(V \otimes V^*)$ be the canonical contraction, so $S^2(V \otimes V^*) \subset \mathrm{End}(V \otimes V^*)$. Write $CP(H(\det_{\mathbf{v}})) = \sum cp_{\mathbf{v}^2-j}y^j$ for the characteristic polynomial. Then*

$$\begin{aligned} cp_0 &= 1 \\ cp_1 &= 0 \\ cp_3 &= \det{}_{\mathbf{v}}\ R_{2\mathbf{v}-6} \\ cp_5 &= \det{}_{\mathbf{v}}\ R_{4\mathbf{v}-10} \\ &\ \vdots \\ cp_{2k} &= \det{}_{\mathbf{v}}^{2(s-\mathbf{v}+1)}\ R_{2(\mathbf{v}^2-2s-\mathbf{v})}\ k > \mathbf{v} \\ cp_{2k+1} &= \det{}_{\mathbf{v}}^{2(s-\mathbf{v})+1}\ R_{2(\mathbf{v}^2-2s-1)}\ k > \mathbf{v} \\ cp_{\mathbf{v}^2-1} &= 2(\det{}_{\mathbf{v}})^{\mathbf{v}(\mathbf{v}-2)-1} Q \\ cp_{\mathbf{v}^2} &= (-1)^{\binom{\mathbf{v}+1}{2}}(\mathbf{v}-1)(\det{}_{\mathbf{v}})^{\mathbf{v}(\mathbf{v}-2)} \end{aligned}$$

where R_k is a polynomial of degree k. Moreover $\det_{\mathbf{v}}$ *does not divide the even cp_s for* $s < 2\mathbf{v} + 1$.

Remark 11.3 The equality $cp_{\mathbf{v}^2} = (-1)^{\binom{\mathbf{v}+1}{2}}(\mathbf{v}-1)(\det_{\mathbf{v}})^{\mathbf{v}(\mathbf{v}-2)}$ is due to B. Segre.

Exercise 11.4 Prove the analog of the B. Segre equality for the discriminant $\Delta \in S^4(S^3\mathbb{C}^2)$ (the equation of the dual variety of $v_3(\mathbb{P}^1)^\vee$). Namely, if one takes $\Delta = 27x_1^2x_4^2 + 4x_1x_3^3 + 4x_2^3x_4 - x_2^2x_3^2 - 18x_1x_2x_3x_4$, then $\det(H(\Delta)) = 3888\Delta^2$.

Problem 11.5 Find all the components of $\mathcal{D}ual_{4,4,1}$, show $\overline{GL_4 \cdot \Delta}$ is an irreducible component of $\mathcal{D}ual_{4,4,1}$, and find defining equations for that component.

11.4 A cousin of $\mathcal{D}et_n$

In GCT one is interested in orbit closures $\overline{GL(W) \cdot [P]} \subset S^dW$ where $P \in S^dW$. One cannot make sense of the coefficients of the characteristic polynomial of $H(P) \in S^2W \otimes S^{d-2}W$ without choosing an isomorphism $Q : W \to W^*$.

If $P = \det_n$ and we choose bases to express elements of W as $n \times n$ matrices, then taking $Q(A) = \mathrm{trace}(AA^T)$ will give the desired identification to enable us to potentially use the equations implied by Theorem 11.2. (Note that taking $Q'(A) = \mathrm{trace}(A^2)$ will not.) However these are equations for $\overline{O(W, Q) \cdot \det_n}$ rather than $\mathcal{D}et_n$.

The proof of Theorem 11.2 used the fact that a Zariski open subset of the space of matrices is diagonalizable under the action of $GL(V)$ by conjugation. We no longer have this action, but instead, writing $W = E \otimes F$, we have the intersection of the stabilizers of $\det_n$ and Q, i.e., $O(W, Q) \cap [(SL(E) \times SL(F))/\mu_n \rtimes \mathbb{Z}_2]$.

Proposition 11.6 *The connected component of the identity of* $O(W, Q) \cap [SL(E) \times SL(F) \rtimes \mathbb{Z}_2]$ *is* $SO(E) \times SO(F)$.

Proof The inclusion $SO(E) \times SO(F) \subseteq O(W, Q) \cap [SL(E) \times SL(F) \rtimes \mathbb{Z}_2]$ is clear. To see the other inclusion, note that over $\mathbb{R}$, $SO(n, \mathbb{R}) \times SO(n, \mathbb{R})$ is a maximal compact subgroup of $SL(n, \mathbb{R}) \times SL(n, \mathbb{R})$. The equations for the Lie algebra of the stabilizer are linear, and the rank of a linear system of equations is the same over $\mathbb{R}$ or $\mathbb{C}$, so the result holds over $\mathbb{C}$. $\square$

Proposition 11.7 *The* $SO(E) \times SO(F)$ *orbit of the diagonal matrices contains a Zariski open subset of* $E \otimes F$.

Proof We show the kernel of the differential of the map $SO(E) \times SO(F) \times D \to E \otimes F$ at (Id_E, Id_F, δ) is zero, where δ is a sufficiently general diagonal matrix. The differential is $(X, Y, \delta') \mapsto \delta' + X\delta + \delta Y$, where δ' is diagonal. The matrix $X\delta + \delta Y$ has zeros on the diagonal and its (i, j)-th entry is $X^i_j\delta_j + \delta_i Y^j_i$. Write out the $2\binom{n}{2}$ matrix in the δ_i for the $2\binom{n}{2}$ unknowns X, Y resulting from the equations $X^i_j\delta_j + \delta_i Y^j_i = 0$. Its determinant is $\Pi_{i<j}(\delta_i^2 - \delta_j^2)$, which is nonzero as long as the δ_j^2 are distinct. $\square$

We apply Theorem 11.2 to obtain:

Theorem 11.8 *Let* $P \in \overline{O(W, Q) \cdot [\det_n]}$, *then* P *divides* $\mathrm{trace}(H(P)^{\wedge j}) \in S^{j(n-2)}W$ *for each odd* $j > 1$ *up to* $j = 2n + 1$. *In particular we obtain modules of equations of degrees* $(j - 1)(d - 1)$ *for* $\overline{O(W, Q) \cdot [\det_n]}$ *for* j *in this range.*

11.5 Relation to GCT?

Since $\dim O(W,Q)$ is roughly half that of $GL(W)$, and it contains a copy of $GL_{\lfloor \frac{\mathbf{w}^2}{2} \rfloor}$, e.g., if $\mathbf{w}=2n$ is even and $Q=x^1y^1+\cdots+x^ny^n$, then

$$\left\{\begin{pmatrix} A & 0 \\ 0 & A^{-1}\end{pmatrix} \mid A\in GL_n\right\}\subset O(W,Q),$$

one might hope to use the variety $\overline{O(W,Q)\cdot[\det_n]}$ as a substitute for $\mathcal{D}et_n$ in the GCT program, since we have many equations for it, and these equations do not vanish identically on cones.

Consider $P\in S^m\mathbb{C}^M$ and $\ell^{n-m}P\in S^n\mathbb{C}^{M+1}\subset S^n\mathbb{C}^N=S^nW$. Taking the naïve coordinate embedding such that Q restricted to $\mathbb{C}^{M+1}$ is nondegenerate gives:

$$\begin{aligned}&\mathrm{trace}(H_N(\ell^{n-m}P)^{\wedge 3})\\ &\quad=\ell^{3(n-m)}\,\mathrm{trace}(H_M(P)^{\wedge 3})+\ell^{3(n-m)-2}[P\,\mathrm{trace}(H_M(P)^{\wedge 2})\\ &\qquad+\sum_{i<j}(2P_iP_jP_{ij}-P_i^2P_{jj}-P_j^2P_{ii})]\end{aligned}$$

where $P_i=\frac{\partial P}{\partial x_i}$ etc... When does $\ell^{n-m}P$ divide this expression? We need that P divides $\mathrm{trace}(H_M(P)^{\wedge 3})$ and $\sum_{i<j}(2P_iP_jP_{ij}-P_i^2P_{jj}-P_j^2P_{ii})$. But these conditions are independent of n,N so there is no hope of getting this condition asymptotically. However, taking a more complicated inclusion might erase this problem.

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12 Appendix: Complexity theory

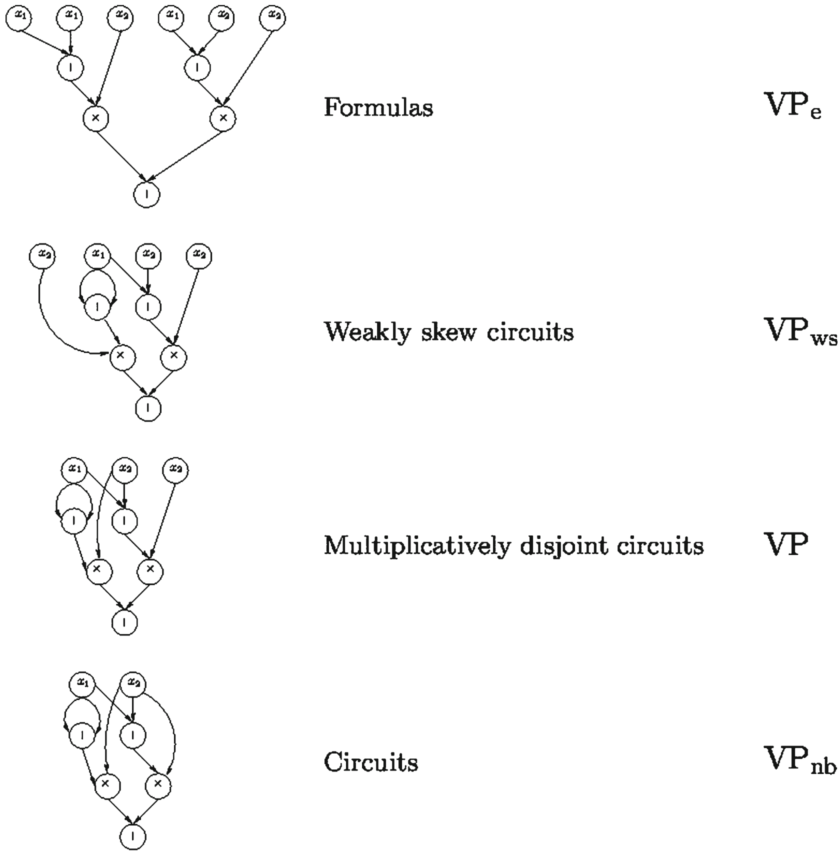
In a letter to von Neumann (see [71, Appendix]) Gödel tried to quantify what we mean by “intuition”, or more precisely the apparent difference between intuition and systematic problem solving. At the same time, researchers in the Soviet Union were trying to determine if “brute force search” was avoidable in solving problems such as the traveling salesman problem, where there seems to be no fast way to find a solution, but a proposed solution can be easily checked. (If I say I have found a way to visit twenty cities by traveling less than a thousand miles, you just need to look at my plan and check the distances.) These discussions eventually gave rise to the complexity classes **P**, which models problems admitting a fast algorithm to produce a solution, and **NP** which models problems admitting a fast algorithm to verify a proposed solution.

The "problems" relevant to us are sequences of polynomials or multi-linear maps (i.e. tensors), and the goal is to find lower bounds on the complexity of evaluating them, or otherwise to find efficient algorithms to do so. Geometry has so far been more useful in determining lower bounds.

12.1 Arithmetic circuits and complexity classes

Recall the definitions regarding circuits from Definition 8.1.

Definition 12.1 A circuit $\mathcal{C}$ is *weakly skew* if for each multiplication gate v, receiving the outputs of gates u, w, one of $\mathcal{C}_u, \mathcal{C}_w$ is disjoint from the rest of $\mathcal{C}$. (I.e., the only output of, say $\mathcal{C}_u$, is the edge entering v.) A circuit is *multiplicatively disjoint* if, for every multiplication gate v receiving the outputs of gates u, w, the subcircuits $\mathcal{C}_u, \mathcal{C}_w$ do not intersect.



Definition 12.2 Let (f_n) be a sequence of polynomials. We say

- $(f_n) \in \mathbf{VP}_e$ if there exists a sequence of formulas $\mathcal{C}_n$ of polynomial size calculating f_n.
- $(f_n) \in \mathbf{VP}_{ws}$ if there exists a sequence of weakly skew circuits $\mathcal{C}_n$ of polynomial size calculating f_n.
- $(f_n) \in \mathbf{VP}$ if there exists a sequence of multiplicatively disjoint circuits $\mathcal{C}_n$ of polynomial size calculating f_n.
- $(f_n) \in \mathbf{VP}_{nb}$ if there exists a sequence of circuits $\mathcal{C}_n$ of polynomial size calculating f_n.

These definitions agree with the standard ones, see [50]. In particular, for the first three, they require $\deg(f_n)$ to grow like a polynomial in n. The class **VNP** has a more complicated definition: (f_n) is defined to be in **VNP** if there exists a polynomial p and a sequence $(g_n) \in \mathbf{VP}$ such that

$$f_n(x) = \sum_{\epsilon \in \{0,1\}^{p(|x|)}} g_n(x, \epsilon).$$

Valiant's conjectures are:

Conjecture 12.3 (Valiant)[75] $\mathbf{VP} \neq \mathbf{VNP}$*, that is, there does not exist a polynomial size circuit computing the permanent.*

Conjecture 12.4 (Valiant)[75] $\mathbf{VP}_{ws} \neq \mathbf{VNP}$*, that is* $dc(\mathrm{perm}_m)$ *grows faster than any polynomial.*

12.2 Complete problems

The reason complexity theorists love the permanent so much is that it counts the number of perfect matchings of a bipartite graph, a central counting problem in combinatorics. It is *complete* for the class **VNP**. A sequence is complete for a class if it belongs to the class and any other sequence in the class can be reduced to it at the price of a polynomial increase in size.

The sequence of polynomials given by iterated matrix multiplication of 3×3 matrices, $IMM_3^n \in S^n(\mathbb{C}^{9n})$ where $IMM_3^n(X_1, \ldots, X_n) = \mathrm{trace}(X_1 \cdots X_n)$ is complete for $\mathbf{VP}_e$, see [4].

The complexity class $\mathbf{VP}_{ws}$ is not natural from the perspective of complexity theory. It exists only because the sequence $(\det_n)$ is $\mathbf{VP}_{ws}$-complete, however, there exists a more natural (from the perspective of complexity theory) class, called **VQP** (see, e.g, [10, §21.5]) for which it is also complete.

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