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GEOMETRIC EXPONENTS FOR HYPERBOLIC JULIA SETS

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Abstract. We show that the Hausdorff dimension of the Julia set associated to a hyperbolic rational map is bounded away from 2, where the bound depends only on certain intrinsic geometric exponents. This result is derived via lower estimates for the iterate-counting function and for the dynamical Poincaré series. We deduce some interesting consequences, such as upper bounds for the decay of the area of parallel-neighbourhoods of the Julia set, and lower bounds for the Lyapunov exponents with respect to the measure of maximal entropy.

1. Introduction

We consider Julia sets $J(T) \subsetneqq \overline{\mathbb{C}}$ of hyperbolic holomorphic endomorphisms $T : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ of the Riemann sphere. It is well-known that these maps form a large open set inside the set of all holomorphic endomorphisms, and it is conjectured that they are in fact dense ('Fatou conjecture'). Furthermore, hyperbolic maps satisfy the 'analogue of the Ahlfors conjecture for Kleinian groups', which asserts that the Julia set has always vanishing 2-dimensional Lebesgue measure. This is an immediate consequence of the well-known fact that for hyperbolic rational maps the Julia set is porous, and hence its Hausdorff dimension h is strictly less than 2 (see, e.g., [13]).

In this paper we refine the latter result by giving an upper bound for h in terms of certain intrinsic parameters. Our estimate clearly reveals the geometric obstacles which prevent a hyperbolic Julia set from having Hausdorff dimension 2. In order to state this estimate, let T have *critical distance* c (the distance of $J(T)$ to the forward orbit of the critical points of T), *core exponent* κ (the inverse of the maximal distortion of T on $J(T)$), and *inner lacunarity exponent* λ (that is, roughly, the area of $U(J(T))/\langle T_*^{-1}\rangle$, where $U(J(T))$

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denotes a suitable neighbourhood of $J(T)$ and $\langle T_*^{-1} \rangle$ is the semi-group generated by the holomorphic inverse branches of T). Our main result, Theorem 4.1, relates these three geometric exponents to the Hausdorff dimension of $J(T)$. Namely, we show that

$$h < 2 - \frac{2\,\lambda\,\kappa^{10}}{\text{area}(U_c(J(T)))},$$

where $U_c(J(T))$ denotes the c-neighbourhood of $J(T)$.

Our motivation for this estimate came from attempts to understand the relationship between the spectrum of the Laplacian and certain intrinsic geometric quantities, such as volume and length-spectrum for hyperbolic manifolds. For geometrically finite, infinite-volume hyperbolic manifolds a lower bound for the bottom of the Laplace spectrum was obtained in [5] in terms of the convex core of the manifold (i.e., its volume and the area of its boundary). In certain cases this leads to upper bounds for the Hausdorff dimension of the associated Kleinian limit sets. In a recent paper [12], we derived by purely geometric means a similar type of bound for all convex cocompact Kleinian groups.

In this paper we adopt the geometric method of [12] and show how to adjust it to the setting of hyperbolic rational maps. We first prove the existence of geometrically well-behaved coverings of $J(T)$ (Lemma 2.2) and of packings of the Fatou set $F(T) := \overline{\mathbb{C}} \setminus J(T)$ (Lemma 2.3). This allows us to introduce the concepts of 'iterate-counting function' and 'dynamical Poincaré series', which are the natural analogues of the 'orbital counting function' and 'Poincaré series' for Kleinian groups. We then derive a more precise estimate for the h-conformal measure (Lemma 2.5), and apply partial summation, a standard technique in number theory, to obtain lower bounds for the iterate-counting function (Lemmas 2.2 and 3.1) and the dynamical Poincaré series (Proposition 3.3). We next interpret these estimates in terms of the fractality of $J(T)$ and the lacunarity of $F(T)$, which then gives our main result, the bound for the Hausdorff dimension of $J(T)$ stated above.

2. Hyperbolic Julia sets

2.1. The geometry of hyperbolic Julia sets. Throughout the paper, let $T : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a hyperbolic holomorphic endomorphism of the Riemann sphere. We assume that $\mathbb{C}$ is equipped with the Euclidean metric d. Let $J = J(T)$ denote the Julia set of T, and $F = F(T) := \overline{\mathbb{C}} \setminus J(T)$ the Fatou set. Furthermore, let $c = c(T)$ denote the *critical distance* of T, that is, c is the distance (with respect to d) between the Julia set J and the forward orbit $\bigcup_{n=1}^{\infty} T^{\ell}(\text{Crit})$ of the critical points Crit of T. Then it is well-known that the hyperbolicity of T is equivalent to the fact that c is strictly positive (cf. [2]). Furthermore, the periodic components of F are basins of (super-)attracting periodic points. Without loss of generality, we assume that ∞

is such a periodic point, so that, in particular, J is contained in $\mathbb{C}$, and that the holomorphic inverse branches $\{T_*^{-1}\}$ of T are strictly contracting on J, i.e., $\max_{z\in J}\{|(T_*^{-1})'(z)|\} < 1$.

DEFINITION 2.1 (CORE EXPONENT). The *core exponent* of T is defined as $\kappa := \min_{z\in J}\{|(T_*^{-1})'(z)|\}$.

We now define the sequences $(\mathcal{C}_n)$ of coverings of J as follows. Fix an optimal initial cover $\mathcal{C}_0$ of J by open discs $D_{0,i}$ of diameter cr_0 centred at points z_i of J, where

$$r_0 = r_0(\kappa) := \frac{1-\sqrt[4]{\kappa}}{1+\sqrt[4]{\kappa}}.$$

Here 'optimal' is to be interpreted in the sense that the cover is of minimal cardinality. For $n \in \mathbb{N}$, let $\mathcal{C}_n = \{D_{n,i}\}$ denote the n-th lexicographical cover of J, given by the family of all n-th inverse images of the elements in $\mathcal{C}_0$. Put $\mathcal{C} := \bigcup_{n\in\mathbb{N}} \mathcal{C}_n$, and order the elements of $\mathcal{C}$ according to their dynamical sizes, defined by $\|D\| := |(T_*^{-n})'(z_i)|$ for $D = T_*^{-n}(D_{0,i}) \in \mathcal{C}_n$. More precisely, let $\mathcal{C} = \{D_0, D_1, D_2, \cdots\}$ be such that $\|D_n\| \geq \|D_{n+1}\|$, for all $n \in \mathbb{N}$.

LEMMA 2.2. *For all positive $t \leq 1$, the set $\{D \in \mathcal{C} : \kappa^2 t < \|D\| < t\}$ covers the Julia set J.*

Proof. We define $r_1 := (1-r_0)^3/(1+r_0)$ and $r_2 := (1+r_0)^3/(1-r_0)$. By the definition of r_0 it follows that $r_1/r_2 = \kappa$. For $z \in J(T)$, there exists $n(z) \in \mathbb{N}$ such that

$$r_1 \cdot \kappa t < |(T^{n(z)})'(z)|^{-1} \leq r_1 \cdot t.$$

Let $D_{0,i}$ be an element of $\mathcal{C}_0$ such that $T^{n(z)}(z) \in D_{0,i}$. If $T_z^{-n(z)}$ denotes the inverse branch such that $T_z^{-n(z)}(T^{n(z)}(z)) = z$, and z_i is the center of $D_{0,i}$, then, by Koebe's theorem (see [11]) we have

$$\frac{\left|(T_z^{-n(z)})'(T^{n(z)}(z))\right|}{r_2} < \left|(T_z^{-n(z)})'(z_i)\right| < \frac{\left|(T_z^{-n(z)})'(T^{n(z)}(z))\right|}{r_1}.$$

This shows that

$$\kappa^2 t = r_1 \cdot \frac{\kappa t}{r_2} < \|T_z^{-n(z)}(D_{0,i})\| < r_1 \cdot \frac{t}{r_1} = t. \qquad \square$$

Let us make the following conventions. For every $D_{0,i} \in \mathcal{C}_0$ fix a disc $F_{0,i} \subset D_{0,i} \cap F(T)$. We associate to each disc D_n an inverse image $F_n \subset D_n$ of one of the discs $F_{0,i}$ obtained by the same backward iterate as D_n. More precisely, if $D_n = T_*^{-i}(D_{0,j})$ then $F_n := T_*^{-i}(F_{0,j})$.

The following lemma shows that we can choose the family $\{F_{0,i}\}$ such that the induced family $\{F_n\}$ serves as an inner approximation for the orbit of a kind of fundamental domain for the 'almost-discontinuous action' of the inverse branches of T. Recall that a *packing* of a given set is a collection of non-empty open subsets which are pairwise disjoint.

LEMMA 2.3. *There exists a choice of* $\{F_{0,i}\}$ *with the following properties (where, as above,* $r_1 = (1-r_0)^3/(1+r_0)$*).*

(i) *The set* $\mathcal{F} := \{F_n : n \in \mathbb{N}\}$ *is a packing of the Fatou set* F.

(ii) *There exists a positive constant* $\lambda_{\mathcal{F}} = \lambda_{\mathcal{F}}(T)$ *such that*

$$\operatorname{area}(F_n) \geq \lambda_{\mathcal{F}} r_1^2 \cdot \|D_n\|^2 \text{ for all } n \in \mathbb{N}.$$

Proof. The hyperbolicity of T implies that the periodic components of F are basins of (super-)attracting periodic points. We fix such a periodic point ω. Let p be the prime period of ω. Note that if T is a polynomial then ω may be chosen to be equal to ∞, and consequently $p = 1$. Thus, there exists a simply connected neighbourhood V of ω such that V is mapped into itself by T^p. Furthermore, for some $0 < s < 1$ and $B_s := \{z : |z| < s\}$, the map $T^p|_V$ is conjugate to $z \mapsto (T^p)'(\omega) \cdot z|_{B_s}$, if ω is an attracting fixed point, and conjugate to $z \mapsto z^\nu|_{B_s}$, if ω is a super-attracting fixed point (so that $(T^p)'$ has a zero of order $\nu - 1$ at ω). Let R denote the ring-domain, defined as the pre-image (under conjugation) of $\{z : |(T^p)'(\omega)|s \leq |z| < s\}$ in the attracting case, and as the pre-image (under conjugation) of $\{z : s^\nu \leq |z| < s\}$ in the super-attracting case. Let G denote a maximal disc in R disjoint from the forward orbit of Crit. Clearly, all inverse branches of T are well-defined on G. Since $T^i(R) \cap T^j(R) = \emptyset$ for $i \neq j$, we deduce that, for any two different inverse branches T_*^{-k} and $T_+^{-\ell}$,

$$T_*^{-k}(G) \cap T_+^{-\ell}(G) = \emptyset.$$

If E is a relatively compact set whose closure does not contain periodic points in F, then it is well-known that $T^{-n}(E)$ converges to J in the Hausdorff topology. (This is an immediate consequence of [3, Th. 6.1].) In particular, this holds for $E = G$. Thus, there exists n_1 such that $T^{-n}(G) \subset \bigcup_i D_{0,i}$ for all $n \geq n_1$. Furthermore, there exists n_2 such that for each $n \geq n_2$ the diameter of every element of $T^{-n}(G)$ is less than the Lebesgue number of the cover $\mathcal{C}_0$. Finally, since $\mathcal{C}_0$ was chosen to be optimal, it follows that each $D_{0,i}$ contains an open subset $D_{0,i}^*$ intersecting J such that $D_{0,i}^* \cap D_{0,j} = \emptyset$ for $j \neq i$. Hence there exists n_3 such that $T^{-n}(G) \cap D_{0,i}^* \neq \emptyset$ for each i and for every $n \geq n_3$.

With $n_0 := \max\{n_1, n_2, n_3\}$, we define $F_{0,i}$ as a maximal disc contained in an element of $T^{-n_0}(G)$ and intersecting $D_{0,i}^*$. Clearly, the induced sets F_n fulfil the first assertion of the lemma, i.e., form a packing (in the above sense) of the Fatou set. To prove the second assertion, note that for $F_n = T_*^{-j}(F_{0,i})$ we have

$$r_1 \cdot \|D_n\| \leq \inf_{z \in F_{0,i}} |(T_*^{-j})'(z)|.$$

Thus, we obtain

$$\text{area}(F_n) \geq \lambda_{\mathcal{F}} r_1^2 \cdot \|D_n\|^2,$$

where $\lambda_{\mathcal{F}} := \min_i\{\text{area}(F_{0,i})\}$. □

DEFINITION 2.4 (INNER LACUNARITY EXPONENT). The *inner lacunarity exponent* λ of T is defined as the supremum of the numbers $\lambda_{\mathcal{F}}$ from Lemma 2.3, taken over all possible choices of families $\mathcal{F}$.

2.2. Conformal measures. Recall from [1], [6] and [13] that, given a hyperbolic rational map with Julia set of Hausdorff dimension h, there exists a unique h-conformal measure m supported on J, i.e., a probability measure satisfying

$$m(T(E)) = \int_E |T'(\xi)|^h dm(\xi)$$

for each Borel set $E \subset J$ on which T is injective. It was shown in [7] that h is the least real number s for which there exists an s-conformal measure. It is known that m is a non-atomic measure which is in the same measure class as the h-dimensional Hausdorff measure on J. In particular, we have $0 < h < 2$.

The following estimate, which will be crucial in the sequel, gives a more precise version of the fact that the h-conformal measure is absolutely continuous with respect to the h-dimensional Hausdorff measure.

LEMMA 2.5. *For each $D \in \mathcal{C}$ we have*

$$m(D) < r_2^2 \kappa^{-4} \, \|D\|^h.$$

Proof. Without loss of generality (note that, by construction, $0 < \|D\| \leq 1$) we can assume that $\kappa^{n+2} < \|D\| \leq \kappa^n$, for some $n \in \mathbb{N}$. By construction we have $D = T_*^{-j}(D_{0,i})$ for some $i, j \in \mathbb{N}$. Recall that $\|D\|$ is given by $|(T_*^{-j})'(z_i)|$. Hence, applying Koebe's theorem as in the proof of Lemma 2.2, we immediately obtain

$$|(T^j)'(\xi)| > \frac{1}{r_2 \cdot \|D\|} \geq \frac{\kappa^{-n}}{r_2}$$

for all $\xi \in D$. By the h-conformality of m, it follows that

$$1 \geq m(T^j(D)) = \int_D |(T^j)'(\xi)|^h dm(\xi) > \frac{\kappa^{-nh}}{r_2^h} \cdot m(D).$$

Since $h < 2$, we deduce that

$$m(D) < r_2^2 \kappa^{-4} \|D\|^h.$$ □

3. Estimates from below for the iterate-counting function and the dynamical Poincaré series

LEMMA 3.1. *The 'iterate-counting function'*

$$N(t) := \operatorname{card}\{D \in \mathcal{C} : t \le \|D\|\}$$

satisfies, for each $t \in (0,1]$,

$$N(t) > K_\kappa \, t^{-h},$$

where $K_\kappa := \kappa^8/r_2^2$,

Proof. For $t \in (0,1]$ define $A(t) := \{D \in \mathcal{C} : t \le \|D\| < t/\kappa^2\}$. By Lemma 2.2, the sets in $A(t)$ form a covering of J. Hence, by Lemma 2.5, it follows that

$$\begin{aligned} 1 = m\left(\bigcup_{D \in A(t)} D\right) &\le \sum_{D \in A(t)} m(D) \\ &\le r_2^2 \kappa^{-4} \left(\frac{t}{\kappa^2}\right)^h \operatorname{card}(A(t)) < r_2^2 \kappa^{-8} t^h \cdot N(t) = K_\kappa^{-1} t^h \cdot N(t). \qquad \square \end{aligned}$$

DEFINITION 3.2 (DYNAMICAL POINCARÉ SERIES). For $s \ge 0$, the *dynamical Poincaré series* for the hyperbolic rational map T is defined by

$$\sum_{D \in \mathcal{C}} \|D\|^s.$$

It is well-known that h is the exponent of convergence of the Poincaré series for T (see [7]). The following proposition makes this more precise by giving an estimate for the rate at which the Poincaré series approaches infinity as s tends to h. An immediate consequence of this result is that hyperbolic rational maps are of 'h-divergence type', that is, the dynamical Poincaré series diverges for $h = s$.

The proof uses partial summation, a standard technique in number theory (see [4]).

PROPOSITION 3.3. *For each $s > h$, we have*

$$\sum_{D \in \mathcal{C}} \|D\|^s > K_\kappa \frac{s}{s-h}.$$

Proof. For $n \in \mathbb{N}$, we define the increasing sequence $t_n := \|D_n\|^{-1}$ and let $g(t) := t^{-s}$. Then $t_0 = 1$, and for $x \in [t_0, \infty)$,

$$N(1/x) = \operatorname{card}\{t_n : t_n \le x\}.$$

Thus, partial summation and Lemma 3.1 yield

$$\begin{aligned}\sum_{\|D_n\|\geq 1/x} \|D_n\|^s &= \sum_{t_n \leq x} g(t_n) \\ &= N(1/x)g(x) - \int_{t_1}^{x} N(1/t)g'(t)dt \\ &= N(1/x)g(x) + s\int_{1}^{x} N(1/t)/t^{s+1}dt \\ &> N(1/x)g(x) + K_\kappa\, s\int_{1}^{x} t^{h-s-1}dt \\ &= N(1/x)g(x) + K_\kappa\, s\left(\frac{x^{-(s-h)}}{h-s} + \frac{1}{s-h}\right).\end{aligned}$$

Letting x tend to infinity then proves the assertion of the proposition. □

4. Upper bounds for the Hausdorff dimension

Theorem 4.1. *Let T be a hyperbolic rational map with Julia set of Hausdorff dimension h. If T has critical exponent c, core exponent κ and lacunarity exponent λ, then*

$$h < 2 - \frac{2\,\lambda\,\kappa^{10}}{\mathrm{area}(U_c(J(T)))}.$$

Proof. Recall that $r_0 := (1-\sqrt[4]{\kappa})/(1+\sqrt[4]{\kappa}) < 1$. By Lemma 2.3 and Proposition 3.3, we have, for any family $\mathcal{F}$ of the type introduced in Section 2.1, with $U_\varrho(J) := \{z : d(z, J) < \varrho\}$ for $\varrho > 0$, that

$$\begin{aligned}\mathrm{area}(U_c(J)) &\geq \mathrm{area}(U_{cr_0}(J)) > \sum_{n\in\mathbb{N}} \mathrm{area}(F_n) \\ &\geq \lambda_{\mathcal{F}} r_1^2 \sum_{n\in\mathbb{N}} \|D_n\|^2 \geq \lambda_{\mathcal{F}} r_1^2 K_\kappa \cdot \frac{2}{2-h}.\end{aligned}$$

Taking the supremum over all $\mathcal{F}$, we obtain

$$\mathrm{area}(U_c(J)) \geq \lambda\, r_1^2 K_\kappa \cdot \frac{2}{2-h}.$$

An elementary rearrangement then gives the theorem. □

5. Some applications

5.1. Upper bounds for the decay of area. Theorem 4.1 may be applied to derive upper bounds for the decay of the area of ε-neighbourhoods of the Julia set.

To this end, consider a compact set $E \subset \mathbb{R}^n$, and let $\mathcal{N}_\varepsilon(E)$ denote the minimal number of sets of diameter at most ε that cover E. For the n-dimensional volume (vol_n) of the ε-neighbourhood of E, we then have the following well-known formula (see, e.g., [8]):

$$\lim_{\varepsilon \to 0} \frac{\log \mathcal{N}_\varepsilon(E)}{-\log \varepsilon} = n - \lim_{\varepsilon \to 0} \frac{\log\ \mathrm{vol}_n(U_\varepsilon(E))}{\log \varepsilon}.$$

Combining Theorem 4.1 with this general result from fractal geometry, we obtain the following result.

PROPOSITION 5.1. *Let $T : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a hyperbolic rational map with Julia set $J(T) \subset \overline{\mathbb{C}}$, critical distance c, core exponent κ and lacunarity exponent λ. Then the area of ε-neighbourhoods of $J(T)$ satisfies*

$$\lim_{\varepsilon \to 0} \frac{\log\ \mathrm{area}(U_\varepsilon(J(T)))}{\log \varepsilon} \geq \frac{2\,\lambda\,\kappa^{10}}{\mathrm{area}(U_c(J(T)))}.$$

Proof. For hyperbolic Julia sets it is well-known that the Hausdorff dimension and the box-counting dimension coincide and are equal to h. In particular, we have

$$\lim_{\varepsilon \to 0} \frac{\log \mathcal{N}_\varepsilon(J(T))}{-\log \varepsilon} = h.$$

Combining this fact, Theorem 4.1, and the above mentioned formula from fractal geometry, the assertion follows. □

5.2. Lower bounds for the Lyapunov exponent. Theorem 4.1 can be used to derive lower bounds for the Lyapunov exponent with respect to the measure of maximal entropy for T.

Recall that there exists a unique measure $\mu = \mu(T)$ of maximal entropy $\log(d)$ for T, where d denotes the degree of T (see [10]). By a result of Ledrappier [9], the Lyapunov exponent χ_μ with respect to μ satisfies

$$\log d = h\,\chi_\mu.$$

Combining this relation with the estimate for h in Theorem 4.1, we immediately obtain the following proposition.

PROPOSITION 5.2. *Let $T : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a hyperbolic rational map with Julia set $J(T) \subset \overline{\mathbb{C}}$, critical distance c, core exponent κ and lacunarity exponent λ. Then the Lyapunov exponent χ_μ satisfies*

$$\chi_\mu \ \geq\ \frac{1}{2} \cdot \frac{\log(d) \cdot \mathrm{area}(U_c(J(T)))}{\mathrm{area}(U_c(J(T))) - \lambda\,\kappa^{10}}.$$ □

5.3. A class of examples. Due to the fractal nature of the Julia set, computing numerical values of the geometric exponents c, κ, λ is, in general, a highly non-trivial task. In particular, the explicit determination of $\lambda_{\mathcal{F}}$ may turn out to be difficult. In this section we illustrate the practical application of our method for a certain easy class of examples.

For $\tau \in \mathbb{R}$, $\tau < -(5+2\sqrt{5})/4$, we consider the family of quadratic maps $T = T_\tau$ given by

$$T : \overline{\mathbb{C}} \to \overline{\mathbb{C}} \quad \text{such that} \quad z \mapsto z^2 + \tau.$$

It is well-known that $J(T)$ is a Cantor set which is a subset of the real line. Let $\Gamma := 1/2 + \sqrt{1/4 - \tau}$ denote the outer escape radius and $\gamma := \sqrt{-\tau - \Gamma}$ the inner escape radius. It is easy to see that $|z| > \Gamma$ implies $|T(z)| > |z|$, and that $|z| < \gamma$ implies $|T(z)| > \Gamma$. Hence it follows that $J(T)$ is contained in the closed annulus $A_{\gamma,\Gamma}$ centred at the origin with inner radius γ and outer radius Γ. Also, note that the critical point 0 satisfies $T(0) = \tau$, $T^2(0) = \tau^2 + \tau$, and that $|T^n(0)| > |\tau^2 + \tau|$ for all $n > 2$. Hence, the critical distance c of T is equal to γ, and the core exponent κ of T turns out to be equal to $(2\Gamma)^{-1}$. Consequently, in this example the constant r_0 of our construction in Section 2 is equal to $(\sqrt[4]{2\Gamma} - 1)/(\sqrt[4]{2\Gamma} + 1)$.

In order to obtain a lower bound for λ, the following observation will be helpful. The proof of this result is straight-forward and we omit it.

LEMMA 5.3. *For $z_0 \in \mathbb{R}, z \in \mathbb{C}$ we have*

$$\begin{aligned} &|2\,T(z) - T(z_0 + |z - z_0|) - T(z_0 - |z - z_0|)| \\ &\qquad \geq 2\,|T(z_0 + |z - z_0|) - T(z_0 - |z - z_0|)|. \end{aligned}$$ □

For a closed interval $I \subset \mathbb{R}$, let $B(I)$ denote the ball centred on the real line such that $B(I) \cap \mathbb{R} = I$. Then $T_*^{-1}(B(I)) \subset B(T_*^{-1}(I))$ holds for each interval $I \subset \mathbb{R}$ and any holomorphic inverse branch T_*^{-1}.

The structure of $J(T)$ as a Cantor set is as follows. $J(T)$ is contained in the two intervals $[-\Gamma, -\gamma]$ and $[\gamma, \Gamma]$ of order 1. The inverse images of these two intervals give rise to four intervals of order 2. By carrying on in this way, we obtain inductively 2^n intervals of order n. Clearly, $(\Gamma - \gamma)/(2\gamma)^{(n-1)}$ is an upper bound for the length of an interval of order n. Hence, there exists n_0 such that $2cr_0$ is greater than the maximal length of an interval of order n_0. Obviously, we have that

$$n_0 \leq \left\lfloor \frac{\log((\Gamma - \gamma)/(2cr_0))}{\log(2\gamma)} \right\rfloor + 2.$$

Now we can construct by induction a covering of $J(T)$ consisting of balls of radius cr_0 as follows. Denote the intervals of order n_0 from left to right by $I_1, \ldots, I_{2^{n_0}}$, and choose $D_{0,1}$ to be a ball of radius cr_0 centred at $J(T)$ such that $D_{0,1}$ contains $I_1, \ldots, I_{i_1}$, where i_1 is chosen to be maximal with respect

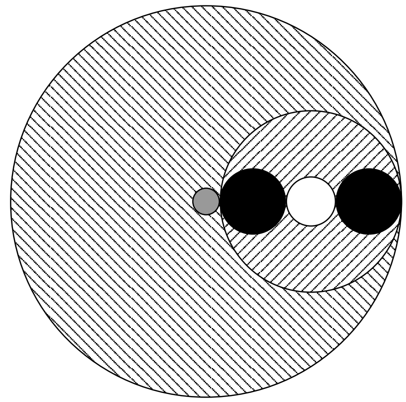

FIGURE 1. Sketch of the regions G and $\tilde{G}$ for $\tau = -2,5$. The shading ▧ corresponds to G; the shading ▨ corresponds to $\tilde{G}$; the white ball is $B_\gamma(0)$; the gray ball is its forward image under T. The black regions are $B([-\Gamma, -\gamma])$ and $B([\gamma, \Gamma])$.

to this property. Next, choose $D_{0,2}$ to be a ball of radius cr_0 centred at $J(T)$ such that $D_{0,2}$ contains $I_{i_1+1}, \ldots, I_{i_2}$, where i_2 is chosen to be maximal with respect to this property. Continuing in this way, we eventually obtain $D_{0,k}$ such that $i_k = 2^{n_0}$. For $\ell = 1, \ldots, k$, it is clear that only $D_{0,\ell}$ entirely contains the interval I_{i_ℓ}. Now, define the simply connected set G by

$$G := B_{\Gamma-\tau}(\tau) \setminus \left(\overline{B_\Gamma(0)} \cup \overline{B_{-(\Gamma+\tau)}(\tau)} \right).$$

By Lemma 5.3, the two inverse images of G contain the set

$$\tilde{G} := B_\Gamma(0) \setminus \left(\overline{B_{(\Gamma-\gamma)/2}(-(\Gamma+\gamma)/2)} \cup \overline{B_{(\Gamma-\gamma)/2}((\Gamma+\gamma)/2)} \right).$$

We have $\operatorname{area}(\tilde{G}) = \pi(\Gamma^2 - (\Gamma-\gamma)^2/2)$. Clearly, to each interval I_{i_ℓ} of order n_0 there corresponds an inverse image of order n_0 of $\tilde{G}$ which is contained in $B(I_{i_\ell})$ and whose area is at least $\pi(\Gamma^2 - (\Gamma-\gamma)^2/2)/(2\Gamma)^{2n_0}$. Thus, it follows that

$$\lambda \geq \pi(\Gamma^2 - (\Gamma-\gamma)^2/2)/(2\Gamma)^{2n_0}.$$

Since $\operatorname{area}(U_c(J(T))) \leq 4\gamma(\Gamma-\gamma) + 2\pi\gamma^2$, we obtain for the Hausdorff dimension of $J(T)$ the bound

$$h < 2 - \frac{2\pi(\Gamma^2 - (\Gamma-\gamma)^2/2)/(2\Gamma)^{2n_0+10}}{4((\Gamma-\gamma)+2\pi\gamma)\gamma}.$$

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