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Geometric Considerations for the Design of Rigid Origami Structures

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Abstract

Transformable polyhedral surfaces with rigid facets, i.e., rigid origami, are useful for designing kinetic and deployable structures. In order to apply rigid origami to various architectural and other engineering design purposes, it is essential to consider the geometry of origami in kinetic motion and provide sufficiently generalized methods to produce controlled variations of shapes that suit the given design conditions. In this paper, we introduce the author's recent studies and their extensions on the geometry of rigid origami for designing transformable and deployable structures.

Keywords: origami, deployable structure, transformable polyhedron, kinematics, form-finding

1 Introduction

Rigid-foldable origami, or rigid origami, is a piecewise linear developable surface that can realize a deployment mechanism if its facets and foldlines are substituted with rigid panels and hinges, respectively. Designing such a deployment mechanism has a significant meaning in an engineering context, particularly in architecture in the following reasons:

1. The structure based on a watertight surface is suitable for constructing an envelope of a space, a roof, or a facade.
2. Purely geometric mechanism that does not rely on the elasticity of materials can realize robust kinetic structure in a larger scale under gravity.
3. The transformation of the configuration is controlled by smaller number of degrees of freedom. This enables a semi-automatic deployment of the structure.

Several designs of rigid-origami structures have been proposed from around 1970's. For example, the developable double corrugation surface, or Miura-ori [1], is a well-known rigid origami structure utilized in the packaging of deployable solar panels for use in space or in the folding of maps (Figure 1). This provides a one-DOF mechanism from a developed state to a flat-folded state. Resch and Christiansen [2] have proposed a kinetic plate mechanism that forms a three-dimensional dome that is folded out of a sheet of a panel and folds into another three-dimensional state without curvature (Figure 2). Hoberman proposed several rigid-foldable surfaces based on symmetric operations, e.g., [3], although he does clearly distinguish rigid and non-rigid foldable patterns.
In spite of these proposals, freely applying rigid origami to actual designs of architectural space has been unachieved thus far, the reason of which includes the lack of designability in the existing methods. Since rigid origami transforms in a

synchronized motion based on multiple non-linear constraints, the design of rigid origami is not a trivial problem given by an arbitrary design approach without geometric considerations. However, starting from a known pattern of origami and just applying them to architectural purposes does not work either because the model cannot sufficiently adapt functional and environmental conditions required by an actual design context. Therefore it is important to show a general geometric methods to find forms based on the kinetic properties of origami, while enabling flexible design variations that preserve that properties, rather than relying on trial-and-error based approach. The objective of this study is to show basic considerations on the geometry of rigid origami and introduce novel methods to obtain the design variations based on them. In this paper, we will show the kinematics of rigid origami, the design methods to allow folding motion, and the design examples. Note that we limit our considerations to geometric ones and do not intend to analyze elastic and plastic behavior of the structure with specific materials.

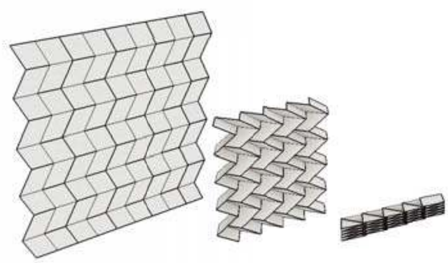


Figure 1: Miura-ori.

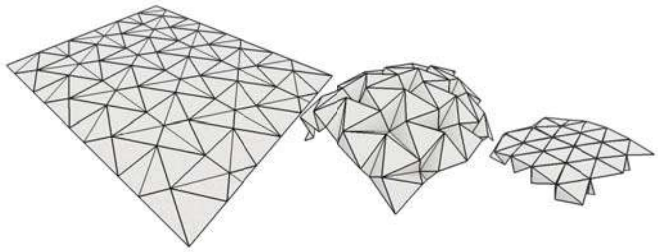


Figure 2: Ron Resch's pattern.

2 Kinematics of Rigid Origami

2.1 Model

The kinematics of origami can be represented by the unstable truss model or the rotational hinges model. The former represents the configuration of the structure by the positions of vertices. The change in the configuration is constrained by length preserving rigid bars along edges (creases and foldlines) and diagonals of facets ($2(k-3)$ bars for a planar k-gonal facet); this model is used by Resch and Christiansen [2], and it is suitable for directly using the points positions in a non-singular state. The latter represents the configuration by the rotational angles of edges and asserts the constraints so that closed loops cannot separate; this model gives comparatively robust simulation method such as *Rigid Origami Simulator* [4].

Here, we use the latter model for understanding the concept of rigid foldability. As described, the configuration is represented by their folding angles $\boldsymbol{\rho}$, which are constrained by any closed strip of facets being not separated by the folding motion. If we assume that a surface is a disk, the closure of any loop can be reduced to the combination of local constraints around interior vertices. For each interior vertex, we can use an (altered) form of the rotational matrix condition introduced by Belcastro and Hull [5]. For each interior vertex and its incident foldlines of fold angles $\rho_1, \ldots, \rho_n$,

$$\mathbf{R}(\rho_1,\cdots,\rho_n)=\chi_1\cdots\chi_{n-1}\chi_n=\mathbf{I} \qquad (1)$$

where 3×3 matrices $\chi_1, \ldots, \chi_n$ represent rotation by fold lines; each of them is represented as

$$\chi_i = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\rho_i & -\sin\rho_i \\ 0 & \sin\rho_i & \cos\rho_i \end{bmatrix} \begin{bmatrix} \cos\theta_i & -\sin\theta_i & 0 \\ \sin\theta_i & \cos\theta_i & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{2}$$

where θ_i is the sector angle between the foldlines i and $i + 1$ (Figure 3 Left). Since i is a rotational matrix, this fundamentally reduces to 3 scalar equations by using elements
$\mathbf{R}(2, 3)$, $\mathbf{R}(3, 1)$, $\mathbf{R}(1, 2)$. The constraints for the global model with N_{Vi} interior vertices can be represented by a $3N_{\mathrm{Vi}}$-vector equation $\mathbf{F} = \mathbf{0}$. Therefore, the infinitesimal motion can be represented by the solution space of the Jacobian matrix, which is a $3N_{\mathrm{Vi}} \times N_{\mathrm{Ei}}$ matrix where N_{Ei} is the number of foldlines (or interior edges).
Note that in a general case of orientable manifold with (possibly multiple) boundary, we can have different number of constraints. If the surface has N_{L} hole(s), we obtain extra N_{L} loop constraints to preserve the connectivity of the loops. (We additionally suggest that in the case of arbitrary manifold, these loops are the cycles that form a homology basis of the manifold.) Each loop constraint is comprised of 6 equations derived from 3 equations of rotational matrix and 3 equations of transition along the rotated edges. The additional 3 equations can be represented as, for a loop of n facets around the hole,

$$\sum_{k=1}^{n}\left(\prod_{i=1}^{k}\chi_i\right)\mathbf{d}_k = \mathbf{0}, \tag{3}$$

where χ_i and $\mathbf{d}_i$ represent the rotation matrix of the i-th facet orientation and the vector representing the difference of i-th and i+1-th origins in a local coordinates (Figure 3 Right). Therefore the number of constraints is $3N_{\mathrm{Vi}} + 6N_{\mathrm{L}}$.

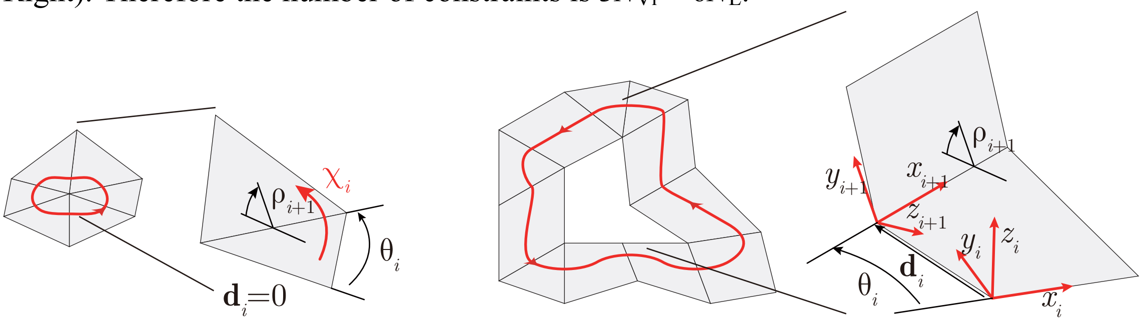


Figure 3: Constraints around loops and vertices

2.2 Rigid Foldability

Infinitesimal Rigid Foldability: The kinetic property of origami can be considered from the viewpoints of stability. By using the Jacobian matrix, we can investigate rigid foldability based on numerically analyzing infinitesimal motion. If the matrix is not singular, the constraint gives under-constrained system with $N_{\mathrm{Ei}} - 3N_{\mathrm{Vi}} - 6N_{\mathrm{L}}$ dimensional solution space. In general, if the number of redundant constraints is N_{S}, the solution space has $N_{\mathrm{Ei}} - 3N_{\mathrm{Vi}} - 6N_{\mathrm{L}} + N_{\mathrm{S}}$ dimensions. The kinetic motion is given by solving the Jacobian as,

$$\Delta\boldsymbol{\rho} = \left[\mathbf{I}_{N_{\mathrm{Ei}}} - \frac{\partial \mathbf{F}}{\partial \boldsymbol{\rho}}^{+} \frac{\partial \mathbf{F}}{\partial \boldsymbol{\rho}} \right] \Delta\boldsymbol{\rho}_0 , \tag{4}$$

where $[\partial\mathbf{F}/\partial\boldsymbol{\rho}]^{+}$ is the Moore-Penrose generalized inverse, or pseudo-inverse of Jacobian matrix. Here, $\Delta\boldsymbol{\rho}_0$ represents an arbitrary infinitesimal folding motion, and the equation solves the solution $\Delta\boldsymbol{\rho}$ closest to $\Delta\boldsymbol{\rho}_0$. In order to simulate the folding motion of rigid origami, we can apply Euler integration of this infinitesimal motion while eliminating the residuals using Newton-Raphson method.

As described, surface attains additional infinitesimal DOFs when the Jacobian matrix is singular. It is known that the constraint vector of one vertex is the direction cosines of crease line incident to the vertex [4, 6]. This indicates that the Jacobian matrix becomes singular when all the vectors are co-planar, which happens in flat states. In fact, every vertex of origami can infinitesimally transform arbitrarily along the normal of the surface in the developed state, although such a transformation is not valid in a finite sense and cannot be applied for structural designs.

Finite Rigid Foldability: Here, in order to use the kinetic behavior for a transformation mechanism, we require that the transformation is finite, i.e., not shaky. The finite folding motion is ensured by keeping the degrees of freedom of the mechanism positive throughout the transformation. If any part of panels is not touching each other, the degrees of freedom is only given by the Jacobean matrix: DOF = $N_{\mathrm{Ei}} - 3N_{\mathrm{Vi}} - 6N_{\mathrm{L}} + N_{\mathrm{S}}$. Here, note that in the discussion of finite foldability, the singularity must be independent of folding configuration and be preserved through the transformation, thus we cannot use singularity coming from co-planarity of edges. If we use Euler's polyhedral formula, this can be written as,

$$\mathrm{DOF} = N_{\mathrm{Eo}} - 3N_{\mathrm{L}} - 3 + N_{\mathrm{S}} - \sum_{k=4} (k-3)\{\text{num of } k\text{ - gon facet}\}, \tag{5}$$

where N_{Eo} is the number of edges on the boundary.

From this consideration, we can understand that the flexibility of rigid origami mainly comes from the flexibility of the boundary of the surface. Therefore, a closed triangulated polyhedron is normally not foldable, since its DOF is N_{S} (its kinetic behavior is equivalent to that of a disk of $N_{\mathrm{Eo}} = 3$). Examples of rigid foldable closed polyhedra [7, 8] are known to have volume preserving singularity as conjectured as the bellows conjecture by Connelly *et al.* [9].

3 Triangle Based Design

The most flexible design comes from a triangular mesh. If we assume no singularity, the degrees of freedom is represented as

$$\mathrm{DOF} = N_{\mathrm{Eo}} - 3N_{\mathrm{L}} - 3 . \tag{6}$$

In this case, the kinetic motion of a mesh is totally controlled by the boundary configuration. For example, a design of a triangular mesh shell with 6 boundary edges and 3 pin hinged legs — let us call this structure *triangulated tripod* — can be useful. The total number of degrees of freedom of the structure is 9 because the structure has 6 degrees of rigid body motion and 3 degrees of transformation mechanism. Since the structure is pinned at their legs, it constrains 3 × 3 degrees of freedom, which makes the overall structure statically determinate. Therefore the structure transforms according to the positions of the legs as shown in Figure 4, and it becomes a static structure once the legs are fixed.

We can design the variations of triangulated tripods in a comparatively easy way since the basic property of the structure is ensured by the number of boundary elements (Figure 5). However, the global behavior such as the possible range of folding and the existence of a path from a state to another is determined by the configuration space of the structure. This is only understood through examining the infinitesimal behavior at every possible state of the specific model through exploring the configuration space with simulational methods. In order to simulate the kinematics of the structure with pin constraints, we can use either the unstable truss model or rotational hinges model with inverse kinematics.

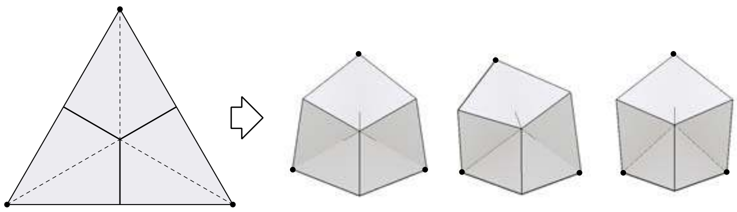
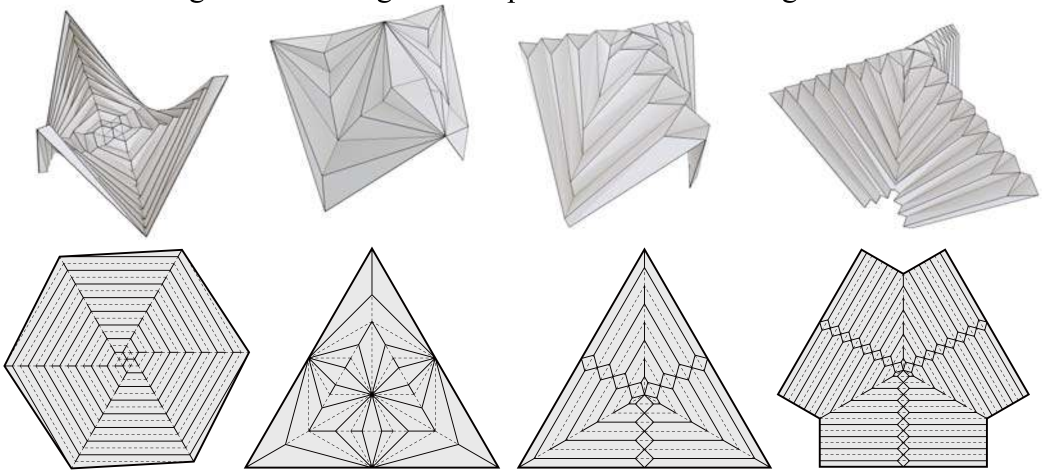


Figure 4: A triangulated tripod of different configurations.



(a) (b) (c) (d)

Figure 5: Examples of triangulated tripods. Quadrilateralpanels are all triangulated.

4 Quadrilateral Based Design

Another approach for designing a rigid-foldable structure is to use a quadrilateral mesh such as Miura-ori. A quadrilateral mesh origami is normally not rigid-foldable since the degrees of freedom is given as

$$\mathrm{DOF} = N_{\mathrm{Eo}} - 3N_{\mathrm{L}} - N_{\mathrm{facets}} + 3 + N_{\mathrm{S}}, \tag{7}$$

Here, the number of facets N_{facets} basically increases proportional to the square of N_{Eo} for a normal two dimensional mesh. Therefore, a rigid-foldable quadrilateral mesh origami must rely on the singularity. Such a structure has an engineering advantage as follows:

1. The mechanism has exactly 1 degree of freedom; thus the transformation can be controlled by one actuator independent of the complexity of the overall surface.
2. The structure is redundant; this enables a robust mechanism that works even when we remove several elements from the surface.

Finding the singularity that works throughout the finite transformation is not a trivial problem, especially when designing a freeform. A general condition for this singularity is not yet been revealed so far. Therefore, we propose a design approach to generalize a pattern known to rigid fold while keeping its intrinsic symmetry. Typical examples of rigid-foldable quadrilateral mesh surfaces are Miura-ori and the “eggbox” pattern [10]. These patterns are known to be generalizable to some extent: the rigid-foldability of the generalized form of the former is investigated by Tachi [11] as a flat-foldable 4-valent mesh origami and the latter by Schief *et al.* [12] as a discrete Voss surface. The mechanisms of the vertices of these rigid-foldable structures are essentially identical, and we can produce a hybrid rigid-foldable surface that is general enough to produce a freeform.

4-Valent Mesh Origami: The condition for the rigid-foldability of quadrilateral mesh flat-foldable origami is investigated [11]. This is written in a general way: a polyhedral surface homeomorphic to a disk composed of planar facets connected by 4-valency developable and flat-foldable vertices is rigid-foldable if and only if there exists a valid state where every foldline is semi-folded (not 0 or $\pm\pi$). In other words, in this type of structure, the existence of a continuous transformation is equivalent to the existence of an intermediate state. This leads to the design method based on obtaining one valid intermediate state that satisfies the following:

1. Every vertex is developable.
2. Every vertex is flat-foldable
3. Every facet is planar.

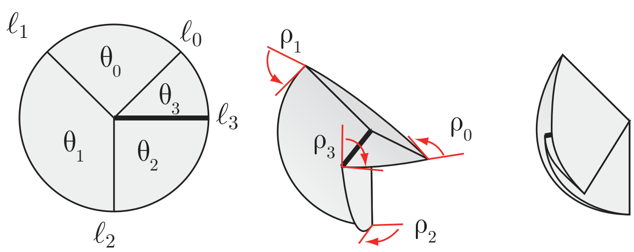


Figure 6: Single vertex of origami.

The former two conditions can be represented as:

$$\theta_0 = \pi - \theta_2 \text{ and } \theta_1 = \pi - \theta_3, \tag{8}$$

where θ_i (i = 0, 1, 2, 3) are the sector angles incident to the vertex. If we obtain one valid configuration represented by folding angles $\{ \rho_i^{\,0} \}$, the continuous transformation is represented by,

$$\left\{ \tan\frac{\rho_i(t)}{2} \right\} = \left\{ \tan\frac{\rho_i^{\,0}}{2} \right\} \frac{\tan(t/2)}{\tan(t_0/2)}, \tag{9}$$

where t ($0 \le t \le \pi$) is the parameter that defines the folding amount ($t = 0$, $t = t_0$, and $t = \pi$ indicate developed, the intermediate, and flat-folded states, respectively).

Discrete Voss Surface: Discrete Voss surface is a planar quadrilateral mesh surface composed of degree-4 vertices each of which satisfies

$$\theta_0 = \theta_2 \text{ and } \theta_1 = \theta_3 . \tag{10}$$

The rigid-foldability of the discrete Voss surface is proved by Schief *et al*. [12]. Here, notice the similarity to Eq. (8). In fact, discrete Voss vertex and flat-foldable vertex are essentially identical in a local sense, and we can construct rigid-foldable hybrid surfaces by combining them. Figure 8 shows an example design of hybrid surface obtained by solving the combined conditions of discrete Voss and flat-foldable origami vertices via the perturbation-based method as used in [11].

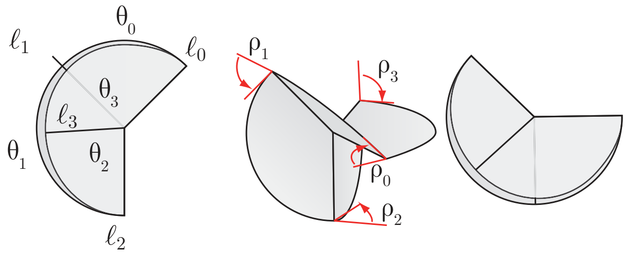


Figure 7: Single vertex of discrete Voss surface.

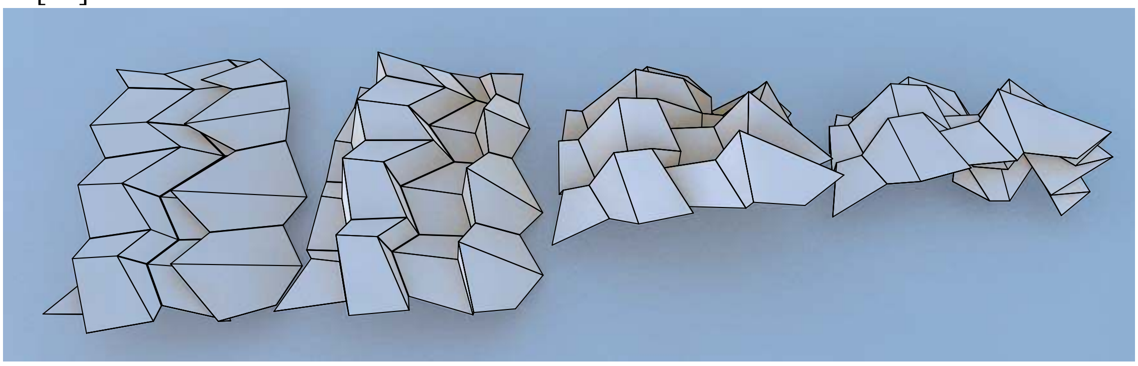


Figure 8: Rigid folding motion of a generalized hybrid of Miura-ori and eggbox pattern.

4.1 Cylindrical Structure: Topological Extension

The rigid-folding motion of a disk surface is ensured by the existence of angle configuration that satisfies the local conditions around each vertex. However, a non-disk surface, such as a cylinder, cannot always rigid fold because of the conditions along the loop around each hole. The exact condition for rigid-foldable loop is not yet revealed in a general way. We thus start from obtaining a valid rigid-foldable cylinder based on symmetry, and then generalize them using the symmetry operations as proposed in [13]. In this proposition of rigid-foldable cylinders, repeating symmetry is used to construct a cylinder from a modular loop. However, we have recently found that a more generalized rigid-foldable cylindrical form can be created using the fact that the rigid folding condition is represented by the combination of local conditions and the loop conditions [14] (Figure 9). The design process is as follows: first, we produce a repeating form of cylinder using the isotropic type proposed in [13], which is composed of flat-foldable origami and discrete Voss vertices; then we fix one of the modules to construct a valid loop; finally, we transform the other part of the pattern under the constraints used to build the rigid-foldable quadrilateral mesh disk.

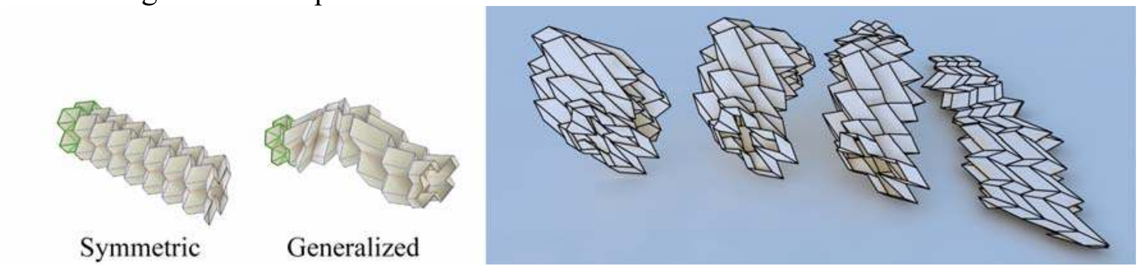


Figure 9: Rigid foldable cylinder.

5 Design Example

The objective of our study is to generalize the geometric conditions required for kinetic structures, thereby enabling a system that a designer can find forms that sufficiently adapt the design context and required functionalities at the same time. We show a hypothetical design example of kinetic architectural space based on generalized degree-4 vertex origami to demonstrate how our approach can potentially useful for architectural design.

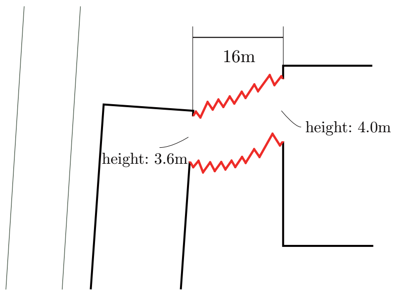


Figure 10: Layout.

Consider building a foldable space by connecting the openings of existing two separate buildings having different sizes and orientations as in Figure 10. Also, the distance between two buildings and the height of the openings are too large for the foldable structure to be made of flexible materials. Therefore, the static form of the structure must flexibly adapt the existing environment (the openings and the ground) while the structure must follow a rigid mechanism (rigid-foldable and flat-foldable).

We solve this problem using the generalized rigid-foldable 4-valent origami. In this method, the problem is translated to a problem of obtaining a valid mesh represented by vertex coordinates **x** that satisfies

1. developable: for each interior vertex, $\theta_0 + \theta_1 + \theta_2 + \theta_3 = 2\pi$
2. flat-foldable: for each interior vertex, $\theta_0 - \theta_1 + \theta_2 - \theta_3 = 0$
3. planar: for each non-foldline edge, $\rho = 0$
4. fixed boundary: for each vertex on the openings, $(x, y, z) = (x_{\text{target}}, y_{\text{target}}, z_{\text{target}})$
5. ground: for each vertex on the boundary, $z = 0$

Let us represent these conditions as a non-linear vector equation $\mathbf{c}(\mathbf{x}) = 0$. We start from a known regular form of origami vault with 68 vertices as shown in Figure 11 (a), which is not a valid form ($\mathbf{c} \neq \mathbf{0}$). Then we modify this shape so that it satisfies the above conditions by optimization based approach (Newton-Raphson method) (Figure 11 (b)). From this valid solution, we can modify the form based on perturbation along the kernel of the Jacobian matrix. The method for obtaining an infinitesimal transformation is the same as the simulation method; however we allow the pattern itself to be changed in this case.

$$\Delta \mathbf{x} = \left[\mathbf{I}_{3N_v} - \frac{\partial \mathbf{c}}{\partial \mathbf{x}}^{+} \frac{\partial \mathbf{c}}{\partial \mathbf{x}} \right] \Delta \mathbf{x}_0 \tag{11}$$

In this design example, the configuration is represented by 68 × 3 = 204 variables, while the developability, flat-foldability, and planarity conditions give 30 + 30 + 48 = 108 equations, and the fixed boundary and ground conditions give 24 + 30 = 54 equations. Therefore, the design space is 42-dimensional, and we can explore the design variations within this space. By specifying $\Delta \mathbf{x}_0$ via a graphical user interface, we obtained variational shapes that follow the designer's preference, in this example we have chosen design (c).

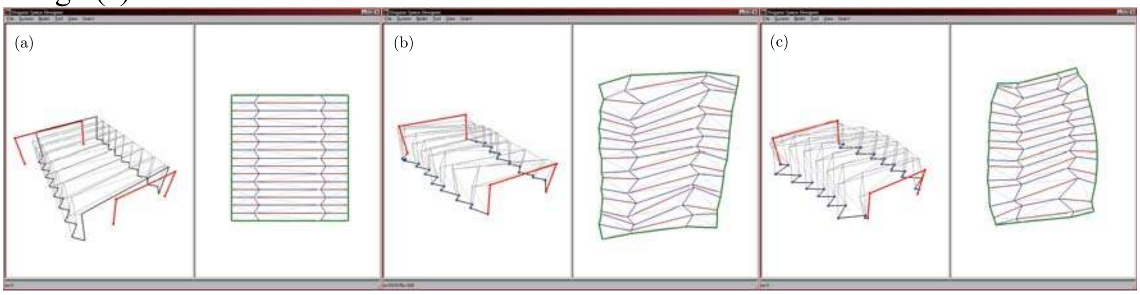


Figure 11: Design Process. (a) Initial regular pattern that does not connect the openings of two buildings shown in red. (b) Transformed pattern that fit the the openings. The boundary points shown in blue are placed on xy-plane. (c) Variational design that satisfies the conditions.

The structure can be manufactured from double layered panels of constant thickness by sufficiently offsetting the boundary shape such that the panels do not collide each other by the folding motion (Figure 12) as presented in [15]. The cutting pattern of the panels is shown in Figure 13. The perspective view of folding motion of the resulting structure is as shown in Figures 14 and 15.

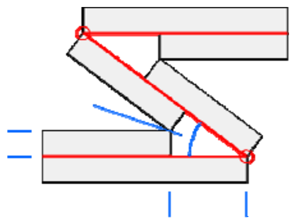


Figure 12: Thickening.

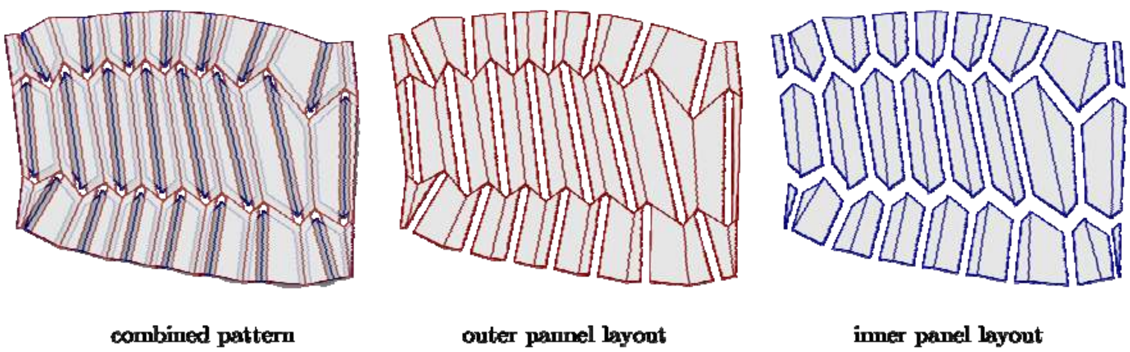


Figure 13: Panel layouts

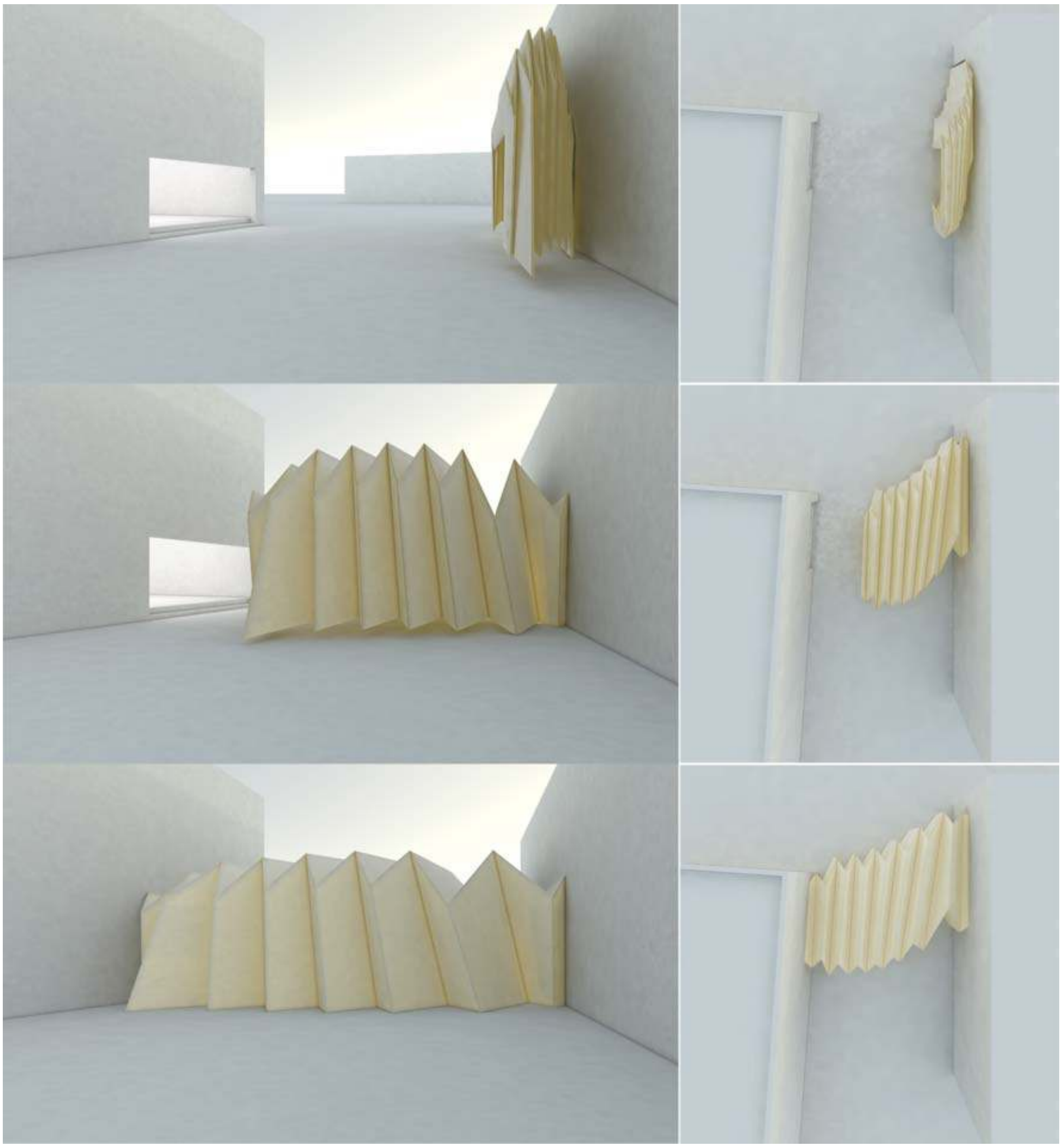


Figure 14: Folding motion of the structure.

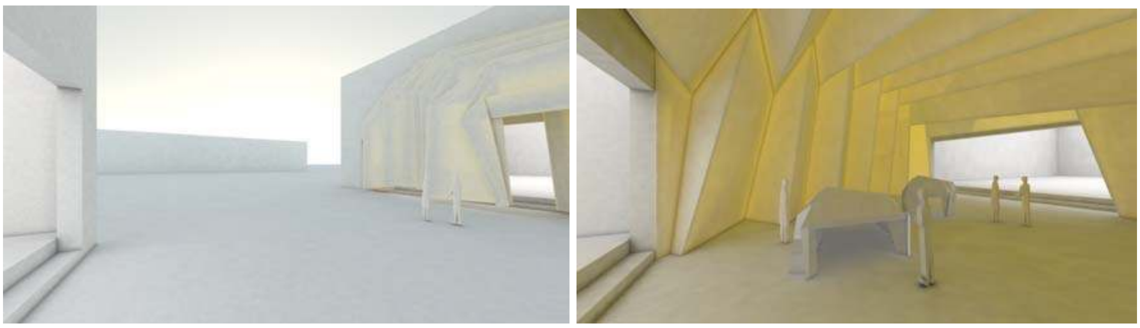


Figure 15: Perspective view.

6 Conclusion and Future Works

In this paper, we presented geometric problems and their solutions for freely designing rigid-foldable structures. Based on the consideration on rigid-foldability, we presented two basic approaches: using triangles and quadrilaterals mesh.

1. Triangular patterns produce kinetic structures whose degrees of freedom are determined by the number of elements on the boundary; using this fact we proposed the design concept of triangulated tripod.
2. Quadrilateral patterns can produce one-DOF kinetic motion based on redundant constraints. Design of such a structure take advantage of the singularity of the pattern. We have shown an approach based on flat-foldable origami and discrete Voss vertices, which can sufficiently generalize flat-foldable disk and cylindrical surfaces.
3. A design example using quadrilateral-based rigid origami is shown to demonstrate the high flexibility of the design method.

At the same time, our study indicates many future studies necessary for making rigid origami structures more designable and realizable. The following shows some of the examples of such studies.

1. The global kinetic behavior of the structures is not fully investigated. In particular, understanding the transformability from one state to the other is very important when applying rigid origami to engineering purposes.
2. Classifying every rigid-foldable quadrilateral mesh is expected to improve the designabilty of rigid origami. This is still an open problem and theoretical studies in this direction can be seen in [15].
3. Investigating rigid-foldability condition around a loop can topologically extend the concept of origami.
4. Enabling kinetic constraints to work with rigid origami structures can contribute to the design of rigid origami combined with different mechanical systems.

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