Proceedings Chapter

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Reference

HAUSMANN, Jean-Claude. Geometric descriptions of polygon and chain spaces. In: Farber, Michael, Ghrist, R., Burger, M., Koditschek, D. *Topology and robotics*. Providence, R.I. : American Mathematical Society, 2007. p. 47-57

arxiv : math/0702521

Geometric descriptions of polygon and chain spaces

Jean-Claude HAUSMANN

Abstract. We give a few simple methods to geometically describe some polygon and chain spaces in $\mathbb{R}^d$. They are strong enough to give tables of m-gons and m-chains when $m \leq 6$.

Introduction

For $a = (a_1, \ldots, a_m) \in \mathbb{R}^m_{>0}$ and d an integer, define the subspace $\mathcal{C}^m_d(a)$ of $\prod_{i=1}^{m-1} S^{d-1}$ by

$$\mathcal{C}^m_d(a) = \big\{ z = (z_1, \ldots, z_{m-1}) \in \prod_{i=1}^{m-1} S^{d-1} \mid \sum_{i=1}^{m-1} a_i z_i = a_m\, e_1 \big\} ,$$

where $e_1 = (1, 0, \ldots, 0)$ is the first vector of the standard basis $e_1, \ldots, e_d$ of $\mathbb{R}^d$. An element of $\mathcal{C}^m_d(a)$, called a *chain*, can be visualized as a configuration of $(m-1)$-segments in $\mathbb{R}^d$, of length $a_1, \ldots, a_{m-1}$, joining the origin to $a_m e_1$. The group $O(d-1)$, seen as the subgroup of $O(d)$ stabilizing the first axis, acts naturally (on the left) on $\mathcal{C}^m_d(a)$. The quotient space by $SO(d-1)$ coincides with the *polygon space*

$$\begin{aligned} \mathcal{N}^m_d(a) &= SO(d-1)\backslash \mathcal{C}^m_d(a) \\ &\approx SO(d) \Big\backslash \big\{ \rho = (\rho_1, \ldots \rho_m) \in (\mathbb{R}^d)^m \mid |\rho_i| = a_i \text{ and } \textstyle\sum_{i=1}^m \rho_i = 0 \big\} . \end{aligned}$$

The notations are that of [**HR04**] where it is emphasized how the union $\mathcal{N}^m_d$ of $\mathcal{N}^m_d(a)$ for all $a \in \mathbb{R}^m_{>0}$ is related to the spaces studied in statistical shape analysis (see, e.g. [**KBCL99**]). An element $a \in \mathbb{R}^m_{>0}$ is *generic* if $\mathcal{C}^m_1(a) = \emptyset$, that is to say there is no lined chain or polygon configuration. When a is generic, $\mathcal{C}^m_d(a)$ is a smooth closed manifold of dimension $(m-2)(d-1)-1$ (see, e.g. [**Ha89**]).

Mathematical robotics is specially interested in the chain and polygon spaces for $d = 2, 3$. When a is generic, the action of $SO(d-1)$ on $\mathcal{C}^m_d$ is then free and therefore $\mathcal{N}^m_2(a)$ and $\mathcal{N}^m_3(a)$ are closed smooth manifolds of dimension $m-3$ and $2(m-3)$ respectively (in addition, $\mathcal{N}^m_3(a)$ carries a symplectic structure, see e.g. [**KM96**]). One has $\mathcal{C}^m_2(a) = \mathcal{N}^m_2(a)$ and $\mathcal{C}^m_3(a) \to \mathcal{N}^m_3(a)$ is a principal circle bundle.

In this paper, we present a few geometrical methods permitting us to describe in some cases the spaces $\mathcal{C}^m_d(a)$ and $\mathcal{N}^m_d(a)$. From the classification results (see

1991 *Mathematics Subject Classification.* Primary 55R80, 70G40; Secondary 57R65.

Section 1), this enables us to describe all the chain or polygon spaces in $\mathbb{R}^d$ when $m \leq 6$ (tables in Section 3).

1. Review of the classification results

The idea of the classification of the polygon and chain spaces goes back to [**Wa85**]. Details may be found in [**HR04**].

1.1. *Short subsets.* Let $a = (a_1, \dots, a_m) \in \mathbb{R}^m_{>0}$. A subset J of $\{1, \dots, m\}$ is called *short* if $\sum_{i \in J} a_i < \sum_{i \notin J} a_i$. Short subsets form, with inclusion, a poset $\mathcal{S}(a)$. Define $\mathcal{S}_m(a) = \{J \in \mathcal{S}(a) \mid m \in J\}$.

Lemma 1.2. *Let a and a' be generic elements in $\mathbb{R}^m_{>0}$. Suppose that $\mathcal{S}_m(a)$ and $\mathcal{S}_m(a')$ are poset isomorphic. Then:*

(i) *$\mathcal{C}^m_d(a)$ and $\mathcal{C}^m_d(a')$ are $O(d-1)$-equivariantly diffeomorphic.*
(ii) *$\mathcal{N}^m_d(a)$ and $\mathcal{N}^m_d(a')$ are diffeomorphic.*

Proof: If $\mathcal{S}_m(a) \approx \mathcal{S}_m(a')$, then there is a poset isomorphism $\varphi\colon \mathcal{S}(a) \xrightarrow{\approx} \mathcal{S}(a')$ with $\varphi(m) = m$ (see [**HK98**, Proposition 2.5]). It is well known that $\mathcal{S}(a) \approx \mathcal{S}(a')$ implies (ii) (see, e.g. [**HK98**, Proposition 2.2] or [**HR04**, Theorem 1.1]). We give however the variation of the proof to get the less classical (stronger) fact that $\mathcal{S}(a) \approx \mathcal{S}(a')$ implies (i).

Let $\mathcal{K}_d(a) = \{z = (z_1, \dots, z_m) \in \prod_{i=1}^m S^{d-1} \mid \sum_{i=1}^m a_i z_i = 0\}$. The group $O(d)$ acts on the left on $\mathcal{K}_d(a)$ and $\mathcal{N}^m_d(a) = SO(d)\backslash\mathcal{K}_d(a)$. The function $F : \mathcal{K}_d(a) \to S^{d-1}$ given by $F(z) = z_m$ is a submersion (since F is $O(d)$-equivariant). One has $\mathcal{C}^m_d(a) = F^{-1}(-e_1)$ with its residual $O(d-1)$-action.

Let σ be the bijection of $\{1, \dots, m-1\}$ giving the poset isomorphism $\mathcal{S}_m(a) \xrightarrow{\approx} \mathcal{S}_m(a')$ and then $\mathcal{S}(a) \xrightarrow{\approx} \mathcal{S}(a')$. Then $(z_1, \dots, z_{m-1}, z_m) \mapsto (z_{\sigma(1)}, \dots, z_{\sigma(m-1)}, z_m)$ induces a $O(d-1)$-equivariant diffeomorphism from $\mathcal{C}^m_d(a_1, \dots, a_{m-1}, a_m)$ onto $\mathcal{C}^m_d(a_{\sigma(1)}, \dots, a_{\sigma(m-1)}, a_m)$. We can therefore suppose that $\mathcal{S}(a) = \mathcal{S}(a')$ and $\sigma = \mathrm{id}$. We claim that $\mathcal{C}^m_d(a)$ and $\mathcal{C}^m_d(a')$ are then canonically diffeomorphic. Indeed, if $\mathcal{S}(a) = \mathcal{S}(a')$, the segment $[a, a']$ contains only generic elements. Hence, the union

$$X = \bigcup_{b \in [a,a']} \big(\mathcal{K}_d(b) \times \{b\}\big) \subset \big(\prod_{i=1}^m S^{d-1}\big) \times [a, a']$$

is an $O(d)$-cobordism between $\mathcal{K}_d(a)$ and $\mathcal{K}(a')$ and the projection $\pi : X \to [a, a']$ has no critical point. One still has the map $F : X \to S^{d-1}$ given by $F(z, t) = z_m$ and $Y = F^{-1}(-e_1)$ is an $O(d-1)$-cobordism between $\mathcal{C}^m_d(a)$ and $\mathcal{C}^m_d(a')$, with again the projection π over $[a, a']$ being a submersion. The standard metric on $\prod_{i=1}^{m-1} S^{d-1}$ induces an $O(d-1)$ invariant Riemannian metric on Y. Following the gradient lines of π for this metric gives the required $O(d-1)$-equivariant diffeomorphism $\Psi : \mathcal{C}^m_d(a) \xrightarrow{\approx} \mathcal{C}^m_d(a')$. □

1.3. *Walls and chambers.* For $J \subset \{1, \dots, m\}$), let $\mathcal{H}_J$ be the hyperplane (*wall*) of $\mathbb{R}^m$ defined by

$$\mathcal{H}_J := \Big\{(a_1, \dots, a_m) \in \mathbb{R}^m \;\Big|\; \sum_{i \in J} a_i = \sum_{i \notin J} a_i\Big\}.$$

The union $\mathcal{H}(\mathbb{R}^m)$ of all these walls determines a set $\mathrm{Ch}((\mathbb{R}_{>0})^m)$ of open *chambers* in $(\mathbb{R}_{>0})^m$ whose union is the set of generic elements. Two generic elements a and

a' are in the same chamber if and only if $\mathcal{S}(a) = \mathcal{S}(a')$. We call $\mathrm{Ch}(a)$ the chamber of a generic element a. If α is a chamber, the poset $\mathcal{S}(a)$ is the same for all $a \in \alpha$ and is denoted by $\mathcal{S}(\alpha)$.

1.4. *Permutations.* Let σ be a permutation of $\{1, \dots, m\}$. The map which sends $(z_1, \dots, z_m)$ to $(z_{\sigma(1)}, \dots, z_{\sigma(m)})$ induces a diffeomorphism from $\mathcal{N}_d^m(a_1, \dots, a_m)$ onto $\mathcal{N}_d^m(a_{\sigma(1)}, \dots, a_{\sigma(m)})$. For the sake of the classification of $\mathcal{N}_d^m(a)$, we may as well assume that $a \in \mathbb{R}^m_{\nearrow}$ where

$$\mathbb{R}^m_{\nearrow} := \{(a_1, \dots, a_m) \in \mathbb{R}^m \mid 0 < a_1 \leq \cdots \leq a_m\}\,.$$

Observe that we then do not classify all the chain spaces $\mathcal{C}_d^m(a)$ but only those for which $a_m \geq a_i$ for $i < m$. Indeed, the permutation σ induces a diffeomorphism from $\mathcal{C}_d^m(a_1, \dots, a_m)$ onto $\mathcal{C}_d^m(a_{\sigma(1)}, \dots, a_{\sigma(m)})$ if and only if $\sigma(m) = m$. We denote by $\mathrm{Ch}(\mathbb{R}^m_{\nearrow})$ the set of chambers determined in $\mathbb{R}^m_{\nearrow}$ by the hyperplane arrangement $\mathcal{H}(\mathbb{R}^m)$.

1.5. *The genetic code of a chamber.* A chamber $\alpha \in \mathrm{Ch}(\mathbb{R}^m_{\nearrow})$ is determined by $\mathcal{S}(\alpha)$ which, in turn, is determined by $\mathcal{S}_m(\alpha)$. Consider the partial order "$\hookrightarrow$" on the subsets of $\{1, \dots, m\}$ where $A \hookrightarrow B$ if and only if there exits a non-decreasing map $\varphi : A \to B$ such that $\varphi(x) \geq x$. For instance $X \hookrightarrow Y$ if $X \subset Y$ since one can take φ being the inclusion. The *genetic code* of α is the set of elements $A_1, \dots, A_k$ of $S_m(\alpha)$ which are maximal with respect to the order "$\hookrightarrow$". Thus, the chamber α is determined by its genetic code; we write $\alpha = \langle A_1, \dots, A_k\rangle$ and call the sets A_i the *genes* of α. As, in this paper $m \leq 9$, we abbreviate a subset A by the sequence of its digits, e.g. $\{6, 2, 1\} = 621$. In [**HR04**], an algorithm is presented to list by their genetic codes all the elements of $\mathrm{Ch}(\mathbb{R}^m_{\nearrow})$ and then all the chambers up to permutation of the components. Tables for $m \leq 6$ are given in [**HR04**] (and in Section 3 below); more tables, for $m \leq 9$, may be found in [**HRWeb**]. The algorithm produces, in each chamber α, a representative $a_{\min}(\alpha) \in \alpha$; though this is not proved theoretically, $a_{\min}(\alpha)$ turned out in all known cases to have integral components a_i and minimal $\sum a_i$. See examples in the tables below.

2. Procedures of description

2.1. Adding a tiny edge. Let $a = (a_2, \dots, a_m)$ be a generic element of $\mathbb{R}^{m-1}_{\nearrow}$. If $\varepsilon > 0$ is small enough, the m-tuple $a^+ := (\delta, a_2, \dots, a_m)$ is a generic element of $\mathbb{R}^m_{\nearrow}$ for $0 < \delta \leq \varepsilon$. This defines a map $\mathrm{Ch}(\mathbb{R}^{m-1}_{\nearrow}) \xrightarrow{+} \mathrm{Ch}(\mathbb{R}^m_{\nearrow})$, sending α to α^+, which is injective (see [**HR04**, Lemma 5.1]). The genetic code of α^+ has the same number of genes than that of α and the correspondence goes as follows. If $\{p_1, \dots, p_r\}$ is a gene of α, then $\{p_1^+, \dots, p_r^+, 1\}$ is a gene of α^+, where $p_i^+ = p_i + 1$. For example: $\langle 631, 65\rangle^+ = \langle 7421, 761\rangle$. The minimal integral representative $a_{\min}(\alpha^+)$ of α^+ is a *conventional representative*: it starts with a 0 followed by the components of $a_{\min}(\alpha)$. Example: as $a_{\min}(\langle 3\rangle) = (1, 1, 1)$, then $a_{\min}(\langle 3\rangle^+) = a_{\min}(\langle 41\rangle) = (0, 1, 1, 1)$, $a_{\min}(\langle 41\rangle^+) = a_{\min}(\langle 521\rangle) = (0, 0, 1, 1, 1)$, *etc.* It has to be understood that these vanishing components stand for small enough positive real numbers, whose sum is less than 1.

Proposition 2.1. *There is a $O(d-1)$-equivariant diffeomorphism*

$$\Phi \colon \mathcal{C}_d^m(\alpha^+) \xrightarrow{\approx} S^{d-1} \times \mathcal{C}_d^{m-1}(\alpha)\,,$$

where $S^{d-1} \times \mathcal{C}_{d-1}^m(\alpha)$ is equipped with the diagonal $O(d-1)$-action.

PROOF: Let $a = (a_2, \dots, a_m) \in \alpha$ and $a^+ = (\varepsilon, a_2, \dots, a_m) \in \alpha^+$. The map Φ is of the form (Φ_1, Φ_2), where $\Phi_1 \colon \mathcal{C}_d^m(a^+) \to S^{d-1}$ and $\Phi_2 \colon \mathcal{C}_d^m(a^+) \to \mathcal{C}_d^{m-1}(a)$ are $O(d-1)$-equivariant maps. The map Φ_1 is just given by $\Phi_1(z_1, \dots, z_m) = z_1$. It remains to define Φ_2.

If $p \in \mathbb{R}^d$ satisfies $p \neq -|p|e_1$, there is a unique $R_p \in SO(d)$ such that $R_p(p) = |p|e_1$ and $R_p(q) = q$ if $q \in \mathrm{EV}(p, e_1)^\perp$, the orthogonal complement to the vector space $\mathrm{EV}(p, e_1)$ generated by p and e_1. In particular, $R_{e_1} = \mathrm{id}$. The map $p \to R_p$ is smooth. We shall apply that to $p = p(z)$, where

$$p(z) = \sum_{i=2}^{m} a_i z_i = a_m e_1 - \varepsilon z_1 \, .$$

We may suppose that $\varepsilon < a_m$, so $p(z) \neq -|p(z)|e_1$. The correspondence $(z_1, \dots, z_m) \mapsto (R_{p(z)} z_2, \dots, R_{p(z)} z_m)$ gives a smooth map

$$\Phi_2' \colon \mathcal{C}_d^m(a^+) \to \mathcal{C}_d^{m-1}(a_2, \dots, a_{m-1}, |p(z)|) \, .$$

The fact that $(\delta, a_2, \dots, a_m)$ is generic when $0 < \delta \leq \varepsilon$ implies that

$$\mathrm{Ch}(a_2, \dots, a_{m-1}, |p(z)|) = \mathrm{Ch}(a) \, .$$

We can then use the canonical $O(d-1)$-equivariant diffeomorphism

$$\Psi \colon \mathcal{C}_d^{m-1}(a_2, \dots, a_{m-1}, |p(z)|) \xrightarrow{\approx} \mathcal{C}_d^{m-1}(a_2, \dots, a_m)$$

constructed in the proof of Lemma 1.2 and define $\Phi_2 = \Psi \circ \Phi_2'$. If $A \in O(d-1)$, the formula $AR_p = R_{A(p)} A$ holds in $O(d-1)$, as easily seen on $\mathrm{EV}(p, e_1)$ and on $\mathrm{EV}(p, e_1)^\perp$. This implies that ϕ_2' is $O(d-1)$-equivariant.

We have thus constructed an $O(d-1)$-equivariant smooth map $\Phi \colon \mathcal{C}_d^m(\alpha^+) \to S^{d-1} \times \mathcal{C}_d^{m-1}(\alpha)$. The reader will easily figure out what the inverse Φ^{-1} of Φ is like, proving that Φ is a diffeomorphism. □

We now turn our interest to $\mathcal{N}_3^m(\alpha^+)$. Let $\mathcal{D}(\alpha)$ be the total space of the D^2-disk bundle associated to $\mathcal{C}_3^{m-1}(\alpha) \to \mathcal{N}_3^{m-1}(\alpha)$. We call *double of* $\mathcal{D}(\alpha)$ the union of two copies of $\mathcal{D}(\alpha)$, with opposite orientations, along their common boundary $\mathcal{C}_3^m(\alpha)$.

PROPOSITION 2.2.
(a) $\mathcal{N}_3^m(\alpha^+)$ *is diffeomorphic to* $S^2 \times_{S^1} \mathcal{C}_3^{m-1}(\alpha)$.
(b) $\mathcal{N}_3^m(\alpha^+)$ *is diffeomorphic to the double of* $\mathcal{D}(\alpha)$.

In Part (a), $S^2 \times_{S^1} \mathcal{C}_3^{m-1}(\alpha)$ denotes the quotient of $S^2 \times \mathcal{C}_3^{m-1}(\alpha)$ by the diagonal action of $S^1 = SO(2)$. The projection $S^2 \times_{S^1} \mathcal{C}_3^{m-1}(\alpha) \to \mathcal{N}_3^{m-1}(\alpha)$ is then the S^2-associated bundle to the $SO(2)$-principal bundle $\mathcal{C}_3^{m-1}(\alpha) \to \mathcal{N}_3^{m-1}(\alpha)$. A direct proof of Part (b) may be found in [**HR04**, Prop. 6.4].
PROOF: For Part (a), we check that the diffeomorphism

$$\Phi \colon \mathcal{C}_3^m(\alpha^+) \xrightarrow{\approx} S^2 \times \mathcal{C}_3^{m-1}(\alpha)$$

of Proposition 2.1 descends to a diffeomorphism from $\mathcal{N}_3^m(\alpha^+)$ to $S^2 \times_{S^1} \mathcal{C}_3^{m-1}(\alpha)$. For Part(b), we observe that $\mathcal{D}(\alpha)$ is the mapping cylinder of the projection $\mathcal{C}_3^{m-1}(\alpha) \to \mathcal{N}_3^{m-1}(\alpha)$. The double of $\mathcal{D}(\alpha)$ is then diffeomorphic to $M = [-1, 1] \times \mathcal{C}_3^{m-1}(\alpha)/\sim$, where "$\sim$" is the equivalence relation generated by $(-1, z) \sim (-1, Az)$ and $(1, z) \sim (1, Az)$ for all $z \in \mathcal{C}_3^{m-1}(\alpha)$ and all $A \in S^1$. Each S^1-orbit of S^2 has an unique point of the form $(u_1, 0, u_3)$. To $(u, z) \in S^2 \times \mathcal{C}_d^{m-1}(\alpha)$ with $u = (u_1, 0, u_3)$, we associate the class $[u_1, z]$ in M and check that this correspondence gives rise to a diffeomorphism from $S^2 \times_{S^1} \mathcal{C}_3^{m-1}(\alpha)$ to the double of $\mathcal{D}(\alpha)$. □

EXAMPLE 2.3. When $m = 3$, there is only one chamber $\alpha = \langle 3\rangle$, with $a_{\min}(\alpha) = (1, 1, 1)$, for which $\mathcal{C}_d^3(\alpha)$ is not empty. Its image under adding tiny edges gives a chamber $\langle\{m, m-3, m-2, \dots, 1\}\rangle \in \mathrm{Ch}(\mathbb{R}^m_\nearrow)$ with $a_{\min} = (0, \dots, 0, 1, 1, 1)$ (conventional representative, § 2.1). As $\mathcal{C}_d^3\langle 3\rangle = S^{d-2}$ with the standard $O(d-1)$-action, Propositions 2.1 and 2.2 give the following

The chamber $\alpha = \langle\{m, m-3, m-2, \dots, 1\}\rangle$

$a_{\min}(\alpha)$	$\mathcal{N}_2^m(\alpha)$	$\mathcal{N}_3^m(\alpha)$	$\mathcal{C}_d^m(\alpha)$
$(0, \dots, 0, 1, 1, 1)$	$T^{m-3} \amalg T^{m-3}$	$(S^2)^{m-3}$	$(S^{d-1})^{m-3} \times S^{d-2}$

REMARK 2.4. Let $A = \{m, m-3, m-2, \dots, 1\}$. We claim that $\alpha = \langle A\rangle$ as above is the only chamber in $\mathbb{R}^m_\nearrow$ having $J \in \mathcal{S}_m(\alpha)$ with $|J| = m-3$. Indeed, let $\beta \in \mathrm{Ch}(\mathbb{R}^m_\nearrow)$ having $J \in \mathcal{S}_m(\beta)$ with $J \neq A$ and $|J| = m-3$. Then $A' = \{m, m-2, m-4, \dots, 1\}$ would satisfy $A' \hookrightarrow J$. Then, $\bar{A}' = \{m-3, m-1\}$ would be long, which contradicts $\{m-3, m-1\} \hookrightarrow \{m-2, m\} \in \mathcal{S}_m(\beta)$. Now, if $A \in \mathcal{S}_m(\beta)$, then $\{m, m-2\}$ is long, since $\overline{\{m, m-2\}} \hookrightarrow A$. Therefore, $A \in \mathcal{S}_m(\beta)$ implies $\beta = \langle A\rangle$. For an application of this remark, see Propositions 2.7 and 2.10.

2.2. The manifold $V_d(a)$. Let $a \in \mathbb{R}^m_{>0}$. Define

$$V_d(a) = \{z = (z_1, \dots, z_{m-1}) \in \prod_{i=1}^{m-1} S^{d-1} \mid \sum_{i=1}^{m-1} a_i z_i = te_1 \text{ with } t \geq a_m\}.$$

Let $f : V_d(a) \to \mathbb{R}$ defined by $f(z) = -|\sum_{i=1}^{m-1} a_i z_i|$. The group $O(d-1)$ acts on $V_d(a)$. The following proposition is proven in [**Ha89**, Th. 3.2].

PROPOSITION 2.5. *Suppose that $a \in \mathbb{R}^m_{>0}$ is generic. Then*

(i) *$V_d(a)$ is a smooth $O(d-1)$-submanifold of $\prod_{i=1}^{m-1} S^{d-1}$, of dimension $(m-2)(d-1)$, with boundary $\mathcal{C}_d^m(a)$.*
(ii) *f is a $O(d-1)$-equivariant Morse function, with one critical point p_J for each $J \in \mathcal{S}_m(a)$, where $p_J = (z_1, \dots, z_{m-1})$ with z_i equal to $-e_1$ if $i \in J$ and e_1 otherwise (aligned configuration). The index of p_J is $(d-1)(|J|-1)$.* □

This permits us to get some information on $\mathcal{C}_d^m(a)$.

EXAMPLE 2.6. *The chamber $\langle m\rangle$.* If $\mathcal{S}_m = \{m\}$, $f : V_d(a) \to \mathbb{R}$ has only one critical point, of index 0. Hence, $\mathcal{C}_d^m(a) \approx S^{(m-2)(d-1)-1}$ and the $O(d-1)$ action is conjugate to that obtained by the embedding $S^{(m-2)(d-1)-1} \subset (\mathbb{R}^d)^{m-2}$ with the standard diagonal action [**Ha89**, Prop. 4.2]. The chamber of a has here genetic code $\langle m\rangle$, with minimal representative $(1, \dots, 1, m-2)$. One then has:

The chamber $\alpha = \langle m\rangle$

$a_{\min}(\alpha)$	$\mathcal{N}_2^m(\alpha)$	$\mathcal{N}_3^m(\alpha)$	$\mathcal{C}_d^m(\alpha)$
$(1, \dots, 1, m-2)$	S^{m-3}	$\mathbb{C}P^{m-3}$	$S^{(m-2)(d-1)-1}$

Another consequence of Proposition 2.5 is the connectivity of $\mathcal{C}_d^m(\alpha)$. We saw in Example 2.3 that $\mathcal{C}_d^m(\beta) = (S^{d-1})^{m-3} \times S^{d-2}$ if $\beta = \langle\{m, m-3, m-2, \dots, 1\}\rangle$.

Thus, $\pi_{d-2}(\mathcal{C}_d^m(\beta)) \approx \mathbb{Z}$ if $d \geq 3$ and $\pi_0(\mathcal{C}_2^m(\beta))$ has 2 elements. But this is an exceptional case:

Proposition 2.7. *Let α be a chamber of $\mathbb{R}^m_{\nearrow}$ with $\alpha \neq \langle\{m, m-3, m-2, \dots, 1\}\rangle$. Then, $\mathcal{C}_d^m(\alpha)$ is $(d-2)$-connected, i.e. $\pi_i(\mathcal{C}_d^m(\alpha)) = 0$ for $i \leq d-2$.*

Proof: Let $a \in \mathbb{R}^m_{\nearrow}$ be a representative of α. By Proposition 2.5, one has that $\pi_i(V_d(a)) = 0$ if $i \leq d-2$. If $\alpha \neq \langle\{m, m-3, m-2, \dots, 1\}\rangle$, then $|J| \leq m-3$ for all $J \in \mathcal{S}_m(a)$ by Remark 2.4. Then $V_d(a)$ has a handle decomposition, starting from $\mathcal{C}_d^m(a)$, with handles of index $\geq (m-2)(d-1) - (m-4)(d-1) = 2(d-1)$. Therefore, $\pi_i(\mathcal{C}_d^m(a)) \approx \pi_i(V_d(a))$ for $i \leq 2d-2 > d-2$. □

Remark 2.8. When $d = 2$, Proposition 2.7 says that $\langle\{m, m-3, m-2, \dots, 1\}\rangle$ is the only chamber β of $\mathbb{R}^m_{\nearrow}$ for which $\mathcal{N}_2^m(\beta)$ is not connected. This was proved by Kapovich and Millson [**KM95**] (see also [**FS06**, Ex.2 in §1]).

2.3. Crossing walls and surgeries.

Proposition 2.9. *Let $J \subset \{1, \dots, m\}$, defining the wall $\mathcal{H}_J$ in $\mathbb{R}^m$. Let α and β be two chambers of $\mathbb{R}^m$, with $\mathcal{S}_m(\beta) = \mathcal{S}_m(\alpha) \cup \{J\}$. Then $\mathcal{C}_d^m(\beta)$ is obtained from $\mathcal{C}_d^m(\alpha)$ by an $O(d-1)$-equivariant surgery of index $A = (d-1)(|J|-1) - 1$:*

$$\mathcal{C}_d^m(\beta) \approx \left(\mathcal{C}_d^m(\alpha) \setminus (S^A \times D^B)\right) \cup_{S^A \times S^{B-1}} \left(D^{A+1} \times S^{B-1}\right),$$

with $B = (m-1-|J|)(d-1)$. The $O(d-1)$-action on D^{A+1} and D^B comes from their natural embedding into a product of copies of $\mathbb{R}^{d-1}$ with the diagonal action.

Proof. Let $a \in \alpha$ and $b \in \beta$. As $\mathcal{S}_m(\beta) = \mathcal{S}_m(\alpha) \cup \{J\}$, the segment $[a, b]$ in $\mathbb{R}^m$ crosses the wall $\mathcal{H}_J$ and has no intersection with any other wall. There exists a vector orthogonal to $\mathcal{H}_J$ with coordinates equal to ± 1. Therefore, e_1 is transverse to $\mathcal{H}_J$ and, by changing a and b if necessary, we assume that $a = b + \lambda e_1$. By Proposition 2.5, the manifold $V_d(b) \setminus \mathrm{int} V_d(a)$ is a $O(d-1)$-equivariant cobordism W from $\mathcal{C}_d^m(a)$ to $\mathcal{C}_d^m(b)$. The map $f : W \to \mathbb{R}$ defined by $f(\rho) = -|\sum_{j=1}^{m-1} a_j \rho_j|$ is an invariant Morse function having a single critical point ρ^0 of index $(d-1)(|J|-1)$; the components $(\rho_1^0, \dots, \rho_{m-1}^0)$ of ρ^0 satisfy $\rho_i^0 = -e_1$ if $i \in J$ and $\rho_i^0 = e_1$ if $i \notin J$. By relabeling the ρ_i if necessary, we assume that $J = \{1, 2, \dots, k, m\}$ and $\bar{J} = \{k+1, \dots, m-1\}$. The index of ρ^0 is then equal to $(d-1)k$. Therefore, W is obtained by adding to a collar neighborhood of $\mathcal{C}_d^m(a)$ an $O(d-1)$-equivariant handle of index $(d-1)k$, whence the surgery assertion. For a reference about equivariant Morse theory, see [**Wn69**, § 4]. By [**Wn69**, Lemma 4.5], the $O(d-1)$-action is determined by the linear isotropy action on $T_{\rho^0} W$, which we shall now describe.

Let $K_\rho = \sum_{j=1}^{k} a_j \rho_j$ and $L_\rho = K_\rho - a_m e_1$. Let $p_1 : \mathbb{R}^d \to \mathbb{R}^1$ and $P : \mathbb{R}^d \to \mathbb{R}^{d-1}$ be the maps $p_1(x_1, \dots, x_d) = x_1$ and $P(x_1, \dots, x_d) = (x_2, \dots, x_d)$ For $\varepsilon > 0$, we consider the following open neighborhood $\mathcal{N}_\varepsilon$ of ρ^0 in W

$$\mathcal{N}_\varepsilon = \{\rho \in W \mid p_1(K_{\rho^0}) - p_1(K_\rho) < \varepsilon \text{ and } |L_\rho| - |L_{\rho^0}| < \varepsilon\}.$$

Consider the unique rotation $R_\rho \in SO(d)$ such that $R_\rho(L_\rho) = -|L_\rho| e_1$ and $R_\rho(q) = q$ if $q \in \mathrm{EV}(e_1, K_\rho)^\perp$ (if ε is small enough, L_ρ is not a positive multiple of e_1 when $\rho \in \mathcal{N}_\varepsilon$, thus R_ρ is well defined). If ε is small enough, we check, as in [**Ha89**, Proof of Theorem 3.2] that the smooth maps $\phi_- : \mathcal{N}_\varepsilon \to (\mathbb{R}^{d-1})^k$ and $\phi_+ : \mathcal{N}_\varepsilon \to (\mathbb{R}^{d-1})^{m-k-2}$ given by

$$\phi_-(\rho) = (P(\rho_1), \dots, P(\rho_k)) \quad \text{and} \quad \phi_+(\rho) = \left(P(R_\rho(-\rho_{k+1})), \dots, P(R_\rho(-\rho_{m-1}))\right)$$

are $O(d-1)$-equivariant and give rise to a $O(d-1)$-equivariant chart

$$\phi = (\phi_-, \phi_+) : \mathcal{N}_\varepsilon \to (\mathbb{R}^{d-1})^k \times (\mathbb{R}^{d-1})^{m-2-k} = (\mathbb{R}^{d-1})^{m-2},$$

where $(\mathbb{R}^{d-1})^n$ is endowed with the diagonal action of $O(d-1)$. One has $\phi(\rho^0) = 0$. The subspaces

$$D_+ = \{\rho \in \mathcal{N}_\varepsilon \mid |L_\rho| = |L_{\rho^0}|\} \text{ and } D_- = \{\rho \in \mathcal{N}_\varepsilon \mid K_\rho = K_{\rho^0}\}$$

are submanifolds of dimensions $k(d-1)$ and $(m-k-2)(d-1)$ respectively, satisfying $\phi(D_+) \subset (\mathbb{R}^{d-1})^k \times 0$ and $\phi(D_+) \subset 0 \times (\mathbb{R}^{d-1})^{m-2-k}$.

As in [**Ha89**, Proof of Theorem 3.2], we prove that f restricts to Morse functions on $D_\pm$. The single critical point ρ^0 is a minimum on D_+ and a maximum on D_-. Therefore, the Hessian form of $f \circ \phi^{-1}$ is positive definite on $T_0(\mathbb{R}^{d-1})^k$ and negative definite on $T_0(\mathbb{R}^{d-1})^{m-2-k}$. We have seen above that the $O(d-1)$-action on these subspaces is the standard diagonal action. By [**Wn69**, Lemma 4.5], this implies the last assertion of Proposition 2.9. □

Proposition 2.10. *Suppose, in Proposition 2.9, that* $|J| = 2$. *Then*

(1) $\mathcal{C}_d^m(\beta) = \mathcal{C}_d^m(\alpha) \sharp (S^{d-1} \times S^{(m-3)(d-1)-1})$
(2) $\mathcal{N}_3^m(\beta) = \mathcal{N}_3^m(\alpha) \sharp \overline{\mathbb{C}P}^{m-3}$

Proof: Since $\mathcal{S}_m(\beta) = \mathcal{S}_m(\alpha) \cup \{J\}$, the chamber α is not $\langle\{m, m-3, m-2, \dots, 1\}\rangle$ by Remark 2.4. By Proposition 2.7, $\mathcal{C}_d^m(\alpha)$ is $(d-2)$-connected. Hence, the sphere $S^{d-2} \subset \mathcal{C}_d^m(\alpha)$ on which the surgery of Proposition 2.9 is performed is null-homotopic. We may assume that $m \geq 4$ and $d \geq 2$ since Proposition 2.10 is empty for $m = 3$ and $\mathcal{C}_1^m(\alpha) = \mathcal{C}_1^m(\beta) = \emptyset$ because of genericity. Therefore $2(d-2) < \dim \mathcal{C}_d^m(\alpha)$, from which we deduce that $S^{d-2} \subset \mathcal{C}_d^m(\alpha)$ is isotopic to a sphere contained in a disk. Observe that we are dealing with stably parallelizable manifolds (for instance, $\mathcal{C}_d^m(-)$ is the pre-image of a regular value of a map from a product of spheres to $\mathbb{R}^d$). Part 1 then follows from standard results in surgery, see e.g. [**Ko93**, Proposition 11.2 and p. 188].

As for Part 2, we have

$$\mathcal{C}_3^m(\beta) \approx \left(\mathcal{C}_3^m(\alpha) \setminus (S^1 \times D^{2(m-3)})\right) \cup_{S^1 \times S^{2(m-3)-1}} \left(D^2 \times S^{2(m-3)-1}\right).$$

The quotient space $S^1 \times D^{2(m-3)}$ by the action of $SO(2)$ is a disk $D^{2(m-3)}$. On the other hand, consider the tautological line bundle $E \to \mathbb{C}P^{m-4}$, where $E = \{(v, \ell) \in \mathbb{C}^{m-3} \times \mathbb{C}P^{m-4} \mid v \in \ell\}$. Seeing D^2 as the unit disk in $\mathbb{C}$, the map $g : D^2 \times S^{2(m-3)-1} \to E$ given by $g(z, w) = (zw, \mathbb{C}w)$ descends to an embedding from $SO(2)\backslash(D^2 \times S^{2(m-3)-1})$ to a neighborhood of the zero section of E. It follows that $\mathcal{C}_3^m(\beta)$ is diffeomorphic to $\mathcal{C}_3^m(\alpha)$ blown up at one point, which implies Assertion 2 (see e.g. [**MDS95**, pp. 214–216]). □

Example 2.11. *The chamber* $\langle\{m, m-3, m-2, \dots, 2\}\rangle$. Let $\alpha = \langle\{m, m-3, m-2, \dots, 2\}\rangle$ and $\beta = \langle\{m, m-3, m-2, \dots, 1\}\rangle$. Then $\mathcal{S}_m(\beta) = \mathcal{S}_m(\alpha) \cup \{m, m-3, m-2, \dots, 1\}$. By Proposition 2.9, $\mathcal{C}_d^m(\beta)$ is obtained from $\mathcal{C}_d^m(\alpha)$ by an $O(d-1)$-equivariant surgery of index $(d-1)(m-3)-1$. Then, conversely, $\mathcal{C}_d^m(\alpha)$ is obtained from $\mathcal{C}_d^m(\beta)$ by an $O(d-1)$-equivariant surgery of index $d-2$. By Example 2.3 and Proposition 2.7, $\mathcal{C}_d^m(\beta) \approx (S^{d-1})^{m-3} \times S^{d-2}$ while $\mathcal{C}_d^m(\alpha)$ is $(d-2)$-connected. This implies that the surgery on $\mathcal{C}_d^m(\beta)$ is performed on a tubular neighborhood of $pt \times S^{d-2}$. Thus, Part 1 of Proposition 2.10 is not true, but one has

$$\mathcal{C}_d^m(\alpha) \approx \Big[\big((S^{d-1})^{m-3} \setminus B\big) \times S^{d-2}\Big] \cup_{\partial B \times S^{d-2}} (\partial B \times D^{d-1})\,, \tag{1}$$

where B is a $((m-3)(d-1))$-disk in $(S^{d-1})^{m-3}$. This is not a very simple expression, except when $d = 2$ where $\mathcal{N}_2^m(\alpha) = \mathcal{N}_2^m(\alpha)$ becomes

$$\mathcal{N}_2^m(\alpha) \approx (S^{d-1})^{m-3} \,\sharp\, (S^{d-1})^{m-3}\,. \tag{2}$$

On the other hand, Part 2 of Proposition 2.10 is valid and we get

$$\mathcal{N}_3^m(\alpha) \approx (S^2)^{m-3} \,\sharp\, \overline{\mathbb{C}P}^{m-3}\,. \tag{3}$$

It was observed by D. Schütz that there are only three chambers $\alpha \in \mathrm{Ch}(\mathbb{R}^m_{\nearrow})$ such that $\mathcal{S}_m(\alpha)$ contains $A = \{m, m-3, m-2, \dots, 2\}$. These are

$$\begin{array}{rcl} \alpha & = & \langle A \rangle \\ \alpha' & = & \langle A, \{m, m-2\} \rangle \\ \alpha'' & = & \langle A, \{m, m-1\} \rangle\,. \end{array} \tag{4}$$

Indeed, if A is short, then $\{m-1, m-2, 1\}$ is long and one cannot add to A a gene containing 3 elements. As $\mathcal{S}_m(\alpha') = \mathcal{S}_m(\alpha) \cup \{m, m-2\}$ and $\mathcal{S}_m(\alpha'') = \mathcal{S}_m(\alpha') \cup \{m, m-1\}$, the chain and polygon spaces for α' and α'' may be obtain from the above using Proposition 2.10.

Example 2.12. *The chamber* $\langle \{m, p\} \rangle$. For $p \geq 2$, one has $\mathcal{S}_m(\langle \{m, p\} \rangle) = \mathcal{S}_m(\langle \{m, p-1\} \rangle) \cup \{m, p\}$ and $\mathcal{S}_m(\langle \{m, 1\} \rangle) = \mathcal{S}_m(\langle m \rangle) \cup \{m, 1\}$. Using Proposition 2.10 and Example 2.6, one sees that

The chamber $\alpha = \langle \{m, p\} \rangle$

$\mathcal{N}_2^m(\alpha)$	$\mathcal{N}_3^m(\alpha)$	$\mathcal{C}_d^m(\alpha)$
$p\,(S^1 \times S^{m-4})$	$\mathbb{C}P^{m-3} \,\sharp\, p\, \overline{\mathbb{C}P}^{m-3}$	$p\,(S^{d-1} \times S^{(m-3)(d-1)-1})$

Here, p times a manifold V means the connected sum of p copies of V (hence, a sphere if $p = 0$). A representative of $\langle \{m, p\} \rangle$ is given by

$$(\underbrace{1, \dots, 1}_{p}, \underbrace{2, \dots, 2}_{m-p-1}, 2m - p - 5)\,.$$

The tables of [**HRWeb**] show that, for $m \leq 9$ (See Section 3 below for $m \leq 6$), this representative is $a_{\min}(\langle \{m, p\} \rangle)$, except for $p = 0, 1$. As $\langle \{m, 1\} \rangle = \langle m-1 \rangle^+$, Proposition 2.2 gives the diffeomorphism

$$\mathbb{C}P^{m-3} \,\sharp\, \overline{\mathbb{C}P}^{m-3} \approx \mathcal{N}_3^m(\langle \{m, 1\} \rangle) \approx \mathcal{N}_3^m(\langle m \rangle^+) \approx S^2 \times_{S^1} S^{2(m-3)-1}\,.$$

In the case $m = 5$, we get the two topological descriptions of the Hirzebruch surface (see, e.g. [**MDS95**, Ex. 6.4]).

3. Tables for $m = 4, 5, 6$

For any m, there is the "trivial" chamber $\langle \rangle$, where a_m is so long that the corresponding chain or polygon spaces are empty. When $m = 4$, Examples 2.6 and 2.3 give the remaining two chambers:

Table A: $m = 4$

	α	$a_{\min}(\alpha)$	$\mathcal{N}_2^4(\alpha)$	$\mathcal{N}_3^4(\alpha)$	$\mathcal{C}_d^4(\alpha)$
1	$\langle\rangle$	$(0,0,0,1)$	$\emptyset$	$\emptyset$	$\emptyset$
2	$\langle 4\rangle$	$(1,1,1,2)$	S^1	$\mathbb{C}P^1$	$S^{2(d-1)-1}$
3	$\langle 41\rangle$	$(0,1,1,1)$	$S^1 \dot\cup S^1$	S^2	$(S^{d-1}) \times S^{d-2}$

Recall that the column $\mathcal{C}_d^4(\alpha)$ does not contain all the 4-chains, only those for which $a_4 \geq a_i$ for $I = 1,2,3$. For example, $\mathcal{C}_d^4(1,1,1\varepsilon) \approx T^1 S^{d-1}$, the unit tangent bundle to S^{d-1}. For a complete classification of 4-chains, see [**Ha89**].

When $m = 5$, there are seven chambers. Lines 2 and 7 come from Examples 2.6 and 2.3. The symbols Σ_g denotes the orientable surface of genus g and T^r is the torus $(S^1)^r$. Within the central block, each line is obtained from the previous one by Proposition 2.10.

Table B: $m = 5$

	α	$a_{\min}(\alpha)$	$\mathcal{N}_2^5(\alpha)$	$\mathcal{N}_3^5(\alpha)$	$\mathcal{C}_d^5(\alpha)$
1	$\langle\rangle$	$(0,0,0,0,1)$	$\emptyset$	$\emptyset$	$\emptyset$
2	$\langle 5\rangle$	$(1,1,1,1,3)$	S^2	$\mathbb{C}P^2$	$S^{3(d-1)-1}$
3	$\langle 51\rangle$	$(0,1,1,1,2)$	T^2	$\mathbb{C}P^2 \sharp \overline{\mathbb{C}P}^2$	$S^{d-1} \times S^{2(d-1)-1}$
4	$\langle 52\rangle$	$(1,1,2,2,3)$	Σ_2	$\mathbb{C}P^2 \sharp 2\,\overline{\mathbb{C}P}^2$	$2\,[S^{d-1} \times S^{2(d-1)-1}]$
5	$\langle 53\rangle$	$(1,1,1,2,2)$	Σ_3	$\mathbb{C}P^2 \sharp 3\,\overline{\mathbb{C}P}^2$	$3\,[S^{d-1} \times S^{2(d-1)-1}]$
6	$\langle 54\rangle$	$(1,1,1,1,1)$	Σ_4	$\mathbb{C}P^2 \sharp 4\,\overline{\mathbb{C}P}^2$	$4\,[S^{d-1} \times S^{2(d-1)-1}]$
7	$\langle 521\rangle$	$(0,0,1,1,1)$	$T^2 \dot\cup T^2$	$S^2 \times S^2$	$(S^{d-1})^2 \times S^{d-2}$

Line 3 in Table B together with Equation (3), re-proves the classical fact that $(S^2 \times S^2) \sharp \overline{\mathbb{C}P}^2$ is diffeomorphic to $\mathbb{C}P^2 \sharp 2\,\overline{\mathbb{C}P}^2$.

When $m = 6$, there are 21 chambers. In order to save space, we did not give $a_{\min}(\alpha)$ (they can be found in [**HR04**, Table 6]). The first line of each block is obtained from § 2.1–2.3. Then, each line is obtained from the previous one by Proposition 2.10.

Table C: $m = 6$

	α	$\mathcal{N}_2^6(\alpha)$	$\mathcal{N}_3^6(\alpha)$	$\mathcal{C}_d^6(\alpha)$
1	$\langle\rangle$	$\emptyset$	$\emptyset$	$\emptyset$
2	$\langle 6\rangle$	S^3	$\mathbb{C}P^3$	$S^{4(d-1)-1}$
3	$\langle 61\rangle$	$S^1\times S^2$	$\mathbb{C}P^3\,\sharp\,\overline{\mathbb{C}P}^3$	$R(d)$ (1)
4	$\langle 62\rangle$	$2(S^1\times S^2)$	$\mathbb{C}P^3\,\sharp\,2\,\overline{\mathbb{C}P}^3$	$2R(d)$
5	$\langle 63\rangle$	$3(S^1\times S^2)$	$\mathbb{C}P^3\,\sharp\,3\,\overline{\mathbb{C}P}^3$	$3R(d)$
6	$\langle 64\rangle$	$4(S^1\times S^2)$	$\mathbb{C}P^3\,\sharp\,4\,\overline{\mathbb{C}P}^3$	$4R(d)$
7	$\langle 65\rangle$	$5(S^1\times S^2)$	$\mathbb{C}P^3\,\sharp\,5\,\overline{\mathbb{C}P}^3$	$5R(d)$
8	$\langle 621\rangle$	T^3	$S^2\times_{S^1}(S^2\times S^3)$	$S^{d-1}\times\mathcal{C}_d^5(\langle 51\rangle)$ (2)
9	$\langle 621, 63\rangle$	$T^3\sharp\,(S^1\times S^2)$	$[S^2\times_{S^1}(S^2\times S^3)]\,\sharp\,\overline{\mathbb{C}P}^3$	$[S^{d-1}\times\mathcal{C}_d^5(\langle 51\rangle)]\,\sharp\,R(d)$
10	$\langle 621, 64\rangle$	$T^3\sharp\,2(S^1\times S^2)$	$[S^2\times_{S^1}(S^2\times S^3)]\,\sharp\,2\,\overline{\mathbb{C}P}^3$	$[S^{d-1}\times\mathcal{C}_d^5(\langle 51\rangle)]\,\sharp\,2R(d)$
11	$\langle 621, 65\rangle$	$T^3\sharp\,3(S^1\times S^2)$	$[S^2\times_{S^1}(S^2\times S^3)]\,\sharp\,3\,\overline{\mathbb{C}P}^3$	$[S^{d-1}\times\mathcal{C}_d^5(\langle 51\rangle)]\,\sharp\,3R(d)$
12	$\langle 631\rangle$	$\Sigma_2\times S^1$	$S^2\times_{S^1}2(S^2\times S^3)$	$S^{d-1}\times\mathcal{C}_d^5(\langle 52\rangle)$ (2)
13	$\langle 631, 64\rangle$	$(\Sigma_2\times S^1)\,\sharp\,(S^1\times S^2)$	$S^2\times_{S^1}2(S^2\times S^3)\sharp\,\overline{\mathbb{C}P}^3$	$[S^{d-1}\times\mathcal{C}_d^5(\langle 52\rangle)]\,\sharp\,R(d)$
14	$\langle 631, 65\rangle$	$(\Sigma_2\times S^1)\,\sharp\,2(S^1\times S^2)$	$S^2\times_{S^1}2(S^2\times S^3)\sharp\,2\,\overline{\mathbb{C}P}^3$	$[S^{d-1}\times\mathcal{C}_d^5(\langle 52\rangle)]\,\sharp\,2R(d)$
15	$\langle 641\rangle$	$\Sigma_3\times S^1$	$S^2\times_{S^1}3(S^2\times S^3)$	$S^{d-1}\times\mathcal{C}_d^5(\langle 53\rangle)$ (2)
16	$\langle 641, 65\rangle$	$(\Sigma_3\times S^1)\,\sharp\,(S^1\times S^2)$	$S^2\times_{S^1}3(S^2\times S^3)\sharp\,\overline{\mathbb{C}P}^3$	$[S^{d-1}\times\mathcal{C}_d^5(\langle 53\rangle)]\,\sharp\,R(d)$
17	$\langle 651\rangle$	$\Sigma_4\times S^1$	$S^2\times_{S^1}4(S^2\times S^3)$	$S^{d-1}\times\mathcal{C}_d^5(\langle 54\rangle)$ (2)
18	$\langle 6321\rangle$	$T^3\,\dot\cup\,T^3$	$(S^2)^3$	$(S^{d-1})^3\times S^{d-2}$
19	$\langle 632\rangle$	$2\,T^3$	$(S^2)^3\,\sharp\,\overline{\mathbb{C}P}^3$	$\mathcal{C}_d^6(\langle 632\rangle)$ (3)
20	$\langle 632, 64\rangle$	$2T^3\sharp\,(S^1\times S^2)$	$(S^2)^3\sharp\,2\,\overline{\mathbb{C}P}^3$	$\mathcal{C}_d^6(\langle 632\rangle)\,\sharp\,R(d)$
21	$\langle 632, 65\rangle$	$2T^3\sharp\,2(S^1\times S^2)$	$(S^2)^3\sharp\,3\,\overline{\mathbb{C}P}^3$	$\mathcal{C}_d^6(\langle 632\rangle)\,\sharp\,2R(d)$

(1) $R(d) = S^{d-1}\times S^{3(d-1)-1}$. (2) see Table B. (3) See Example 2.11

The list for $\mathcal{N}_2^6(\alpha)$ is present in [**Wa85**] with some short-hand justification. A version of the column for $\mathcal{N}_3^6(\alpha)$ is in [**HR04**].

For $m \geq 7$, the above procedure fails to give all the chambers, since surgeries of higher index are needed. For example, for $m = 7$, only 49 chambers out of 135 are reached.

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